*Actes, Congrès intern. Math.,* 1970. Tome 2, p. 307 à 312.

# SEMISIMPLE GROUP SCHEMES OVER CURVES AND AUTOMORPHIC FUNCTIONS

by G. HARDER

Let $k$ denote a field and let $K/k$ denote a function field of one variable over $k$. We assume $K/k$ is a regular extension, i.e. $K \otimes_k \bar{k}$ is a field ($\bar{k}$ = algebraic closure of $k$). Let $Y/k$ denote a projective, smooth model of $K/k$.

I want to study semisimple affine groupschemes $G/Y$ ; a satisfactory theory of such groupschemes over $Y$ has implications for the arithmetic of semisimple algebraic groups which are defined over the function field $K/k$. A semisimple groupscheme $G/Y$ is called rationally trivial if its generic fiber $G \underset{Y}{\times} K = G_K$ is a Chevalley group ; then $G/Y$ is locally split for the Zariski topology on $Y$. By $X/Y$ I denote the scheme of Borel subgroups of $G/Y$, this is a smooth projective scheme over $Y$ (Compare [2], Exp. XXII). From the projectivity of this scheme follows that a Borel subgroup $B_K \subset G_K$ can be extended in a unique way to a Borel subgroup of $G/Y$ :

$$\Gamma(X/Y) = \mathrm{Hom}_Y(Y, X) = \Gamma(X_K/\mathrm{Spec}(K)) .$$

If $B \subset G$ is a Borel subgroup of $G/Y$ we denote its unipotent radical by $B_u$. The quotient $B/B_u = T$ is a split torus.

Let $\Delta$(resp. $\Delta^+$) be the set of roots (res. positive roots) in the charactermodule $X(T) = \mathrm{Hom}(T, G_m)$. By $\pi = \{\alpha_1 \ldots \alpha_r\}$ I denote the set of simple roots in $\Delta^+$. There is a natural filtration of the unipotent radical

$$B_u = U_0 \supset U_1 \ldots U_{\nu+1} \ldots \supset U_n = \{e\}$$

by smooth subschemes which are normal in $B$ such that the quotients $U_\nu/U_{\nu+1}$ are line bundles, i.e. they are locally isomorphic to $G_a/Y$. The action of $T$ on $U_\nu/U_{\nu+1}$ is given by multiplication with a root $\alpha \in \Delta^+$. This yields a one-to-one correspondence between the roots $\alpha \in \Delta^+$ and the quotients $U_\nu/U_{\nu+1}$. If $\alpha$ corresponds to $U_\nu/U_{\nu+1}$ we put $W_\alpha = U_\nu/U_{\nu+1}$, and call $W_\alpha$ the line bundle associated to the root $\alpha \in \Delta^+$. If $W_{\alpha_1} \ldots W_{\alpha_r}$ are the line bundles associated to the simple roots $\{\alpha_1 \ldots \alpha_r\} = \pi$ we put

$$n_i(B) = \mathrm{degree}(W_{\alpha_i}) = c(W_{\alpha_i})$$

Thus we assigned to any Borel subgroup $B$ of $G/Y$ a vector

$$n(B) = (n_1(B) \ldots n_r(B)) \in \mathbf{Z}^r .$$

This makes sense because we can canonically identify the set of simple roots of two different Borel subgroups. If $\alpha_{i_0} \in \pi$ is a simple root and $B \subset G/Y$ a Borel subgroup then $P^{(i_0)} \supset B$ is the maximal parabolic subgroup of type $\pi - \{\alpha_{i_0}\}$ containing $B$ ([1], § 4). The root system of the semisimple part of $P^{(i_0)}$. is of type $\pi - \{\alpha_{i_0}\}$. The unipotent radical $R_u(P^{(i_0)})$ is contained in $B_u$. The intersection of the filtration above with $R_u(P^{(i_0)})$ yields a filtration of $R_u(P^{(i_0)})$

$$R_u(P^{(i_0)}) \supset U'_1 \supset U'_2 \supset \ldots \supset U'_{d_{i_0}} = \{e\} .$$

The quotients are line bundles which correspond to the roots in

$$\Delta^+_{i_0} = \left\{ \alpha \in \Delta^+ \mid \alpha = \sum_{i=1}^{r} m_i \alpha_i \ ; \ m_{i_0} > 0 \right\}$$

Now we assign a second vector $\mathbf{p}(B) = (p_1(B) \ldots p_r(B))$ to our Borel subgroup $B \subset G/Y$ by putting

$$p_i(B) = \sum_{\alpha \in \Delta^+_i} c(W_\alpha)$$

The elements $\widetilde{\chi}_i = \sum_{\alpha \in \Delta^+_i} \alpha$ form a basis of $X(T) \otimes \mathbf{Q}$, in fact the $\widetilde{\chi}_i$ are multiples of the fundamental weights $\chi_i$, so we get $\widetilde{\chi}_i = f_i \chi_i$ where the $f_i$ are positive integers. We express the characters $\widetilde{\chi}_i$ in terms of the simple roots :

$$\widetilde{\chi}_i = \Sigma a_{ij} \alpha_j \qquad a_{ij} \in \mathbf{N}$$

and vice versa

$$\alpha_i = \Sigma b_{ij} \widetilde{\chi}_j \qquad b_{ij} \in \mathbf{Q} .$$

From this we get the following relations for the vectors $\mathbf{n}(B)$ and $\mathbf{p}(B)$ :

$$p_i(B) = \sum_{j=1}^{r} a_{ij} n_j(B)$$

$$(*)$$

$$n_i(B) = \sum_{j=1} b_{ij} p_j(B)$$

It is an easy but important observation that for a given group scheme $G/Y$ the numbers $p_i(B)$ are bounded from above as $B$ is running over the set of Borel subgroups of $G/Y$. We call a vector $\mathbf{p}(B) = (p_1(B) \ldots p_r(B))$ maximal for a given $G/Y$, if there is no Borel subgroup $B'$ of $G/Y$ such that $p_i(B) \leqslant p_i(B')$ for all $\alpha_i \in \pi$ and $p_{i_0}(B) < p_{i_0}(B')$ for some $\alpha_{i_0} \in \pi$.

Let $g$ denote the genus of $Y/k$, let $h > 0$ denote the g.c.d. of all degrees of positive divisors on $Y$. Then we have ([5], Satz 2.2.6 und Kor. 2.2.14).

THEOREM 1. — *If $G/Y$ is a semisimple rationally trivial groupscheme and if $B \subset G/Y$ is a Borel subgroup such that* $\mathbf{p}(B)$ *is maximal, then*

$$n_i(B) \geqslant -2g - 2(h-1) \quad \textit{for all} \quad \alpha_i \in \pi$$

We call a Borel subgroup $B \subset G$ *reduced* if $n_i(B) \geqslant -2g - 2(h-1)$ for all $\alpha_i$.

THEOREM 2. — *There exists a constant $M$ which only depends on $g$, $h$ and on the Dynkin diagram of $G/Y$ such that the following statement holds : If $B \subset G$*

*is reduced and if for $\alpha_{i_0}$ we have $n_{i_0}(B) > M$ then the maximal parabolic subgroup $P^{(i_0)} \supset B$ of type $\pi - \{\alpha_{i_0}\}$ contains all reduced Borel subgroups of $G/Y$.*

To any vector $\mathbf{n} = (n_1 \dots n_r)$ we may associate a quasiprojective scheme

$$\Gamma_{\mathbf{n}}(X/Y) \to \mathrm{Spec}(k)$$

the points of which are the Borel subgroups of $G$ satisfying $\mathbf{n}(B) = \mathbf{n}$. To be more precise we put for any scheme $S \to \mathrm{Spec}(k)$

$$\Gamma_{\mathbf{n}}(X/Y)(S) = \{B \subset G \underset{Y}{\times} (Y \underset{k}{\times} S) \mid n_i(B \underset{Y}{\times} k(s)) = n_i \quad \text{for any point} \quad s \in S\} .$$

The functor $S \to \Gamma_{\mathbf{n}}(X/Y)(S)$ is representable by a quasiprojective scheme over $k$ (Comp. [3]). This functor can be defined for all groupschemes of inner type. Analagously we define for any vector $\mathbf{p} = (p_1 \dots p_r)$ the scheme $\Gamma^{\mathbf{p}}(X/Y)/k$ of Borel subgroups $B$ satisfying $p_i(B) = p_i$. Of course we have $\Gamma_{\mathbf{n}}(X/Y) = \Gamma^{\mathbf{p}}(X/Y)$ if the relation (*) holds between n and p. If $\mathbf{n} = (n_1 \dots n_r)$ is a vector whose components satisfy $n_i \leqslant -2g + 1$ and if $\Gamma_{\mathbf{n}}(X/Y)$ is not empty then the scheme $\Gamma_{\mathbf{n}}(X/Y)$ is smooth over $k$. Moreover we can calculate the dimension of this scheme. For this purpose we consider the corresponding vector $\mathbf{p} \leftrightarrow \mathbf{n}$. Then the dimension of $\Gamma_{\mathbf{n}}(X/Y) = \Gamma^{\mathbf{p}}(X/Y)$ is given by

$$\dim \Gamma^{\mathbf{p}}(X/Y) = -2 \sum_{i=1}^{r} \frac{p_i}{f_i} + (1-g).\#\Delta^+$$

The following theorem 3 seems to be deeper than the preceding ones. It is only formulated in the case of a finite ground field $k = \mathbf{F}_q$, but I believe it can derived from this special case by general theorems in algebraic geometry.

THEOREM 3. — *Let $k = \mathbf{F}_q$ be a finite field and $Y/k$ a smooth projective curve. Let $G/Y$ be a semisimple group scheme of inner type. If the components of the vector $\mathbf{p} = (p_1 \dots p_r)$ are sufficiently small and if $\Gamma^{\mathbf{p}}(X/Y)$ is not empty then we have*

$$\dim \Gamma^{\mathbf{p}}(X/Y) = -\sum_{i=1}^{r} \frac{2p_i}{f_i} + (1-g)\#\Delta^+$$

*and there is exactly one irreducible component of this dimension.*

I want to give an idea of the proof. Before doing this I introduce some notation. I put

$$l_i(B) = \frac{p_i(B)}{f_i}$$

These numbers are not necessarily integers. This is due to the fact that in general the roots do not generate the lattice spanned by the fundamental characters. But under a certain assumption on the isomorphism type of $G/Y$ they are integers, and I will explain the idea only in this special case. For any vector $\mathbf{l} = (l_1 \dots l_r)$ I put

$$n(G, l_1 \dots l_r) = \#\Gamma^{\mathbf{p}}(X/Y)(\mathbf{F}_q)$$

where $p_i = -f_i l_i$. Then I consider the Laurent series

$$E(G,t)=\sum_{\mathbf{l}} n(G,l_1\dots l_r)\,.\,q^{-\Sigma l_i}\cdot t_1^{l_1}\dots t_r^{l_r}$$

It will be shown in [6] that $E(G,t)$ is a rational function, and can be written in the following form

$$E(G,t)=\frac{P(G,t)}{Q(t)\cdot\prod_{\nu=1}^{r}(1-qt_\nu)}$$

Here $P(G,t)$ is a polynomial in the variables $t_i$, $t_i^{-1}$ and $Q(t)$ is a polynomial in the variables $t_i$ depending only on the Dynkin diagram of $G/Y$. Moreover the polynomial $Q(t)$ has no zeroes in the disc $D\left(\frac{1}{\sqrt{q}}\right)=\left\{(t_1\dots t_r)\mid |t_i|<\frac{1}{\sqrt{q}}\right\}$. We also know the residue of $E(G,t)$ at the point $(q^{-1}\dots q^{-1})$. Here the residue is defined by

$$\operatorname*{Res}_{(t_1\dots t_r)=(q^{-1}\dots q^{-1})}E(G,t)=\prod_{\nu=1}^{r}(1-qt_\nu)\,E(G,t)|_{(t_1\dots t_r)=(q^{-1}\dots q^{-1})}$$

It can be expressed in terms of values the $\zeta$-function of our field $K/F_q$ and we obtain

$$\operatorname*{Res}_{(t_1\dots t_r)=(q^{-1}\dots q^{-1})}E(G,t)=q^{-(g-1)}\ \#\ \Delta^+ + O(q^{-(g-1)\#\Delta^+-1/2})$$

(Here the Riemannian hypothesis comes in!). This yields an estimate

$$|\#\,\Gamma^{\underline{p}}(X/Y)\,(\mathbf{F}_q)-q^{2\sum_{i=1}^{r}l_i+(1-g).\#\Delta^+}|\leqslant C\,q^{2\sum_{i=1}^{r}l_i+(1-g).\#\Delta^+-1/2}$$

if the vector $\mathbf{l}=(l_1\dots l_r)$ has sufficiently large components, say $l_i>n_0(G)=n_0$. It can be shown that this estimate also holds with the same constant $C$ and under the same conditions $l_i>n_0$ on $\mathbf{l}$ if we extend our ground field $\mathbf{F_q}$ to $\mathbf{F}_{\mathbf{q}^n}$. Of course we have to substitute $q^n$ for $q$. Then our theorem 3 is a consequence of a theorem of Lang and Weil [10].

Of course the properties of $E(G,t)$ I need are not at all abvious. This function depends on the isomorphism type $[G]$ of $G/Y$ and for the investigation of $E(G,t_1\dots t_r)$ one has to consider this function as a function of $[G]$. Let $G_0/\mathbf{F}_q$ be a Chevalley group of the same type as $G/Y$, let $T_0$(resp. $B_0\supset T_0$) be a maximal torus (resp. a Borel subgroup containing $T_0$). Then $G_1=G_0\underset{\mathbf{F}}{\times}Y$ is a Chevalley scheme over $Y$. Let $G_0(A)$ be the adele group of $G_{1,K}$ and $\mathcal{R}=\prod_{\mathfrak{y}}G_0(\sigma_{\mathfrak{y}})$ the canonical maximal compact subgroup. Then we may identify

$$H^1(Y_{\text{zar}}\,,\,G_1)\xrightarrow{\sim}\mathcal{R}\backslash G_0(A)/G_0(K)$$

From the Iwasawa decomposition we get $G_0(A)=\mathcal{R}.B_0(A)$ so for $\mathbf{x}\in G_0(A)$ we can write $\mathbf{x}=\mathbf{k_x}\,.\,\mathbf{b_x}$ (not unique).

If $(s_1,\dots,s_r)=s$ is a vector whose components are complex numbers we put $\eta_s(\mathbf{x})=\prod_{i=1}|\chi_i(\mathbf{b_x})|^{-1-s_i}$. Then the following series

$$E(\mathrm{x}\,,s) = \sum_{\gamma \in G_0(K)/B_0(K)} \eta_s(\mathrm{x}\gamma)$$

converges for $\mathrm{Re}(s_i) > 1$. It is analogous to the Eisenstein series considered by Langlands in the number field case (Compare [7], [8]), and I will show in [6] that Langland's theory can be carried over to the function field case.

To $\mathrm{x} \in G_0(A)$ corresponds a cohomology class in $H^1(Y\,, G_1)$ and this cohomology class defines an isomorphism class of twisted semisimple group schemes over $Y$ and I assume that my given group scheme $G/Y$ is in this class. Then it is clear that

$$E(\mathrm{x}\,, s) = E(G\,, q^{-s_1} \ldots q^{-s_r})$$

and all desired properties of $E(G\,, t_1 \ldots t_r)$ can be derived from the theory of Eisenstein series. For the estimations of the Eisenstein series which are needed in the proof the theorem 3 the theorems 1 and 2 are important. The theorem 3 has nice consequences :

THEOREM 4. – *Let $G/K$ be a simply connected semisimple algebraic group over the function field $K/\mathrm{F}_q$. Then $H^1(K\,, G) = 0$.*

This theorem follows from theorem 3 in the case where $G/K$ is a Chevalley group with out any case by case discussion (Comp. [6]). For the general case one has to use the methods in [4].

Finally I want to mention that the calculation of the residue of $E(x\,, s)$ at $(1\,, \ldots 1)$ yields.

THEOREM 5. – *The Tamagawa number of a semisimple simply connected Chevalley group $G/K$ is one.*

This is proved by the same method as Langland's in the numberfield case (Comp. [9].

REFERENCES

[1] BOREL A. et TITS J. — Groupes réductifs, *Publ. Math. I.H.E.S.*, 27, 1965, p. 55-151.
[2] DEMAZURE M. et GROTHENDIECK A. — Schémas en groupes, *Séminaire I.H.E.S.*, Bures-sur-Yvette, 1963-64.
[3] GROTHENDIECK A. — Les schémas de Hilbert, *Séminaire Bourbaki*, Exp. 221, Paris, 1960-61.
[4] HARDER G. — Uber die Galoiskohomologie halbeinfacher Matrizengruppen, *Math. Zeitschrift*, 90, 1968, p. 404-428.
[5] HARDER G. — Halbeinfache Gruppenschemata über vollständigen Kurven, *Inv. math.*, 6, 1968, p. 107-149.
[6] HARDER G. — *Chevalleygruppen über Funktionenkörpern und automorphe Funktionen* (in Preparation).
[7] HARISH-CHANDRA. — Automorphic forms on semisimple Lie groups, *Springer Lecture notes*, 62, 1968.

[8] Langlands R. — *On the functional equation satisfied by Eisenstein series,* (Preprint).
[9] Langlands R. — The volume of the Fundamental Domain for Some Arithmetical Subgroups of Chevalley groups, *Proceedings of Symposia in Pure Mathematics,* AMS, Providence, Vol. IX, 1966.
[10] Lang S., Weil A. — Number of points of varieties over finite fields, *Am. Journ. of Math.,* 76, 1954, p. 819-827.

Mathematisches Institut der Universitat
Wegelerstrasse 10
53 - Bonn
République Fédérale Allemande