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REPRESENTATIONS OF BINARY FORMS BY QUINARY QUADRATIC FORMS

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Abstract. In this article, we survey our recent results about the representations of (positive definite integral) binary forms by quinary quadratic forms. In particular, we will give various examples of quinary forms that are 2-universal, even 2-universal, almost 2-universal and its candidates.

1. Introduction

The famous *four square theorem* of Lagrange[L] says that the quadratic form $x^2+y^2+z^2+u^2$ represents all positive integers. In the early 20-th century, Ramanujan [R] extended Lagrange's result by listing all 54 positive definite integral quaternary diagonal forms, up to equivalence, that represent all positive integers. Dickson [D] called such forms *universal* and confirmed Ramanujan's list. Willerding [W] found 124 non-diagonal quaternary universal forms. Recently, Conway and Schneeberger [CSc] found all quaternary universal forms. They also announced the so called 15-theorem, which implies that every quadratic form that represents $1, 2, 3, 5, 6, 7, 10, 14$ and 15 can represent all positive integers(see also [Du]).

In 1926, Kloosterman [Kl] determined all positive definite diagonal quaternary quadratic forms that represent all sufficiently large integers, which we call *almost universal forms,* remaining only four as candidates. Pall [P] showed that the remaining candidates are almost universal, so there are exactly 199 almost universal quaternary diagonal quadratic forms that are anisotropic over some p-adic integers. Furthermore Pall and Ross [PR] proved that there exist only finitely many almost universal quaternary quadratic forms that are anisotropic over some p-adic integers. In fact, every positive definite quaternary quadratic form L such that $L_p := L \otimes \mathbb{Z}_p$ represents all p-adic integers and is isotropic over $\mathbb{Z}_p$ for all primes p is almost universal by Theorem 2.1 of [HJ]. Therefore there are infinitely many almost universal quaternary quadratic forms.

In his book [K], Kitaoka conjectured that both $\mathbb{Z}$-lattices in the following genus $\{A_4 \perp \langle 4\rangle, D_4 20[2\frac{1}{2}]\}$ represent all except only finitely many binary $\mathbb{Z}$-lattices. The discriminant of this genus, which is 20, is the smallest among the genera of quinary positive even $\mathbb{Z}$-lattices with class number bigger than 1.

In this article, we will consider the representation problems of **binary** $\mathbb{Z}$-lattices by quinary $\mathbb{Z}$-lattices. In particular, we will extend the above results and give an answer to Kitaoka's above conjecture.

We shall adopt lattice theoretic language. A $\mathbb{Z}$*-lattice* L is a finitely generated free $\mathbb{Z}$-module in $\mathbb{R}^n$ equipped with a non-degenerate symmetric bilinear form B,

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such that $B(L,L) \subseteq \mathbb{Z}$. The corresponding quadratic map is denoted by Q. Let L be a $\mathbb{Z}$-lattice. L is called *even* if $Q(L) \subseteq 2\mathbb{Z}$. We define $L_p := L \otimes \mathbb{Z}_p$ the localization of L at prime p. If L admits an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, we call L *diagonal* and simply write

$$L \simeq \langle Q(\mathbf{e}_1), \ldots, Q(\mathbf{e}_n)\rangle.$$

We always assume the following unless stated otherwise:

Every $\mathbb{Z}$-lattice is positive definite.

The set of all $\mathbb{Z}$-lattices K such that $L_p \cong K_p$ for all primes p (including ∞) is called the *genus* of L, denoted by $\mathrm{gen}(L)$. The number of classes in a genus is called the *class number* of the genus (or of any $\mathbb{Z}$-lattice in the genus), which is always finite. For any $\mathbb{Z}$-lattice L, it is well known that every $\mathbb{Z}$-lattice which is locally represented by L is represented by some $\mathbb{Z}$-lattices in the genus of L. Therefore if the class number of L is 1, then the global representation can be reduced to the local representation, which is completely known (see [O'M2]).

Let $\mathfrak{P}_k$ be the set of all $\mathbb{Z}$-lattices of rank k. For a $\mathbb{Z}$-lattice L, we define

$$\begin{aligned} Repn(k; gen(L)) &:= \{\ell \mid \ell_p \to L_p \text{ for all } p, \ \ \text{rank } (\ell) = k\}, \\ Repn(k; L) &:= \{\ell \mid \ell \to L, \ \ \text{rank } (\ell) = k\}. \end{aligned}$$

If $Repn(k; gen(L)) = \mathfrak{P}_k$, L is called *locally k-universal* and if $Repn(k; L) = \mathfrak{P}_k$, L is called *k-universal.* If L is locally k-universal and $|Repn(k; gen(L)) - Repn(k; L)|$ is finite up to isometry, L is called almost k-universal. The definitions of *locally even k-universal* $\mathbb{Z}$-lattice and *even k-universal* $\mathbb{Z}$-lattice are similar to the above ones. We set

$$[a, b, c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

for convenience. For unexplained terminologies, notations, and basic facts about $\mathbb{Z}$-lattices, we refer the readers to O'Meara [O'M1] and Conway-Sloane [CS1,2].

2. (Even) 2-universal quinary $\mathbb{Z}$-lattices

In 1930, Mordell [M] proved that $I_5 := x^2 + y^2 + z^2 + u^2 + v^2$ can represent all binary $\mathbb{Z}$-lattices, that is, I_5 is 2-universal. If a $\mathbb{Z}$-lattice L is locally 2-universal and has class number 1, then L is 2-universal as mentioned above. Note that every quaternary $\mathbb{Z}$-lattice cannot be 2-universal by a local property. The complete list of 2-universal quinary $\mathbb{Z}$-lattices are the following:

Theorem 2.1 [KKR],[KKO]. *The number of 2-universal quinary $\mathbb{Z}$-lattices is 11. They are:*

$$I_5, \ \ I_4 \perp A_1, \ \ I_4 \perp \langle 3\rangle, \ \ I_3 \perp A_1 \perp A_1, \ \ I_3 \perp A_1 \perp \langle 3\rangle, \ \ I_3 \perp A_2,$$
$$I_3 \perp A_1 10[1\tfrac{1}{2}], \ \ I_2 \perp A_2 \perp A_1^{\dagger}, \ \ I_2 \perp A_2 \perp \langle 3\rangle, \ \ I_2 \perp A_3, \ \ I_2 \perp A_2 21[1\tfrac{1}{3}].$$

In fact, all $\mathbb{Z}$-lattices except $K := I_2 \perp A_2 \perp A_1$ have class number 1, so the proof of 2-universality is very easy. But the proof of 2-universality of K is a little difficult (see [KKO]). As an analogue of Conway and Schneeberger's 15-theorem, the following theorem can be proved:

Theorem 2.2 [KKO]. *A $\mathbb{Z}$-lattice is 2-universal if and only if it represents the following 6 binary $\mathbb{Z}$-lattices:*

$$\langle 1,1\rangle,\ \langle 2,3\rangle,\ \langle 3,3\rangle,\ A_2,\ A_1 10[1\frac{1}{2}],\ A_1 14[1\frac{1}{2}].$$

This is slightly different from 15-theorem in the following sense. The $\mathbb{Z}$-lattice $\langle 1,2,5,5,15\rangle$ is 1-universal but it doesn't contain any quaternary 1-universal sublattice. But every 2-universal $\mathbb{Z}$-lattice must contain one of 2-universal quinary $\mathbb{Z}$-lattices listed in Theorem 2.1.

In the remaining of this section, we consider only even $\mathbb{Z}$-lattice. To find all even 2-universal quinary $\mathbb{Z}$-lattices, the following *escalation method* is very useful: Using the binary $\mathbb{Z}$-lattices which have small successive minima, for example $A_2, A_1 \perp A_1$, determine the upper bounds of the successive minima of quinary $\mathbb{Z}$-lattices to show the finiteness of even 2-universal quinary $\mathbb{Z}$-lattices. For each $\mathbb{Z}$-lattice, check the locally even 2-universality of it by [O'M2] and exhibit all lattices satisfying the above conditions up to isometry. Now among the remaining lattices, find all $\mathbb{Z}$-lattices which have class number 1 by using the various tables such as [CS2],[N]. Clearly these $\mathbb{Z}$-lattices are even 2-universal. Lastly, for the other $\mathbb{Z}$-lattices, either find an exceptional binary $\mathbb{Z}$-lattice or prove the universality by using various techniques.

Theorem 2.3 [KO2]. *All even 2-universal $\mathbb{Z}$-lattices of rank 5 and the candidates are the followings:*

(i) Even 2-universal quinary $\mathbb{Z}$-lattices

$$A_5,\ D_5,\ A_1\perp D_4,\ A_1\perp A_4,\ A_2\perp A_3,\ A_2\perp A_2\perp A_1,\ A_2\perp A_1\perp A_1\perp A_1^{\dagger},$$
$$A_3\perp A_1\perp A_1,\ A_4 70[2\frac{1}{5}],\ A_2A_2 24[11\frac{1}{3}],\ \ D_4 12[2\frac{1}{2}],\ A_1\perp A_3\perp <4>,$$
$$A_2\perp A_2 30[1\frac{1}{3}]^{\dagger},\ A_1\perp A_3 12[2\frac{1}{2}],\ A_1A_3 44[11\frac{1}{4}],\ A_1A_3 10[12\frac{1}{2}],\ A_1\perp A_3 20[2\frac{1}{2}]^{\dagger},$$
$$A_3\perp A_1 14[1\frac{1}{2}]^{\dagger},\ A_1A_1A_2 84[111\frac{1}{6}],\ A_2\perp A_1A_1 12[11\frac{1}{2}],\ A_1\perp A_3\perp <6>^{\dagger},$$
$$A_2\perp A_1\perp A_1\perp <4>^{\dagger},\ A_1\perp A_3 52[1\frac{1}{4}]^{\dagger},\ [1,1,-1],\ [2,2,0],$$
$$[01,11,1],\ [01,11,-1],\ [01,01;-1]^{\dagger}.$$

(ii) Candidates

$$A_3\perp A_1 22[1\frac{1}{2}],\ A_1A_3 76[11\frac{1}{4}],\ A_1\perp A_1\perp A_2 30[1\frac{1}{3}],\ A_1\perp A_1A_2 102[11\frac{1}{6}],$$
$$A_1\perp A_1A_2 174[11\frac{1}{6}],\ A_2\perp A_1A_1 20[11\frac{1}{2}],\ A_1\perp A_2\perp A_1 14[1\frac{1}{2}],$$
$$A_1A_1A_2 156[111\frac{1}{6}],\ [0,1,2],\ [0,2,1],\ [1,2,0],\ [00,01,2],\ [00,10,2],$$
$$[00,11,0],\ [00,11,1],\ [00,11,2],\ [01,10,2],\ [10,11,0],\ [11,11,1],\ [01,0],$$
$$[01,-1],\ [00,0],\ [00,1],\ [00,2],\ [10,0],\ [11,1].$$

The notations of each $\mathbb{Z}$-lattice are given by [CS1,2] and [O1]. In (i), the class number of the $\mathbb{Z}$-lattice with †-mark is bigger than 1 and all candidates are locally even 2-universal $\mathbb{Z}$-lattice with class number bigger than 1.

3. Almost 2-universal quinary $\mathbb{Z}$-lattices

Let L be an almost 2-universal quinary $\mathbb{Z}$-lattice. Clearly L is locally 2-universal and represents primitively all positive integers. The following lemma is very useful to show whether a quinary $\mathbb{Z}$-lattice is almost 2-universal or not.

Lemma 3.1 [O3]. *Let L be any locally 2-universal $\mathbb{Z}$-lattice of rank 5 and for all prime p, let $d(L_p) = p^{u_p}\alpha_p$, where α_p is a unit in $\mathbb{Z}_p$ and u_p is a non-negative integer. Then there exists a prime p dividing $2dL$ such that L cannot primitively represent binary $\mathbb{Z}$-lattices ℓ of the form*

$$\ell_p \simeq \langle p^{\epsilon_p}\alpha_p, p^k\beta_p\rangle,$$

where ϵ_p is 0 or 1, respectively the parity of u_p, β_p is any unit in $\mathbb{Z}_p$ and $k \geq 2$ if p is odd and $k \geq 7$ otherwise.

Therefore L represents all binary $\mathbb{Z}$-lattices ℓ satisfying the above property. From these conditions, one can prove the following theorem, which is quite different from the rank 1 case.

Theorem 3.2 [O3]. *The number of almost 2-universal quinary $\mathbb{Z}$-lattices is finite.*

Now we consider diagonal quinary $\mathbb{Z}$-lattices. As a natural generalization of Halmos' result [H], Hwang [Hw] proved that there are exactly 3 quinary diagonal $\mathbb{Z}$-lattices that represent all binary $\mathbb{Z}$-lattices except only one.

Theorem 3.3 [O3]. *All of diagonal almost 2-universal quinary $\mathbb{Z}$-lattices and its exceptions are the followings:*

(i) 2-universal $\mathbb{Z}$-lattices

$$\langle 1,1,1,1,a\rangle \quad a=1,2,3, \qquad \langle 1,1,1,2,b\rangle \quad b=2,3.$$

(ii) Almost 2-universal quinary $\mathbb{Z}$-lattices and its exceptions

$$\begin{array}{llll} \langle 1,1,1,2,4\rangle : [3,0,3] & \langle 1,1,1,1,5\rangle : [2,1,4],\ [4,1,4],\ [8,1,8], \\ \langle 1,1,1,2,5\rangle : [3,0,3], & \langle 1,1,1,2,7\rangle : [3,0,3],\ [6,0,6], \\ \langle 1,1,2,2,3\rangle : [2,1,2], & \langle 1,1,2,2,5\rangle : [2,1,2],\ [2,1,4],\ [4,1,4],\ [8,1,8]. \end{array}$$

(iii) Candidates

$$\langle 1,1,1,3,7\rangle, \quad \langle 1,1,2,3,5\rangle, \quad \langle 1,1,2,3,8\rangle.$$

As an answer of the Kitaoka's question, we proved the following:

Theorem 3.4 [KKO2]. *For two $\mathbb{Z}$-lattices in the following $gen(A_4 \perp \langle 4\rangle) = \{A_4 \perp \langle 4\rangle, D_4 20[2\frac{1}{2}]\}$, the former represents all binary even $\mathbb{Z}$-lattices except $[4,2,4]$ and the latter represents all except $[2,1,4],[4,1,4]$, and $[8,1,8]$.*

In [KKO2], there are various examples of representations of binary $\mathbb{Z}$-lattices by some particular quinary $\mathbb{Z}$-lattices with class number 2. To prove the (almost)

2-universality of each $\mathbb{Z}$-lattice, some particular methods are needed for each $\mathbb{Z}$-lattice. As an example, we show that $L = A_2 \perp A_2 30[1\frac{1}{3}]$ is even 2-universal. Let ℓ be any binary even $\mathbb{Z}$-lattice. Since $A_4^{\perp}$ in E_7 is $A_2 30[1\frac{1}{3}]$,

$$\ell \to L \text{ if and only if } A_4 \perp \ell \to E_7 \perp A_2.$$

Note that

$$gen(E_7 \perp A_2) = \{E_7 \perp A_2, E_8 \perp \langle 6 \rangle\}.$$

We can easily show that if $A_4 \perp \ell \to E_8 \perp \langle 6 \rangle$, then $A_4 \perp \ell \to E_7 \perp A_1 \perp \langle 6 \rangle \to E_7 \perp A_2$ by considering the local property (see also [O1]). Therefore L is even 2-universal.

As a refinement of Hsia, Kitaoka and Kneser's result [HKK], Jöchner [J] proved that every $\mathbb{Z}$-lattice L of rank 6 can represent all binary $\mathbb{Z}$-lattices that are locally represented by L and whose minimum is sufficiently large. We give some quinary $\mathbb{Z}$-lattices satisfying these properties. Let L be a quaternary $\mathbb{Z}$-lattice with class number 1. For a positive integer k, we define

$$L(\delta k) := L \perp \langle \delta k \rangle,$$

where $\delta = 1$ if L is odd, and $\delta = 2$ if L is even.

Theorem 3.5 [KO1]. *Assume that $d := dL(\delta k)$ is a (odd, if $\delta = 1$) squarefree integer. Let ℓ be a binary $\mathbb{Z}$-lattice such that*

$$\ell_p \to L(\delta k)_p$$

primitively at all p. Then for any $\epsilon > 0$, there exists a constant $C > 0$ depending only on ϵ such that

$$\textit{if } \min(\ell) > C \cdot d^{5+\epsilon}, \textit{ then } \ell \to L(\delta k).$$

Furthermore, the primitive condition cannot be omitted.

Remark. For $n \geq 3$, see [O2] for n-universal $\mathbb{Z}$-lattices with minimum rank. See also [CKR] for totally positive ternary 1-universal $\mathfrak{O}_K$-lattices, where $\mathfrak{O}_K$ is the ring of integers of real quadratic field K.

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