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The Classification of the Finite Simple Groups

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Abstract. This is the first monograph in a series devoted to a revised proof of the classification of the finite simple groups.

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To the memory of

Pearl Solomon (1918–1978)

and

Marvin Lyons (1903–1975)

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Preface

Though elated at the successful completion of the classification of finite simple groups in the early 1980's, Danny Gorenstein nevertheless immediately appreciated the urgency of a "revision" project, to which he turned without delay. Before long we had joined him in this effort, which is now into its second decade. Danny always kept the project driving forward with relentless energy, contagious optimism and his unique global vision of finite simple group theory. He inspired other collaborators—Richard Foote and Gernot Stroth—to contribute theorems designed to fit our revised strategy. Their work forms a vital part of this project. The conception of these volumes is unmistakably Danny's, and those who know his mathematics and his persuasive way of explaining it should recognize them everywhere. Since his death in 1992, we have tried to maintain his standards and we hope that whatever changes and additions we have made keep the vigorous spirit which was his trademark. Of course the responsibility for any stumbling or errors must remain with us. To accompany him on this mathematical journey was a privilege and as we continue, our debt to our teacher, colleague and loyal friend is hard to measure. Thanks, Danny.

This monograph is Number 1 of a projected dozen or so volumes, and contains two of the roughly thirty chapters which will comprise the entire project. Not all of the chapters are completely written at this juncture, but we anticipate that the publication process, now begun, will continue at a steady pace. In the first section of Chapter 1, we discuss the current status of the work in some more detail; later in that chapter we delineate the background results which form the foundation for all the mathematics in the subsequent volumes.

When Danny began, there was already a well-established tradition of "revisionism" in finite group theory. Indeed, beginning in the late 1960's, Helmut Bender produced a series of "revisions" whose beauty and depth profoundly influenced finite group theory. During the final decade of the classification proof, several more group-theorists showed how to develop deep mathematics while re-addressing some of the fundamental theorems in the theory of simple groups. While it is sometimes difficult to draw a distinct line between revisionist and other mathematics, the clear successes of Bender, Michel Enguehard, George Glauberman, Koichiro Harada, Thomas Peterfalvi, and Bernd Stellmacher in various revisionist projects have inspired us.

We are extremely grateful to Michael Aschbacher, Walter Feit, George Glauberman, Gary Seitz, Stephen Smith, and John Thompson for their enthusiastic support and interest in our endeavor, and specifically for having read some of the manuscripts and made valuable comments. To our collaborators Richard Foote and Gernot Stroth, our special thanks. The ideas and comments of many colleagues have been most useful and will appear in some form in the volumes to come. Indeed, this work has ingredients of several types: there is new mathematics; there is exposition of unpublished work of our colleagues, ranging from short arguments to entire case analyses which they have generously shared with us, and which we shall acknowledge specifically as we go along; and finally there are reworkings of published papers. For now, in addition to all the people already mentioned, we must also thank Jonathan Alperin, Michel Broué, Andrew Chermak, Michael Collins, K. M. Das, Alberto Delgado, Paul Fong, Robert Gilman, David Goldschmidt, Kensaku Gomi, Robert Griess, Robert Guralnick, Jonathan Hall, William Kantor, Martin Liebeck, Ulrich Meierfrankenfeld, Michael O'Nan, Lluis Puig, Geoffrey Robinson, Jan Saxl, Ernie Shult, Franz Timmesfeld, Jacques Tits, John Walter, Richard Weiss, and Sia K. Wong.

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Finally, to our wives Lisa and Myriam, to our families, and to Helen Gorenstein, we gratefully acknowledge the love and commitment by which you have endured the stresses of this long project with good humor, grace and understanding.

Richard Lyons and Ronald Solomon
June, 1994

Preface to the Second Printing

Six years have passed since the first printing of this book and sixteen since the detailed classification strategy presented in Chapter 2 was first adumbrated. At present the first four books in our series have been published and the fifth and sixth are well on their way to completion. Since the first printing, some important second-generation pieces of the classification of the finite simple groups have also appeared, in particular two volumes [**BG1,P4**] presenting the revised proofs by Bender, Glauberman and Peterfalvi of the Odd Order Theorem of Feit and Thompson. We have taken the Feit-Thompson Theorem as a Background Result, but now could therefore remove [**FT1**] from the list of Background Results (see pp. 48–49) and replace it with Peterfalvi's revision of Chapter V of that paper [**P4**]. In addition, Peterfalvi's treatment of the Bender-Suzuki Theorem (Part II of [**P1**]) has now appeared in an English version[**P5**]. We have taken the opportunity of this second printing to add these new references, and to correct some misprints and other minor slips in the first printing.

We are nearing a milestone in our series – the completion of our revised proof of the Classification Theorem for $\mathcal{K}$-proper simple groups of *odd type* following the strategy carefully outlined in this volume, and in particular in our Theorems $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_7$.

During these same years there has been considerable research activity related to simple groups of *even type* from the so-called "unipotent" point of view, that is, by means of the analysis of the structure of 2-constrained 2-local subgroups and their interplay with one another. With the maturing of the theory of amalgams, such notable results have emerged as Stellmacher's second-generation proofs of the classifications of N-groups and thin groups, first proved by Thompson and Aschbacher, respectively. Furthermore an ambitious program proposed by Meierfrankenfeld is currently being pursued by Meierfrankenfeld, Stellmacher, Stroth, Chermak and others; it is directed towards the classification of groups of characteristic p-type for an arbitrary prime p, excepting those satisfying uniqueness conditions akin to the existence of a strongly p-embedded subgroup. This promises to remain an area of intense activity for another decade and one can at best speculate from this distance concerning what major theorems will emerge once the dust settles. Nevertheless, since it is the amalgam approach which we have proposed for our Theorem $\mathcal{C}_4$, recent results have naturally had an impact on the appropriateness of the strategy presented in this volume.

In the seminal work of Thompson on N-groups, the parameter $e(G)$ was introduced to denote the maximum value of $m_p(H)$ as p ranges over all odd primes and H ranges over all 2-local subgroups of G. A crucial division in the set of all groups of characteristic 2-type was made between those groups G with $e(G) \leq 2$

and those with $e(G) \geq 3$, the former being dubbed *quasithin*[1] groups. This subdivision persisted in the full classification of finite simple groups of characteristic 2-type (simple groups G of 2-rank at least 3 such that $F^*(H) = O_2(H)$ for every 2-local subgroup H of G) undertaken during the 1970's. It was however discovered by Gorenstein and Lyons in the work leading up to their Memoirs volume [**GL1**] that the case $e(G) = 3$ presented unique difficulties deserving special treatment. Hence [**GL1**] treats simple groups of characteristic 2-type with $e(G) \geq 4$, and the case $e(G) = 3$ was handled independently by Aschbacher in [**A13**]. Both treatments took a "semisimple" rather than a "unipotent" approach, studying p-signalizers and p-components of p-local subgroups for primes p other than the "characteristic", in these cases for $p > 2$.

The Quasithin Case ($e(G) \leq 2$) was subdivided further into the case $e(G) = 1$ (the "Thin Case") and the case $e(G) = 2$. The former was treated by Aschbacher[**A10**], while the latter was undertaken and almost completed by G. Mason. A unified treatment of the Quasithin Case is nearing completion by Aschbacher and S. D. Smith and will be published in this AMS series. When published, the Aschbacher-Smith volumes will represent a major milestone: the completion of the first published proof of the Classification of the Finite Simple Groups. But it is also pertinent to our strategy, since Aschbacher and Smith have relaxed the hypotheses somewhat. Rather than restricting themselves to the set of all simple groups of characteristic 2-type, they have partly accommodated their work to the strategic plan outlined in this volume by having their theorem encompass the larger class of all simple groups of even type. Indeed their main theorems have the following immediate consequence, in our terminology.

THE $e(G) \leq 2$ THEOREM. *Let G be a finite $\mathcal{K}$-proper simple group of even type. Suppose that every 2-local subgroup H of G has p-rank at most 2 for every odd prime p. Then one of the following conclusions holds:*

(a) *$G \in \mathcal{C}hev(2)$, and G is of twisted Lie rank at most 2, but G is not isomorphic to $U_5(q)$ for any $q > 4$; or*

(b) *$G \cong L_4(2)$, $L_5(2)$, $Sp_6(2)$, A_9, $L_4(3)$, $U_4(3)$, $G_2(3)$, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, HS, He, or Ru.*

However it must be noted that the Aschbacher-Smith Theorem will not complete the classification of "quasithin" groups of even type as this term is defined in the current volume (p. 82), the reason being that during the 1980's we chose to redefine "quasithin" to include the $e(G) = 3$ case for groups of even type. This was motivated both by the peculiar difficulties of the $e(G) = 3$ case noted above and by conversations concerning the Amalgam Method which suggested to us that treating groups with $e(G) \leq 3$ via the Amalgam Method would not be substantially more difficult than treating groups with $e(G) \leq 2$.

In any event the recent work of Aschbacher and Smith has now reopened the possibility of restoring "quasithin" more or less to its original meaning, or more precisely to the hypotheses of the $e(G) \leq 2$ Theorem above. The groups of even type with $e(G) = 3$ then could be treated by a "semisimple" strategy akin to the one for the proof of Theorem $\mathcal{C}_6$ discussed briefly in Section 24 of Chapter 2.

Such a change in strategy would entail a different case division defining rows 4, 5 and 6 of the classification grid (p. 85), i.e., altered hypotheses for Theorems

[1] The word "quasithin" has been used in a number of distinct senses by various authors. In this paragraph and the next, we stick to the original meaning $e(G) \leq 2$.

$\mathcal{C}_4$, $\mathcal{C}_5$ and $\mathcal{C}_6$ (pp. 105–106). Namely the $e(G) \leq 2$ Theorem would take the place of Theorem $\mathcal{C}_4$, and in rough terms the dichotomy $m_p(M) \geq 4$ or $m_p(M) \leq 3$ for certain 2-local subgroups would be replaced throughout by the dichotomy $m_p(M) \geq 3$ or $m_p(M) \leq 2$. Consequently there would be changes in the sets $\mathcal{K}_4$, $\mathcal{K}_5$ and $\mathcal{K}_6$ of target groups, the first one shrinking and the other two expanding (and all would be nonempty). In addition the uniqueness theorems $\mathcal{M}(S)$ and $U(\sigma)$ would have to be strengthened to accommodate weakened hypotheses on the p-ranks of 2-local subgroups of G. The authors are now contemplating and investigating the possibility of adopting this different approach.

Richard Lyons and Ronald Solomon
August, 2000

Part I

PRELIMINARIES

PART I, CHAPTER 1

OVERVIEW

Introduction to the Series

The existing proof of the classification of the finite simple groups runs to somewhere between 10,000 and 15,000 journal pages, spread across some 500 separate articles by more than 100 mathematicians, almost all written between 1950 and the early 1980's. Moreover, it was not until the 1970's that a global strategy was developed for attacking the complete classification problem. In addition, new simple groups were being discovered throughout the entire period—in succession the Chevalley groups, the Steinberg variations, the Suzuki and Ree groups and the twenty-one "modern" sporadic groups—so that it was not even possible to state the full theorem in precise form before the constructions of the last two sporadic groups F_1 and J_4 in the early 1980's. Then, too, new techniques for studying simple groups were steadily being developed; but earlier papers, essential to the overall proof, were of necessity written without benefit of many of these later methods.

Under such circumstances it is not surprising that the existing proof has an organizational structure that is rather inefficient and that its evolution was somewhat haphazard, including some duplications and false starts. As a result of these various factors, it is extremely difficult for even the most diligent mathematician, not already versed in its intricacies, to obtain a comprehensive picture of the proof by examining the existing literature.

Considering the significance of the classification theorem, we believe that the present state of affairs provides compelling reasons for seeking a simpler proof, more coherent and accessible, and with clear foundations. This series of monographs has as its purpose an essentially self-contained presentation of the bulk of the classification of the finite simple groups–primarily of that portion of the total proof that can be achieved by what is known as local group-theoretic analysis. The arguments we give will depend only on existing techniques and the complete project will cover between 3,000 and 4,000 pages.

Our major simplification is obtained as a result of a global strategy that differs in several key respects from that of the existing proof. This strategy enables us, on the one hand, to bypass completely a number of general and special classification theorems that were integral to the present proof and, on the other, to treat in a uniform way certain portions of the proof that were originally handled by separate arguments. A second simplification occurs because we assume in effect at the very outset that the simple group G under investigation is a minimal counterexample to the classification theorem, which implies that all proper simple sections of G are of known type. In the earlier period, when there was a prevalent belief that additional simple groups remained to be discovered, such an initial hypothesis would have been entirely inappropriate.

Finally we note that we are intentionally adopting a fuller style of exposition than would be appropriate for a journal article, for our primary aim is to provide as readable a proof as we are able to attain. Some reduction in our projected 3,000–4,000 pages could certainly have been made (perhaps as much as 20%) had we chosen a more compact style; and the argument would still have been viewed as self-contained. On the other hand, it is our belief that it will not be possible to construct a substantially shorter classification of the finite simple groups without first developing some radically new methods.

This first monograph is preliminary, presenting the conceptual framework underlying the simplified proof of the classification of the finite simple groups that will appear in the series. Chapter 1 is a relatively non-technical overview. After the statement of the Classification Theorem, it gives a description of important general aspects of the structure of finite groups particularly relevant to the classification proof. (Though some of these properties have been proved only assuming that all simple sections of the groups in question are of known type, they become properties of all finite groups after the proof of the classification theorem.) This is followed by some general comments about classifying simple groups and a short discussion and critique of the existing classification proof. The Background Results which we permit ourselves to assume—including some portions of the classification proof itself—are then listed. The chapter concludes with a brief sketch of the proof to be presented in this series and some miscellaneous remarks.

In Chapter 2 we provide a more detailed outline of the proof of the Classification Theorem. The summary includes definitions of the key terms in which the proof is expressed, a description of the major case divisions, and a diagrammatic chart of the various stages of the analysis for each case. Finally, we briefly discuss the principal group-theoretic techniques underlying the proof.

This series of monographs is divided into five major parts:

Part I:	Preliminaries
Part II:	Uniqueness theorems
Part III:	Generic simple groups
Part IV:	Special odd simple groups
Part V:	Special even simple groups.

Moreover, each part is itself subdivided into a number of chapters and its full exposition will require several monographs.

In particular, Part I consists of four chapters. The two chapters of the current monograph will be designated [I_1] (this Overview) and [I_2] (the Outline of Proof), respectively. Chapter 3 of Part I, designated [I_G], will cover essentially all the general (primarily local) group-theoretic results needed for the classification proof, while Chapter 4, designated [I_A], will cover basic properties of almost simple $\mathcal{K}$-groups[1], which underlie the more detailed results about $\mathcal{K}$-groups needed for the analysis of Parts II, III, IV, and V. Both of these chapters will of course rely heavily on the Background Results named in the Overview.

[1]A $\mathcal{K}$-group is a group such that all the composition factors of all its subgroups are known simple groups—i.e., simple groups in the conclusion of the Classification Theorem. See Section 2 of Chapter 1.

The proof of the Classification Theorem proceeds by induction, focusing on a minimal counterexample G. All proper subgroups of G are then $\mathcal{K}$-groups, and the proof requires a very elaborate theory of $\mathcal{K}$-groups. Each of Parts II, III, IV, and V will contain initial chapters establishing the $\mathcal{K}$-group results needed specifically for that part. To enhance the organizational structure each part will have two chapters consisting of

1. Principal properties of $\mathcal{K}$-groups
2. Specialized properties of $\mathcal{K}$-groups

to be designated [II_P], [II_S], [III_P], etc. By "principal properties" we mean those more central conceptually to that part.

Some general group-theoretic results will be omitted from [I_G] and deferred to the parts of the proof in which they are used. This material will be included in a preliminary chapter of the respective part designated [II_G], [III_G], etc. Similarly there will be chapters [II_A], etc.

Our intention is to make explicit the statements and proofs of all group-theoretic and $\mathcal{K}$-group results needed for the proof of the classification theorem, using only material from the explicitly listed Background Results as references.

As has been indicated, the proof of the Classification Theorem will run to several thousand pages. Because of this, the principal authors have been able to induce some of our colleagues with expertise in specific areas to work on certain major portions of the proof, and we gratefully acknowledge their collaboration in the project at the outset. The topics covered by their contributions are the following. The terms involved and the meaning of the results are discussed more fully in the two chapters to follow.

A. In Part II, the classification of groups of even type in which $|\mathcal{M}(G;S)| = 1$ is joint work with Richard Foote, who wrote Chapter 4 of Part II.
B. In Part II, the nonexistence of groups of even type that contain a p-uniqueness subgroup for each $p \in \sigma(G)$ is established in Chapter 6 of Part II, written by Gernot Stroth.

It is appropriate here to comment on the status, at the moment of this writing, of the mathematics and the manuscripts for this lengthy project. There is one major case, whose proof is visualized by the amalgam method, which has not yet been fully analyzed—the quasithin type case. We should perhaps call it the "revised" quasithin case because it is not the same as the quasithin case ($e(G) \leq 2$, G of characteristic 2 type) of the original classification proof, but is defined by a different condition ($\sigma(G) = \emptyset$, G of even type) to be made precise below, and it includes in particular the case $e(G) \leq 3$, G of characteristic 2 type. The envisioned analysis of this case is rather complex, but the results obtained so far, including a global $C(G,S)$-theorem to appear in Part II, and amalgam-theoretic results of Stellmacher and Delgado, make us reasonably confident that it will be completed along the lines described below in Chapter 2 (in row 5 of the classification grid and in section 22). Nevertheless, for the present the reader should regard the strategy described there as provisional. It would be possible, if absolutely necessary, to assemble parts of the original classification proof—including the "original" quasithin case—to analyze our "revised" quasithin case, but the newer amalgam method is appropriate and extremely well-suited to the problem. We note that because the theory of amalgams, or "weak (B, N)-pairs", is presently an active field of general interest, it is likely that a substantial portion of the work on the quasithin case will be published outside of this series; if so, it will simply be added to the Background Results.

Turning to the project as a whole, many of the chapters exist now in final drafts, but not all; some exist only in earlier drafts. Of course, if changes turn out to be necessary upon final revision, the effects upon the global strategy and the grids of Chapter 2 will be duly noted. Nevertheless, we are ready to present the whole strategy at this time, confident that it will stand essentially as is.

A. The Finite Simple Groups

1. Simple groups

It is our purpose in these monographs to prove the following theorem:

Classification Theorem. *Every finite simple group is cyclic of prime order, an alternating group, a finite simple group of Lie type, or one of the twenty-six sporadic finite simple groups.*

A precise statement of the Background Results which we allow ourselves to assume in the course of the proof will be given in sections 15–18. This material includes some particular cases of the Classification Theorem as well as a considerable amount of general finite group theory and the theory of the particular groups in the conclusion of the theorem. In addition the current status of the proof of the theorem is discussed in the preceding Introduction to the Series.

We shall presently discuss the structure of arbitrary finite groups in the context of the Classification Theorem, but first we shall give a brief introductory sketch of the principal characters of our story, the finite simple groups. A more detailed description of these groups without proofs is given in [**G3,** Chapter 2].

The most familiar examples of finite simple groups are the cyclic groups Z_p of prime order and the alternating groups A_n consisting of all the even permutations of the symmetric groups Σ_n, $n \geq 5$. However, the bulk of the set of finite simple groups consists of finite analogues of Lie groups, including analogues of real forms. These are called the finite simple groups of **Lie type**, and naturally form 16 infinite families. In addition, there exist precisely twenty-six **sporadic** finite simple groups that are not members of any of these or any reasonably defined infinite families of simple groups.

The oldest of the finite groups of Lie type are the **classical** groups: **linear**, **symplectic**, **orthogonal**, and **unitary** [**Di1, D1, Ar2, A1, KlLi1, Ch1**]. The **general linear group** $GL_n(q)$ is the multiplicative group of all $n \times n$ nonsingular matrices over the finite field $\boldsymbol{F}_q$ with q elements, q a prime power. The **special linear group** $SL_n(q)$ is its normal subgroup consisting of all matrices of determinant 1, and the **projective special linear group** $L_n(q) = PSL_n(q)$ is the quotient group of $SL_n(q)$ by its central subgroup of scalar matrices. If $n \geq 2$, then the group $PSL_n(q)$ is simple except for the two cases $n = 2$, $q = 2$, and $n = 2$, $q = 3$, and $SL_n(q)$ is the commutator subgroup of $GL_n(q)$ except for the case $n = 2$, $q = 2$.

The symplectic and orthogonal groups can be defined as subgroups of $GL_n(q)$ consisting of the matrices leaving invariant a given nondegenerate alternating bilinear form or quadratic form, respectively, on the underlying n-dimensional vector space over $\boldsymbol{F}_q$ on which $GL_n(q)$ naturally acts. By passage to the commutator subgroup and the quotient by scalar matrices, they lead to further families $PSp_n(q)$ and $P\Omega_n^{\pm}(q)$ of finite simple groups. We note that over the complex numbers all nondegenerate quadratic forms in a given dimension are equivalent, but over arbitrary fields, this is far from the case; for example, in any even dimension n over $\boldsymbol{F}_q$, there are exactly two such inequivalent forms, so we obtain two families of finite orthogonal groups in this way. (When n is odd only one group arises, $P\Omega_n(q)$.)

The complex unitary group GU_n can be defined as the subgroup of the complex general linear group GL_n consisting of all matrices fixed by the automorphism a of GL_n defined by

$$a(X) = ((\bar{X})^t)^{-1},$$

where $\bar{X}$ denotes the matrix obtained from $X \in GL_n$ by replacing each entry by its complex conjugate and X^t denotes the transpose of the matrix X. The finite unitary groups $GU_n(q)$ are obtained in the same way from $GL_n(q^2)$ as the fixed points of the corresponding automorphism a, where now $\bar{X}$ denotes the matrix obtained from $X \in GL_n(q^2)$ by raising each of its entries to the q^{th} power. Again this leads to the family of simple groups $U_n(q) = PSU_n(q)$ via the special unitary group $SU_n(q) = GU_n(q) \cap SL_n(q^2)$.

Some of these finite classical groups, namely the linear groups, the orthogonal groups $P\Omega_{2n+1}(q)$, the symplectic groups and the orthogonal groups $P\Omega_{2n}^{+}(q)$, are analogues of the complex Lie groups A_n, B_n, C_n and D_n. Although finite analogues of some of the exceptional Lie groups had been constructed by Dickson by the early part of the century, it was not until 1955 that Chevalley [**Ch2**] showed by a general Lie-theoretic argument that there exist finite analogues $\mathcal{L}(q)$ of every semisimple complex Lie group $\mathcal{L}$ for every finite field $\boldsymbol{F}_q$. In particular, he proved simultaneously the existence of the five families $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$ of exceptional simple groups of Lie type. At approximately the same time, Tits was giving geometric constructions of several of these families (see [**Ti1**]). These groups $\mathcal{L}(q)$, $\mathcal{L} = A_n$, B_n, C_n, D_n, G_2, F_4, E_6, E_7 and E_8, are called the **untwisted** groups of Lie type. They are also called the **Chevalley groups**, although this term sometimes is applied to all groups of Lie type.

The general construction of the **twisted groups** followed soon after. The complex unitary groups are often referred to as **compact real forms** of the linear groups. It was known also that real forms of the orthogonal groups $P\Omega_{2n} = D_n$ and E_6 exist, connected with a "graph" automorphisms τ of period 2, induced from a symmetry of the associated Dynkin diagram of type D_n and E_6, respectively. In 1959 Steinberg [**St3**] showed that Chevalley's construction of the finite analogues of complex Lie groups could be extended to give analogues of their real forms, thus obtaining systematically the families ${}^2D_n(q)$ (the second family of even-dimensional orthogonal groups) and ${}^2E_6(q)$ over $\boldsymbol{F}_q$ for every q, as well, of course, as the unitary groups, which in the Lie notation are the twisted linear groups ${}^2A_{n-1}(q)$.

The classical orthogonal group D_4 is the only Lie group having a graph automorphism σ of period 3. However, as the complex field possesses no automorphism

TABLE I — THE FINITE SIMPLE GROUPS

Group (notes)	Other names	Order
Z_p		p
A_n, $n \geq 5$	Alt_n	$\frac{1}{2}\, n!$
$A_n(q)$, $n \geq 1$ (1)	$PSL_{n+1}(q) = L_{n+1}(q) = L^+_{n+1}(q) = A^+_n(q)$	$\frac{1}{(n+1,q-1)}\, q^{\binom{n+1}{2}} \prod_{i=2}^{n+1} (q^i - 1)$
${}^2A_n(q)$, $n \geq 2$ (1)	$PSU_{n+1}(q) = U_{n+1}(q) = L^-_{n+1}(q) = A^-_n(q)$	$\frac{1}{(n+1,q+1)}\, q^{\binom{n+1}{2}} \prod_{i=2}^{n+1} (q^i - (-1)^i)$
$B_n(q)$, $n \geq 2$ (2)	$P\Omega_{2n+1}(q) = \Omega_{2n+1}(q)$	$\frac{1}{(2,q-1)}\, q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$
${}^2B_2(q)$ (1),(3)	$Sz(q) = {}^2B_2(\sqrt{q})$	$q^2(q-1)(q^2+1)$
$C_n(q)$, $n \geq 2$ (2)	$PSp_{2n}(q)$	$\frac{1}{(2,q-1)}\, q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$
$D_n(q)$, $n \geq 3$	$P\Omega^+_{2n}(q) = D^+_n(q)$	$\frac{1}{(4,q^n-1)}\, q^{n(n-1)}\, (q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$
${}^2D_n(q)$, $n \geq 2$	$P\Omega^-_{2n}(q) = D^-_n(q)$	$\frac{1}{(4,q^n+1)}\, q^{n(n-1)}\, (q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$
${}^3D_4(q)$		$q^{12}(q^2-1)(q^8+q^4+1)(q^6-1)$
$G_2(q)$ (2)		$q^6(q^2-1)(q^6-1)$
${}^2G_2(q)$ (2),(4)	$R(q) = {}^2G_2(\sqrt{q})$	$q^3(q-1)(q^3+1)$
$F_4(q)$		$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
${}^2F_4(q)$ (2),(3)	${}^2F_4(\sqrt{q})$	$q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$
$E_6(q)$	$E^+_6(q)$	$\frac{1}{(3,q-1)}\, q^{36}(q^2-1)(q^5-1)(q^6-1)\cdot (q^8-1)(q^9-1)(q^{12}-1)$
${}^2E_6(q)$	$E^-_6(q)$	$\frac{1}{(3,q+1)}\, q^{36}(q^2-1)(q^5+1)(q^6-1)\cdot (q^8-1)(q^9+1)(q^{12}-1)$
$E_7(q)$		$\frac{1}{(2,q-1)}\, q^{63}(q^2-1)(q^6-1)\cdot (q^8-1)(q^{10}-1)(q^{12}-1)\cdot (q^{14}-1)(q^{18}-1)$
$E_8(q)$		$q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)\cdot (q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$

Continued

TABLE I — CONTINUED

Group	Other names	Order
M_{11}		$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}		$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}		$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1		$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	HJ	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	HJM	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4		$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS		$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
He	$HHM = F_7$	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Mc		$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Suz	Sz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Ly	LyS	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
Ru		$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$O'N$	$O'S$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	$\cdot 1$	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	$\cdot 2$	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	$\cdot 3$	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	$M(22)$	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	$M(23)$	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	$M(24)'$	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
F_5	HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
F_3	Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
F_2	$B = BM$	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
F_1	M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot$ $47 \cdot 59 \cdot 71$

Notes: (1) $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$, and ${}^2B_2(2)$ are solvable.
(2) For $G = B_2(2) = C_2(2)$, $G = G_2(2)$, $G = {}^2G_2(3)$, and $G = {}^2F_4(2)$, $G' = [G, G]$ is simple and its index in G is $q = 2$, 2, 3, 2, resp.
(3) $q = 2^{2n+1}$ only (4) $q = 3^{2n+1}$ only

TABLE II

ISOMORPHISMS AMONG THE GROUPS IN TABLE I

$B_2(q) \cong C_2(q)$
$D_3(q) \cong A_3(q)$
${}^2D_3(q) \cong {}^2A_3(q)$
${}^2D_2(q) \cong A_1(q^2)$
$B_n(2^m) \cong C_n(2^m)$
$A_5 \cong A_1(4) \cong A_1(5) \cong {}^2D_2(2)$, of order 60
$A_1(7) \cong A_2(2)$, of order 168
$A_1(8) \cong {}^2G_2(3)'$, of order 504
$A_6 \cong A_1(9) \cong B_2(2)' \cong C_2(2)'$, of order 360
${}^2A_2(3) \cong G_2(2)'$, of order 6048
$A_8 \cong A_3(2)$, of order 20160
${}^2A_3(2) \cong {}^2D_3(2) \cong B_2(3) \cong C_2(3)$, of order 25920

of period 3, it is not possible to use this automorphism to construct a twisted form of D_4. On the other hand, in the finite case, one can define an automorphism a of $D_4(q^3)$ by the equation $a(X) = (\bar{X})^\sigma$, where now $\bar{X}$ denotes the matrix obtained from $X \in D_4(q^3)$ by raising each of its entries to the q^{th} power. Again the fixed elements of a form a simple group, ${}^3D_4(q)$, known as "**triality** D_4". Steinberg's construction covered this case as well. The groups ${}^2A_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, and ${}^2E_6(q)$ are sometimes called the **Steinberg variations**.

Although Suzuki's discovery [**Su2, Su3**] of his family of simple groups $Sz(2^n)$, n odd, $n > 1$, was not made Lie-theoretically, but rather in the course of solving a group-theoretic classification problem, Ree noted that they could in fact be obtained by a variant of the Steinberg procedure as a twisted form of the orthogonal groups $B_2(2^n)$. Indeed, it was known earlier that because of certain degeneracies in the multiplication coefficients, the groups $B_2(2^n)$, $G_2(3^n)$ and $F_4(2^n)$ possess an "extra" automorphism. In the case that n is odd, this automorphism can be taken to have order 2. The Suzuki groups are then twisted versions of B_2 obtained using this automorphism: $Sz(2^n) = {}^2B_2(2^n)$. In 1961 Ree carried through his variation of the Steinberg construction to produce the **Ree groups**, which were two new families of twisted groups, ${}^2G_2(3^n)$ and ${}^2F_4(2^n)$, n odd [**R1, R2**].

There are then 16 families of finite groups of Lie type altogether, twisted and untwisted. The theories of linear algebraic groups and of (B, N)-pairs, which were also flowering during this postwar period, gave uniform viewpoints for all the families. A general proof of simplicity was given by Tits [**Ti3**] (noting the handful of exceptions, including the fact that ${}^2F_4(2)$ is not simple, but its commutator subgroup ${}^2F_4(2)'$, which has index 2, is simple). In 1968, Steinberg [**St2**] gave a uniform construction and characterization of all the finite groups of Lie type as groups of fixed points of endomorphisms of linear algebraic groups over the algebraic closure of a finite field.

The twenty-six sporadic groups (see [**A2, CCPNW1**]) arose from a variety of group-theoretic contexts.[2] The first five were constructed by Mathieu in the 1860's as highly transitive permutation groups [**M1, M2, M3**]. The remaining twenty-one were discovered between the mid-1960's and 1980. Moreover, apart from the three Conway groups [**Co1**], which are connected with the automorphism group of the 24-dimensional Leech lattice, all the others emerged either directly or indirectly from one of the classification theorems being considered during this period.

The first of the twentieth-century sporadic groups was constructed by Janko [**J1**] as an exceptional solution to the problem of determining all simple groups in which the centralizer of every **involution** (i.e., element of order 2) has the form $Z_2 \times PSL_2(q)$, q odd. The general solution is Ree's twisted group ${}^2G_2(q)$, $q = 3^n$, n odd, $n > 1$, and the single exception is Janko's group J_1 for $q = 5$.

Altogether the initial evidence for eight sporadic groups came from an analysis of centralizers of involutions: each of Janko's four groups, the Held, Lyons and O'Nan groups, and the Fischer-Griess Monster group F_1 [**J2, J3, HiMcK1, No1, He1, He2, L2, Si1, ON3, An1, Gr3**].

Janko's second group J_2 or HJ was ultimately constructed by M. Hall and D. Wales [**HaWa1**] as a rank 3 primitive permutation group (i.e., a transitive permutation group in which a one-point stabilizer is a maximal subgroup and possesses precisely 3 transitive constituents). This led to an intensive investigation of such groups and the resulting discovery of four more sporadic groups: the Higman-Sims, Suzuki, McLaughlin and Rudvalis groups [**HiSi1, Su5, McL1, Ru1, CoWa1**].

Fischer's three groups Fi_{22}, Fi_{23} and Fi'_{24} were discovered in the course of his characterization of the groups which are generated by a conjugacy class of involutions the product of any two of which has order 1, 2, or 3 [**Fi1, Fi2**]. In addition to the previously known examples (which include the symmetric groups and certain classical groups), he found the three additional ones, Fi_m, $m = 22$, 23, 24, which he constructed as rank 3 primitive permutation groups. Fischer's "Baby Monster" F_2 arose in turn from his study of the broader class of groups generated by a conjugacy class of involutions the product of any two of which has order 1, 2, 3 or 4 [**LeSi1**].

Finally, the initial evidence for the Thompson and Harada groups F_3 and F_5 came from the structure of the centralizers of elements of order 3 and 5, respectively, in the Monster [**H1, T3**]. However, the starting point for their actual analysis was the structure of the centralizers of their involutions.

The finite simple groups are listed in Table I. The isomorphism question among all these groups was largely settled by Artin [**Ar1**] in the mid-1950's, who determined all coincidences of orders of the simple groups known to exist at that time. Moreover, Artin's methods were extendable without difficulty to include the newer groups of Lie type as they were discovered (see for example [**Ti1**] or [**KLST1**]). The isomorphism question for sporadic groups was settled by proofs of the uniqueness of each group as it was discovered, subject to various conditions—order, local structure, existence of a certain permutation representation, etc. The complete list of isomorphisms among the groups in Table I is given in Table II.

[2] We give only an early reference or two for each group. Extensive bibliographies can be found in [**A2**] and [**CCPNW1**].

2. $\mathcal{K}$-groups

At the time the classification effort was drawing to a close—the late 1970's and early 1980's—the groups in Table I were naturally referred to as the "known simple groups" or the "simple $\mathcal{K}$-groups", and we shall continue to use this terminology for convenience. Of course, the gist of the Classification Theorem is that the word "known" can be omitted.

In attempting to prove the classification, it is natural to proceed indirectly by induction, focusing attention on a **minimal counterexample**—that is, on a simple group G of least order not isomorphic to one of the known simple groups. Establishing the theorem then becomes equivalent to reaching a contradiction by showing that G itself must be isomorphic to a known simple group.

If H is a proper subgroup of such a minimal counterexample G, then every simple section[3] of H has order less than that of G and hence by the minimality assumption on G is necessarily isomorphic to one of the known simple groups. This leads to the following fundamental definitions.

Definition 2.1. A **$\mathcal{K}$-group** is a group X such that every simple section of X is isomorphic to a known simple group, that is, to a group in Table I.

Definition 2.2. A group G is **$\mathcal{K}$-proper** if and only if every proper subgroup of G is a $\mathcal{K}$-group.

We have adopted the letter "$\mathcal{K}$" as an abbreviation for "known". The first definition implies that if X is a $\mathcal{K}$-group and X is simple, then X is a known simple group.

In this terminology, the statement of the Classification Theorem is proved in the following equivalent form:

If G is a $\mathcal{K}$-proper finite simple group, then G is a $\mathcal{K}$-group.

The proof of the Classification Theorem involves a comprehensive study of the proper subgroups of G. Because of this, it is necessary to develop an elaborate and detailed theory of $\mathcal{K}$-groups.

On the other hand, clearly the term "$\mathcal{K}$-group" is of significance only as part of the proof the classification theorem. Indeed, that theorem states that every simple group is isomorphic to a known simple group and it follows that *every finite group is a $\mathcal{K}$-group.* Thus the theory of $\mathcal{K}$-groups developed for the classification proof ultimately becomes incorporated into the general theory of finite groups.

B. The Structure of Finite Groups

3. The Jordan-Hölder theorem and simple groups

The Jordan-Hölder theorem, one of the oldest results in the theory of finite groups[4], gives a procedure for associating with an arbitrary finite group X a finite

[3] A **section** of a group H is a quotient of a subgroup of H.

[4] Most of the results stated in the next 6 sections, and in particular in sections 6–8, will be discussed in more detail in subsequent chapters covering basic group-theoretical material.

sequence of finite simple groups, known as the **composition factors** of X, which are uniquely determined by X up to isomorphism and the order in which they are specified. Since every finite group is thereby a "composite" of finite simple groups, the theorem shows that an understanding of simple groups[5] is of fundamental importance for understanding the structure of arbitrary groups.

The composition factors of a given group X are constructed from a **composition series** of X, which is defined to be a sequence of subgroups $1 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ of X such that

(3.1)
(1) X_i is a proper normal subgroup of X_{i+1}, $0 \leq i \leq n-1$; and
(2) Each X_i is maximal in X_{i+1}, subject to (1).

The maximality of X_i implies that the n factor groups X_{i+1}/X_i, $0 \leq i \leq n-1$, are all simple. These factor groups are called the **composition factors** of the given series and the integer n is called the **length** of the series.

Thus the composition factors of X are sections of X obtained via a composition series of X. The Jordan-Hölder theorem asserts the following:

THEOREM 3.1. *For any group X, any two composition series of X have the same lengths, and for suitable orderings of the sequences of composition factors, corresponding factors are isomorphic.*

In particular, if every composition factor of X is of prime order, X is called a **solvable** group.

At the other extreme, a group X is simple if $X \neq 1$ and the identity subgroup is the only proper normal subgroup of X, in which case X possesses a unique composition series $1 = X_0 < X_1 = X$.

The theorem has a natural extension to groups with operators that enables one to obtain a fuller picture of the role of simple groups in determining the structure of an arbitrary finite group X. Indeed, if A is any set of endomorphisms of X, then an **A-composition series** of X is by definition a sequence of subgroups X_i of X, $0 \leq i \leq n$, such that $X_i \lhd X_{i+1}$ for all $0 \leq i < n$, each X_i is A-invariant, and each X_i is maximal in X_{i+1} subject to these conditions. The groups X_{i+1}/X_i are called the **A-composition factors** of X.

The general form of the Jordan-Hölder theorem asserts that for any set A of endomorphisms of X, any two A-composition series of X have the same lengths, and if the lists of A-composition factors arising from the two composition series are suitably reordered, then corresponding A-composition factors are A-isomorphic.

Thus the special form of the theorem stated above corresponds to the case in which A is empty.

Of special importance to us here is the additional case in which A is the group $Inn(X)$ of all inner automorphisms of X. The condition that each subgroup X_i be A-invariant is then equivalent to the assertion that each X_i be normal in X. Such A-composition series are called **chief series** of X and the corresponding factor groups X_{i+1}/X_i are called the **chief factors** of X. Thus likewise the chief factors of X are uniquely determined by X up to isomorphism (preserving the conjugacy action of X) and the sequence in which they occur in a chief series.

Moreover, any terms not defined here will be defined later. Much of the material can be found in basic texts [**A1, G1, Hu1, HuB1, Su1**] and the papers [**GW2, GW4, GL1**].

[5]To avoid endless repetition, let us agree that henceforth in this series, "group" means "finite group" unless explicitly stated otherwise.

Furthermore, in a chief series, it follows from the maximality of X_i in X_{i+1} that each X_{i+1}/X_i is a **minimal normal subgroup** of X/X_i and in particular, X_1 is a minimal normal subgroup of X. On the other hand, one easily establishes the following result concerning the structure of minimal normal subgroups.

PROPOSITION 3.2. *Every minimal normal subgroup of a group X is the direct product of isomorphic simple groups.*

It follows therefore that each chief factor of X is the direct product of isomorphic simple groups. If these direct factors have prime order p, then the given chief factor is elementary abelian and can be identified with the additive group of a finite-dimensional vector space over the field $\boldsymbol{F}_p$ of p elements. In particular, if X is solvable, then every chief factor of X has this form. We use the symbol E_{p^n} to denote an elementary abelian p-group of rank n. In the case $n = 1$, we generally use the more customary term Z_p.

Likewise, it is easy to show by induction that every normal subgroup of X can be included in a chief series of X, and, in particular, any minimal normal subgroup can be so included. Furthermore, by considering the action of X on such a minimal normal subgroup V by conjugation, one obtains an initial picture of the general shape of the group X.

Indeed, suppose first that V is an elementary abelian p-group for some prime p and hence a vector space over $\boldsymbol{F}_p$, in which case for each $x \in X$, the conjugation map $Int(x) : v \mapsto v^x = x^{-1}vx$, $v \in V$, is in fact a linear transformation on V. Moreover, Int is a homomorphism of X into $GL(V)$ whose kernel is precisely the subgroup $C_X(V)$ consisting of all elements of X that centralize V. Thus we have

PROPOSITION 3.3. *If V is a minimal normal subgroup of X and V is abelian, then via the conjugation action of X on V, $X/C_X(V)$ is identified with a subgroup of the general linear group $GL(V)$.*

In this case, since V itself is contained in $C_X(V)$, the group $X/C_X(V)$ is a proper homomorphic image of X.

On the other hand, if the minimal normal subgroup V of X is the direct product of nonabelian simple groups V_i, $1 \le i \le r$, then it is straightforward to establish that the V_i are the only minimal normal subgroups of V and consequently each $x \in X$ induces by conjugation a permutation of the set $\mathcal{V} = \{V_i \,|\, 1 \le i \le r\}$. Thus there is a natural homomorphism β of X into the symmetric group on the set $\mathcal{V}$. The kernel Y of β is the subgroup consisting of all elements of X leaving each V_i invariant. Clearly $VC_X(V) \le Y$ and $Y/C_X(V)$ is isomorphic to a subgroup of $Aut(V)$ leaving each V_i invariant. It follows easily from this last condition that $Y/C_X(V)$ is in fact isomorphic to a subgroup of the direct product $Aut(V_1) \times Aut(V_2) \times \cdots \times Aut(V_r)$. Moreover, as $V \le Y$, the image of V in $Y/C_X(V)$ is isomorphic to the group $Inn(V)$ of inner automorphisms of V. Hence in this case we have

PROPOSITION 3.4. *Let V be a minimal normal subgroup of X which is the direct product of nonabelian simple groups V_i, $1 \le i \le r$. Let Y be the subgroup consisting of all elements of X leaving each V_i invariant, $1 \le i \le r$. Then*

(i) *Through the action of X permuting $V_1, \ldots, V_r$ by conjugation, X/Y is identified with a subgroup of Σ_r; and*
(ii) *Through the action of Y on V by conjugation, $Y/C_X(V)$ is identified with a subgroup of $Aut(V_1) \times Aut(V_2) \times \cdots \times Aut(V_r)$ containing $Inn(V)$.*

4. The generalized Fitting subgroup and quasisimple groups

Unfortunately, the pictures of X given by the approaches just discussed have serious inadequacies. On the one hand, even if one knows the isomorphism types of the composition factors of the group X, the isomorphism type of X is far from determined, and the "extension problem"—determining all X with given composition factors—has such depth and subtlety that to date it has been solved in only a handful of special cases. On the other hand, if one tries to understand X by means of its action on a minimal normal subgroup V, one has no information about the subgroup $C_X(V)$ whose action on V is trivial. Moreover, $C_X(V)$ may include a large portion of X. Indeed, V may well be in the center $Z(X)$ of X, in which case $C_X(V) = X$. For example, this will occur if V is an abelian direct factor of X (i.e., $X = V \times X_0$ for some subgroup X_0 of X).

As a first attempt to eliminate this deficiency, one might replace V by the subgroup V^* generated by *all* minimal normal subgroups of X and then consider the action of X on V^* (this would at least exclude the preceding example, since clearly V^* will never be a proper direct factor of X). Moreover, the structure of V^* is very similar to that of a minimal normal subgroup. Indeed, if V and W are any two distinct minimal normal subgroups of X, then as $V \cap W$ is also normal in X, their minimality forces $V \cap W = 1$, and it follows that $\langle V, W \rangle = VW = V \times W$. This argument is easily extended to yield:

PROPOSITION 4.1. *If V^* is the subgroup of X generated by all minimal normal subgroups of X, then*

$$V^* = V_1 \times V_2 \times \cdots \times V_m$$

for suitable minimal normal subgroups $V_1, \ldots, V_m$ of X.

However, it still may happen that V^* is in the center of X. For example, if $X = SL_2(q)$, the group of 2×2 matrices of determinant 1 with coefficients in $\boldsymbol{F}_q$, and q is odd, then X possesses a unique minimal normal subgroup, which is generated by the scalar matrix $-I$. Thus in this case $V^* = V = Z(X)$.

On the other hand, there is an important special case in which $C_X(V^*) = 1$, in which case X does act faithfully on V^*. Indeed, if X contains no nontrivial solvable normal subgroups, then every minimal normal subgroup is a direct product of nonabelian simple groups and so therefore is V^*. But then $C_{V^*}(V^*) = 1$ and consequently $C_X(V^*) \cap V^* = 1$. Since $C_X(V^*) \lhd X$, the definition of V^* now forces $C_X(V^*) = 1$, as asserted.

This leads to the following natural question: *Does there exist a canonically defined normal subgroup W of X having both a relatively uncomplicated structure and the property $C_X(W) \leq W$?*

As the preceding discussion makes clear, central subgroups of X are initial obstructions to locating such a subgroup W in X. Moreover, the class of groups constructed inductively from central subgroups is that of the nilpotent groups, which suggests that nilpotent normal subgroups may well be a key to the question's answer.

There are several equivalent definitions of a nilpotent group. In the spirit of the present discussion, we take the following: X is said to be **nilpotent** if X possesses a chief series $1 = X_0 \leq X_1 \leq \cdots \leq X_n = X$ such that

$$X_{i+1}/X_i \leq Z(X/X_i) \qquad \text{for all } i,\ 0 \leq i \leq n-1.$$

In particular, each X_{i+1}/X_i must then be abelian. In fact, as X_{i+1}/X_i is a minimal normal subgroup of X/X_i, each X_{i+1}/X_i must be of prime order. It follows therefore that a nilpotent group is necessarily solvable.

We state two basic facts concerning the structure of nilpotent groups.

PROPOSITION 4.2. *Every group of prime power order is nilpotent.*

PROPOSITION 4.3. *X is nilpotent if and only if X is the direct product of its Sylow subgroups.*

Considering nilpotent normal subgroups of the group X, it is not difficult to show that the subgroup generated by any two such subgroups is itself a nilpotent normal subgroup of X, which immediately yields the following results.

PROPOSITION 4.4. *X contains a unique normal subgroup maximal subject to being nilpotent.*

This maximal nilpotent normal subgroup of X is called the **Fitting subgroup of** X and is denoted by $F(X)$. Thus $F(X)$ is the subgroup of X generated by all nilpotent normal subgroups of X. In particular, $F(X)$ contains every abelian normal subgroup of X. Clearly $F(X)$ is **characteristic** in X, that is, invariant under every automorphism of X.

The following theorem of Philip Hall shows that the Fitting subgroup provides a satisfactory answer to the above question in solvable groups.

THEOREM 4.5. *If X is solvable, then $C_X(F(X)) \leq F(X)$.*

Note that if X is solvable (and nontrivial), then $F(X) \neq 1$.

Bender has extended Hall's theorem to arbitrary finite groups. The form of his result reveals the fundamental role that perfect central extensions (or **covering groups**) of simple groups play in the theory of groups. We make the following definition:

DEFINITION 4.6. The group X is **quasisimple** if and only if X is **perfect** (i.e., $X = [X, X]$) and $X/Z(X)$ is simple.

In particular, quasisimple groups are necessarily nonsolvable. Also it is obvious that nonabelian simple groups are quasisimple. The groups $SL_2(q)$, q odd, $q \geq 5$, afford examples of quasisimple groups which are not simple; $Z(SL_2(q))$ is of order 2, generated by the matrix $-I$. We note that the group $SL_2(3)$ is solvable and hence not quasisimple.

It is natural to consider central products of quasisimple groups. Thus X is said to be **semisimple** if X is the product of quasisimple groups X_i, $1 \leq i \leq m$, such that

$$[X_i, X_j] = 1 \quad \text{for all } i \neq j,$$

that is, X_i and X_j centralize each other elementwise for $i \neq j$.

For completeness, the identity group is included among the semisimple groups.

If X is semisimple, then the quasisimple factors X_i of X can be characterized as the minimal nonsolvable normal subgroups of X and so are in fact uniquely determined by X.

We mention two important properties of a semisimple group X. First, if $Y \leq X$ and Y covers $X/Z(X)$, then $Y = X$. Second, if a group A acts on the quasisimple group X, and acts trivially on $X/Z(X)$, then it acts trivially on X. This is a simple consequence of the three subgroups lemma[6].

There exists a direct analogue of Proposition 4.4 for semisimple normal subgroups of a group X, since it easily shown that the subgroup generated by any two such subgroups is itself a semisimple normal subgroup of X. Thus we have

PROPOSITION 4.7. *X contains a unique normal subgroup which is maximal subject to being semisimple.*

This maximal semisimple normal subgroup is called the **layer** of X and is denoted by $E(X)$. Clearly likewise $E(X)$ is characteristic in X.

If $E(X) \neq 1$, the uniquely determined quasisimple factors of X are called the **components** of X. In particular, it follows therefore that each element of X induces by conjugation a permutation of the set of components of X. Moreover, the components can be equivalently described as the quasisimple subnormal subgroups of X. Furthermore, $Z(E(X))$ is an abelian normal subgroup of X, and $E(X)/Z(E(X))$ has no trivial normal nilpotent subgroups, so $E(X) \cap F(X) = Z(E(X))$. Since $E(X)$ and $F(X)$ are both normal in X, $[E(X), F(X)] \leq Z(E(X))$, so by the second property above, $E(X)$ centralizes $F(X)$.

Bender has called the group $E(X)F(X)$ the **generalized Fitting subgroup** of X and denoted it by $F^*(X)$. His theorem provides an elegant extension of Hall's theorem to arbitrary finite groups, thereby justifying use of this terminology.

THEOREM 4.8. (Bender) *For any group X, we have*
(i) *$C_X(F^*(X)) \leq F^*(X)$; and*
(ii) *If W is a normal subgroup of X such that $C_X(W) \leq W$, then $E(X) \leq W$.*

Statement (ii) is included only to show the inevitability of considering $E(X)$. Statement (i) is the main point, and implies that

$$C_X(F^*(X)) = Z(F^*(X)) = Z(E(X))Z(F(X)) = Z(F(X)),$$

so that by the conjugation action, $X/Z(F(X))$ embeds in $Aut(F^*(X))$. In particular, if X is nontrivial, then so is $F^*(X)$.

Of course, if X is solvable, then $E(X) = 1$, whence $F^*(X) = F(X)$. At the opposite extreme, if X contains no nontrivial solvable normal subgroups, then $F^*(X) = E(X)$ is the direct product of nonabelian simple groups.

Because of Theorem 4.8, for many purposes the analysis of a group X reduces to an analysis of the structure of $F^*(X)$ and of its automorphism group. The group extensions essential to understanding $F^*(X)$ are of limited types, since $F^*(X)$ is the product of pairwise commuting p-groups, for various primes p, and quasisimple groups. For this reason, the problem of determining all quasisimple groups is a critical one, and so the extension problem which is arguably the most vital in finite group theory is *to determine all perfect central extensions of a given simple group*. This subtle problem has been solved for all the known simple groups by the combined work of many authors, principally Schur, Steinberg and Griess. Indeed, Schur showed that a (finite) simple group has a **universal** covering group which is itself finite—that is, a finite perfect central extension of which all others are homomorphic images. For all the simple groups, the universal covering groups and

[6]This lemma states that if A, B, and C are subgroups of a group and $[A, B, C] = [B, C, A] = 1$, then $[C, A, B] = 1$. Here as always, $[A, B, C] = [[A, B], C]$ and $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$.

their centers (called the **Schur multipliers**) have been calculated (see [**Sch1, St5, Gr1, Gr2, CCNPW1**].)

For reasons to be discussed presently, although finite nilpotent groups can be the direct product of p-groups for arbitrarily many primes p, for most of the groups X playing a critical role in the classification theory either $E(X) \neq 1$ or $F^*(X)$ is a p-group for some prime p. In any case, since p-groups can be very complicated, it is desirable to obtain faithful action of $X/Z(F(X))$ on a suitable homomorphic image of $F^*(X)$ having simpler p-subgroup structure than $F^*(X)$ itself.

Recall that in any group X, the intersection of all maximal subgroups of X is called the **Frattini subgroup** of X and is denoted by $\Phi(X)$. It, too, is clearly a characteristic and hence normal subgroup of X. For groups of prime power order, one has the following basic properties of the Frattini subgroup.

PROPOSITION 4.9. *If X is a p-group for some prime p, then we have*
(i) *$X/\Phi(X)$ is an elementary abelian p-group; and*
(ii) *If α is an automorphism of X whose order r is relatively prime to p, then α induces an automorphism of $X/\Phi(X)$ of order r.*

Since a nilpotent group is the direct product of its Sylow subgroups, it follows if X is nilpotent that $X/\Phi(X)$ is an abelian group of square-free exponent, isomorphic to the direct product of the Frattini factor groups of its Sylow subgroups.

We have noted already that if X is a semisimple group and α is an automorphism of X of order r, then α induces an automorphism of $X/Z(X)$ of order r. Combining this observation with Proposition 4.9, one can establish the following result.

PROPOSITION 4.10. *For any group X, if we set $\overline{X} = X/Z(E(X))\Phi(F(X))$, then*

$$F^*(\overline{X}) = \overline{F^*(X)}.$$

(Here we have used the **bar convention**, that the image of a subset or element Y of X under the canonical projection $X \to \overline{X}$ is denoted by $\overline{Y}$. We shall use this convention throughout.)

By definition of $\overline{X}$, $E(\overline{X})$ is the direct product of nonabelian simple groups and $F(\overline{X})$ is the direct product of elementary abelian groups. Thus $C_{\overline{X}}(F^*(\overline{X})) = F(\overline{X})$ and $\overline{X}/F(\overline{X})$ ($\cong X/F(X)$) is faithfully represented as a group of automorphisms on $F^*(\overline{X}) = E(\overline{X}) \times F(\overline{X})$.

Through Bender's theorem, many questions about the structure of the general finite group X reduce to questions about subgroups of X that leave invariant a component of X and in turn to the study of groups Y such that $F^*(Y)$ is quasisimple.[7] Aschbacher has termed such a group Y **almost simple**.

If X is an almost simple group, Proposition 4.10 implies that $F^*(\overline{X})$ is simple and so by Theorem 4.8, $\overline{X}$ is isomorphic to a subgroup of $Aut(F^*(X))$ containing $Inn(F^*(\overline{X}))$. Thus, an almost simple group is constructed from a simple group by "decorating" at the bottom and the top, i.e., by permitting perfect central extensions and outer automorphisms.

[7]There are analogous structure theorems of Aschbacher, O'Nan and Scott in the theories of permutation groups and linear groups, leading to useful reduction paradigms in these theories [**A17, Sc1**].

5. p'-cores and p-components

We have taken considerable length to motivate the generalized Fitting subgroup $F^*(X)$ of group X. One must now ask how effective this subgroup is for studying the group X itself.

On the positive side, one can at least assert that $F^*(X)$ bounds the order of X inasmuch as $X/Z(F(X))$ is isomorphic to a subgroup of $Aut(F^*(X))$. On the other hand, the structure of $X/F^*(X)$ may be arbitrarily complex. We illustrate the point with solvable groups.

Suppose then that X is solvable and $X \neq 1$ and define the **Fitting series** $F_i(X)$ of X inductively as follows:

(1) $F_1(X) = F(X)$; and
(2) $F_{i+1}(X)/F_i(X) = F(X/F_i(X))$.

Thus $F_{i+1}(X)$ is the complete inverse image of $F(X/F_i(X))$ in X. Since homomorphic images of solvable groups are solvable and since the Fitting subgroup of a nontrivial solvable group is nontrivial, it is immediate that the Fitting series terminates in X itself. We therefore obtain a series:

$$F(X) = F_1(X) < F_2(X) < \cdots < F_h(X) = X.$$

The integer h is called the **Fitting length** of X. By the following simple result, solvable groups exist with arbitrarily large Fitting length.

PROPOSITION 5.1. *If Y is a group, then there exists a group X such that $Y \cong X/F(X)$.*

Despite the possible complexity of the structure of the group $X/F^*(X)$, the significance of the generalized Fitting subgroup rests on the fact that $F^*(X)$ itself controls a key portion of the **local** structure of X. In the study of a simple group G the application of this fact to local subgroups makes possible the transfer of information among various local subgroups in identifiable chief factors of these subgroups. We shall describe this control and transfer of information in the next two sections. Here we introduce the general terms in which that description will be expressed. We first formalize the key definition of a local subgroup.

DEFINITION 5.2. A **local subgroup** of X is a subgroup Y of X which is the normalizer $Y = N_X(Q)$ of some nontrivial solvable subgroup Q of X. If Q is a nontrivial p-group for some prime p, Y is called a p**-local** subgroup of X.

Often, we shall simply say "Y is a local" to abbreviate "Y is a local subgroup of X", and we use a similar contraction with the expression "Y is a p-local".

Next, for any set π of primes, X contains a unique normal subgroup $O_\pi(X)$ that is maximal subject to the condition that $O_\pi(X)$ has order divisible only by primes in π. Likewise X contains a unique normal subgroup $O^\pi(X)$ minimal such that $X/O^\pi(X)$ has order divisible only by primes in π. We also denote by π' the complementary set of primes to π, and if $\pi = \{p\}$, we write $O_p(X)$, $O_{p'}(X)$, $O^p(X)$, $O^{p'}(X)$ for $O_{\{p\}}(X)$, $O_{\{p\}'}(X)$, $O^{\{p\}}(X)$, $O^{\{p\}'}(X)$, respectively. $O_{p'}(X)$ is called the p'**-core** of X. When $p = 2$, we call $O_{2'}(X)$ the **core** of X and denote it by $O(X)$. Thus $O(X)$ is the unique largest normal subgroup of X of **odd** order. By the Feit-Thompson Odd Order Theorem, $O(X)$ is solvable.

The following facts are direct consequences of the definitions.

LEMMA 5.3. *For any set of primes π, we have*
(i) $O_\pi(X/O_\pi(X)) = 1$*; and*
(ii) *If $O_\pi(X) = 1$, then $F(X)$ has order divisible only by primes in π' and every component of X has order divisible by some prime in π'.*

In particular, let X be a group with $O_{p'}(X) = 1$. Then $F(X) = O_p(X)$ and every component of X has order divisible by p. In this case we say that X is p-**constrained** if $E(X) = 1$, i.e.,

$$F^*(X) = O_p(X).$$

Hence when $O_{p'}(X) = 1$, p-constraint can be equivalently defined by the condition

$$C_X(O_p(X)) \leq O_p(X).$$

More generally, an arbitrary group X is said to be p-constrained if and only if $X/O_{p'}(X)$ is p-constrained.

Clearly solvable groups are p-constrained for every prime p.

Concerning components of $X/O_{p'}(X)$, we have the following basic result.

PROPOSITION 5.4. *Let X be a group, p a prime, and set $\overline{X} = X/O_{p'}(X)$. Then X contains a unique normal subgroup L minimal subject to the condition $\overline{L} = E(\overline{X})$. Moreover, if $E(\overline{X}) \neq 1$, then L is the product of normal subgroups L_i, $1 \leq i \leq r$, having the following properties:*
(i) *$\overline{L}_i$ is a component of $\overline{X}$, $1 \leq i \leq r$;*
(ii) *$L_i = O^{p'}(L_i)$, $1 \leq i \leq r$;*
(iii) *$[L_i, L_i] = L_i$, and $[L_i, L_j] \leq O_{p'}(X)$ for all $i \neq j$, $1 \leq i, j \leq r$;*
(iv) *The L_i are the only normal subgroups of L—and the only subnormal subgroups of X— satisfying (i) and (ii); and*
(v) *Each element of X induces a permutation of the set $\{L_1, \ldots, L_r\}$ by conjugation.*

For completeness we set $r = 0$ if $E(\overline{X}) = 1$. L is called the p-**layer** of X and is denoted by $L_{p'}(X)$. Furthermore, the subgroups L_i, $1 \leq i \leq r$, are called the p-**components** of X.

It is immediate from the definitions that X is p-constrained if and only if X possesses no p-components (equivalently, $L_{p'}(X) = 1$).

The connection between components and p-components is straightforward.

PROPOSITION 5.5. *Let X be a group and p a prime, and let K be a p-component of X. Then the following conditions are equivalent: K is a component of X; K is quasisimple; K centralizes $O_{p'}(K)$; K centralizes $O_{p'}(X)$. Conversely, any component of X of order divisible by p is a p-component of X.*

6. The embedding of p'-cores and p-components

The critical property of the generalized Fitting subgroup is its control of the embedding in X of the p'-cores and p-layer of every p-local subgroup Y of X—more precisely, it is the generalized Fitting subgroup of $X/O_{p'}(X)$ that exercises this control. It is this control that allows one to analyze effectively the subgroup structure of arbitrary finite simple groups and ultimately force key portions of that structure to approximate closely the corresponding subgroup structure in one of the known simple groups.

We fix the group X and the prime p and first consider the embedding of p'-cores when X is p-constrained (in particular, when X is solvable).

PROPOSITION 6.1. *If X is p-constrained, then $O_{p'}(Y) \leq O_{p'}(X)$ for every p-local subgroup Y of X.*

This result is easily proved by passing to $\overline{X} = X/O_{p'}(X)$ and invoking Thompson's $A \times B$ lemma to conclude that the image $\overline{V}$ of $O_{p'}(Y)$ in $\overline{X}$ centralizes $O_p(\overline{X})$. Since $F^*(\overline{X}) = O_p(\overline{X})$ and $C_{\overline{X}}(F^*(\overline{X})) \leq F^*(\overline{X})$, this forces $\overline{V} = 1$, whence $O_{p'}(Y) \leq O_{p'}(X)$. [The $A \times B$ lemma states that if P is a p-group on which $A \times B$ acts, with A a p'-group and B a p-group, then A centralizes P if and only if A centralizes $C_P(B)$.]

In the general case, the same argument yields that $\overline{V}$ centralizes $O_p(\overline{X})$ and hence that $\overline{V} \leq \overline{X}_0 = C_{\overline{X}}(O_p(\overline{X}))$. Again as $C_{\overline{X}}(F^*(\overline{X})) \leq F^*(\overline{X})$ and $F^*(\overline{X}) = E(\overline{X})O_p(\overline{X})$, it follows that $\overline{V}$ acts faithfully on $E(\overline{X})$. Ideally, one would like to be able to assert that $\overline{V} \leq E(\overline{X})$. One can in fact prove that $\overline{V}$ normalizes each component of $\overline{X}$; however, in general, $\overline{V}$ need not induce inner automorphisms on $E(\overline{X})$. This will explain the need to introduce the following term.

Define $\hat{L}_{p'}(X)$ to be the subgroup of X consisting of the elements of X that leave each p-component of X invariant and centralize $O_p(X/O_{p'}(X))$. Clearly $O_{p'}(X)L_{p'}(X) \leq \hat{L}_{p'}(X)$ and $\hat{L}_{p'}(X)$ is characteristic and hence normal in X.

THEOREM 6.2. *For any p-local subgroup Y of X, $O_{p'}(Y) \leq \hat{L}_{p'}(X)$.*

The corresponding general result for p-layers, known as $L_{p'}$-**balance**, has been given a proof independent of the Classification Theorem only for $p = 2$. For $p > 2$, it has not yet been proved without some assumption on the components K of $\overline{X} = X/O_{p'}(X)$. The most convenient such assumption is that each $K/Z(K)$ has the **Schreier property**—that is, its outer automorphism group is solvable[8]. Now the automorphism groups of the known simple groups have all been calculated, and observed to have this property conjectured by Schreier. Of course the case in which the components of $\overline{X}$ are all covering groups of known simple groups is the only case that we need, since we shall apply the result only to proper sections of a $\mathcal{K}$-proper simple group.

THEOREM 6.3. *($L_{p'}$-balance) If the components of $X/O_{p'}(X)$ have the Schreier property, then $L_{p'}(Y) \leq L_{p'}(X)$ for every p-local subgroup Y of X.*

[In this connection, we note that Theorem 6.2 can also be slightly sharpened when the components of $\overline{X}$ are $\mathcal{K}$-groups. In that case, for example, the image of $O_{p'}(Y)$ in $\overline{X}_0/E(\overline{X})$ is actually contained in $O_{p'}(\overline{X}_0/E(\overline{X}))$, where $\overline{X}_0 = C_{\overline{X}}(O_p(\overline{X}))$ (cf. Theorem 30.3 of Chapter 2).]

The effect of Theorems 6.2 and 6.3 is that a very large part of the p-local analysis of a $\mathcal{K}$-proper simple group focuses on the sections $\hat{L}_{p'}(Y)/O_{p'}(Y)$ for p-local subgroups Y of G. Although the structure of Y may be extremely complicated, this section is an an extension of a central p-group by a subgroup of $Aut(L_1) \times \cdots \times Aut(L_r)$ containing $Inn(L_1) \times \cdots \times Inn(L_r)$, where $L_1, \ldots, L_r$ are the components of $Y/O_{p'}(Y)$.

[8] Weaker conditions will do, and when $p = 2$ a theorem of Glauberman [**Gl3**] shows that no extra assumption is necessary.

7. Terminal and p-terminal p-components

In the study of the p-local structure of a $\mathcal{K}$-proper simple group X, the centralizers of elements of order p play an especially important role. Let us suppose for this section that the centralizers of some such elements have p-components. In such a case, those p-components dominate the analysis.

In particular, Theorem 6.3 takes a more precise form for centralizers of commuting elements of order p in an arbitrary group X. However, for it to be applicable for odd p, it is again necessary to assume that

$$\begin{aligned} &\overline{K} \text{ has the Schreier property for every } x \in X \text{ of order } p \text{ and every} \\ &\text{component } \overline{K} \text{ of } \overline{C_X(x)} = C_X(x)/O_{p'}(C_X(x)). \end{aligned} \tag{7.1}$$

We therefore make this assumption throughout the section.

PROPOSITION 7.1. *If x and y are commuting elements of order p in a group X and X satisfies (7.1), then the following conditions hold.*

(i) $L_{p'}(L_{p'}(C_X(x)) \cap C_X(y)) = L_{p'}(L_{p'}(C_X(y)) \cap C_X(x)) = L_{p'}(C_X(\langle x, y\rangle))$; *and*

(ii) *If I is a p-component of $L_{p'}(C_X(x)) \cap C_X(y)$ and J is the normal closure of I in $L_{p'}(C_X(y))$ (note that $I \leq L_{p'}(C_X(y))$ by (i)), then we have*
  (1) *J is either a single x-invariant p-component of $C_X(y)$ or the product of p p-components of $C_X(y)$ cycled by x; and*
  (2) *In either case, I is a p-component of $C_J(x)$.*

One refers to J as the **pumpup** of I in $C_X(y)$. According as J is a single p-component or a product of p p-components, the pumpup is called **vertical** or **diagonal**. Moreover, in the vertical case, either x centralizes or does not centralize $J/O_{p'}(J)$, and correspondingly the pumpup is said to be **trivial** or **proper**.

If the pumpup J of I in $C_X(y)$ is trivial, then x centralizes $J/O_{p'}(J)$ and it follows that I maps onto the quasisimple group $J/O_{p'}(J)$. Hence in this case, $J = IO_{p'}(J)$ and consequently $I/O_{p'}(I) \cong J/O_{p'}(J)$.

Proposition 7.1(ii) links the p-components in the centralizer of one element of order p in X to those in the centralizer of another (commuting) element of order p, and enables one to analyze the configuration of p-components of the centralizers of all the elements of order p in X by means of the pumping-up process. That analysis in fact revolves around such p-components that are maximal in the sense that they possess only trivial pumpups. Such p-components are referred to as p-**terminal**, a term we now define precisely.

However, to do so requires preliminary notation. First, we use the notation $C_X(A/B)$, whenever A and B are subgroups of X with $B \lhd A$, to mean the subgroup consisting of all $y \in X$ such that $[y, A] \leq B$; equivalently, $C_X(A/B)$ is the largest subgroup of $N_X(A) \cap N_X(B)$ acting trivially by conjugation on A/B. Next, for any element x of order p in X and any p-component K of $C_X(x)$, we let $C(K, x) = C_{C_X(x)}(K/O_{p'}(K))$, the subgroup of $C_X(x)$ consisting of all elements leaving K invariant and centralizing $K/O_{p'}(K)$. Note that if K is quasisimple (equivalently, if K is a component of $C_X(x)$), then $C(K, x) = C_X(K) \cap C_X(x)$.

If $y \in C(K, x)$ is of order p, then as y centralizes $K/O_{p'}(K)$ and has order prime to $|O_{p'}(K)|$, it can be shown that $K = C_K(y)O_{p'}(K)$. Hence if we set $I = L_{p'}(C_K(y))$, then likewise I maps onto $K/O_{p'}(K)$ and $K = IO_{p'}(K)$, whence

$$I/O_{p'}(I) \cong K/O_{p'}(K).$$

Furthermore, we see that I is in fact a p-component of $L_{p'}(C_X(x)) \cap C_X(y)$. In particular, if K is quasisimple, then clearly $I = K$.

Thus in any case, Proposition 7.1(ii) applies to I. In this case we write K_y for the pumpup of I in $C_X(y)$, and sometimes call K_y a pumpup of K rather than of I. In particular, if K_y is a trivial pumpup of I, it follows that I maps onto both $K/O_{p'}(K)$ and $K_y/O_{p'}(K_y)$, so that

$$K/O_{p'}(K) \cong I/O_{p'}(I) \cong K_y/O_{p'}(K_y).$$

Note also that if $a \in C(K, x)$ centralizes y, then a leaves I invariant and as I covers $K/O_{p'}(K)$, a centralizes $I/O_{p'}(I)$. Moreover, since K_y is the normal closure of I in $L_{p'}(C_X(y))$, a must leave K_y invariant. But now if K_y is a trivial pumpup, then I covers $K_y/O_{p'}(K_y)$, and so it follows that a centralizes $K_y/O_{p'}(K_y)$, whence $a \in C(K_y, y)$. Hence in the trivial pumpup case, we conclude that

$$C(K, x) \cap C_X(y) \leq C(K_y, y).$$

With all these preliminaries, we are now ready to introduce the notion of a p-terminal p-component.

Definition 7.2. If K is a p-component of $C_X(x)$ for some $x \in X$ of order p, we say that K is p-**terminal** in X (more properly, the **pair** (K, x) is p-terminal in X) if and only if the following two conditions hold for every $y \in C(K, x)$ of order p:

(1) $L_{p'}(C_K(y))$ has a trivial pumpup K_y in $C_X(y)$; and
(2) If $Q \in Syl_p(C(K, x))$ and $y \in Z(Q)$, then $Q \in Syl_p(C(K_y, y))$.

The preceding discussion shows that if (1) holds, and Q and y satisfy the hypothesis of (2), then $Q \leq C(K_y, y)$. Also, we use the notation $Syl_p(Y)$ for the set of Sylow p-subgroups of the group Y.

One establishes the following general result.

Theorem 7.3. *If the centralizer of some element of X of order p has a p-component, then there exists an element $x \in X$ of order p and a p-component K of $C_X(x)$ with K p-terminal in X.*

One can in fact prove a much sharper result. Indeed, beginning with any $x_1 \in X$ of order p and p-component K_1 of $C_X(x_1)$, one can reach x and K with K p-terminal in X by following a suitable sequence of pumpups. Thus every p-component of the centralizer of an element of order p in X can be "pumped up" to a p-terminal p-component in X. Special versions of Theorem 7.3, with extra hypotheses and stronger conclusions, will be established in later volumes of this series.

A considerable portion of the p-local analysis of the general simple group X is devoted to eliminating the p'-core "obstruction" of p-terminal p-components and their pumpups. The desired conclusions are incorporated into the following definition.

Definition 7.4. If K is a p-terminal p-component of $C_X(x)$ for some $x \in X$ of order p, we say that K is **terminal** in X if and only if K is a *component* of $C_X(y)$ for every element $y \in C(K, x)$ of order p.

In particular, the definition implies (with $y = x$) that K itself must then be a component of $C_X(x)$. It also implies that K is a component of $C_X(y)$ for every

$y \in C_X(K)$ of order p. Indeed, let Q and R be Sylow p-subgroups of $C(K,x)$ and $C_X(K)$, respectively, with $Q \le R$. Since $C(K,x) = C_X(K) \cap C_X(x)$, we have $Q = C_R(x)$. Choose $z \in Z(R)$ of order p. Then $z \in Q$, so that by definition of p-terminality, Q is a Sylow subgroup of $C(K,z)$. This implies that $Q = R$. Hence the desired assertion follows from the definition of terminality. [We prefer "terminal" to the current term "nonembedded" to describe p-components satisfying the given conditions, because of the number of distinct usages of "embedded" in finite group theory.]

The following result, which is a direct consequence of the definitions and Proposition 5.5, expresses the operational conditions for a p-terminal p-component to be terminal.

PROPOSITION 7.5. *If K is a p-component of $C_X(x)$ for some $x \in X$ of order p and K is p-terminal in X, then K is terminal in X if and only if K_y centralizes $O_{p'}(C_X(y))$ for every element $y \in C(K,x)$ of order p.*

The existence of terminal p-components is closely linked to the so-called B_p-property. A group X with $O_{p'}(X) = 1$ is said to have the **B_p-property** if and only if every p-component of the centralizer of every element of order p in X is quasisimple. This general property has been verified directly for almost simple $\mathcal{K}$-groups, and in this form is needed for the classification proof. Moreover, it has the following immediate consequence, analogous to $L_{p'}$-balance. For any group X define $B_{p'}(X)$ to be the product of all p-components L of X such that L is not quasisimple (equivalently: $L \not\le E(X)$).

THEOREM 7.6. *Let X be a group in which all almost simple sections with trivial p'-core have the B_p-property and the Schreier property. Then $B_{p'}(Y) \le B_{p'}(X)$ for every p-local subgroup Y of X.*

As noted, once the Classification Theorem is proved, it implies that the hypothesis of Theorem 7.6 holds for any group X.

8. p-constrained and p-solvable groups

As the discussion of the generalized Fitting subgroup suggests, the study of an arbitrary group X has a natural division into the analysis of its solvable substructure on the one hand, and its quasisimple substructure on the other. Thus, in order to analyze the p-local subgroups of X for some prime p, it is necessary to consider not only the p-components of its p-locals, but also its **p-solvable** substructure, p-solvability being a form of solvability "localized at the prime p."

By definition, a group X is **p-solvable** if and only if every composition factor of X has order either equal to p or relatively prime to p. Clearly every solvable group is p-solvable for any prime p. Conversely, in view of the solvability of groups of odd order, every 2-solvable group is solvable. Furthermore, any p-solvable group is p-constrained.

In any group X, the subgroup $O_{p'p}(X)$ is p-solvable. [By definition, $O_{p'p}(X)$ is the full preimage in X of $O_p(X/O_{p'}(X))$.] The study of the p'-cores of p-locals relies on various basic results about p-groups acting on p'-groups. Underlying many of these results is the fundamental Schur-Zassenhaus theorem.

THEOREM 8.1. *If Y is a normal subgroup of X whose order is relatively prime to its index in X, then*

(i) *Y possesses a complement in X (i.e., there is $A \leq X$ such that $X = YA$ and $Y \cap A = 1$), and*
(ii) *Any two complements to Y are conjugate in X.*

The only known proof of (ii) uses the solvability of one of the groups Y and X/Y, which in turn depends on the solvability of groups of odd order.

The key properties of p-solvable groups are contained in the following variation of Sylow's theorem, due to Philip Hall. If σ is a set of primes, we define a **σ-subgroup** of H to be a subgroup whose order is divisible only by primes in σ. We define a **Hall σ-subgroup** of H to be a σ-subgroup W such that $|H : W|$ has no prime divisors in σ.

THEOREM 8.2. *If $X = YA$ with Y normal in X and $(|Y|, |A|) = 1$, then*
(i) *For any prime q,*
  (1) *A leaves invariant a Sylow q-subgroup of Y;*
  (2) *Any two A-invariant Sylow q-subgroups of Y are conjugate by an element of $C_Y(A)$; and*
  (3) *Any A-invariant q-subgroup of Y is contained in an A-invariant Sylow q-subgroup of Y.*
(ii) *If Y is solvable and σ is any set of primes, then the assertions of (i) hold with "(Hall) σ-subgroup" in place of "(Sylow) q-subgroup".*

One also has the following key generational result in the above situation.

PROPOSITION 8.3. *Assume that $X = YA$ with Y normal in X and $(|Y|, |A|) = 1$. If A is abelian but not cyclic, then*

$$Y = \langle C_Y(a) \,|\, a \in A^{\#} \rangle.$$

In the analysis of p-local structure, Theorem 8.2 and Proposition 8.3 are applied with A a p-group.

Finally, we consider the structure of p-constrained groups Y such that $O_{p'}(Y) = 1$, that is, groups such that $F^*(Y) = O_p(Y)$. By a theorem of Borel and Tits [**BuWi1**], all p-locals in simple groups of Lie type defined with respect to fields of characteristic p enjoy this property. Accordingly, when all p-locals in an arbitrary simple group X have this property, X is said to have characteristic p-type[9]. If X is simple of characteristic p-type, then the study of p-locals by their p'-cores and p-components that we have been discussing is vacuous. Instead, the focus of analysis is on chief factors $V = Y_1/Y_2$ of p-local subgroups Y with $Y_1 \leq O_p(Y)$ (we say that V is "within" $O_p(Y)$). The aim is to show that unless certain specific types of chief factors occur, then the structure of Y is controlled by the normalizers of certain characteristic subgroups of a Sylow p-subgroup P of Y. The purpose is to be able to link the structures of various p-local subgroups.

Again we have to introduce some terminology.

DEFINITION 8.4. If X is a group with $O_p(X) = 1$ and V is a faithful $\boldsymbol{F}_pX$-module, then V is a **quadratic $\boldsymbol{F}_pX$**-module with respect to the elementary abelian p-subgroup A of X if and only if $A \neq 1$ and $[V, A, A] = 1$ in the semidirect product of V by X.

Returning to a chief factor V of the group Y within $O_p(Y)$, we have that V is a faithful irreducible $\boldsymbol{F}_p\overline{Y}$-module for the group $\overline{Y} = Y/C_Y(V)$, which quickly implies

[9]The complete definition requires the p-rank to be at least 2, and at least 3 if $p = 2$.

that $O_p(\overline{Y}) = 1$. Thus we shall call V quadratic with respect to the elementary abelian p-subgroup $\overline{A}$ of $\overline{Y}$ if and only if V is a quadratic $\boldsymbol{F}_p\overline{Y}$-module with respect to $\overline{A}$ in the sense of Definition 8.4; and we simply call V **quadratic** if and only if it is quadratic with respect to some elementary abelian p-subgroup $\overline{A} \neq 1$ of $\overline{Y}$.

If $A = \langle g \rangle$ has order p in Definition 8.4, then that definition requires that the minimal polynomial of g in its action on V be $(t-1)^2$, rather than $(t-1)^p = t^p - 1$, as one might expect. Of course if $p = 2$ and X has even order, any faithful $\boldsymbol{F}_2X$-module is quadratic with respect to any subgroup of X of order 2.

Thus the notion is nontrivial only for odd p or $|A| \geq 4$. In these cases, quadratic modules are rather rare among all isomorphism types. For example, if $p \geq 5$ and G is solvable with $O_p(G) = 1$, then G has no quadratic modules. This is a corollary of P. Hall and G. Higman's Theorem B [**HaHi1**].

At the start of the theory of simple groups of characteristic p-type, one has the following two celebrated theorems. For any p-group P, we define the **Thompson subgroup** $J(P)$ to be the subgroup of P generated by all the elementary abelian subgroups of P of largest possible order. We also define $\Omega_1(B)$, for any abelian p-group B, to be the largest subgroup of B of exponent p. Then $J(P)$ and $\Omega_1(Z(P))$ are characteristic subgroups of P, and both are nontrivial if P is nontrivial.

The ZJ-theorem has a number of closely related versions, one of which is the following.

THEOREM 8.5. (Glauberman's ZJ-theorem) *Let p be an odd prime and let Y be a group such that $F^*(Y) = O_p(Y)$. Let P be a Sylow p-subgroup of Y. If no chief factor of Y within $O_p(Y)$ is quadratic, then $\Omega_1(Z(J(P))) \lhd Y$.*

If one weakens the hypotheses of this theorem by allowing $p = 2$ or more generally by allowing quadratic modules, one cannot hope to name a single characteristic subgroup of P which will be necessarily normal in Y. However, the Thompson factorization theorem shows that under some conditions two characteristic subgroups exist whose normalizers together cover Y. For that theorem, the obstructions are fewer: they form a subset of the full set of quadratic modules. If V is a faithful $\boldsymbol{F}_pX$-module, call V a **failure of factorization module** if and only if X has an elementary abelian p-subgroup $A \neq 1$ such that

$$|A| \geq |V/C_V(A)|. \tag{8.1}$$

THEOREM 8.6. (Thompson) *If $F^*(Y) = O_p(Y)$ and $V = \Omega_1(Z(O_p(Y)))$ is not a failure of factorization module for $Y/C_Y(V)$, then $Y = N_Y(J(P))C_Y(\Omega_1(Z(P)))$.*

On the other hand, the Thompson Replacement Theorem implies that if V is a failure of factorization module for $\overline{Y} = Y/C_Y(V)$, then V is a quadratic $\boldsymbol{F}_p\overline{Y}$-module with respect to some elementary abelian p-subgroup $\overline{A} \neq 1$ satisfying (8.1). This shows the connection between failure of factorization and quadratic modules.

The presence of quadratic or failure of factorization modules complicates the theory of groups of characteristic p-type—unavoidably so, since quadratic modules are ubiquitous in the parabolic subgroups of groups of Lie type. In their presence, the more subtle method of amalgams can be used to link the structures of the p-local subgroups. This method is discussed in some detail in section 33 of the next chapter.

C. Classifying Simple Groups

9. Internal analysis: targeting local structure

From the more general structural properties of finite groups which we have been discussing, we begin in this section to narrow our focus toward the classification. We discuss some important aspects of the way in which the simple groups were actually classified, including mention of ways to recognize them. We also preview our general strategy.

At the outset of the proof of the classification theorem, one is faced with a $\mathcal{K}$-proper simple group G. Because of Sylow's theorem, G has an abundance of p-subgroups for various primes p, and hence (by taking normalizers) an abundance of local subgroups. The bulk of the analysis of G then focuses on the subgroup structure of G, and in particular on the structure of local subgroups and the relations among their embeddings in G. In a few places certain objects external to G, namely linear representations, must be considered. Although these situations are crucial, they are isolated and occur in cases where G is "small" in some sense. Overwhelmingly, the classification proof, both in its original form and as revised in this series, consists of an analysis of the local subgroups of G.

The goal is to show that for some known simple group G^*, the local subgroup structures of G and G^* resemble each other so much that one can then prove, as will be discussed in Section 11, that $G \cong G^*$.

Because of very strong family similarities among the local structures of all the simple groups of Lie type, and also among the alternating groups, the "target" subgroup structure of G^* is extremely restricted in nature. Therefore it is a massive step to pass from the initial data that all proper subgroups of G are $\mathcal{K}$-groups to the goal that G and G^* locally resemble each other.

We illustrate the magnitude of this passage with some comments about the structure of centralizers of involutions and other local subgroups in the known simple groups. If C is the centralizer in the $\mathcal{K}$-proper simple group G of an involution x, then *a priori*, C may have arbitrarily many nonsolvable composition factors of arbitrarily varied isomorphism types and, in addition, C may contain normal subgroups of odd order of arbitrarily large Fitting lengths.

On the other hand, consider the centralizer C^* of an involution x^* in one of the known simple groups G^*. As a first example, we take $GL_n(q)$, q odd, with x^* in diagonal form with k eigenvalues -1 and $n-k$ eigenvalues $+1$. [It is easier for expository purposes to make the calculations in $GL_n(q)$; to obtain the centralizer in the simple group $PSL_n(q)$ of the image of x^*, one must of course modify this calculation slightly.] One computes that the elements of C^* are the block matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A \in GL_k(q)$ and $B \in GL_{n-k}(q)$, which implies that

$$C^* \cong GL_k(q) \times GL_{n-k}(q).$$

In particular, we see that C^* has at most 2 nonsolvable composition factors, which if they exist are isomorphic to $PSL_k(q)$ and $PSL_{n-k}(q)$. This latter statement is true as well for $G^* = PSL_n(q)$.

Similarly in the case of the alternating group A_n with x^* the "short" involution $(12)(34)$, one computes that C^* contains a normal subgroup C_0^* of index 2 of the form

$$C_0^* \cong E_4 \times A_{n-4}.$$

Hence in this case, C^* has at most 1 nonsolvable composition factor, which if it exists is isomorphic to A_{n-4}.

The following general result gives the structure of centralizers of involutions in arbitrary known simple groups; it will be verified in later volumes dealing with properties of $\mathcal{K}$-groups.

THEOREM 9.1. *If G^* is a simple $\mathcal{K}$-group, x^* is an involution in G^*, and $C^* = C_{G^*}(x^*)$, then the following conditions hold:*

(i) *C^* has at most 2 nonsolvable composition factors, unless G^* is an orthogonal group, in which case C^* has at most 4 nonsolvable composition factors;*

(ii) *If G^* is of Lie type of characteristic r, then each nonsolvable composition factor of C^* is of Lie type of characteristic r; and*

(iii) *$O(C^*)$ is cyclic.*

[The doubling factor in the orthogonal case arises because one of the 4-dimensional orthogonal groups is itself a central product of two copies of $SL_2(q)$.]

Since the automorphism group of a cyclic group is abelian, the cyclicity of $O(C^*)$ yields the B_2-property as an important corollary.

COROLLARY 9.2. *If C^* is the centralizer of an involution x^* in one of the known simple groups G^*, then*

$$L_{2'}(C^*) = E(C^*).$$

Other local subgroups of the known simple groups similarly are restricted in their structures. Indeed, many of the local subgroups X of G^* crucial for the classification have one of the two following general forms:

(1) X has a subgroup X_0 of very small index (usually 1 or 2) such that the components of $F^*(X)$ are normal subgroups of X_0 and $X_0/F^*(X)$ is abelian; or

(2) $F^*(X) = R$ is a p-group for some prime p (so that $X/F^*(X)$ acts faithfully by conjugation on the elementary abelian p-group $R/\Phi(R)$).

In (1), X_0 has structure analogous to that of a closed connected reductive subgroup of a linear algebraic group; in (2), X is analogous to a parabolic subgroup of a linear algebraic group.

Given the extremely strong implications that the isomorphism between the $\mathcal{K}$-proper simple group G and the known simple group G^* has for the local structure of G, it is reasonable to ask how likely it will ever be to establish an isomorphism between G and G^* *without first forcing critical portions of the subgroup structure of G to closely approximate those of the target group G^*.*

However this obviously rhetorical question is eventually resolved, at present the only known method for achieving the desired isomorphism is by means of a very long and detailed investigation of the local subgroups of G and their interrelations. That analysis depends upon a combination of general results of finite group theory coupled with a veritable "dictionary" of properties of $\mathcal{K}$-groups. As has been indicated, only after this analysis has forced G to have a local subgroup structure approximating that of one of the target groups G^* has it been possible until now to produce a presentation for G identical to one for G^*.

10. Internal analysis: passing from global to local information

How is one to use the *global* hypothesis that G is simple to analyze the *local* structure of the minimal counterexample G to the classification theorem? The negative flavor of the hypothesis—G has no nontrivial quotients—makes it difficult to use. The initial observation that G is a $\mathcal{K}$-proper group is vital but insufficient in itself. Beyond that, the following obvious consequences of simplicity affect the local structure of G by means of some important techniques which we shall discuss. Here $S(G)$ is the largest solvable normal subgroup of G.

(10.1)
(1) G has no proper abelian quotient, i.e., $G = [G, G]$;
(2) $S(G) = 1$; and
(3) $E(G)$ has no proper direct factor.

Before discussing the techniques to exploit these, we make two digressions. First we note that in hindsight these three properties form a complete characterization of simplicity, in the sense that a $\mathcal{K}$-group G (and hence an arbitrary group) is simple if and only if it satisfies (10.1). Indeed, conditions (2) and (3) imply that if we set $K = F^*(G)$, then $K = E(G)$ is simple, so that G/K embeds in the outer automorphism group $Out(K)$ of K. As noted previously, it has been verified that every simple $\mathcal{K}$-group has the Schreier property, so G/K is solvable. However, G/K is perfect by (1), so $G = K$ is a simple group.

Second, we briefly recall the effects of the conditions analogous to (10.1) in another theory—the Killing-Cartan classification of finite-dimensional simple Lie algebras over the complex field. If $\mathfrak{g}$ is such an algebra, then Cartan's criterion translates semisimplicity, the analogue of (2), into the nondegeneracy of the Killing form κ. When κ is restricted to a Cartan subalgebra $\mathfrak{h}$, the resulting form $\kappa_{\mathfrak{h}}$ is shown to be nondegenerate and the analysis of the root system begins. One consequence which is analogous to the B_p-property is that for any $x \in \mathfrak{h}$, the centralizer $\mathfrak{Z}_{\mathfrak{g}}(x)$ is reductive, so that it is the direct sum of simple algebras (including 1-dimensional ones). On the other hand, the power of Cartan's criterion makes the analogue of (1) superfluous in the presence of (2), since it forces a semisimple algebra to be the direct sum of simple nonabelian ideals. Finally, the analogue of (3), that $\mathfrak{g}$ is indecomposable, implies that the root system is indecomposable.

Now we return to the techniques for exploiting (10.1). The oldest of these, the theory of transfer and fusion, shows how for each prime p, the structure of the largest abelian p-factor group of G is completely determined by the structure and embedding of the normalizers of certain nontrivial subgroups of a fixed Sylow p-subgroup of G. Accordingly, (10.1)(1) forces restrictions on those local subgroups. Results of this kind were obtained by Frobenius and Burnside; for example, Burnside proved that if G is simple then $|G|$ is divisible by 4 or p^3 for some odd prime p. The subject has been developed steadily by many authors through this century, more recently with the notable theories of Alperin and Yoshida [**Al1, Y1**]. In the original classification proof as well as in our revision, however, transfer comes into play only in low rank situations and in situations where considerable local information about G has already been amassed.

By contrast, exploiting the condition $S(G) = 1$ is a highly technical and delicate matter, and central to our problem. The principal method for exploiting it has been the **signalizer functor** method pioneered by Gorenstein and Walter [**G2,**

GW3, GW4]. It can be attempted whenever G contains an elementary abelian p-subgroup A whose p-rank[10] $m_p(A)$ is at least 3. The subgroup A may be viewed as analogous to a Cartan subalgebra of a Lie algebra, although the analogy is weak in some respects. Under certain conditions, one can associate to each $a \in A^\#$ an A-invariant p'-subgroup $\theta(C_G(a))$ of $C_G(a)$ satisfying the "balance" condition

$$\theta(C_G(a)) \cap C_G(b) = \theta(C_G(b)) \cap C_G(a) \quad \text{for all } a,\, b \in A^\#.$$

The mapping θ is then called an A**-signalizer functor on** G, and there are various signalizer functor theorems asserting that under certain hypotheses, the closure $\theta(G; A) = \langle \theta(C_G(a)) \mid a \in A^\# \rangle$ is a p'-group. For example, if $\theta(C_G(a))$ is a $\mathcal{K}$-group for all $a \in A^\#$, a theorem of McBride [**McB2**] gives this conclusion. Furthermore, in many situations it can also be shown that $\theta(G; A)$, or some appropriately defined subgroup of it, is normal in G. By the simplicity of G, that subgroup is trivial, which in turn yields consequences for the structure of $C_G(a)$, $a \in A^\#$, because of the way θ was originally defined. When the method succeeds, the result generally is that for every $a \in A^\#$, $C_G(a)$ is "reductive" in the sense that $L_{p'}(C_G(a)) \le E(C_G(a))$, or equivalently every p-component of $C_G(a)$ is actually a component. This is a partial B_p-property, just for the elements of A. [The signalizer functor method really uses the condition $O_{p'}(G) = 1$ rather than $S(G) = 1$. However, this distinction is minor, and indeed when $p = 2$, $S(G) = 1$ implies $O(G) = 1$ by the Odd Order Theorem.]

The signalizer functor method cannot be undertaken when G has p-rank at most 2 for all primes p. In this case and in other low-rank situations, the **Bender method** is often useful (e.g,. [**Be2, Be5**]). It analyzes the structure and embedding of certain maximal subgroups of G and their generalized Fitting subgroups—particularly those maximal subgroups containing the centralizer of an involution. For example this method underlies an elegant proof of Burnside's theorem that groups of order $p^a q^b$ are solvable, not using the theory of characters as Burnside did (for example, see [**Su1**]). Until the 1960's no such proof was known.

A further vital way to exploit $S(G) = 1$ is to use Glauberman's Z^***-theorem** [**Gl2**], which can be rephrased in the following way. If $O(G) = 1$ and an involution $z \in G$ lies in $Z(N)$ for certain 2-local subgroups N of G containing z, then $z \in Z(G)$. Again, in our simple group G, $Z(G) = O(G) = 1$ (by the Feit-Thompson theorem), so the Z^*-theorem gives information about various 2-local subgroups N.

The third condition (10.1)(3), that $E(G)$ has only one component, is exploited primarily through p**-component uniqueness theorems**, developed by Aschbacher and others [**A4, Gi1, S1, PoTh1**] for $p = 2$ and to be generalized by us in [II_2] to arbitrary primes p for $\mathcal{K}$-proper simple groups. When successfully combined with the signalizer functor method, these theorems often produce an element x of G of order p and a component L of $E(C_G(x))$ such that $C_G(L)$ has very small p-rank, and such that x has a G-conjugate normalizing but not centralizing L.

These p-component uniqueness theorems, as well as certain parts of the signalizer functor method and the Bender method, rely ultimately on the construction of a **strongly embedded subgroup** and then on the classification theorem due to Bender and Suzuki [**Be3, Su4**] of finite groups with such a subgroup. By definition,

[10]The p**-rank** $m_p(X)$ of a group X is the largest nonnegative integer n such that X contains an elementary abelian subgroup of order p^n.

the subgroup H of G is strongly embedded in G if and only if $H < G$, H has even order, and $H \cap H^x$ has odd order for all $x \in G - H$.

The Bender-Suzuki theorem asserts that if G possesses a strongly embedded subgroup and a subgroup isomorphic to E_4, then G has a unique composition factor L of even order, and L is a group of Lie type of Lie rank 1 and characteristic 2.

The importance of the Bender-Suzuki theorem resides in the following fact. Analysis of a group by its p-local subgroups is most effective when G is generated by a set of p-local subgroups having a large p-subgroup in common; but is ineffective at the other extreme, when for some Sylow p-subgroup T of G and some subgroup $H < G$, H contains $N_G(Q)$ for every nonidentity p-subgroup Q of T. When this occurs for $p = 2$, however, H is strongly embedded in G (as is readily seen from the definition) and the Bender-Suzuki theorem provides a critical step. Indeed, such "strongly p-embedded" subgroups H are encountered for odd primes p as well, but when p is odd their existence can actually be shown in crucial situations to lead to the existence of strongly embedded subgroups.

Thus the Bender-Suzuki theorem is a pillar of the classification proof. Its proof is deep and difficult, and like all of the techniques for exploiting simplicity of the group G, it eventually relies on the basic property that every nontrivial homomorphism on G has trivial kernel. In particular, it relies on character theory, that is, analysis of the linear representations of the group in question. Likewise the proofs of some other basic results, such as the Feit-Thompson theorem and the Z^*-theorem, rely on representation theory, usually in combination with local analysis.

Other homomorphisms on G, such as transfer homomorphisms or permutation representations, can be regarded as internal or external to G, depending on one's point of view. To the extent that linear representations are used, one cannot call the classification proof "internal." But as already observed, the uses of representation theory are limited to a few well-defined situations, such as the theorems mentioned above and a part of the analysis of groups of 2-rank two. Such situations are even more limited in our revised strategy than in the original proof. Beyond those situations, the classification proof is truly internal, an intricate study of the local subgroups of G.

11. Identifying simple groups

The final stage of every classification theorem involves an identification of the group G under investigation with some known simple group G^* by means of a set of intrinsic conditions which characterize G^*. A description of the various sets of conditions for achieving this recognition appears in [**G3**]. Here we present only a summary of that discussion.

There are three primary methods of identifying the simple groups:

A. By presentations (i.e., by generators and relations).
B. By the action of the groups on a suitable geometry.
C. By a primitive permutation representation.

For example, the symmetric group Σ_n is generated by the transpositions $x_1 = (12)$, $x_2 = (23), \ldots, x_{n-1} = (n-1, n)$, which satisfy the conditions

$$(x_i x_j)^{r_{ij}} = 1, \text{ where } \begin{cases} r_{ij} = 1 \text{ if } i = j, \\ r_{ij} = 2 \text{ if } |j-i| > 1, \text{ and} \\ r_{ij} = 3 \text{ if } |j-i| = 1. \end{cases} \tag{11.1}$$

Moreover, it can be shown that every relation among the x_i's is a consequence of the relations (11.1), so that by definition the x_i's together with the relations (11.1) provide a presentation for Σ_n.

This means that if G is an arbitrary group with generators $x_1', x_2', \ldots, x_{n-1}'$ satisfying the corresponding relations (11.1), then the mapping $x_i \mapsto x_i'$, $1 \le i \le n-1$, extends to a homomorphism of Σ_n onto G. Thus the existence of the given presentation for Σ_n in terms of its generating involutions $x_1, x_2, \ldots, x_{n-1}$ enables one to identify G as a symmetric group. A slight variation of the conditions (11.1) provides a similar presentation for alternating groups.

Every group $G(q)$ of Lie type, where $q = p^n$, p a prime, has a distinguished presentation [**St1, St4**] called the **Steinberg presentation**, expressed in terms of relations (the "Steinberg relations") among elements of so-called **root subgroups** X_α These X_α are p-groups.

In particular, when $G(q)$ is an untwisted group, the X_α are isomorphic to the additive group of $\boldsymbol{F}_q$ and so can be described parametrically as

$$X_\alpha = \{x_\alpha(t) \mid t \in \boldsymbol{F}_q\},$$

with $x_\alpha(t+u) = x_\alpha(t)x_\alpha(u)$. Furthermore, the indices α run over the **roots** of the indecomposable **root system** Σ of the associated complex finite-dimensional Lie algebra. There is a (non-canonical) ordering $<$ on Σ, which in particular splits Σ into sets of positive and negative roots Σ^+ and Σ^-, respectively, so that Σ is the disjoint union $\Sigma^+ \cup \Sigma^-$ and $\Sigma^- = -\Sigma^+$. The indecomposable elements of Σ^+ form a **fundamental system** Π for Σ. The Dynkin diagram has nodes labelled by the elements of Π; it encodes the geometry of Π, which in turn determines Σ completely. In particular, Π is a basis of $\boldsymbol{R}\Sigma$. (When $G(q)$ is a twisted group, the X_α have more complicated structures and Σ is a possibly "non-reduced" root system.)

The **Lie rank**[11] of $G(q)$ is defined as the rank of Σ, which is the dimension of the ambient Euclidean space $\boldsymbol{R}\Sigma$. Whenever the Lie rank is at least 2 the key Steinberg relation, other than the relations defining the individual subgroups X_α, is the **Chevalley commutator formula**. It applies to any linearly independent $\alpha, \beta \in \Sigma$ and to each $x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$. These relations have the following form:

$$[x_\alpha, x_\beta] = \prod_\gamma x_\gamma. \tag{11.2}$$

[11] The Lie rank is sometimes also called the **twisted** Lie rank. There is a second notion of Lie rank, sometimes called the **untwisted** Lie rank; the two notions coincide for the untwisted groups. The untwisted Lie rank of a twisted group $G(q)$ is the subscript in the Lie notation for $G(q)$, or equivalently the Lie rank of the ambient algebraic group; it is the Lie rank of the untwisted group which was twisted to form $G(q)$.

Here γ runs over all elements of Σ of the form $\gamma = i\alpha + j\beta$, with i and j are positive; the x_γ are suitable elements of X_γ whose parameters are given explicitly in terms of those of x_α and x_β, and the order of the product is given by $<$.

For groups of Lie rank at least 2, the conditions (11.2) actually provide a presentation not of the simple group $G(q)$, but of its **universal version** $\hat{G}(q)$, which is a central extension of $G(q)$. In all but a few cases $\hat{G}(q)$ is actually the universal covering group of $G(q)$. Thus it follows that an abstract group G which contains subgroups $\hat{X}_\alpha$ isomorphic to X_α and indexed by the given root system Σ, and satisfying the corresponding relations (11.2) is necessarily isomorphic to a homomorphic image of $\hat{G}(q)$; if G is simple, then $G \cong G(q)$.

It is the latter fact that explains the procedure for identifying the abstract simple group as a given group $G(q)$ of Lie type. Sufficient information about the subgroup structure of G must be established to enable one to specify a prime p and to associate with G a root system Σ of type $G(q)$ and p-subgroups $\hat{X}_\alpha$ of G that behave like the root subgroups of $G(q)$ relative to the root system Σ—i.e., which satisfy the corresponding relations of (11.2).

[In practice, the analysis leads to the construction of a subgroup

$$G_0 = \langle \hat{X}_\alpha \,|\, \alpha \in \Sigma \rangle$$

of G isomorphic to the target group $G(q)$ or to a homomorphic image of $\hat{G}(q)$. There remains the entirely separate problem of proving that $G = G_0$. This remark applies equally well to the case in which the target group is an alternating group. This final step is generally accomplished with the aid of the Bender-Suzuki theorem.]

The X_α (and likewise the $\hat{X}_\alpha$) satisfy many conditions that are consequences of the Steinberg relations. Thus

$$U = \langle X_\alpha \,|\, a \in \Sigma^+ \rangle \text{ and } V = \langle X_\alpha \,|\, a \in \Sigma^- \rangle$$

are each Sylow p-subgroups of $G(q)$. Also for any $\alpha \in \Sigma$, $\langle X_\alpha, X_{-\alpha} \rangle$ is a rank one group of Lie type; in the untwisted case,

$$\langle X_\alpha, X_{-\alpha} \rangle \cong SL_2(q) \text{ or } PSL_2(q).$$

Furthermore, a **Cartan subgroup** H of $G(q)$ can be expressed in terms of the X_α's. In particular, in the untwisted case if one sets

$$n_\alpha(t) = x_\alpha(t) x_{-\alpha}(t^{-1}) x_\alpha(t) \text{ and } h_\alpha(t) = n_\alpha(t) n_\alpha(1)^{-1}, \qquad t \in \boldsymbol{F}_q^\times,$$

then

$$H = \langle h_\alpha(t) \,|\, \alpha \in \Sigma, t \in \boldsymbol{F}_q^\times \rangle.$$

The group H is abelian and isomorphic to the factor group of the direct product of m (= rank of Σ) copies of $\boldsymbol{F}_q^\times$ by a subgroup isomorphic to $Z(\hat{G}(q))$.

In addition, in general

$$N_{G(q)}(U) = UH \text{ and } N_{G(q)}(V) = VH, \text{ with } U \cap H = V \cap H = 1.$$

Likewise the **Weyl group** W of Σ is recoverable from the Steinberg relations. Again in the untwisted case if one sets

$$N = \langle n_\alpha(t) \,|\, \alpha \in \Sigma, t \in \boldsymbol{F}_q^\times \rangle,$$

one has

$$H \lhd N \text{ and } N/H \cong W.$$

Here W is a finite **Coxeter group** and thus a finite group **generated by reflections**, i.e., W is generated by m involutions w_i, $1 \le i \le m$, where m is the rank of Σ, and a presentation for W is afforded by $w_1, \ldots, w_m$ and a set of relations analogous to (11.1), but with the r_{ij} allowed to take values other than 1, 2, and 3. In particular, the symmetric group Σ_{n+1} is the Weyl group of $A_n(q) \cong PSL_{n+1}(q)$. We note also that W acts as a group of isometries of Σ.

N is often called a **monomial** subgroup of $G(q)$.

We call the subgroup

$$R = \langle n_\alpha(1) \mid \alpha \in \Sigma \rangle$$

a **reduced** monomial subgroup of $G(q)$. If q is even, $R \cong W$, while if q is odd, $R/E \cong W$, where $E = R \cap H$ is a normal elementary abelian 2-subgroup of R. In either case we have

$$N = HR.$$

Also R induces a group of permutations on the set Σ. Analogous results hold for the twisted groups.

Finally, it follows that $G(q)$ has the **Bruhat decomposition**

$$G(q) = BNB,$$

where $B = N_{G(q)}(U) = UH$; B is a **Borel subgroup** of $G(q)$.

Indeed, Tits has used the subgroups B, N, and W of $G(q)$ to give an alternative description and characterization of the groups of Lie type based on the notion of a (B, N)-pair.

DEFINITION 11.1. A group G is said to be a (B, N)-**pair** or have a **Tits system** of rank m if and only if G has subgroups B and N such that:

(i) $G = \langle B, N \rangle$;
(ii) $H := B \cap N$ is normal in N;
(iii) $W := N/H$ is generated by involutions $w_1, \ldots, w_m$; and
(iv) If v_i is a representative of w_i in N, then for each $v \in N$ and every i,
$$vBv_i \subseteq BvB \cup Bvv_iB \text{ and } v_iBv_i \not\subseteq B.$$

Moreover, G is said to be **split** if $B = (B \cap N)U = HU$, where U is a normal nilpotent subgroup of B.

A group $G(q)$ of Lie type is a split (B, N)-pair with respect to the subgroups B, N, H mentioned above.

All simple split (B, N)-pairs G (with G finite) have been determined, which enables one to identify G (more precisely, a particular subgroup G_0 of G) as a group of Lie type by means of its generation by such a pair of subgroups B and N. When the rank of G is at least 3, this is a consequence of a geometric result of Tits, classifying suitable **buildings** [**Ti4**]. Tits's result can be viewed as a broad generalization of the fact that for $n \ge 3$, any finite n-dimensional projective space is n-dimensional projective space over $\boldsymbol{F}_q$, for some q. Partial extensions of Tits's results have been obtained for (B, N)-pairs of rank 2, but the complete classification of split (B, N)-pairs of ranks 1 and 2 has been obtained only by permitting group-theoretic and generator-relation type arguments[12].

Curtis and Tits [**Cu1, Ti2**] have obtained a more efficient method of identifying the groups of Lie type of rank ≥ 3. In effect, their results assert that a suitable

[12]For rank 2, the reference is [**FoSe1**]. For rank 1, the much longer story culminates in the papers [**HeKaSe1, Sh1, Bo1**] and is told in [**G4**] and [**Su6**].

subset of the Steinberg relations (11.2) imply the complete set of relations (11.2). The vital relations are between two rank 1 subgroups, each generated by root subgroups corresponding to a fundamental root $\alpha \in \Pi$ and its negative. In the untwisted case, for example, these rank 1 subgroups are isomorphic to $SL_2(q)$ or $PSL_2(q)$, and in practice, it is often easier to verify these $(P)SL_2(q)$ conditions than the full set of Steinberg relations, particularly when q is odd. Other authors have investigated analogous presentations of certain groups of Lie type by other types of rank 1 subgroups, which in the classical cases are stabilizers of nonsingular subspaces on the natural module, for example [**Ph1, Ph2, Wo1, Da1**]. Again the relevant relations concern pairs of such subgroups.

Yet a third procedure for establishing the Steinberg relations has been used by Gilman and Griess in the case $q = 2^n$ [**GiGr1**]. Their result is expressed in terms of the structure and embedding of a component K in the centralizer of a suitable element u of odd prime order p of $G(q)$ (usually, u lies in a Cartan subgroup H) and a reduced monomial subgroup R of $G(q)$ (R is isomorphic to the Weyl group of $G(q)$, as $q = 2^n$). Under certain compatibility assumptions between K and R, the Steinberg relations for $G(q)$ are shown to be determined solely from these two subgroups.

In characterizations of groups $G(2)$ of Lie type over $\boldsymbol{F}_2$, Finkelstein, Frohardt and Solomon used either the Curtis-Tits theorem or a variant of the Gilman-Griess theorem [**FinFr1, FinS1, FinS2, FinS3**].

The three approaches—by (B, N)-pairs, by generation by rank 1 subgroups, and by a component and Weyl group, are closely linked, as one would expect. For simplicity of exposition, we shall refer here to verification of the Steinberg relations without regard to the particular method used to substantiate them.

Finally, identifications of the sporadic groups have been achieved by a variety of methods: as highly transitive permutation groups, as primitive rank 3 permutation groups, as suitable primitive permutation groups of rank > 3, as groups of automorphisms of the Leech lattice, as groups of automorphisms of suitable algebras, as groups of matrices over suitable fields, and more recently in terms of their 2-local geometries. The identification of such a group G as a primitive permutation group has sometimes required computer calculations to force the uniqueness of its generating permutations. In the existing classification proof, each sporadic group has also been characterized in terms of the structure of the centralizers of its involutions, this information sufficing to yield the above more basic uniqueness conditions.

12. A capsule summary of this series

At this point it is appropriate to give a cursory view of our overall plan for this series. This view will, for the purpose of brevity, be somewhat imprecise; the reader must refer to subsequent sections and the succeeding Outline chapter for a more precise and detailed description as well as a discussion of the ways in which it differs from the original proof of the classification.

Let G then be a $\mathcal{K}$-proper simple group. We must prove that $G \cong G^*$ for some known simple group G^*. For this we make a broad case division depending on the internal structure of G. In the "generic case" there is a prime p such that G possesses an elementary abelian p-subgroup T of rank at least 3 (or 4 if $p > 2$), an element $x \in T^{\#}$ and a component $\overline{L}$ of $\overline{C_G(x)} = C_G(x)/O_{p'}(C_G(x))$ such that $\overline{L}$

is "p-generic". This means that $\overline{L}$ is neither a group of Lie type in characteristic p, nor a member of a specified family of other small troublesome groups including among others almost all groups of p-rank 1. This case leads to the conclusion $G \cong G^*$ where G^* is a group of Lie type in characteristic other than p and of sufficiently high Lie rank or G^* is a sufficiently large alternating group. Indeed, if $\overline{L}$ is chosen as large as possible, then the isomorphism type of $\overline{L}$ already determines the isomorphism type of the target group G^* to within a few possibilities.

In the generic case, the subgroup T is analogous to a toral subalgebra of a Lie algebra. The basic idea in this case is (1) to use the signalizer functor method to show that for all $g \in T^{\#}$, $C_G(g)$ is "reductive" in the sense that $\overline{E(C_G(g))} = E(\overline{C_G(g)})$, where $\overline{C_G(g)} = C_G(g)/O_{p'}(C_G(g))$ and bars denote reduction modulo $O_{p'}(C_G(g))$; (2) to use the Steinberg presentation (or a presentation of the alternating groups) to identify the isomorphism type of $G(T) = \langle E(C_G(g)) \mid g \in T^{\#}\rangle$ as that of a group of Lie type or an alternating group; and (3) to prove that $G = G(T)$, eventually by using the Bender-Suzuki theorem on groups with a strongly embedded subgroup. We carry out essentially this strategy in Part III. It should be pointed out that we use the prime $p = 2$ if possible. Because of this preference, our overall design is constructed so that the generic case is encountered for an odd prime p only when it is also known that G is of **even type**[13]. As a consequence, in this case G turns out to be a group of Lie type in characteristic 2, as opposed to some other characteristic.

Now consider the case where G is not generic. One extreme case which might occur is that G is what might be called "quasi-unipotent", that is, for every prime p and element $g \in G$ of order p, $C_G(g)/O_{p'}(C_G(g))$ has no component at all. This is a very difficult case; by contrast, the roughly analogous case of finite-dimensional complex Lie algebras without semisimple elements is handled elegantly by Engel's theorem. One prototype of a quasi-unipotent group is a simple split (B,N)-pair of rank one, and the recognition of such (B,N)-pairs ($G \cong L_2(q)$, $q > 3$, $U_3(q)$, $q > 2$, $Sz(2^{2n+1})$ or ${}^2G_2(3^{2n+1})$, $n > 0$) is one of the most difficult chapters in the current classification proof. (Actually ${}^2G_2(3^{2n+1})$ is not quasi-unipotent.) Likewise the Feit-Thompson Odd Order Theorem may be regarded as another hard chapter in the classification of quasi-unipotent groups. We have no design for improving these two important results, and take those theorems—whose proofs have already been improved and streamlined by others—as assumptions for our work. Our entire set of assumed Background Results will be discussed in Sections 15–18.

Given these assumptions, we may now describe our strategy for the non-generic case, or the **special** case, as we call it. First, assume that there exists an elementary abelian 2-subgroup T of G of rank at least 3, an involution $x \in T$, and a component $\overline{L}$ of $\overline{C_G(x)} = C_G(x)/O(C_G(x))$ such that $\overline{L}$ is not a Chevalley group in characteristic 2, nor one of a small finite number of specified characteristic 2-like groups. As $\overline{L}$ is not 2-generic, $\overline{L}$ has one of several "thin" isomorphism types of small 2-rank, and the principal case here is $\overline{L} \cong L_2(q)$ or $SL_2(q)$, q odd. It is convenient to include with this analysis all cases in which G has 2-rank 2, regardless of the structure of centralizers of involutions of G. (By the Feit-Thompson and Z^*-theorems, any nonabelian simple G has 2-rank at least 2.)

[13]The definition of "even type"is given in Section 21. A sufficient condition for G to be of even type is for G to be of characteristic 2-type, that is, $m_2(G) \geq 3$ and $F^*(N) = O_2(N)$ for every 2-local subgroup N of G.

The lengthy analysis of this case comprises Part IV of our work. The principal outcome is to recognize G as a split (B,N)-pair of rank 1 or 2 over a field of odd characteristic, although there are a few further possibilities. In the case of (B,N)-pairs of rank 2, so much additional information is obtained en route that the final recognition of G is fairly easy.

Second, we may now assume that G has elementary abelian 2-subgroups T of order 8, and that for any such T, any $x \in T^{\#}$, and any component $\overline{L}$ of $\overline{C_G(t)} = C_G(t)/O(C_G(t))$, $\overline{L}$ is a Chevalley group in characteristic 2 or one of the finite list of characteristic 2-like groups referred to above. In this case the signalizer functor method yields that $O(C_G(x)) = 1$ for every involution x, and then that G is of even type.

Since we are in the special case, if E is an elementary abelian p-subgroup of G, p odd, and $m_p(E) \geq 4$, and if $g \in E^{\#}$ and $\overline{L}$ is a component of $\overline{C_G(g)} = C_G(g)/O_{p'}(C_G(g))$, then $\overline{L}$ is not p-generic. A refinement of this statement is established in Part V, ruling out in addition the existence of such $\overline{L}$ of "p-thin" type, and as a result, the only possible isomorphism type for such a component $\overline{L}$ is a Chevalley group in characteristic p or one of finitely many characteristic p-like groups.

The analysis of this case hinges on the existence of 2-local subgroups with large elementary abelian subgroups of odd order. The following set is basic for the case subdivision.

DEFINITION 12.1.

$$\sigma(G) = \{p \mid p \text{ is an odd prime, and for some 2-local subgroup } N \text{ of } G,\ |G:N|_2 \leq 2 \text{ and } m_p(N) \geq 4\ \}.$$

We consider the cases

$$(a)\quad \sigma(G) = \emptyset \quad \text{(the "quasithin case")}$$
$$(b)\quad \sigma(G) \neq \emptyset \quad \text{(the "large sporadic case")}.$$

This "revised" quasithin problem is more general than the "classical" quasithin problem suggested by Thompson and originally considered by Aschbacher and Mason [**A10, Ma1, A18**], in which it was assumed that for any 2-local subgroup N of G, $F^*(N) = O_2(N)$ and $m_p(N) \leq 2$ for all odd primes p. Our assumptions are less restrictive; though G has even type, it may be that $F^*(N) \neq O_2(N)$; and we only assume that $m_p(N) \leq 3$, and for that matter, only for certain 2-local subgroups N.[14]

The "revised" quasithin problem is currently being investigated by the amalgam method[15]. Stellmacher and Delgado have made inroads; but considerable parts of the problem remain to be done. Nevertheless, as noted in the Introduction, we are reasonably confident that this approach will succeed, and if necessary, one could

[14]The rather technical restriction on the size of a Sylow 2-subgroup of N in the definition of $\sigma(G)$ is chiefly for the sake of facilitating the amalgam method in the "revised" quasithin case. But for that, one could take the definition to include the condition that N contain a Sylow 2-subgroup of G.

[15]The suitability of the amalgam method is evident in Stellmacher's reworking of the original "thin" problem $m_p(N) \leq 1$ for all odd p and all 2-locals N [**Ste2**]. Over the last fifteen years, the method has steadily been strengthened by the work of Gomi, Hayashi, Rowley, Stellmacher, Stroth, Tanaka, Timmesfeld and others.

reassemble theorems from a number of papers from the original proof[16] to cover the "revised" quasithin case. In Part II, we supply the first step of the analysis. It is a "global $C(G,T)$-Theorem", proving that a Sylow 2-subgroup of G lies in at least two maximal 2-local subgroups of G. This result, obtained in collaboration with Richard Foote, is one of several "uniqueness" theorems comprising Part II.

In the large sporadic case, the signalizer functor method yields one of two possibilities for each $p \in \sigma(G)$; either G satisfies an analogue of the even type condition for the prime p, or G has a maximal 2-local subgroup M_p with p-uniqueness properties; among other things, M_p contains a Sylow p-subgroup of G and contains $N_G(Q)$ for every noncyclic p-subgroup Q of M_p.

In the first case, close analysis of 2-local and p-local subgroups following the ideas of Klinger and Mason [**KMa1**] leads to the conclusion $p = 3$ and $G \cong G^*$, where G^* is one of the sporadic groups F_1, F_2, Fi'_{24}, Fi_{23}, Fi_{22}, Co_1, or one of six groups of Lie type defined over the field of 2 or 3 elements. The analysis of this case is in Part V.

The remaining case is that for each $p \in \sigma(G)$, G has a p-uniqueness subgroup M_p as described above. This uniqueness case is shown to lead to a contradiction in a chapter in Part II written by Gernot Stroth, using techniques of the amalgam method.

As indicated above, we have collected in Part II a variety of uniqueness theorems which provide the underpinnings for the subsequent argument. In addition to the global $C(G,T)$-theorem and Stroth's uniqueness theorem already mentioned, there is a treatment of p-component uniqueness theorems; a proof for $\mathcal{K}$-proper simple groups of the Bender-Suzuki strongly embedded subgroup theorem, quoting Suzuki's work on split (B,N)-pairs of rank 1 (see [**P1**]); several related theorems about involutions and 2-local subgroups; and some more technical results needed in Part V connecting uniqueness theorems for $p = 2$ and odd p.

13. The existing classification proof

Over the thirty-year intensely active period in which the classification of the finite simple groups was achieved, varied techniques were gradually developed for studying their local structure, beginning with Brauer's theory of blocks of characters and the Brauer-Suzuki theory of exceptional characters. The Feit-Thompson Odd Order Theorem and Thompson's subsequent determination of simple groups each of whose local subgroups is solvable (known as N**-groups**) provided the initial impetus for what was to become an elaborate theory of local group-theoretic analysis.

At the core of that theory were two basic methods that provided effective extensions of Feit and Thompson's ideas to a general setting. The Bender method enabled one to study the structure of maximal subgroups containing the centralizer of a given involution, while the Gorenstein-Walter signalizer functor method and the related theory of balanced groups provided a general technique for studying centralizers of involutions (and later centralizers of elements of arbitrary prime order) in simple groups. Two results of Glauberman—his fundamental Z^*-theorem for investigating the fusion of involutions and his ZJ-theorem for analyzing p-local structure—became critical auxiliary tools for applications of either method, as did

[16]principally [**AGL1, A13, Ma1, A18, A10**], together with solutions of certain involution standard form problems

several basic results of Aschbacher, Gilman and Solomon dealing with terminal and 2-terminal 2-components of the centralizers of involutions.

Bender's classification of groups G with a strongly embedded subgroup H and its various extensions by Aschbacher were built upon Suzuki's earlier investigations of doubly transitive groups in which a one-point stabilizer contains a regular normal subgroup. These results provided critical underpinnings for all subsequent broad local group-theoretic analyses. Aschbacher's theory and applications of groups G with a **tightly embedded** subgroup H (i.e., a proper subgroup H of G of even order that intersects each of its distinct G-conjugates in a group of odd order) provided the basis for investigating arbitrary simple groups in which the centralizer of some involution is not 2-constrained.

At the same time, Fischer's general theory of groups generated by a conjugacy class of transpositions, which had its origins in basic properties of transpositions in symmetric groups, not only led to the discovery of seven sporadic simple groups, but also was expanded and developed by Timmesfeld to reach important characterizations of the groups of Lie type of characteristic 2 by internal properties.

During the final years, three important closely related techniques were introduced for the analysis of groups targeted as groups of Lie type of characteristic 2: (1) the Baumann-Glauberman-Niles theory of **pushing up**, emerging from a study of maximal 2-local subgroups of even index in the given group G; (2) Aschbacher's theory of χ-**blocks**, an extension of properties of 2-components to certain configurations of 2-constrained subgroups; and (3) Goldschmidt's theory of **amalgams**, dealing with the structure of two subgroups H, K sharing a Sylow 2-subgroup but with $O_2(\langle H, K\rangle) = 1$. The last of these had its beginnings in a result of Sims on primitive permutation groups in which a point stabilizer has an orbit of length 3.

Despite the many duplications and false starts alluded to in the Introduction, a strategy for classifying the simple groups seemed to evolve inexorably as we proceeded step by step from one partial classification result to the next, each time constructing a platform from which to jump off to a greater level of generality. One can already discern a pattern from four of the first major classification theorems to be established:

1. The Feit-Thompson proof of the solvability of groups of odd order [**FT1**].
2. Thompson's classification of simple N-groups [**T2**].
3. The Alperin-Brauer-Gorenstein-Walter classification of simple groups containing no E_8 subgroups [**ABG2, ABG1, GW1, L1**].
4. The Gorenstein-Harada classification of simple groups containing no sections isomorphic to E_{32} [**GH1**].

The Odd Order Theorem showed that every (nonabelian) simple group must have even order and hence necessarily must contain involutions. The N-group theorem demonstrated the power and the potential of local group-theoretic analysis for treating broad classification theorems. The "2-rank $\leq$ 2" theorem disposed once and for all of the "smallest" simple groups. The inductive family of groups of "sectional 2-rank $\leq$ 4" had been introduced [**MacW1**] as a means of treating the noninductive problem of determing the simple groups with a nonconnected Sylow 2-subgroup (a group S is said to be **connected** if any two four-subgroups[17] of S can be included in a chain of four-subgroups of S in which every two successive

[17] A **four-group** is a group isomorphic to $Z_2 \times Z_2$.

members centralize each other). At the time, nonconnected Sylow 2-subgroups presented a serious barrier to effective application of the signalizer functor method.

Together these four theorems put group theorists in a position to mount a full-scale attack on (connected) simple groups in which some involution has a non-2-constrained centralizer. The 2-rank ≤ 2 and sectional 2-rank ≤ 4 theorems had already made evident the primary obstructing role of $2'$-cores in preventing centralizers of involutions from approximating those in one of the known target groups. Attention therefore became focused on establishing the B_2-property in arbitrary simple groups, this being the operational meaning of the removal of $2'$-core obstruction. This turned out to be a very difficult undertaking requiring many years and involving the combined efforts of a considerable number of group theorists (see [**G4, Wa2**]). However, once it was achieved, Aschbacher's component theorem [**A4**] then directly yielded the desired approximation of some non-2-constrained centralizer of an involution to that in one of the target groups, and the path was now set for the complete classification of all such simple groups. But these centralizers possessed such a variety of distinct shapes that it required more than fifty separate, often highly technical analyses over a number of years to finish this major subproblem of the project. In fact, some of these analyses were among the last papers written as part of the complete classification of the finite simple groups.

The success of this attack meant that in a minimal counterexample G every 2-local subgroup is 2-constrained. This represented a watershed on the path towards a full classification of the finite simple groups, for it meant that the only remaining known targets G^* for G were the groups of Lie type of characteristic 2 and a few sporadic groups. Since solvable groups are 2-constrained, N-groups necessarily satisfy the specified conditions; and it turned out that Thompson's proof of the N-group theorem provided a framework for the analysis that lay ahead.

First, direct applications of signalizer functor theory eliminated all $2'$-cores, so that $F^*(H) = O_2(H)$ for every 2-local subgroup H of G, and hence by definition of the term the minimal counterexample G to the classification theorem was a group of characteristic 2 type. Just as in Thompson's N-group analysis, so the analysis of the general characteristic 2 type group can be split into the following three major subcases:

(1) For some maximal 2-local subgroup H of G, $O_2(H)$ contains no noncyclic characteristic abelian subgroups;
(2) No 2-local subgroup of G contains a E_{p^3}-subgroup for any odd prime p;
(3) Some 2-local subgroup contains a E_{p^3}-subgroup for some odd prime p,

with the solutions of (2) and (3) presupposing the solution of (1). For (1) the target groups G^* are most of the groups of Lie type defined over the prime field $\boldsymbol{F}_2$ and a few characteristic 2-type sporadic groups; for (2) (excluding the results of (1)) they are the groups of Lie type of characteristic 2 and low Lie rank (plus the remaining characteristic 2-type sporadic groups); and for (3) (again excluding the results of (1)) they are the "generic" groups of Lie type of characteristic 2.

Despite the complexity of each of these classification problems, the analysis proceeded much more quickly than in earlier stages of the classification proof, for most of the necessary tools were in place and finite group theorists had already had considerable practice in their use. Fischer's theory of transpositions was the key underlying technique for investigating the "$GF(2)$-type" groups, the general case treated by Timmesfeld [**Tim3, Tim1**], with important subcases treated in the work of Aschbacher, S. Smith, F. Smith, and others [**A7, A8, Sm1, Sm2, Sm3, Smi1**]. Likewise Aschbacher treated the major "thin" subcase of the "quasithin" group problem, while G. Mason, in work eventually completed by Aschbacher, extended Aschbacher's analysis to the much more difficult general quasithin problem [**A10, Ma1, A18**]. The theories of χ-blocks, pushing up and amalgams were all critical for treating certain minimal configurations that arose in the course of these analyses.

The "generic" characteristic 2-type case involved a study of the centralizers in G of elements of odd prime order by means of the signalizer functor method, very similar in spirit to the treatment accorded centralizers of involutions in the generic but non-2-constrained case. The analysis of this case was primarily the work of Aschbacher, Gorenstein and Lyons, and Gilman and Griess [**A13, AGL1, GL1, GiGr1**].

Finally there was a rich interplay between the developing classification proof and the discovery and construction of the sporadic groups during the 1960's and 1970's, which provided an added dimension to the entire endeavor.

14. Simplifying the classification proof

As a result of the evolutionary character of the existing classification proof, the focus of attention remained localized until the final years. Because a complete classification of the simple groups was viewed as too vast to be considered as a single problem and its attainment well beyond reach, individual facets were almost always investigated without reference to their global implications. In particular, theorems were formulated in such a way that they could be established independently of the full classification theorem. Even within these theorems, individual cases were often treated as independent results.

For example, a result of Alperin asserts that a simple group of 2-rank ≤ 2 has a Sylow 2-subgroup that is either quaternion, dihedral, semidihedral, wreathed ($Z_{2^n} \wr Z_2$), homocyclic abelian, or isomorphic to a Sylow 2-subgroup of $PSU_3(4)$. In the original classification proof, each of these five cases was given an entirely separate treatment. Verification of the B_2-property constitutes perhaps the most extreme illustration of this phenomenon. The general case was covered by Aschbacher's "classical involution theorem"; however, treatment of the various residual cases spread over nearly a dozen additional difficult papers (see [**A9**], Wa2).

Beyond this, classification results established in the early years were of course derived without benefit of techniques developed later. Under all these circumstances, it was hardly suprising that the complete classification proof would be inefficient and accessible only to experts already versed in its intricacies. Whether new techniques can be developed or a new approach discovered for drastically simplifying the classification theorem remains to be seen. The only question there is any hope of settling at present is the following: *Is it possible with existing group-theoretic techniques to construct a coherent, efficient, and accessible proof of the classification of the finite simple groups*? It is the purpose of this series of monographs to provide an affirmative answer to this question.

Even a superficial examination of the present proof will reveal many areas capable of consolidation and simplification. Consider, for example, the construction of the sporadic groups. Originally nineteen of the twenty-one groups other than the five Mathieu groups were constructed by separate means. However, the Fischer-Griess monster F_1 is known to contain twenty of the twenty-six sporadic groups as either subgroups or sections. Construction of F_1 depends only on the existence of Conway's largest group Co_1, and construction of Co_1 only on the existence of the largest Mathieu group M_{24}. Hence once M_{24}, Co_1, and F_1 have been constructed, one obtains the existence of 17 additional sporadic groups as a corollary.

Although centralizers of involutions and centralizers of elements of odd prime order p were each investigated by the signalizer functor method, there are considerable differences in the strategies that were employed. For example, the odd p analysis was carried out on the basis of only a *partial* B_p-property and also without any analogue of the sectional 2-rank ≤ 4 theorem for dealing with nonconnected Sylow p-subgroups. Then, too, in the involution case, the identification of G as a group of Lie type of odd characteristic was established by means of its "intrinsic $SL_2(q)$ 2-components" via Aschbacher's classical involution theorem, whereas in the odd p case, the identification of G as a group of Lie type of characteristic 2 was achieved via a terminal p-component and a "proper neighbor." It is therefore natural to ask whether this entire analysis can be carried out uniformly for all primes simultaneously.

As a still further illustration, we note that in the present proof many of the groups of Lie type of characteristic 2 are identified *three or four* times, from each of the following points of view:

1. From suitable terminal 2-components in the centralizers of involutions. (The only target groups arising here have non-inner automorphisms of order 2.)
2. As groups of $GF(2)$-type (only certain groups over $\boldsymbol{F}_2$ arise here).
3. From Timmesfeld's "root involution" theorem [**Tim2**].
4. From suitable terminal p-components in the centralizers of elements of odd prime order p [**GiGr1**].

[Almost all of the groups G^* of Lie type of characteristic 2 are generated by a conjugacy class $\mathcal{C}^*$ of involutions with the property that the order of the product of any two elements of $\mathcal{C}^*$ is 1, 2, 4 or odd and in addition, if the product has order 4, then its square is an element of $\mathcal{C}^*$. The class $\mathcal{C}^*$ consists of **root** elements in the Lie sense. Timmesfeld's root involution theorem gives a complete determination of all simple groups generated by a conjugacy class of involutions satisfying these conditions.]

One can thus ask whether such duplication of effort is necessary—equivalently, whether it is possible to organize the classification proof in such a way that each known target group is identified *just once* in the course of the analysis. Our aim has been to provide an affirmative answer to this question as well.

Considered from the most general perspective, the overall strategy of our "second generation" classification proof is essentially the same as that of the original proof: a division of groups into "odd" and "even" types, special treatments of the low rank groups of each type, and a systematic treatment of terminal p-component problems for centralizers of prime order p.

However, by treating the classification theorem as a single result rather than as a collection of more or less independent subsidiary theorems, as had been necessary in the original proof, we are able to effect a considerable simplification and consolidation in its organizational structure. In particular, this approach allows us to make more systematic use of the $\mathcal{K}$-group assumption on proper subgroups. Moreover, it leads to significant differences in the way the component parts of the two proofs are "packaged," the effect of which is to enable us to bypass a number of important subsidiary theorems of the present proof.

Indeed, the proof we shall present here does not directly involve the general form of any of the following results that were used in the existing classification proof:

1. Sectional 2-rank ≤ 4 classification theorem [**GH1**].
2. Classical involution theorem [**A9**].
3. Solutions of a large number of the terminal 2-component centralizer of involution problems (see [**Se1; Se2,** pp. 47–53]).
4. Structure of a Sylow 2-group of a tightly embedded subgroup [**ASe1, GrMaSe1, A6**].
5. $GF(2)$-type classification theorem [**Tim3, Sm3**].
6. Root involution theorem [**Tim2**].
7. Strongly closed abelian 2-subgroup classification theorem [**Go5**].

[Goldschmidt has generalized the Z^*-theorem by determining the simple groups G in which a Sylow 2-subgroup S contains a nontrivial strongly closed abelian subgroup A, that is, $A^g \cap S \leq A$ for all $g \in G$.]

In the previous paragraph, the terms "directly involve" and "general form" have been carefully chosen, for it should be clear that any classification of the simple groups based upon presently available techniques will encounter many of the same critical configurations. Nor is it reasonable to expect that the proof can be so organized that *all* such configurations can be bypassed. Indeed, we are forced to consider many that were involved in the original proof.

In particular, that is the case for a number of configurations arising in the sectional 2-rank ≤ 4 analysis. In some cases, our present approach yields a modest simplification, but in others the initial treatment seems to be optimal. The proof of the classical involution theorem enters in an even more significant way, for our analysis of centralizers of involutions and more generally of elements of prime order p in the case of generic simple groups G utilizes "3/2-balanced" signalizer functors. These objects, developed by Goldschmidt and Aschbacher [**A9, Go4**], formed the basis for Aschbacher's elimination of $2'$-core obstruction in the classical involution context.

Furthermore, the bulk of Goldschmidt's proof of the strongly closed abelian theorem is taken up with elimination of $2'$-core obstruction in the centralizers of involutions. However, we require the result only in the case that the $2'$-core of the centralizer of every involution is assumed to be trivial and therefore need only the remaining small part of his argument. On the other hand it is precisely the Bender method for eliminating $2'$-core obstruction, as elaborated by Goldschmidt, which plays a crucial role in the analysis of special groups of odd type. Thus much of Goldschmidt's proof appears in our work in cannibalized form.

Just as we use the Bender method more extensively than the original proof did, we should also point out that the proposed treatment of the revised quasithin problem as well as Stroth's new elimination of the "uniqueness case" for odd primes makes considerably greater use of the Goldschmidt amalgam method than occurred in the original proof, for the full power of the method was not developed until after the classification theorem had been completed. The original treatments of the quasithin and uniqueness cases used the method of weak closures, invented by Thompson in the N-group paper [**T2, A14, A16, Ma1**].

Finally, we do not wish to leave the reader with the impression that our approach is without its own liabilities. One of these occurs precisely because of our strategy of bypassing many terminal 2-component centralizer of involution problems. Indeed, although this results in a considerable net saving, it forces us to permit non-2-constrained 2-locals to occur in situations that in the prior analysis had been restricted to 2-constrained 2-locals, thereby increasing the number of configurations to be investigated. To be sure, there are also the liabilities that come with an induction hypothesis as massive as the one we use. Not only does a mathematical argument hang by an inductive thread for thousands of pages, but also a vast and sometimes tedious theory of $\mathcal{K}$-groups is required. We can hope that future revisionists will find a tidier and more robust proof; for now we have taken what seems to us the most direct approach.

D. The Background Results

15. Foundational material

This brings us to a further aspect of the existing classification proof. Because of its extreme length—both *in toto* and in many of the individual papers—finite group theorists were strongly motivated to quote available results whenever possible. With a few notable exceptions, little effort was made to search for self-contained arguments. As a consequence, the papers making up the classification proof, together with the papers to which they refer, together with those to which they refer, and so on, form a web whose ultimate foundations are difficult to trace. They include not only group-theoretic results from the turn of the century, but also early papers in geometry and number theory. Thus not only is the existing proof itself of inordinate length, but the foundational material on which it is based is presently in an unorganized state.

Any simplified proof of the classification theorem must also deal with this foundational problem. Clearly for a project of this magnitude ever to reach completion, some body of group-theoretic material must be accepted at the outset and permitted to be quoted as needed. On the other hand, it is essential that this foundational background material be made explicit if the resulting proof is to be placed on a firm base.

Furthermore, as we have already observed, the existing proof depends on a very large number of properties of $\mathcal{K}$-groups and especially simple $\mathcal{K}$-groups. Some of these are very general, but many involve detailed properties of individual simple groups or families of simple groups. Similarly the classification theorem requires a great many properties of general finite groups, ranging from the very broad to the very specialized.

Indeed, the opening sections of a paper on the classification of the simple groups typically consist of the verification of those $\mathcal{K}$-group and general group-theoretic properties required for the classification theorem under consideration. For example, the 55-page first installment of Thompson's N-group paper [**T2**] is largely devoted to establishing such results, in this case primarily properties of solvable and nilpotent groups. The most extreme illustration of this phenomenon occurs in the Gorenstein-Lyons work on groups of characteristic 2 type [**GL1**], in which the first Part (pages 15–425) is taken up essentially solely with the development of the various $\mathcal{K}$-group and general group-theoretic results needed for the subsequent analysis. In particular, it includes (without proofs) lists of local properties of each of the twenty-six sporadic groups [**GL1,** Part I, §5] (many of which were worked out for us by O'Nan).

It seems inevitable that any classification of the finite simple groups based on an internal analysis of the subgroup structure of a minimal counterexample will require a vast body of such preliminary results. This is therefore as much a feature of our revised proof as it is of the original one.

One of the most difficult decisions the authors have had to make in formulating an overall strategy has concerned these preliminary properties of $\mathcal{K}$-groups and of arbitrary finite groups:

(A) What should we be allowed to quote?
(B) What should we be required to prove?
(C) How should the material be organized?

Ideally, our preference would have been the following:

(I) Quote only results available in standard books, monographs, or lecture notes.
(II) Establish the needed $\mathcal{K}$-group and general finite group properties on the basis solely of results available in (I).
(III) To achieve maximum coherence and efficiency of development, defer organization of the material in (II) until the revised proof has been completed.

Unfortunately, we have reluctantly concluded that this strategy would delay publication of any portion of our revision beyond the foreseeable future. Indeed, it would force us to include several additional volumes pertaining to this preliminary material.

The most serious problem concerns the sporadic groups, whose development at the time of the completion of the classification theorem was far from satisfactory. The existence and uniqueness of the sporadic groups and the development of their properties form a very elaborate chapter of simple group theory, spread across a large number of journal articles. Moreover, some of the results are unpublished (e.g. Sims's computer calculations establishing the existence and uniqueness of the Lyons group Ly). Furthermore, until very recently, the two principal sources for properties of the sporadic groups were [**CCNPW1**] and [**GL1,** Part I, §5] consisting only of statements of results without proofs.

However, over the past few years attempts to rectify this situation have been begun by a number of authors, initially focusing on existence and uniqueness of the sporadic groups. Moreover, Aschbacher [**A2**] has recently begun a more systematic development of the sporadic groups which illustrates the verification of many of their properties as well as existence and uniqueness.

One faces a somewhat different problem regarding the finite groups of Lie type. Carter's two books and Steinberg's lecture notes [**Ca1, Ca2, St1**] provide an excellent treatment of many of their general properties, but they do not cover in sufficient detail certain bread-and-butter material required for the classification proof, such as the structure of the centralizers of semisimple elements, questions about balance and generation, or Schur multipliers. The Schur multipliers are treated in the general case by Steinberg [**St1, St4, St5**], with a number of residual cases computed by several authors, most of them by Griess [**Gr1**] (see also [**Gr2**]).

Although most of the standard repertoire of finite group theory needed for local group-theoretic analysis can be found in books by Gorenstein, Huppert, Blackburn, Suzuki, and Aschbacher [**G1, Hu1, HuB1, Su1, A1**], nevertheless many important local results exist only in the journal literature. A similar situation prevails regarding representation theory and Brauer's theory of blocks of characters. Most but not all of the needed background material is covered in standard references such as those of Feit and Isaacs [**F1, Is1**].

Thus a systematic treatment of all this preliminary material would require volumes concerning (a) the sporadic groups, (b) the finite groups of Lie type, and (c) basic local group theory and character theory. However, we prefer to focus our efforts on the classification proof *per se*, for it is here that our major revisions and simplifications are to be made. We have therefore decided not to proceed in quite so ambitious a fashion.

16. The initial assumptions

On the other hand, to follow the opposite path with respect to this large body of preliminary material, allowing ourselves to quote the literature at will, would defeat one of our primary objectives, which is to provide as self-contained a proof of the classification theorem as we can feasibly achieve.

We have therefore settled on a middle course—on the one hand practicable, and on the other, consistent with this objective. We have decided to operate on the basis of the following two principles. First, in addition to the material contained in certain published books, monographs, and lecture notes, listed below in section 17, we shall assume a number of further basic results contained in a *selected list of papers* to be specified in section 18. This list will be comparatively short; moreover, within those papers we shall quote only basic results.

Second, regarding the sporadic groups, we shall proceed essentially as in [**GL1**], permitting ourselves to quote any of their properties in tables like those of [**GL1,** Part I, §5] and to be given in [I_A]. In addition, we shall make an assumption concerning their existence and uniqueness. As observed in section 1, a variety of paths has been followed for constructing and identifying the sporadic groups. However, an identification of each group has been achieved in terms of the centralizers of its involutions. To avoid listing the large number of papers on which this assertion is based, we shall simply assume here that[18]

(16.1) *the twenty-six sporadic groups exist, and each of them (except M_{11}) is uniquely determined up to isomorphism as the only simple group with its centralizer of involution pattern.*

[In the case of M_{11}, one must add its order to the given conditions to distinguish it from $L_3(3)$. Because of this disparity with the other sporadic groups, we prefer to identify M_{11} in the text from the given conditions.]

[18] By definition, two groups G and G^* have the same centralizer of involution pattern if and only if there is an isomorphism $x \mapsto x^*$ from a Sylow 2-subgroup S of G to a Sylow 2-subgroup S^* of G^* such that for all involutions x and y in S, the following two conditions hold: (a) x and y are G-conjugate if and only if x^* and y^* are G^*-conjugate; and (b) $C_G(x) \cong C_{G^*}(x^*)$. If only condition (a) is imposed, G and G^* are said to have the same involution fusion pattern.

Aschbacher's recent volume goes some distance toward organizing the proof of (16.1), by constructing the twenty sporadic groups involved in the Monster F_1 and proving uniqueness for five of these groups from conditions even weaker than centralizer of involution patterns. It also documents some of the background properties we assume.

The complete list of our assumed material will, for brevity, be referred to as the **Background Results**, and is an implicit hypothesis of the main theorem. This material consists, of course, of proved results; we use the term "hypothesis" here solely to emphasize the "rules of the game" under which we plan to operate. Thus, subject to these specific background results, all proofs are intended to be self-contained, including not only the classification proof proper but all further necessary supporting properties of $\mathcal{K}$-groups and general finite groups. That is, the only results we may quote either must come from the Background Results or else must have been previously established in the course of the proof.

To summarize: the Background Results consist of (16.1) and the references listed in the next two sections. These references, upon which our proofs depend, are also listed in a separate bibliography as the Background References. All other references, included only for the sake of exposition, are listed in a bibliography of Expository References, and this practice will be followed in all the monographs.

17. Background Results: basic material

We now list the basic books and monographs (including lecture notes and memoirs) from which we shall be permitted to quote in the course of the classification proof. The aim is to keep this reference list as short as feasible, consistent with covering the needed background material.

(1) General group theory

[**A1**] M. Aschbacher, *Finite group theory*, Cambridge University Press, Cambridge, 1986.

[**G1**] D. Gorenstein, *Finite groups*, Second Edition, Chelsea, New York, 1980.

[**Hu1**] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, New York, 1967.

[**HuB1**] B. Huppert and N. Blackburn, *Finite groups* III, Springer-Verlag, Berlin, New York, 1985.

[**Su1**] M. Suzuki, *Group Theory I, II*, Springer-Verlag, Berlin, New York, 1982, 1986.

(2) Character theory

[**F1**] W. Feit, *The Representation Theory of Finite Groups*, North-Holland, Amsterdam, 1982.

[**Is1**] I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.

(3) Groups of Lie type

[**Ca1**] R. Carter, *Simple groups of Lie type*, Wiley-Interscience, New York, 1972.

[**D1**] J. Dieudonné, *La géometrie des groupes classiques*, Springer-Verlag, Berlin, 1955.

[**St1**] R. Steinberg, *Lectures on Chevalley groups*, Lecture Notes, Yale University, 1967-68.

[**St2**] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. 80 (1968).

(4) Sporadic groups

[**A2**] M. Aschbacher, *Sporadic Groups*, Cambridge University Press, Cambridge, 1994.

The principal reference for general group theory is [**G1**] and to a lesser extent [**A1**]; the remaining listed books, and the character theory references, will be referred to only for certain specific results. For example, the subgroups of $PSL_2(q)$ are determined in [**Hu1**], the Bender-Thompson odd prime signalizer lemma is proved in [**HuB1**], and Glauberman's Z^*-theorem in [**F1**]. The material from the Lie type references relevant to finite groups will be used freely. Finally, since we are taking (16.1) as given, we use [**A2**] only for certain structural properties of sporadic groups.

The final reference to be listed will be the source for considerable local group theory and for many properties of the groups of Lie type as well as for structural assertions concerning the sporadic groups not covered in [**A2**].

[**GL1**] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic* 2 *type* (Part I, pp. 15–425), Mem. A. M. S. 276 (1983).

Some comments concerning our use of [**GL1**] are in order, for it is only by means of this memoir that we are able to keep the list of background references as compact as it is. For example, our source for the Schur multipliers of the simple groups of Lie type will be found in the tables on page 72 of [**GL1**]. However, the validity of this table is based on the results of at least a dozen papers. This is the most serious example of this phenomenon, though not the only one. On the other hand, we supply our own proof of Seitz's fundamental theorem on the generation of groups of Lie type [**Se3, L3**].

18. Background Results: revised portions of the proof and selected papers

Revisions of certain portions of the classification proof have already appeared in print. To avoid undue duplication of effort we shall include a selected list of such references within our Background Results.

The most important such revisions concern the solvability of groups of odd order and the recognition of split (B, N)-pairs of rank 1.

A. **Solvability of groups of odd order:** Bender and Glauberman have revised Chapter IV of the original Feit-Thompson proof [**FT1**] and Peterfalvi [**P3**] has shortened Chapter VI. Both of these are included in:

[**BeGl1**] H. Bender and G. Glauberman, *Local Analysis for the Odd Order Theorem*, L. M. S. Lecture Notes Series 188, Cambridge Univ. Press, 1994.

In addition, Sibley has revised substantial parts of Chapter V, but his work is not published. Hence for completeness, we include in the Background Results[19]

[**FT1**] W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math. 13 (1963), 775–1029,

but *only* for the material in Chapter V and the preceding material which supports it, namely, Chapter III and a few elementary lemmas from Chapters I and II.

[19] See the Preface to the Second Printing: this can be replaced by Peterfalvi's revision [**P4**].

B. **Recognition of split (B, N)-pairs of rank 1:** Peterfalvi has revised both Bender and Suzuki's original work [**Be3, Su4**] in the characteristic 2 case and O'Nan's characterization of the unitary groups in odd characteristic [**ON1, ON2**], and Enguehard has revised the corresponding characterizations by H. Ward, Janko-Thompson, Thompson, and Bombieri [**War1, JT1, T1, Bo1**] of the Ree groups of characteristic 3. We only require Part II of Peterfalvi's work on the Bender-Suzuki theorem, which is a treatment of Suzuki's original work; we shall establish separately the main result of Part I under our $\mathcal{K}$-group hypotheses[20].

[**E1**] M. Enguehard, *Obstructions et p-groupes de classe 3; Caracterisation des groupes de Ree*, Astérisque 142–143 (1986), 3–139.

[**P1**] T. Peterfalvi, *Le Théorème de Bender-Suzuki* (Part II *only*), Astérisque 142–143 (1986), 235–295.

[**P2**] T. Peterfalvi, *Sur la caracterisation des groupes* $U_3(q)$*, pour q impair*, J. Algebra 141 (1991), 253–264.

For a complete treatment of the Ree group problem we must also include the appendices to Bombieri's paper [**Bo1**] by Hunt and Odlyzko. These independently each contain the final step of the characterization of the Ree groups—reports of machine computations eliminating a finite number of exceptional cases of fields of characteristic 3 not included in Bombieri's or Enguehard's work.

[**Od1**] A. Odlyzko, *The numerical verification of Thompson's identity*, Invent. Math. 58 (1980), 97–98.

[**Hun1**] D. Hunt, *A check of the Ree group conjecture for small fields*, Invent. Math. 58 (1980), 99.

C. Signalizer functor theorems: The theory of signalizer functors, and in particular the signalizer functor theorems asserting the completeness of various signalizer functors, play a crucial role in the proof of the classification. The solvable signalizer functor theorem was perfected in the papers of Goldschmidt, Bender, and Glauberman [**Go2, Go3, Be4, Gl1**], and building on those, the nonsolvable case is covered by two papers of McBride [**McB1, McB2**]. For the solvable theorem, there is a choice of references, and we choose the original paper of Glauberman; the theorem is treated also in [**A1**].

In addition, the authors [**GL3, GL4**] have recently established a nonsolvable signalizer functor theorem, which although less general than McBride's is considerably easier to prove and is entirely sufficient for application in the classification proof. Nevertheless, we have in the end decided to use McBride's papers as background; though his proofs are more difficult than ours, the statement of the result itself is cleaner and stronger and its application requires no technical hypotheses to be verified beyond certain simple properties of $\mathcal{K}$-groups.

[**Gl1**] G. Glauberman, *On solvable signalizer functors on finite groups*, Proc. London Math. Soc. 33 (1976), 1–27.

[**McB1**] P. McBride, *Near solvable signalizer functors on finite groups*, J. Algebra 78 (1982), 181–214.

[**McB2**] P. McBride, *Nonsolvable signalizer functors on finite groups*, J. Algebra 78 (1982), 215–238.

[20] In Part II, Peterfalvi makes occasional reference to technical lemmas from Part I and we include these implicitly in the above reference.

D. Near components and local $C(G,T)$-theorem: Underlying Aschbacher's fundamental local $C(G,T)$-theorem is a basic result of Baumann, Glauberman, and Niles [**Ba1, GlNi1, Ni1**] dealing with the following special configuration: $F^*(G) = O_p(G)$, p a prime, and $G/O_p(G) \cong SL_2(p^n)$. Their result gives the precise structure of G when no nontrivial characteristic subgroup of a Sylow p-subgroup T of G is normal in G. Using the amalgam method, Stellmacher has obtained a considerably shorter proof of a version of the Baumann-Glauberman-Niles theorem, and we include his paper as a background reference. In addition, we have separately written a revised proof of the local $C(G,T)$-theorem, under a $\mathcal{K}$-group hypothesis appropriate for its applications here. We shall therefore also take that proof as a background reference.

[**Ste1**] B. Stellmacher, *Pushing up*, Arch. Math. 46 (1986), 8–17.
[**GL2**] D. Gorenstein and R. Lyons, *On Aschbacher's local $C(G,T)$ theorem*, Israel J. Math. 82 (1993), 227–279.

E. Character theory: We have had some difficulty deciding how to handle the character-theoretic portions of the classification proof not available in standard texts. Character theory is used only in the analysis of simple groups of 2-rank at most 3 apart from its use in establishing some of the Background Results, such as the recognition of split (B,N)-pairs of rank 1 or existence and uniqueness of certain sporadic groups—primarily those having one conjugacy class of involutions. In the end, we have decided to include complete proofs involving only ordinary character theory (for example, the contents of the Bender-Glauberman paper [**BeGl2**], which provides the character-theoretic basis for the classification of groups with dihedral Sylow 2-groups), but to leave the development of certain basic modular character-theoretic material to the Background Results.

This material consists of the analysis of p-blocks with certain specified defect groups, aimed at certain explicit order formulas for G in terms of the structure of centralizers of involutions. The cases in question are the principal 2-block in groups with semidihedral, wreathed or E_8 Sylow 2-groups; blocks with defect group E_4 and inertial index 1; and the principal p-block (p odd) in case the Sylow p-subgroup is abelian with automizer of order 2. Enguehard's revision of the Ree group problem covers the E_8 case. For the other cases, and just in connection with these specific preliminary modular character-theory results, we include the following papers.

[**Br1**] R. Brauer, *Some applications of the theory of blocks of characters of finite groups III*, J. Algebra 3 (1966), 225-255.
[**Br2**] R. Brauer, *Some applications of the theory of blocks of characters of finite groups IV*, J. Algebra 17 (1971), 489–521.
[**Br3**] R. Brauer, *Character theory of finite groups with wreathed Sylow 2-subgroups.* J. Algebra 19 (1971), 547-592.

Finally, it may be necessary or desirable to add a few additional papers to our background list as we proceed. This remark applies as well to possible further $\mathcal{K}$-group references. For example, it is quite likely that verification of a few needed results about sporadic groups will depend on basic properties not listed in [**GL1,** Part I, §5], but which appear in the Conway-Curtis-Norton-Parker-Wilson *Atlas of Finite Groups* [**CCNPW1**]. If so, we shall include these further properties in our Background Results and add the *Atlas* to our reference list.

E. Sketch of the Simplified Proof

19. Centralizers of semisimple elements

In 1954, Richard Brauer proposed that an inductive attack on the classification of the finite simple groups might proceed by passing from an inductive knowledge of the centralizers of involutions to a determination of the groups themselves. Bolstered by the Odd Order Theorem, this philosophy prevailed until around 1974, when in the final years of the classification program it became clear that the central significance of the prime 2 resided primarily in the following facts.

Most finite simple groups are groups of Lie type defined over a field of odd characteristic r. As such their most natural matrix representations are as matrix groups over finite fields $\boldsymbol{F}_q$ of characteristic r. In such groups, elements of order 2 are **semisimple** and indeed the largest rank among the semisimple abelian subgroups is achieved, or very nearly achieved, by certain 2-subgroups. Moreover, if the group has Lie rank at least 3, the centralizers of the (commuting semisimple) elements in such a 2-group contain subnormal subgroups which are quasisimple or isomorphic to $SL_2(3)$ and which together generate the entire group, and from which the Steinberg presentation of the group can be recovered.

In groups of Lie type over fields of characteristic 2, it is elements of odd prime order p that are semisimple, and if one chooses p appropriately (usually to divide the order of a Cartan subgroup) and an appropriate abelian semisimple p-subgroup, then the above assertions for commuting involutions carry over to commuting elements of order p. In most of the alternating groups we shall regard elements of order 2 as semisimple and proceed as in Lie type groups of odd characteristic to obtain the Coxeter presentation for the groups.

From our point of view, then, *the primary objects of study in the classification of finite simple groups should be the centralizers of "semisimple" elements of appropriate prime order* p*, and more precisely the components of these centralizers.*

A group of Lie type is, of course, defined over some field, so at the outset one knows that elements of order coprime to its characteristic — in particular, of prime order dividing the order of a Cartan subgroup — are semisimple. On the other hand, if G is an arbitrary finite simple group, even a minimal counterexample to the classification theorem, then no underlying field is specified *a priori*, let alone a Cartan subgroup, so that determination of the appropriate prime p on which to focus attention is itself one of the central problems within the analysis.

There is the even more basic decision of whether to take p to be 2 or an odd prime — i.e., whether the ultimate presentation for G is to emerge from a study of the centralizers of involutions or from the centralizers of elements of odd prime order. In general, we know in the abstract how we hope to make this decision: namely, apart from low rank cases, if G is to turn out to be a group of Lie type of odd characteristic or an alternating group, we want to take $p = 2$, whereas if it is to turn out to be a group of Lie type of characteristic 2, we want to take p to be odd. On the other hand, for the sporadic groups, low rank groups of Lie type and small alternating groups, the situation is more complex but generally the identification of these groups will proceed from 2-local data. What is required then are criteria on the *internal* subgroup structure of G that will enable us to determine in advance which of these possibilities occurs in a given situation (see section 21 below).

20. Uniqueness subgroups

As has been discussed earlier, centralizers of involutions (and more generally of elements x of any prime order p) in an arbitrary $\mathcal{K}$-proper simple group may *a priori* have a structure far more complex than that of such centralizers in any of the known target groups G^*. Moreover, a large part of the existing proof as well as of our simplified classification proof is taken up with forcing an approximation of the structure of these centralizers for appropriate p and x. The strategy for carrying this out is quite involved, utilizing a number of distinct techniques, the appropriate one depending upon the ultimate known target group G^*.

Despite these differences, however, there is a common feature to all the methods, which we digress briefly to discuss. The existence of some obstruction to the similarity of G and G^* leads to the construction of a "large" maximal subgroup M of G — large in the sense that M contains the normalizer in G of many nonidentity p-subgroups of M, in some cases for a single prime p and in others for all primes p in some set. Usually, these conditions equivalently assert that for each Q in some suitable collection of p-subgroups of M, M is the unique G-conjugate of M containing Q. For this reason, such subgroups M are frequently called p-uniqueness subgroups of G. In the classification proof at various points we construct such p-uniqueness subgroups both for $p = 2$ and for all p in the set $\sigma(G)$ of odd primes (Definition 12.1).

Harada has colorfully referred to p-uniqueness subgroups as "black holes," in the sense that they ensnare all subgroups in their neighborhood. This is certainly an apt description of their basic properties.

The primary example of a p-uniqueness subgroup is that of a **strongly p-embedded subgroup** M, which by definition is a proper subgroup H of G of order divisible by p such that $H \cap H^g$ is a p'-group for all $g \in G - H$. For $p = 2$, this is the usual notion of a strongly embedded subgroup. If M is strongly p-embedded in G, then $M = N_G(M)$ and M is the unique G-conjugate of M containing any nontrivial p-subgroup Q of M. It follows that $N_G(Q) \leq M$ for each such Q. In fact, more generally, any p-local subgroup of G containing Q lies in M.

We shall require a slight weakening of this notion for odd primes and introduce the idea of a **strong p-uniqueness subgroup**. For the purposes of this chapter, the reader should think of this as a strongly p-embedded subgroup. The precise definition, only required for certain odd primes p, will be given in the next chapter.

Underlying the classification proof is a network of results related to uniqueness subgroups, involving, on the one hand, a complete determination of the possibilities for G when G is assumed to have a uniqueness subgroup of some type and on the other, various sufficient conditions for G to possess a uniqueness subgroup.

The Bender-Suzuki theorem [**Be3**] is the most fundamental example of the former. Because of our $\mathcal{K}$-proper hypothesis we need only the following weaker theorem which will be proved in [II_1]. In the form stated here, it incorporates the Brauer-Suzuki theorem [**BrSu1, Gl4**] on groups of 2-rank 1, a special case of the Z^*-theorem.

THEOREM 20.1. *If G is a $\mathcal{K}$-proper simple group containing a strongly embedded subgroup M, then $G \cong L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$, $n \geq 2$, and M is a Borel subgroup of G.*

An example of the latter type of uniqueness theorem is the following result concerning **component uniqueness subgroups**, which extends to arbitrary primes earlier work of Aschbacher, Gilman and Solomon for the prime 2 [**A4, Gi1, S1**], and will be proved in [II_2].

THEOREM 20.2. *Let G be a $\mathcal{K}$-proper simple group, p a prime, and M a maximal subgroup of G. If p is odd, assume that M has p-rank at least 4. Let $\overline{K}$ be a component of $\overline{M} = M/O_{p'}(M)$, of p-rank at least 2. If $C_G(x) \leq M$ for every element x of order p in $C_M(\overline{K})$, then one of the following holds:*
(i) *$\overline{K}$ is normal in $\overline{M}$; or*
(ii) *M is strongly p-embedded in G.*

In particular, for $p = 2$, the theorem yields as a corollary the basic result that terminal components of centralizers of involutions in G are necessarily standard in G, if they have 2-rank at least 2. By definition, a component K in the centralizer of an involution of G is **standard** in G if and only if K is terminal and commutes elementwise with none of its G-conjugates.

A third example of a uniqueness result is a particular case of the $C(G, S)$ theorem (for $S \in Syl_2(G)$), in which it is assumed that $N_G(S)$ is contained in a *unique* maximal 2-local subgroup of G and various additional side conditions are satisfied. As in the original $C(G, S)$-theorem, the classification of such groups G is achieved by first arguing that G contains an Aschbacher χ-block; the subsequent analysis involves the study of groups containing such a χ-block. It is the latter investigation that has been carried out by Foote. Chapters 3 and 4 of Part II are devoted to this result. The remainder of Part II is taken up with work on the "uniqueness" case for groups of even type. This is in two parts. The first consists of some theorems about 2-local properties of p-uniqueness subgroups, analogous to [**AGL1**]; in particular it is shown that for $p \in \sigma(G)$, if M is a strong p-uniqueness subgroup of G, and Q is a noncyclic p-subgroup of M, then any *2-local* subgroup of G containing Q lies in M. The second, which uses these technical results, is Stroth's proof that the uniqueness case does not occur, an analogue of Aschbacher's original work [**A16**] but proved by a different approach.

Thus Part II: Uniqueness Theorems should be viewed as a collection of tool theorems needed for the classification *per se*.

21. The sets $\mathcal{L}_p(G)$ and groups of even type

We return to the discussion of section 19, which clearly indicates that in a $\mathcal{K}$-proper simple group G the following sets will be of critical importance for the analysis.

DEFINITION 21.1. For each prime divisor p of the order of G,

$$\mathcal{L}_p(G) = \{K \mid K \text{ is a component of } C_G(x)/O_{p'}(C_G(x)) \\ \text{for some } x \in G \text{ of order } p\}$$

Since G is $\mathcal{K}$-proper, every element of $\mathcal{L}_p(G)$ is a quasisimple $\mathcal{K}$-group — that is, a perfect central extension of a known simple group — whose center is a p-group. It is important for us to consider all isomorphism types of such groups:

DEFINITION 21.2. $\mathcal{K}$ is the set of all quasisimple $\mathcal{K}$-groups, and

$$\mathcal{K}_p = \{K \in \mathcal{K} \mid K \text{ is quasisimple and } O_{p'}(K) = 1\}.$$

Thus, $\mathcal{L}_p(G) \subseteq \mathcal{K}_p$. Of course, $\mathcal{L}_p(G)$ may be empty for a given p. For example, this will certainly be the case if the centralizer in G of every element of order p is solvable.

The question of which elements of G will play the role of semisimple elements—involutions or elements of odd prime order—can be almost settled by the nature of the sets $\mathcal{L}_p(G)$ for various primes p. Indeed, in a group G of Lie type of characteristic r, the composition factors (if any) of the centralizers of semisimple elements are themselves groups of Lie type of the same characteristic r, while the centralizers of elements of order r are r-constrained and so have no r-components. Therefore for any prime p, $\mathcal{L}_p(G)$ consists of groups of Lie type in the same characteristic r, and is actually empty if $p = r$. On the other hand, apart from a few low degree alternating groups and a few sporadic groups, standard components of the centralizers of involutions in all other known simple groups are not groups of Lie type of odd characteristic.

Thus, the existence of a single element of $\mathcal{L}_2(G)$ that is a group of Lie type of odd characteristic in our $\mathcal{K}$-proper simple group G should in general be sufficient to guarantee that G will turn out to be a group of Lie type of odd characteristic. Hence when $\mathcal{L}_2(G)$ has a component of such an isomorphism type, we shall consider involutions to be semisimple elements and attempt to recover a Steinberg presentation for G by studying centralizers of involutions. As remarked earlier, we also plan to consider involutions to be "semisimple" elements when the target group for G is an alternating group. On the other hand, for most target sporadic groups G, even though $\mathcal{L}_2(G) \neq \emptyset$, we do not wish to consider involutions to be "semisimple" elements of G.

To distinguish the two outcomes we divide the set $\mathcal{K}_2$ into two subsets: one called $\mathcal{C}_2$, consisting of those elements of $\mathcal{K}_2$ which are groups of Lie type of characteristic 2 as well as a few other "characteristic 2-like" groups, and the complementary set consisting of the "odd characteristic-like" groups, including sufficiently large alternating groups. In an analogous way, for each prime p we define a subset $\mathcal{C}_p$ of $\mathcal{K}_p$ consisting of the groups of Lie type of characteristic p as well as a few "characteristic p-like" groups[21]. These sets $\mathcal{C}_p$ are defined precisely in the next chapter.

In this terminology, we shall thus regard involutions as semisimple elements whenever some element of $\mathcal{L}_2(G)$ is not in $\mathcal{C}_2$.

There is a further situation in which involutions should be viewed as semisimple elements. Indeed, the only known simple groups in which the centralizer C of some involution has the property $O_{2'}(C) \neq 1$ are certain alternating groups and groups of Lie type of odd characteristic. Hence we also want to view involutions as semisimple elements when $O_{2'}(C_G(x)) \neq 1$ for some involution x of our $\mathcal{K}$-proper simple group G.

Thus, only when the above two conditions fail is it appropriate to focus attention on centralizers of elements of odd prime order. Furthermore, the analysis of groups of 2-rank at most 2 also concentrates on centralizers of involutions. This leads to a fundamental division of all simple groups that is basic to our proof of the classification theorem.

[21]Actually, for $p = 2$ and 3, a handful of bizarre covering groups of groups of Lie type of characteristic p are excluded from $\mathcal{C}_p$.

DEFINITION 21.3. A $\mathcal{K}$-proper simple group G is said to be of **even type** if and only if

(1) Every element of $\mathcal{L}_2(G)$ lies in $\mathcal{C}_2$; and
(2) $O_{2'}(C_G(x)) = 1$ for every involution x of G; and
(3) G has 2-rank at least 3.[22]

This gives a purely internal set of conditions on G that enables one to decide in advance whether to take $p = 2$ or p odd; namely, if G is not of even type, focus on centralizers of involutions, while if G is of even type, focus on centralizers of elements of odd prime order. In the latter case, however, the question of which odd prime p to choose remains open. In fact the question is moot in the important quasithin case ($\sigma(G) = \emptyset$) where the analysis remains focused on 2-local subgroups throughout. When $\sigma(G) \neq \emptyset$, the choice of odd prime is resolved only at a later stage of the analysis in terms of a suitable maximality condition on elements of $\mathcal{L}_p(G)$. [In groups of Lie type of characteristic 2 — the primary target groups when G is of even type — the maximal p-rank for odd primes p is realized by primes dividing the order of either a Cartan subgroup or one other closely related torus.]

It should be noted that if G is of even type with $\mathcal{L}_2(G)$ empty, then G is of **characteristic** 2 **type** — i.e., $F^*(H) = O_2(H)$ (equivalently, $C_H(O_2(H)) \leq O_2(H)$) for every 2-local subgroup H of G. Moreover, in the original classification proof, it was the latter set of conditions which determined the point at which one shifted attention from centralizers of involutions to centralizers of odd prime order. The advantage of separating groups of even type from the others rather than drawing the dividing line at groups of characteristic 2 type is that we are thereby able to bypass a large number of technically difficult standard component problems.

Finally, we organize the global strategy for classifying groups of even type so that, if possible, we only examine centralizers of elements x of odd prime order p when these centralizers have p-rank ≥ 4. We thus make the following definitions.

DEFINITION 21.4. For any prime p,

$$\begin{aligned}
&\mathcal{I}_p(G) = \{x \in G \mid x \text{ has order } p\},\\
&\mathcal{I}_p^o(G) = \{x \in \mathcal{I}_p(G) \mid m_p(C_G(x)) \geq 4\} \quad \text{if } p \text{ is odd},\\
&\mathcal{I}_2^o(G) = \{x \in \mathcal{I}_2(G) \mid m_2(C_G(x)) \geq 3\}, \text{ and}\\
&\mathcal{L}_p^o(G) = \{L \mid L \text{ is a component of } C_G(x)/O_{p'}(C_G(x)) \text{ for some } x \in \mathcal{I}_p^o(G)\}.
\end{aligned}$$

The definition of $\mathcal{I}_2^o(G)$ has been made in this fashion for uniformity. In reality, an application of the Thompson transfer lemma shows that if $m_2(G) \geq 3$, then $m_2(C_G(x)) \geq 3$ for every involution $x \in G$. Thus $\mathcal{I}_2^o(G)$ equals $\mathcal{I}_2(G)$ if $m_2(G) \geq 3$, and is empty otherwise. Likewise $\mathcal{L}_2^o(G) = \mathcal{L}_2(G)$ if $m_2(G) \geq 3$, and otherwise $\mathcal{L}_2^o(G)$ is empty.

22. Generic simple groups and neighborhoods

As observed in section 19, a presentation of a group of Lie type of odd characteristic and Lie rank at least 3 can be recovered from the components of the centralizers of suitable commuting involutions, and a corresponding assertion holds

[22]Condition (1) is the primary one; in fact (2) turns out to be redundant. Were it not for pathologies in some small $\mathcal{C}_2$-groups, the redundancy of (2) would be easy to prove.

in characteristic 2 for components of the centralizers of commuting elements of suitable odd prime order p.

In fact, the centralizers of families of commuting semisimple elements contain redundant information, and in many cases the desired presentation is completely determined from only *two* appropriately chosen commuting elements x, y of order p and single components K, J in their respective centralizers with $K \cap J$ a "large" subgroup of both K and J. An analogous assertion holds for alternating groups of sufficiently high degree (with $p = 2$) and leads to their classical generation by involutions.

However, there are examples of distinct groups of Lie type that possess isomorphic pairs K, J. In such cases it is necessary to consider further elements $u \in \langle x, y\rangle$ and components L of their centralizers to distinguish the two groups. The resulting configuration of components is the basis of the key notion of a **neighborhood of** (x, K). To avoid technicalities, we shall not attempt to make the definition precise here, but limit ourselves to two illustrative examples in which the two components K, J suffice for the desired presentation. Further discussion of neighborhoods appears in sections 26 and 27 below.

First, consider $G = SL_n(q)$, $q = r^m$, r a prime, with $n \geq 4$ and $q \geq 3$. If r is odd, take $p = 2$, while if $r = 2$, take p to divide $q - 1$. For such a choice of p, commuting elements of order p can be simultaneously diagonalized. Let us assume for simplicity that $(n, q-1) = 1$. Then G is simple, and there exist λ, $\mu \in \boldsymbol{F}_q^\times$ with $\lambda \neq \mu$, $\lambda\mu^{n-1} = 1$, and $\lambda^p = \mu^p = 1$. The diagonal elements

$$x = \begin{pmatrix} \lambda & & & & \\ & \mu & & & \\ & & \mu & & \\ & & & \ddots & \\ & & & & \mu \end{pmatrix} \text{ and } y = \begin{pmatrix} \mu & & & & \\ & \mu & & & \\ & & \ddots & & \\ & & & \mu & \\ & & & & \lambda \end{pmatrix}$$

have order p and determinant 1 and so lie in G. Moreover, one computes that $C_G(x)$ and $C_G(y)$ have respective components K and J satisfying the following conditions:

(1) $K \cong J \cong SL_{n-1}(q)$;
(2) $I := E(K \cap J) \cong SL_{n-2}(q)$;
(3) $C_G(K)$ and $C_G(J)$ have cyclic Sylow p-subgroups; and
(4) I is contained in a single component of $C_G(u)$ for all $u \in \langle x, y\rangle$.

A large part of a set of defining relations for G is already visible in the three groups K, J, $K \cap J$, and it is possible to recover the rest from corresponding data in the centralizers of G-**conjugates** (indeed $\langle K, J\rangle$-conjugates) of x and y. In this situation, the 5-tuple (x, y, K, J, I) suffices to describe the neighborhood of (x, K). The inclusion mappings of I into J and K are considered part of the information contained in the 5-tuple.

Next, consider the case $G = A_n$, $n \geq 13$, and take $x = (n-3, n-2)(n-1, n)$ and $y = (12)(34)$, so that x and y are involutions of G. This time $C_G(x)$ and $C_G(y)$ have respective components K and J satisfying the following conditions:

(1) $K \cong J \cong A_{n-4}$;
(2) $I = E(K \cap J) \cong A_{n-8}$;
(3) $C_G(K)$ and $C_G(J)$ have E_4 Sylow 2-subgroups; and
(4) I is a component of $C_G(xy)$.

Again the desired presentation for G is determined from $K, J, K \cap J$ and corresponding information in the centralizers of G-conjugates of x and y. Here (x, y, K, J, I) suffices to describe the neighborhood of (x, K).

Note that in the $SL_n(q)$ case, conditions (1), (2) and (3) also hold in the wreath product $G = SL_{n-1}(q) \wr Z_p$ with x, y suitable commuting elements of order p in G that cycle the p $SL_{n-1}(q)$ components of G. But in that case, $\langle x, y\rangle \cap E(G) \neq 1$, and for any $u \in \langle x, y\rangle \cap E(G)$, $C_G(u)$ contains a subgroup isomorphic to $SL_{n-2}(q) \wr Z_p$, with I embedded diagonally, so condition (4) fails in this example. Thus in the general definition an analogue of condition (4) is critical for avoiding wreathed solutions for G, and is included in the definition of what we call a **vertical** neighborhood.

It should also be clear that this approach will be effective only for sufficiently large values of n. Indeed, if $K \cap J$ is too small (solvable, for example), there may be insufficient interplay between the groups K and J to recover the presentation for G. For $SL_n(q)$, we require $n \geq 4$ for $K \cap J$ to have a component, while for A_n, we require $n \geq 13$ to achieve this condition. In terms of the components K and J, it means that correspondingly we must have $K \cong SL_k(q)$, $k \geq 3$, or A_k, $k \geq 9$. Thus, the definition of a neighborhood of (x, K) must also include some minimal size restrictions on the given components K and J.

What we have just described is the "end game" for reaching presentations of the generic known simple groups. It provides a target for the global strategy of the analysis of our $\mathcal{K}$-proper simple group G: generally, our aim will be to prove the existence of a neighborhood in G of an appropriate (x, K), where x is an element of G of suitable prime order p and K is a component of $C_G(x)$; and by means of this neighborhood to demonstrate that G is isomorphic to the corresponding known simple group. However, as the above discussion has indicated, this strategy can be successful only if $\mathcal{L}_p(G)$ includes "sufficiently large" groups not in $\mathcal{C}_p$. For example, if $\mathcal{L}_2(G)$ contains no alternating groups of degree $k \geq 9$, it will obviously be impossible to construct an alternating type neighborhood within G.

This leads to a further fundamental division, to be made precise in the next chapter, of the set of groups not in $\mathcal{C}_p$ into the set $\mathcal{G}_p$ of "p-generic groups", i.e., those that are large enough to eventually yield presentations for G via the existence of appropriate neighborhoods, and the complementary set $\mathcal{T}_p$ of "p-thin groups". Recall that for each prime p, $\mathcal{K}_p$ is the set of all known quasisimple groups K with $O_{p'}(K) = 1$. Thus we actually have a *tripartite* partition

$$\mathcal{K}_p = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{G}_p.$$

Furthermore, groups in $\mathcal{C}_p$, $\mathcal{T}_p$ and $\mathcal{G}_p$ will often be called $\mathcal{C}_p$-groups, $\mathcal{T}_p$-groups, and $\mathcal{G}_p$-groups. For example, for $p = 2$, the definition will stipulate that all groups of Lie type of odd characteristic are $\mathcal{G}_2$-groups other than $SL_2(q)$ and $L_2(q)$, q odd, $L_3(3)$, $U_3(3)$, $PSp_4(3)$, $L_4(3)$, $U_4(3)$, $2U_4(3)$, ${}^2G_2(3)'$ and $G_2(3)$. $SL_2(q)$ is an $\mathcal{T}_2$-group for all odd q and $L_2(q)$ is an $\mathcal{T}_2$-group except for q a Fermat or Mersenne prime or 9, and the rest of the listed exceptions are $\mathcal{C}_2$-groups.

We can now define the basic notion of a group of $\mathcal{L}_p$**-generic type**. We emphasize that it requires that the group be of even type if p is odd and not of even type if $p = 2$. Recall also that $\mathcal{L}_2^o(G) = \mathcal{L}_2(G)$ or $\emptyset$ according as $m_2(G) \geq 3$ or not.

DEFINITION 22.1. Let p be a prime. We say that G is of $\mathcal{L}_p$**-generic** type if and only if one of the following holds:

(1) $p = 2$ and some element of $\mathcal{L}_2^o(G)$ is a $\mathcal{G}_2$-group (whence by definition $m_2(G) \geq 3$ and G is not of even type); or
(2) p is odd, G is of even type, and some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group.

Thus it is for the groups of $\mathcal{L}_p$-generic type that the neighborhood strategy described above is applicable. In case (2), p will in fact be further restricted to the set $\sigma(G)$ defined in Definition 12.1.

23. The main case division

We have now introduced the elements of the main logic of the proof of the Classification Theorem. As the discussion of the preceding section indicates, the proof divides into a "generic" case and the residual "special" cases. Moreover, our arguments to classify $\mathcal{L}_p$-generic groups work most smoothly when their p-rank is large; and one alternative conclusion of the analysis is always that the group has a strong p-uniqueness subgroup.[23] Accordingly the generic argument deals with a $\mathcal{K}$-proper simple group G and a prime p for which G has no strong p-uniqueness subgroup and for which the p-rank is large in a suitable sense which we now define.

For any $S \in Syl_2(G)$ let $\mathcal{M}_1(S)$ be the set of all maximal 2-local subgroups M of G such that $|S : M \cap S| \leq 2$. Recall from Section 12 that $\sigma(G)$ is the set of all odd primes p such that $m_p(M) \geq 4$ for some $M \in \mathcal{M}_1(S)$. This definition is of interest only when G is of even type. It is obviously independent of S, by Sylow's theorem.

Definition 23.1.

$$\sigma_0(G) = \{p \in \sigma(G) \mid G \text{ has no strong } p\text{-uniqueness subgroup}\}.$$

The term "generic" is then made precise as follows:

Definition 23.2. A $\mathcal{K}$-proper simple group G is **generic**, or **of generic type**, if and only if one of the following holds:
(a) G is of $\mathcal{L}_2$-generic type;
(b) G is of even type[24], $\sigma_0(G) \neq \emptyset$, and G is of $\mathcal{L}_p$-generic type for all $p \in \sigma_0(G)$.
Finally, G is of **special type** if and only if it is not generic. (Of course this term has nothing to do with the common notion of a "special" p-group.)

For the special cases we introduce the following terminology. We say that G is of **$\mathcal{L}_2$-special type** if and only if no element of $\mathcal{L}_2^o(G)$ is a $\mathcal{G}_2$-group. For an odd prime p, we say that G is of **$\mathcal{L}_p$-special type** if and only if G is of even type and no element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group. Thus a group is either of $\mathcal{L}_2$-generic or $\mathcal{L}_2$-special type (but not both). Likewise, for any odd prime p, a group G of even type is either of $\mathcal{L}_p$-generic or $\mathcal{L}_p$-special type, but not both.

Therefore in considering our $\mathcal{K}$-proper simple group G, there are four mutually exclusive and exhaustive cases, as indicated by Table III. We call G of **odd type**

[23] See Section 20.

[24] We shall actually introduce in the next chapter a notion of "restricted" even type, a slight and technical strengthening of the notion of even type; and our analysis will divide into cases according as G is or is not of restricted even type. Thus in our definitions here "even type" should be replaced by "restricted even type". However, for the expository purposes of this chapter we shall ignore this minor distinction. For the case division for generic groups, moreover, there is no particular advantage to the refined definition; it is of value only for special groups.

TABLE III: THE FOUR-PART CASE DIVISION

	ODD TYPE	**EVEN TYPE**
GENERIC TYPE	G has $\mathcal{L}_2$-generic type (PART III)	$\sigma_0(G) \neq \emptyset$, and G has $\mathcal{L}_p$-generic type for all $p \in \sigma_0(G)$ (PART III)
SPECIAL TYPE	G has odd type and $\mathcal{L}_2$-special type (PART IV)	$\sigma_0(G) = \emptyset$, or G has $\mathcal{L}_p$-special type for some $p \in \sigma_0(G)$ (PART V)

if and only if it is not of (restricted) even type. The four cases are to be considered in the indicated parts of this series, with the two generic type cases unified to the extent possible.

We emphasize that this entire analysis proceeds with the benefit of our uniqueness results from Part II. For example, we may assume the following:

(1) G contains no strongly embedded subgroup.
(2) If G is of even type and $\sigma(G) \neq \emptyset$, then $\sigma_0(G) \neq \emptyset$.

In addition to (1), there are other types of "uniqueness subgroups" which G is forbidden to contain because of results in the first two chapters of Part II. Statement (2) will be established in Chapters 5 and 6 of Part II.

24. Special simple groups

We now describe briefly the analysis of groups of special type. Although most of the known simple groups fall within the generic category, unfortunately the largest portion of the proof is devoted to the analysis of the special cases (which, in particular, include all sporadic groups). In fact, it is necessary to make a careful subdivision of these non-generic cases according to the method to be used for investigating the subgroup structure of G and establishing an appropriate set of internal conditions in terms of which G is ultimately to be identified.

Moreover, these internal conditions themselves differ widely from group to group. To emphasize their importance in the analysis, we introduce the symbol

$$G \approx G^*$$

to indicate the fact that the $\mathcal{K}$-proper simple group G under investigation satisfies the same set of internal conditions as the known simple group G^*.

Thus the meaning of "$\approx$" is dependent on the group G^*. For instance, when G is of generic type, G^* is an alternating group or a group of Lie type and the definition of $G \approx G^*$ requires G and G^* to have isomorphic neighborhoods (see Section 26 below). On the other hand, for a sporadic group G^* it is precisely the structure of the centralizers of its involutions together with the involution fusion pattern that defines the term $G \approx G^*$ (so that once it is proved that $G \approx G^*$, our Background Results imply immediately that $G \cong G^*$).

The point is that a presentation for each finite simple group, generic or special, is to be derived via an appropriate set of conditions on its subgroup structure, these forming the basis for the definition of the term $G \approx G^*$. In all cases, the global strategy of the classification proof is designed to force G to satisfy one of these possible sets of conditions.

The analysis of groups of special type in fact splits into six major cases, three for groups of odd type and three for groups of even type. To describe these, we require some more terminology.

First, if $S \in Syl_2(G)$, then $\mathcal{M}(S) = \mathcal{M}(G; S)$ is the set of maximal 2-local subgroups of G containing $N_G(S)$. Next, the groups $SL_2(q)$, q odd, $2A_n$, $7 \leq n \leq 11$, (the coverings of A_n by a center of order 2) and the coverings of $L_3(4)$ by a center of exponent 4 are called **$\mathcal{B}_2$-groups** ("$\mathcal{B}$" for Bender, since when these groups occur the "Bender method" is used). With this terminology, the special type case division is as follows.

<u>Special odd type</u>: G is of $\mathcal{L}_2$-special type.

Case 1. G has odd order.

Case 2. Either G has 2-rank at most 3 or some element of $\mathcal{L}_2(G)$ is a $\mathcal{B}_2$-group.

Case 3. G has 2-rank at least 4 and no element of $\mathcal{L}_2(G)$ is a $\mathcal{B}_2$-group.

<u>Special even type</u>: $\sigma_0(G) = \emptyset$, or G is of $\mathcal{L}_p$-special type for some $p \in \sigma_0(G)$.

Case 4. $\sigma(G)$ is empty. (By the uniqueness result at the end of the last section, this is equivalent to $\sigma_0(G) = \emptyset$.)

Case 5. For some $p \in \sigma_0(G)$, every element of $\mathcal{L}_p^o(G)$ is a $\mathcal{C}_p$-group.

Case 6. Case 5 does not hold, but for some $p \in \sigma_0(G)$, some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{T}_p$-group.

Likewise no fewer than six major methods enter into the analysis of these various cases (some used in more than one case)—three limited to groups that are of odd type, two to groups of even type, and one applicable to both types. We limit ourselves to listing them by name, adding a few brief comments concerning where they are used—a more detailed discussion appears in the next chapter.

<u>Special odd type</u>:

A. Character theory
B. Analysis of 2-fusion
C. The Bender method

<u>Special even type</u>:

D. The Goldschmidt amalgam method
E. The Klinger-Mason method

<u>Both types</u>:

F. k-balanced signalizer functors

Exceptional character theory is used in both the Odd Order Theorem and in the analysis of Case 2; while block theory is used in Case 2.

In Case 2 an analysis of 2-fusion forces the possible isomorphism types of a Sylow 2-subgroup of G as well as the involution fusion pattern in G.

The Bender method is used in Case 2 to eliminate $2'$-core obstruction and is also used in the revised version of the Odd Order Theorem to study the maximal subgroups of G.

Case 4 is the **quasithin** case, or more distinctly the "revised" quasithin case. Here the possible structures of the elements of $\mathcal{M}(S)$ are to be determined by the amalgam method. One of the uniqueness theorems of Part II allows us to conclude that $|\mathcal{M}(S)| > 1$, so the method is indeed applicable.

In the even type case, once the quasithin case has been treated, one is reduced to the case in which $\sigma_0(G) \neq \emptyset$. Now G contains a 2-local subgroup of p-rank at least 4 for every $p \in \sigma(G)$, so in particular $\mathcal{I}_p^o(G) \neq \emptyset$. It is for this reason that we are able to restrict the analysis in Cases 5 and 6 (as well as in the $\mathcal{L}_p$-generic type case for odd p) to the study of centralizers of elements of $\mathcal{I}_p^o(G)$.

The Klinger-Mason method [**KMa1, GL5**] is the least known of the six listed techniques. It is applicable in groups G that are simultaneously "characteristic 2-like" and "characteristic p-like" for some odd prime p, under the assumption that G contains a "large" $\{2, p\}$-subgroup—thus, in particular, it is applicable in Case 5. This is an interesting case, for p is forced to be 3 and the largest sporadic groups, including the Monster, occur as solutions as well as a half-dozen moderately large groups of Lie type over the fields of 2 and 3 elements. Furthermore, the local analysis involves the structure of the centralizers of commuting elements of order 2 and 3, and by analogy with the notion of a neighborhood for the prime p these form the basis of the definition of a $Z_6 \times Z_2$**-neighborhood** in G.

Finally, the signalizer functor method is used in both Cases 3 and 6 to eliminate p'-core obstruction. It is also used in Case 2 to treat the subcase in which a Sylow 2-subgroup contains a nontrivial strongly closed abelian subgroup.

25. Stages of the proof

The analysis in each of the six special type cases just described as well as in the generic type case involves several distinct *stages*, the total number varying from case to case. These can be viewed as landmarks on the road to the identification of G. It is our intention to make these critical junctures in the proof explicit, so that a flow chart of the entire classification proof will appear in the next chapter as a *grid* whose rows consist of the seven different cases and whose columns consist of the conclusions of the separate stages of the analysis. An analogous flow chart will be provided for the various uniqueness results which underlie the classification proof.

In the next chapter, we describe the individual stages for the grids. Actually the stages for the generic type case will be discussed as well in the next section. Here as an illustration we give a sketchier description of one of the special type cases. We consider the following subcase of Case 2 above:

1. Every element of $\mathcal{L}_2(G)$ is either a $\mathcal{C}_2$-group or isomorphic to $PSL_2(q)$ or $SL_2(q)$, q odd; and
2. Some element of $\mathcal{L}_2(G)$ is isomorphic to $SL_2(q)$ for some odd q.

In this situation the target groups G^* are the following groups (here $q = p^n$, p an odd prime, with $q > 3$ except in the case of ${}^3D_4(q)$):

$$G^* = L_3(q), U_3(q), PSp_4(q), G_2(q), {}^3D_4(q), \text{ or } L_4^{\epsilon}(q), q \not\equiv \epsilon (\text{mod } 8), \epsilon = \pm 1.$$

If we ignore certain groups defined over $\boldsymbol{F}_3$, then these families, together with the groups $L_2(q)$ and ${}^2G_2(q)$, q odd, consist in fact precisely of those simple groups of Lie type of odd characteristic in which no element of $\mathcal{L}_2(G)$ is a $\mathcal{G}_2$-group. Note, too, that they include the groups $L_4(q)$, $q \not\equiv 1 \mod 8$, which are of Lie rank 3. On the other hand, when $G \cong L_4(q)$, $q \equiv 1 \mod 8$, some element of $\mathcal{L}_2(G)$ is isomorphic to $L_3(q)$, which is a $\mathcal{G}_2$-group. This illustrates how delicately one must make the divisions among the various cases.

At the other end of the spectrum, the groups $U_3(q)$ are the only ones on the list of Lie rank 1. Because groups of Lie rank 1 have a much tighter internal structure than do groups of higher Lie rank, the configurations leading to the $U_3(q)$ solutions require very lengthy analysis.

All results stated below are derived using the uniqueness results of Part II. In particular, we may assume that G does not contain a strongly embedded subgroup.

Stage 1. On the basis of an analysis of 2-fusion in G, establish the existence of a target group G^* on the above list with the following properties:

1. A Sylow 2-subgroup S of G is isomorphic to one of G^*;
2. G and G^* have the same involution fusion pattern; and
3. If z is an involution of $Z(S)$ and $C = C_G(z)$, then the product of the components of $C/O_{2'}(C)$ is approximately the same as that in $C^* = C_{G^*}(z^*)$, z^* a 2-central involution of G^*.

It is during this stage that signalizer functor theory is used to eliminate a particular configuration in C involving a nontrivial strongly closed abelian 2-subgroup.

Stage 2. Use the Bender method to study a maximal subgroup M of G containing C and prove that one of the following holds:

1. G and G^* have 2-rank at least 3, C has the B_2-property[25], and $O_{2'}(C)$ is cyclic of order dividing $|O_{2'}(C^*)|$;
2. $M = C$, $G^* = U_3(q)$, and C has the B_2-property;
3. $M > C$, $G^* = L_3(q)$, $q = p^n$, p a prime, and $O^{p'}(M)$ is isomorphic to $O^{p'}(M^*)$ for some maximal parabolic subgroup M^* of G^*.

In case (3), we have shown at this point that both the centralizer of an involution and a maximal p-local subgroup of G approximate the corresponding subgroups in the groups $L_3(q)$, and in addition that G has only one conjugacy class of involutions. We take these conditions as the definition of the term $G \approx L_3(q)$, from which the presentation for G is to be derived.

Therefore, at the end of stage 2 one of the target families has already emerged from the analysis. A further bifurcation of the flow chart occurs at this juncture, one path leading to the family $U_3(q)$, the other to the remaining families.

Stage 3. Assume that $G^* = U_3(q)$ and use local group-theoretic methods to pin down the structure of $O_{2'}(C)$ and its embedding in G to the extent possible.

[Without character theory, it does not seem to be possible to prove that $O_{2'}(C)$ must be cyclic, as it is in the groups $U_3(q)$.]

Stage 4. Assume that $G^* = U_3(q)$ and use block-theoretic methods to establish the following results:

1. C has approximately the same structure as $C^* = C_{G^*}(z^*)$, z^* an involution of G^*;
2. If $P \in Syl_p(G)$, then $B = N_G(P)$ has approximately the same structure as a Borel subgroup of G^*; and
3. G acts doubly transitively on the set Ω of right cosets of B with P acting regularly on $\Omega - \{B\}$.

[25]In this context, this means that every component of $C/O(C)$ is the image of a component of C. See Section 7.

[It is essentially these conditions that form the basis of the term $G \approx U_3(q)$. In particular, G is then a split (B, N)-pair of rank 1. Now it follows from the Background Results that $G \cong U_3(q)$ in this case.]

Stage 5. Assume that $G^* \not\cong L_3(q)$ or $U_3(q)$ and use a combination of local analysis and further block-theoretic calculations to prove that G has subgroups H, N and M of approximately the same structure as a Cartan subgroup H^* of G^*, its normalizer N^*, and a maximal parabolic subgroup M^* of G^* (except when $G^* = L_4(q)$ or $U_4(q)$, in which case M approximates the centralizer of an involution in G^*.)

[These conditions together with the given information about C (plus the already determined involution fusion pattern) define the term $G \approx G^*$ in these cases. It is analogous to the definition of the term $G \approx L_3(q)$.]

Stage 6. Assume that $G^* \not\cong U_3(q)$ and prove next that (1) if $G^* \cong L_4(q)$ or $U_4(q)$, then G is generated by subgroups L_i, $i = 1, 2, 3$ such that $\langle L_i, L_2 \rangle \cong (S)L_3(q)$ or $(S)U_3(q)$ for $i = 1$ and 3, and $[L_1, L_3] = 1$; (2) otherwise, G is a split (B, N)-pair of rank 2 of the same general shape as G^*. Now one is in a position to identify G, using the Curtis-Tits Theorem[26] or a unitary variation of it in (1), and proving in (2) that the multiplication table of B is uniquely determined and in turn uniquely determines that of G, forcing $G \cong G^*$. In case (2), much less elaborate analysis is needed than in the Fong-Seitz classification of split (B, N)-pairs of rank 2, since so much additional information is already known.

26. Generic simple groups

Because most of the known finite simple groups are of $\mathcal{L}_p$-generic type (groups of Lie type of large Lie rank or alternating groups of degree at least 13), it is especially important to describe the various phases of the analysis that lead to their presentations by generators and relations.

We thus assume that G is a $\mathcal{K}$-proper simple group satisfying the following conditions:

1. Either $p = 2$ and some element of $\mathcal{L}_2(G)$ is a $\mathcal{G}_2$-group, or G is of even type, $p \in \sigma(G)$ and some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group; and
2. G does not contain a strong p-uniqueness subgroup.

For brevity, for any $x \in \mathcal{I}_p(G)$, we write C_x for $C_G(x)$. Also, recall that $\mathcal{L}_2^o(G) = \mathcal{L}_2(G)$ if $\mathcal{L}_2^o(G)$ is not empty.

Because some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group, our general results on p-terminal p-components yield the existence of $x \in \mathcal{I}_p^o(G)$ such that C_x contains a p-component K with K p-terminal in G and $K/O_{p'}(K)$ a $\mathcal{G}_p$-group. We call such an (x, K) a **p-terminal $\mathcal{G}_p$-pair**. Moreover, among all such pairs, we choose x and K so that $K/O_{p'}(K)$ is maximal in a suitable ordering of the elements of $\mathcal{L}_p^o(G)$ according to isomorphism type; and our analysis focuses on such a pair (x, K).

The case $p = 2$, $K/O_{p'}(K) \cong 2A_m$, is an anomaly which is easily eliminated by an analysis of 2-fusion.

The global strategy is then aimed at showing for suitable p, x and K that G possesses a neighborhood of (x, K) of the same shape as in one of the target groups G^*. To describe the process by which the desired neighborhood emerges from the analysis, we need some preliminary terminology.

Set $\overline{C}_x = C_x/O_{p'}(C_x)$, so that $\overline{K}$ is a component of $\overline{C}_x$.

[26]See section 27.

DEFINITION 26.1. Let $y \in \mathcal{I}_p(N_{C_x}(K))$ with $\overline{y}$ acting nontrivially on $\overline{K}$ and let I be a p-component of $C_K(y)$. If the image of I in $K\langle y\rangle/C_{K\langle y\rangle}(\overline{K})$ is terminal in $K\langle y\rangle/C_{K\langle y\rangle}(\overline{K})$, we call (y, I) a **subterminal** (x, K)**-pair**.

[When $p = 2$, there are small cases in which one must extend the definition to allow I to be a solvable 2-component, so that $I/O(I) \cong SL_2(3)$.]

DEFINITION 26.2. If (y, I) is a subterminal (x, K)-pair, then for any $u \in \langle x, y\rangle^{\#}$, the pumpup I_u of I in C_u (that is, the subnormal closure of I in C_u) is called a **neighbor** of K (more precisely, a (y, I)**-neighbor of** (x, K)**)**. Moreover, if each p-component of I_u is quasisimple, we call the neighbor I_u **semisimple**.

Note that in particular, K itself is a neighbor of K (arising from $u = x$). Furthermore, by $L_{p'}$-balance, each I_u is either a single x-invariant p-component of C_u or the product of p such p-components cycled by x.

The goal of the first two stages of the analysis is to show that every neighbor of K is in fact semisimple, thus establishing a partial B_p-property for centralizers of suitable elements of order p in G. This is achieved via signalizer functor theory, using our assumption that G does not possess a strong p-uniqueness subgroup. We focus on a suitably chosen E_{p^3}-subgroup A of C_x called an (x, K) p**-source**, and in the first stage of the analysis verify that G is 3/2-**balanced** with respect to A—that is, for all $a, a' \in A^{\#}$ and all $B \leq A$ with $B \cong E_{p^2}$, we have

1. $\Delta_G(B) \cap C_a \leq O_{p'}(C_a)$, where $\Delta_G(B) = \bigcap_{b \in B^{\#}} O_{p'}(C_b)$; and
2. $[O_{p'}(C_{a'}) \cap C_a, A] \leq O_{p'}(C_a)$.

[See section 29 of the next chapter for a fuller discussion.]

When $p = 2$ and $\overline{K} = K/O_{2'}(K)$ is of Lie type over $\boldsymbol{F}_q$, q odd, of sufficiently high Lie rank, $\overline{K}$ contains the direct product $\overline{H}$ of three "root" $SL_2(q)$-subgroups, in which case we take A to map onto $Z(\overline{H})$. It is then straightforward to verify 3/2-balance on the basis of $L_{2'}$-balance and the B_2-property in proper sections of G. However, when such a choice of A is not available, verification of 3/2-balance is considerably more complicated. This occurs when $\overline{K} \cong A_n$ or when p is odd, and especially when $p = 2$ and $\overline{K}$ is of Lie type of low Lie rank, the most troublesome case. The precise definition of an (x, K) p-source for all $\overline{K}$ is too technical to present here.

We are now ready to state the objectives of the various stages of the analysis, but in somewhat simplified form.

Stage 1. Establish the following two results:
1. By an analysis of 2-fusion, show that $K/O_{2'}(K) \not\cong 2A_n$ when $p = 2$; and
2. In the remaining cases, verify that G is 3/2-balanced with respect to any (x, K) p-source A.

REMARK. For $2A_n$ to be a $\mathcal{G}_2$-group, one must have $n \geq 12$ by the definition of $\mathcal{G}_2$.

Stage 2. Using signalizer functor theory, relative to A, show that suitable neighbors of K are semisimple. In particular, K itself is quasisimple.

REMARKS. In Stage 2, on the basis of the signalizer functor theorem, it follows from Stage 1 that the associated "3/2-functor" $\Theta_{3/2}(G; A)$ on E_{p^2}-subgroups of A

is a p'-group (see section 29 of the next chapter). The analysis splits into two parts according as $\Theta_{3/2}(G;A) = 1$ or $\Theta_{3/2}(G;A) \neq 1$. In the first case, we argue that suitable neighbors of K are semisimple, as desired, while in the second, with the aid of our p-component uniqueness theorems, we show that G possesses a strong p-uniqueness subgroup, contrary to assumption.

In Stage 3, we establish a variety of results, which taken together show that the internal structure of G approximates that of one of the desired target groups. We need the following definition.

DEFINITION 26.3. Let (y, I) be a subterminal (x, K)-pair and for $u \in \langle x, y\rangle^{\#}$, let I_u be the pumpup of I in C_u. The **(y, I)-neighborhood $\mathcal{N}$ of (x, K)** consists of the group $\langle x, y\rangle$, the group I, and the groups I_u, $u \in \langle x, y\rangle^{\#}$. Furthermore, we say that $\mathcal{N}$ is **vertical** provided:
(1) I_u is quasisimple for every $u \in \langle x, y\rangle^{\#}$; and
(2) $I < I_u$ for some $u \in \langle x, y\rangle - \langle x\rangle$.

Stage 3. For some choice of the prime p and of the p-terminal $\mathcal{G}_p$-pair (x, K) and the subterminal (x, K)-pair (y, L), show that the (y, L)-neighborhood $\mathcal{N}$ of (x, K) is vertical and that one of the following holds:

1. $p = 2$, $K \cong A_n$, $n \geq 9$, and either
   (a) $C_G(K) \cong E_4$ and a root four-subgroup[27] of K centralizes a G-conjugate of K; or
   (b) $m_2(C_G(K)) = 1$;
2. K is of Lie type defined over $\boldsymbol{F}_q$ and the following conditions hold:
   (a) $m_p(C_G(K)) = 1$;
   (b) If $p = 2$, then q is odd, and if p is odd, then $q = 2^n$;
   (c) p divides $q \pm 1$;
   (d) Either every component of $\mathcal{N}$ is defined over $\boldsymbol{F}_q$ or K is of low Lie rank; and
   (e) If K is a linear or unitary group, then every component of $\mathcal{N}$ of the same untwisted Lie rank as K is a linear or unitary group; or
3. $p = 2$, K is sporadic, and $m_2(C_G(K)) = 1$.

REMARKS. We make a number of comments concerning the rather elaborate conclusions of Stage 3. First, these results together yield that $m_p(C_G(K)) = 1$ unless $p = 2$ and $K \cong A_n$. The proof of this assertion depends on generational properties of K together with our p-component uniqueness theorems. Likewise if p is odd, K is necessarily of Lie type of characteristic 2, the proof of which depends critically on the fact that G is of even type when p is odd.

The assertion that the neighbors in $\mathcal{N}$ are quasisimple is a consequence of our maximal choice of (x, K). Indeed, if false for some $u \in \langle x, y\rangle^{\#}$, we pump up a component of I_u to a p-terminal $\mathcal{G}_p$-pair (x^*, K^*) and compare (x^*, K^*) to (x, K) in our ordering for a contradiction. The proof that $\mathcal{N}$ is in fact vertical (whence $I < I_u$ for some $u \in \langle x, y\rangle - \langle x\rangle$) depends on the following generational property of centralizers of elements of order p acting on $\mathcal{G}_p$-groups: If $B \cong E_{p^2}$ acts faithfully on the (quasisimple) $\mathcal{G}_p$-group L and $b \in B^{\#}$, and we set $L_0 = \langle C_L(b') | b' \in B - \langle b\rangle\rangle$, then generally $\langle b\rangle$ is not weakly closed in a Sylow p-subgroup of L_0 with respect to L_0.

[27] A root four-subgroup of A_n is a four-subgroup of A_n fixing $n - 4$ points.

It is in the verification of (2c) (which we express by saying that p **splits** K) that a choice of the prime p is made when G is of even type. To prove that p can be chosen to split K, it is necessary to consider the set $\gamma(G)$ of all odd primes p for which no p-uniqueness subgroup exists, but for which there exists a p-terminal $\mathcal{G}_p$-pair (x, K), and then choose $p \in \gamma(G)$, x and K so that K is as large as possible in our ordering, with p splitting K if possible. If, for such a choice of p, x and K, it turns out that p does not split K, we shift attention to an odd prime r that does split K and argue on the basis of our maximal choice of p, x and K that G contains a r-uniqueness subgroup M. Using this result, we go on to show that M is in fact a p-uniqueness subgroup, contrary to hypothesis.

Likewise the proof of (2d) (which we express by saying that $\mathcal{N}$ is **level**) depends on our maximal choice of (x, K). Furthermore, if (2e) fails, so that some neighbor I_u of the same untwisted Lie rank as K is not linear or unitary, we argue that the pair (u, I_u) is also maximal in our ordering and then replace (x, K) by (u, I_u).

Finally, each target group possesses a neighborhood satisfying the conclusions of Stage 3. However, there exist possibilities for K and $C_G(x)$ in Stage 3 that correspond to no target group G^*. For example, if $p = 2$, there is no G^* if $K \cong A_n$ with $m_2(C_G(K)) = 1$, or if K is sporadic, or if K is one of certain proper homomorphic images of $SL_n(q)$, q odd. Thus it is not until the next stage of the analysis that we are able to associate a specific target group G^* with G.

On the other hand, when K is of Lie type, the fact that $\mathcal{N}$ is vertical and (in most cases) level does enable us to eliminate the troublesome nonsimple "wreathed" and "field automorphism" cases, which in the original classification proof required rather elaborate analyses, with a final contradiction to the simplicity of G obtained by a transfer argument.

Indeed, in the wreathed case G locally resembles $J\langle x\rangle$, where $J = J_1 \times J_2 \times \cdots \times J_p$ with $J_1 \cong K$, x cycling the J_i, $1 \leq i \leq p$, and $K = C_J(x)$ (whence $C_{G^*}(x) = K \times \langle x\rangle$). However, in this case one checks that for suitable u, the pumpup I_u of I in $C_G(u)$ is the product of p components cycled by x, contrary to the fact that I_u is quasisimple as $\mathcal{N}$ is vertical. Hence G^* does not have this form.

Similarly in the field automorphism case, G resembles $J\langle x\rangle$, where J is defined over $\boldsymbol{F}_{q^p}$, x induces a nontrivial field automorphism on J, and $K = E(C_J(x))$. This time one sees that I_y is defined over an extension field of $\boldsymbol{F}_{q^p}$, contrary to the fact that $\mathcal{N}$ is level, so G^* does not have this form either.

Now fix the prime p, the p terminal $\mathcal{G}_p$-pair (x, K), the subterminal (x, K) pair (y, I) and the corresponding neighborhood $\mathcal{N}$ to satisfy the conclusion of Stage 3. [There are cases in which there is more than one choice for (y, I). Usually y is taken to lie in K, but in some situations the appropriate (y, I) requires $y \notin K$.]

We define the **span** of $\mathcal{N}$ to be the subgroup

$$G_0(\mathcal{N}) = \langle I_u \mid u \in \langle x, y\rangle^{\#}\rangle.$$

The next major step in the analysis is to identify $G_0(\mathcal{N})$ as a quasisimple $\mathcal{K}$-group.

Since the Lie type case is considerably more complicated than the alternating and sporadic case, we treat it separately.

Stage 4a. If K is of Lie type defined over $\boldsymbol{F}_q$, show that $G_0(\mathcal{N})$ is a central extension of G^* for some group G^* of Lie type defined over $\boldsymbol{F}_q$.

Remarks. The goal of the analysis is to show that $G_0(\mathcal{N})$ satisfies the same set of Steinberg relations as hold in the target group G^*, from which the desired isomorphism follows. Moreover, this is to be achieved either via the Curtis-Tits or Gilman-Griess approach. Many of the desired relations are already visible in K inasmuch as the Lie rank of K is either 1 or 2 less than that of G^*; and the remaining rank 1 subgroups needed to identify $G_0(\mathcal{N})$ are constructed by an analysis of the neighbors in $\mathcal{N}$ (in some cases also of components of centralizers of other elements of order p in $C_G(\langle x, y\rangle)$). Although it is probably possible to follow either approach in proving that $G_0(\mathcal{N})$ satisfies the Steinberg relations for G^*, it appears to be most efficient to use the Curtis-Tits theorem or a variant of it when $p = 2$ (and correspondingly K is of odd characteristic) and the Gilman-Griess theorem when p is odd (and K is of characteristic 2). There are several reasons why the first method is better adapted for the $p = 2$ analysis and the second for the p odd analysis.

First, if $p = 2$ and v is a classical involution in K (i.e., $C_K(v)$ contains a component or solvable component $J \cong SL_2(q)$ with $v \in S$), then we are able to "keep J in the picture" and to determine its embedding in G. There is no analogous result for odd p.

On the hand when p is odd, there exists a unique (up to conjugacy) elementary abelian p-subgroup B of maximal rank in $C_G(\langle x, y\rangle)$, and using the embedding of B in both K and a suitable neighbor of K one can determine a subgroup R of $N_G(B)$ generated by involutions acting as reflections on B and such that R is isomorphic to the Weyl group W^* of G^* (to only a proper subgroup of W^* in some cases when $q = 2$).

A corresponding result for $p = 2$ would be much more difficult to derive since one would then have to show that R is isomorphic to a reduced monomial subgroup R^* of G^* (R^* contains a nontrivial elementary abelian normal 2-subgroup H^* and $R^*/H^* \cong W^*$). This would involve (a) identification of an appropriate elementary abelian 2-subgroup B of $C_G(\langle x, y\rangle)$ with $B \cong H^*$; (b) proof that $R/B \cong W^*$; and (c) determination of the isomorphism type of the extension of R/B by B.

In the course of the foregoing analysis, we eliminate those possibilities for K of Lie type that correspond to no target simple $\mathcal{K}$-group G^*. Also the low Lie rank generic cases require special treatment.

We express the conclusion of Stage 4a by writing $G \approx G^*$.

We carry out a similar analysis in the alternating and sporadic cases.

Stage 4b. If $p = 2$ and K is not of Lie type, show that
(i) $K \cong A_n$, $n \geq 9$;
(ii) $C_G(K) \cong E_4$; and
(iii) $G_0(\mathcal{N}) \cong A_{n+4}$.

Remarks. If $K \cong A_n$ and $C_G(K) \cong E_4$, we identify $G_0(\mathcal{N})$ via the classical presentation for A_{n+4}. Hence to establish the conclusions of Stage 4b, it remains to eliminate the cases $m_2(C_G(K)) = 1$ with K alternating or sporadic.

If $K \cong A_n$, one argues first that $G_0(\mathcal{N})\langle x\rangle \cong \Sigma_{n+2}$, this time using the classical presentation for Σ_{n+2} (see the next section) and then uses transfer to contradict the simplicity of G. [The case $n = 10$ is exceptional since $Aut(F_5)$ possesses an outer automorphism with a $Z_2 \times \Sigma_{10}$ centralizer. This case is also eliminated by transfer.]

On the other hand, if K is sporadic, then as K is a $\mathcal{G}_2$-group,

$$K \cong J_1, Mc, Ly, ON, He, Co_3 \text{ or } F_5.$$

In each case, one derives a contradiction by playing off the existence of the vertical neighborhood $\mathcal{N}$ against the structure of appropriate local subgroups of G.

We express the conclusion of Stage 4b by writing $G \approx A_{n+4}$.

Thus Stages 4a and 4b together yield that $G \approx G^*$ for some G^*.

Finally in Stage 5, we argue that $G_0(\mathcal{N}) = G$, thereby completing the proof of our theorem in the generic case.

Stage 5. Show that $G \approx G^*$ implies $G \cong G^*$ for the G^* arising in Stage 4.

Remarks. In the contrary case, $G_0(\mathcal{N}) < G$ as $G_0(\mathcal{N})$ is a central extension of G^*; and we argue that $M = N_G(G_0(\mathcal{N}))$ is a strong p-uniqueness subgroup of G, contrary to our hypothesis. The first step in the proof is the assertion that $C_G(u) \leq M$ for every $u \in \langle x, y\rangle^{\#}$, which follows readily from the definition of $G_0(\mathcal{N})$ and the structure of $C_G(u)$ for $u \in \langle x, y\rangle^{\#}$. [It is at this step that the $Aut(F_5)$ and Σ_{12} cases separate.]

We hope that this rather detailed outline provides a useful summary of the generic case analysis.

27. The identification of G

As repeatedly emphasized, the overall objective of the classification proof is to derive a set of internal conditions on the group G under investigation that will lead to a presentation for it identical to a presentation in one of the known simple groups G^*, thereby enabling us to "recognize" G as the target group G^*. There is great variation in the complexity of the passage from such internal conditions to the desired presentation. At one end of the spectrum, the existence of a vertical neighborhood in G leads to the identification of G as a generic group of Lie type via the Steinberg relations or to an alternating group via the classical presentation. At the other end, corresponding presentations for some sporadic groups such as Ly and ON have required computer calculations, as noted earlier.

We conclude this sketch of the classification proof by first outlining the pasage from subgroup information to generators and relations data in a specific example in which the steps are particularly transparent. Following that, we state the Curtis-Tits and Gilman-Griess theorems. We shall take as G^* the almost simple group Σ_n, $n \geq 11$, and we shall assume that our $\mathcal{K}$-proper simple group G possesses a "neighborhood" (x, y, K, J, I) of "symmetric" type—i.e., satisfying the following conditions:

(27.1)

1. $C_x = \langle x\rangle \times K,$ where $K \cong \Sigma_m, m \geq 9$;
2. $C_y = \langle y\rangle \times J,$ where $J \cong \Sigma_m$;
3. $y \in K$ is a transposition and $x \in J$ is a transposition;
4. $C_x \cap C_y = \langle x, y\rangle \times I,$ where $I = K \cap J \cong \Sigma_{m-2}$; and
5. I' is a component of C_{xy}.

Note that if $G \cong \Sigma_{m+2}$, $m \geq 9$, and x and y are the transpositions (12) and $(m+1\ m+2)$, then $\langle x, y\rangle$ is a four-group satisfying the conditions (27.1). [Since $m \geq 9$, all automorphisms of K and J are inner, so it is a well-defined notion for an element of one of these groups to be a transposition.]

Although Σ_{m+2} is not simple, the specified configuration of subgroups corresponds nevertheless to a "real-life" situation. Indeed, it seems that the best way to eliminate the possible existence in G of such a configuration is first to prove that G has a presentation identical to the classical presentation of Σ_{m+2} by transpositions, whence $G \cong \Sigma_{m+2}$; and only after this has been established to use the simplicity of G to force a contradiction.

Observe that Σ_{m-2} has two embeddings in Σ_m up to conjugacy: one has image in A_m, fixing none of the m points, while the other, which we call "canonical", has image fixing 2 points. At the outset, the precise nature of the embeddings of I in J and K needs to be settled; questions of this nature about embeddings are typical at this stage of the proof. In this case, a brief fusion argument using $m \geq 9$ shows that

(27.2) $\qquad$ I is canonically embedded in both J and K.

We shall outline the proof of the following illustrative result.

Theorem 27.1. *If x, y, K, J, I satisfy (27.1) and (27.2), then*

$$\langle K, J\rangle \cong \Sigma_{m+2}.$$

As indicated, the goal of the proof will be to show that the subgroup $\langle K, J\rangle$ of G has a presentation identical to that of the classical presentation of Σ_{m+2} in terms of its transpositions stated in section 11.

Thus our task is to produce nontrivial elements a_i, $1 \leq i \leq m+1$, which together generate $\langle K, J\rangle$ and satisfy the corresponding conditions $(a_i a_j)^{r_{ij}} = 1$ where $r_{ij} = 1$ if $i = j$, $r_{ij} = 2$ if $|j - i| > 1$, and $r_{ij} = 3$ if $|j - i| = 1$. Indeed, it will then follow that the map $(i\ i+1) \mapsto a_i$, $1 \leq i \leq m+1$, extends to an isomorphism of Σ_{m+2} with $\langle K, J\rangle$.

Identifying $K \cong \Sigma_m$ with the symmetric group on the letters $\{1, 2, \ldots, m\}$, we have the corresponding canonical generation of K by transpositions $a_i = (i\ i+1)$, $1 \leq i \leq m-1$. Moreover, as y is a transposition in K, we can order the underlying set so that $y = a_1 = (12)$.

Then since $C_K(y) = \langle y\rangle \times I$ and I is canonically embedded in K, I is generated by elements $a_3, a_4, \ldots, a_{m-1}$ corresponding to the transpositions (34), (45), $\ldots$, $(m-1\ m)$.

On the other hand, $I \times \langle x\rangle \leq J$ with this embedding being canonical. We can therefore similarly view J as acting on the set $\{3, 4, \ldots, m+2\}$ with x corresponding to the transposition $a_{m+1} = (m+1\ m+2)$. Furthermore, there exists a transposition $a_m = (m\ m+1)$ in J such that the elements $a_3, a_4, \ldots a_{m+1}$ are a canonical generating set for J.

In particular, clearly $\langle K, J\rangle = \langle a_1, a_2, \ldots, a_{m+1}\rangle$. Furthermore, as $\langle x, y\rangle$ centralizes I with x centralizing K and y centralizing J, all relations $(a_i a_j)^{r_{ij}} = 1$ hold whenever both a_i and a_j lie in $K \times \langle x\rangle$ or both lie in $J \times \langle y\rangle$, since they hold in these groups. However, the only a_i from $K \times \langle x\rangle$ not contained in $J \times \langle y\rangle$ is a_2, while the only one from $J \times \langle y\rangle$ not contained in $K \times \langle x\rangle$ is a_m. Thus to show that the a_i, $1 \leq i \leq m+1$, form a canonical generating set yielding the desired presentation for $\langle K, J\rangle$, *it remains only to prove that the order of the product* a_2a_m *is the same as that for the corresponding generators of* Σ_{m+2}. That is, $(a_2a_m)^2 = 1$, or equivalently, a_2 centralizes a_m.

Thus the following lemma will complete the proof of the theorem.

LEMMA 27.2. a_2 *centralizes* a_m.

For simplicity of notation, put $u = a_2$, $v = a_m$, $t = a_3$, and $w = a_{m-1}$. Then by observation of $K \times \langle x\rangle$ and $J \times \langle y\rangle$, we know all the relations among y, u, t and x, v, w, respectively, and, in addition that $\langle y, t\rangle$ centralizes $\langle x, v, w\rangle$ while $\langle x, w\rangle$ centralizes $\langle y, u, t\rangle$. Furthermore, these six elements are all distinct.

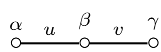
We now choose k with $4 < k < m - 2$, and consider the transposition $z = a_k$. Such a choice of k is possible as $m \geq 9$. Furthermore, the six involutions x, v, w, y, u, t are all contained in $C_G(z)$. On the other hand, J has only one conjugacy class of transpositions so $z = x^a$ for some $a \in J$. Set $L = K^a$, so that

$$C_G(z) = \langle z\rangle \times L, \text{ where } L \cong \Sigma_m.$$

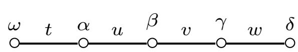
Since $a \in J$, $y = y^a$, and as y is a transposition in K, it is then also one in L.

Now $C_K(z) \cong \Sigma_{m-2} \times Z_2$ (as z is a transposition of $K \cong \Sigma_m$). Furthermore, y, u and t are all $C_K(z)$-conjugate, and so all are transpositions in L. Similarly, considering $C_J(z)$, we obtain that also x, v, w map to transpositions in $L\langle z\rangle/\langle z\rangle \cong L$. Observe that by inspection of C_y and C_x, the six elements y, u, t, x, v, w remain distinct modulo $\langle z\rangle$ (except for a possible coincidence of u and v, which would in any case imply $[u, v] = 1$).

To prove that u and v centralize each other we work in the natural representation of $L\langle z\rangle/\langle z\rangle$ on m letters and show that their supports are disjoint. Suppose false. Since $u \neq v$, their supports thus have a point β in common, as indicated by the following diagram (i.e., uv is a 3-cycle):



On the other hand, in K we see that tu has order 3, whence the support of t has a point in common with the support of u. But t centralizes v, and t and v are distinct modulo $\langle z\rangle$, so this common point must be α. Similarly w and v have the point γ as common support, thus giving the following diagram (where $\omega \neq \delta$ since $[t, w] = 1$, as can be seen in K):



Finally consider the elements y and x. Since yu has order 3, the support of y has a point in common with that of u. But y centralizes t, and y and t are distinct modulo $\langle z\rangle$, so this common point is not α. Hence it must be β. Similarly it follows that β is one of the points of the support of x (as xv has order 3 and x centralizes w). However, since $x \neq y \mod \langle z\rangle$, we conclude that the projection of xy on L is a 3-cycle, contrary to the fact that x and y are commuting involutions.

This contradiction establishes that u and v commute modulo z. Therefore $[u, v] \in \langle z\rangle \cap (L\langle z\rangle)' = 1$, which proves the lemma and completes the proof of the theorem.

Finally we turn to the Curtis-Tits and Gilman-Griess theorems. The precise statement of the former [**Cu1, Ti2**] is expressed in terms of relations among certain root subgroups of a group of Lie type.

THEOREM 27.3. (Curtis-Tits) *Let G^* be a finite group of Lie type. In a Bruhat decomposition of G^*, let Σ be the root system and X_α ($\alpha \in \Sigma$) the corresponding root subgroups. Let Π be a fundamental system in Σ and for each $\alpha \in \Pi$ let $G^*_\alpha = \langle X_\alpha, X_{-\alpha}\rangle$. Assume that $|\Pi| \geq 3$. Then the relations among the elements of the groups G^*_α holding in the groups $\langle G^*_\alpha, G^*_\beta\rangle$, α, $\beta \in \Pi$, form a set of defining relations for a central extension of G^*. Consequently, if G is any group generated by subgroups G_α ($\alpha \in \Pi$), and if there are homomorphisms $\phi_\alpha : G^*_\alpha \to G_\alpha$ and $\phi_{\alpha\beta} : \langle G^*_\alpha, G^*_\beta\rangle \to \langle G_\alpha, G_\beta\rangle$ for all α, $\beta \in \Pi$, which are coherent in the sense that $\phi_{\alpha\beta} = \phi_{\beta\alpha}$ and $\phi_{\alpha\beta}|_{G^*_\alpha} = \phi_\alpha$ for all α and β in Π, then $G/Z(G)$ is a homomorphic image of $G^*/Z(G^*)$.*

In the applications, G is simple and G^* quasisimple, and so $G \cong G^*/Z(G^*)$.

In practice, hypotheses which are superficially somewhat weaker are sufficient to establish the same conclusion. For example, in the simplest case of untwisted groups with only single bonds in the Dynkin diagram, the theorem has the following corollary.

COROLLARY 27.4. *Let q be an odd prime power and let G^* be one of the simple groups $A_n(q)$, $D_n(q)$, $E_6(q)$, $E_7(q)$ or $E_8(q)$, so that the Dynkin diagram Δ of G^* has only single bonds. Suppose that G is a simple group generated by subgroups $G_\alpha \cong SL_2(q)$, one for each node $\alpha \in \Delta$, and let z_α be the involution in the center of G_α. Assume that the following conditions hold.*

(1) *$\langle G_\alpha, G_\beta\rangle \cong (P)SL_3(q)$ or $[G_\alpha, G_\beta] = 1$ according as α and β are joined or are not joined in Δ;*
(2) *$[z_\alpha, z_\beta] = 1$ for all $\alpha \neq \beta \in \Pi$; and*
(3) *If we set $H_0 = \langle z_\alpha \mid \alpha \in \Delta\rangle$ (so that H_0 is an elementary abelian 2-group), then for each α, $|H_0 : C_{H_0}(G_\alpha)| \leq 2$.*

Then $G \cong G^*$.

In the applications, when the G_α are constructed in a natural way from a terminal component and its neighbors, the second and third conditions above emerge naturally. We shall also use variants of the above result applicable to twisted groups G^* and allowing the possibility $\langle G_\alpha, G_\beta\rangle \cong (P)SU_3(q)$. Theorems of this type were investigated by Phan and more recently by Das [**Ph1, Ph2, Da1**].

To state the Gilman-Griess theorem [**GiGr1**], we first make the following definition.

DEFINITION 27.5. Two groups $G(q)$ and $G_1(q)$ of Lie type of characteristic 2 are said to be **Y-compatible** if and only if the following conditions hold:

(1) $G(q)$ has Lie rank at least 4 and $G_1(q)$ has Lie rank at least 2;
(2) $G(q)$ and $G_1(q)$ are either both twisted or both untwisted;
(3) If Σ and Σ_1 are root systems of $G(q)$ and $G_1(q)$, respectively, then Σ_1 can be identified with a proper subsystem (also called Σ_1) of Σ; and
(4) If $\alpha \in \Sigma_1$, then the root subgroup X_α of $G(q)$ is isomorphic to the root subgroup X_α of $G_1(q)$;
(5) Y is a subgroup of the Weyl group of Σ, and for every pair of roots $\alpha, \beta \in \Sigma$, there is an element $w \in Y$ such that $\{\alpha, \beta\}^w \subseteq \Sigma_1$.

THEOREM 27.6. (Gilman-Griess) *Let $G(q)$ and $G_1(q)$ be Y-compatible groups of Lie type of characteristic 2, as in the preceding definition. Identify Σ_1 with a subset of Σ, and let W_1 and W be their Weyl groups, so that W_1 is identified with a subgroup of W and $Y \leq W$. In a Bruhat decomposition of $G(q)$, let $\{X_\alpha \mid \alpha \in \Sigma_1\}$*

be the set of root subgroups generating $G_1(q)$, with H_1 and N_1 the corresponding Cartan and monomial subgroups of $G_1(q)$.

Suppose that G_0 is a group generated by $G_1(q)$ and a subgroup R satisfying the following conditions:

(a) *$R \cong Y$;*

(b) *When R is identified with Y as in (a), $R \cap G_1(q) = Y \cap W_1$; and*

(c) *For each $\alpha \in \Sigma_1$, the stabilizer R_α of α in R normalizes X_α, and the group $R_\alpha / C_{R_\alpha}(X_\alpha)$ is abelian.*

Under these conditions, $G_0/Z(G_0) \cong G(q)/Z(G(q))$.

F. Additional Comments

28. The length of the proof

The preceding sketch provides a summary of our several-thousand-page multi-volume proof of the classification of the finite simple groups. It is reasonable to ask why any single mathematical theorem should require such an excessively long argument. The only answer we can give is that the present proof consists of the shortest argument we have been able to devise. Perhaps the matter should rest there, but this response seems somehow inadequate, especially as the general mathematical community believes on the basis of the historical record that in due course this result, like other major mathematical theorems, will undergo dramatic simplification. Even though our "second generation" classification represents roughly a three-fold reduction of the original proof, it is far too long to satisfy this requirement.

Obviously it is impossible to predict the minimum length of a proof of any mathematical theorem. Nevertheless, one may ask whether there are features of the classification of the finite simple groups which indicate heuristically whether or not a short proof—on the order of a few hundred pages—is attainable.

Consider first the very existence of the known simple groups. Elegant constructions of the groups of Lie type exist via the theory of algebraic groups. Moreover, the existence of M_{24} suffices for the construction of Co_1 and that of Co_1 in turn for the existence of F_1, and embedded within these three groups are twenty of the twenty-six sporadic groups. However, a complete proof of these two assertions, plus the construction of the remaining six sporadic groups, requires a substantial body of argument, and we have not yet begun the main body of the classification proof.

But existence is not the central issue—in fact, a reasonable case can be made that the question of existence is entirely separate from that of the classification of the finite simple groups, which should be concerned solely with proving that the list has not missed any groups. Rather the key factor that will dominate any inductive proof of the classification theorem is the *a priori* structure of a minimal counterexample G. Why should the structure of G bear any resemblance to that of some known simple group G^*; yet how is one to imagine a proof that G is isomorphic to G^* which does not require one to show first that G and G^* have comparable subgroup structures?

The most common suggestion of an alternative classification strategy is to try to associate some mathematical object on which G acts as a group of automorphisms and then to derive the classification theorem from a classification of the corresponding "geometries". It certainly sounds plausible, particularly as there exist important geometric classification theorems, notably Tits's classification of spherical buildings of rank at least 3, yielding as a corollary a characterization of the finite groups of Lie type of Lie rank at least 3.

Perhaps we can provide insight into the classification of the simple groups by examining this idea in some detail. For simplicity we limit the discussion to an important but very special case. We consider the problem of determining the *minimal* simple groups, which by definition consist of the nonabelian simple groups G each of whose proper subgroups is solvable. Our remarks apply also to the slightly more general class of N-groups, in which each *local* subgroup is assumed solvable. Moreover, by replacing the term "solvable" by "$\mathcal{K}$-group" and the short list of groups in (28.1) below by the full set of known simple groups, our remarks can be readily adapted to a minimal counterexample to the complete classification theorem.

Whatever approach one may take in defining an associated geometry $\Gamma(G)$ of G, in order for G to act as a group of automorphisms of $\Gamma(G)$, clearly some structural properties of G must be involved in the definition. However, until one proves otherwise, why doesn't the subgroup structure of G resemble that of an *arbitrary* solvable group G^*? If this is the case, then $\Gamma(G)$ will resemble $\Gamma(G^*)$, the associated geometry of the general solvable group!

On the other hand, the goal is to prove that G is isomorphic to one of the known simple N-groups:

$$L_2(q),\ Sz(2^n),\ L_3(3),\ U_3(3),\ {}^2F_4(2)',\ A_7,\ \text{or}\ M_{11}, \tag{28.1}$$

excluding the solvable exceptions $L_2(2)$, $L_2(3)$, and $Sz(2)$. Equivalently, in our geometric formulation, our goal is to show that $\Gamma(G)$ is isomorphic to the geometry of one of these groups. [Note that, in particular, G must have even order, so that there are no minimal simple groups of odd order. Thus the group-theoretic or geometric analysis must include a proof of the Feit-Thompson Odd Order Theorem!]

Observe that apart from A_7 and M_{11}, the groups in (28.1) are of Lie type of Lie rank at most 2 (or of index 2 in such a group), and the remaining two groups are doubly transitive permutation groups. Thus the desired structure of $\Gamma(G)$ is extremely restricted compared to that of the geometry of an arbitrary solvable group. The restricted nature of $\Gamma(G)$ for the groups in (28.1) is reflected in their subgroup structure. Consider, for example, a Sylow p-subgroup of G for some prime p. If $p = 2$, then except for ${}^2F_4(2)'$, P is either abelian, of nilpotency class 2, or of 2-rank 2; if $p \geq 7$, then P is cyclic; and if $p = 3$ or 5, then P has rank at most 2.

Thus introduction of $\Gamma(G)$ has shifted the focus of the classification problem but has not necessarily eased the task. Clearly the assumption of *simplicity* must be used to show that the geometry of G in no way approximates that of a general solvable group, but is in fact close to that of one of the groups in $(*)$. This would appear to require an effort comparable to showing group-theoretically that the subgroup structure itself of G is severely limited, approximating that of one of the above groups.

In fact, the group-theoretic approach has a decided advantage at the present time since powerful tools have been developed over the past century for studying the subgroup structure of G:

A. There exists a well-developed theory of solvable groups.
B. Representation and character theory (ordinary and modular) have considerable consequences for the subgroup structure (and order) of a simple group.
C. There exist many non-simplicity criteria established by means of the transfer homomorphism.
D. Deep investigations have been made in the theory of doubly transitive permutation groups.

No corresponding theory has as yet been developed in connection with the geometry of a general solvable group.

This discussion has, of course, been predicated on the assumption that the classification of the finite simple groups is to be carried out by induction, with the group G under investigation taken as a counterexample of least order. However, for some time finite group-theorists (and perhaps the mathematical community as well) have dreamed of a non-inductive proof that depended solely on the simplicity of G, independent of any $\mathcal{K}$-group conditions on its proper subgroups. At least two appealing ideas have been floated:

1. The Sims conjecture, which asserts that in a primitive permutation group G (i.e., a transitive group on a set Ω in which the stabilizer G_α of a point $\alpha \in \Omega$ is maximal in G) the order $|G_\alpha|$ is bounded above by a function of the minimum length of an orbit of G_α in its action on $\Omega - \{\alpha\}$.
2. The conjecture that there is an *a priori* way to establish an upper bound to the order of a sporadic simple group (which would, of course, imply that there are only a finite number of sporadic groups).

Despite some effort, a viable independent approach to the Sims conjecture has not been developed. A proof of either conjecture would very likely yield a much shorter proof of the classification theorem, and one can never preclude the possibility that one of them will be verified some time in the future. [The Sims conjecture has been shown to be a consequence of the classification theorem [**CPSS1**].]

However, the more likely prognosis is that simplifications in the classification proof will continue to occur as they have in the past, relative to specific subcases. For example, in the years since the Odd Order Theorem was first established in 1962, its proof has been reduced by a factor of two with the aid of new techniques and some improved arguments. On the other hand, the original Feit-Thompson overall strategy has remained intact. Likewise, Bender and Glauberman have achieved considerable simplifications of the the original classifications of simple groups with dihedral or abelian Sylow 2-subgroups [**GW1, Wa1, BeGl2, Be2**].

The proof we present here is in the same spirit. Our global strategy is essentially the same as that of the existing proof, but with some important repackagings and consolidations which produce their own substantial simplifications. However, most of our improvements are in the treatment of various major subcases of the theorem.

No doubt this process will continue long after our classification proof has been published, steadily chipping away at its overall length. But the number of subcases seemingly requiring separate treatment is so large that no matter how successful this effort turns out to be, it is difficult to visualize that it will ever reduce the classification of the finite simple groups to the desired several-hundred-page argument.

29. The $\mathcal{K}$-group environment

As has been repeatedly stressed, our intent is to establish the Classification Theorem as a *single* result rather than as a consequence of a collection of largely independent subsidiary theorems, as was true of the original proof. The implication of this viewpoint is that we make no special attempt to derive any such subsidiary result in any more general form than needed for our arguments. On the contrary, the aim throughout will be to strive for simplicity rather than generality of exposition. Certainly other approaches to the classification of the simple groups are equally valid, but we feel that an attempt to construct a "geodesic" proof is justified by the fact that even then the argument will require several thousand pages to elucidate.

The principal manner in which this single-minded perspective manifests itself is that in our set-up, all proper subgroups of G are $\mathcal{K}$-groups. We invoke this continually throughout the analysis. To illustrate the point, consider the case in which the simple group G under investigation contains a strongly embedded subgroup M. The Bender-Suzuki theorem gives a complete classification of such simple groups G without any $\mathcal{K}$-group assumptions on its proper subgroups. On the other hand, if one assumes that M is a $\mathcal{K}$-group, one can prove at the outset that one of the following holds:

(29.1)
(1) M is solvable; or
(2) $E(M/O_{2'}(M)) \cong L_2(q)$ for some $q \equiv 3 \mod 4$

[In particular, G has dihedral Sylow 2-subgroups in the latter case.]

Indeed, a preliminary property of groups with a strongly embedded subgroup asserts the following:
(29.2)
(1) M has a single conjugacy class of involutions; and
(2) $M = C_M(z)D$ for some involution $z \in M$ and subgroup D of odd order.

But if one analyzes $\mathcal{K}$-*groups* satisfying the conditions of (29.2), one finds that M must have one of the two forms specified in (29.1). Hence our Classification Theorem requires the Bender-Suzuki theorem only under the assumption that the strongly embedded subgroup M possesses one of these two structures, in which case the analysis is easier than in the general case; and it is this approach that will be presented here, in [II_1].

This $\mathcal{K}$-group approach has certain significant consequences. First, by shifting portions of the argument to preliminary $\mathcal{K}$-group properties, as with the strongly embedded subgroup theorem above, it makes the essential lines of the proof of a particular subsidiary result more readily transparent. Second, again as in the above example, it leads to the investigation of properties of $\mathcal{K}$-groups which may be of independent interest. Finally, it often allows for extensions to arbitrary primes of results previously proved only for the prime $p = 2$.

An example of the latter is the Aschbacher-Gilman theorem [**A4, Gi1**] that a terminal (i.e., nonembedded) component K of 2-rank at least 2 is necessarily standard in a simple group G. The original result, for general G but $p = 2$, depends upon special properties of involutions. On the other hand, under the assumption that G is $\mathcal{K}$-proper, an analogous assertion holds for odd primes p with K a p-component of the centralizer of some element of G of order p and K terminal in

G. Moreover, the proof for odd p is equally valid for $p = 2$, thus leading to a considerable conceptual simplification of the global strategy.

The strongly embedded subgroup theorem is an example of a simplification resulting solely from our $\mathcal{K}$-group hypothesis. Other simplifications occur because we permit ourselves to derive subsidiary results in less than full generality, using any convenient extra hypotheses which will be available when we apply them. Examples of this are our treatment of Goldschmidt's strongly closed abelian 2-subgroup theorem and of the classification of split (B, N)-pairs of rank 2 (see sections 14 and 25).

Furthermore, as indicated, an important effect of our global $\mathcal{K}$-group assumption is that it enables us to package the classification proof in a manner quite distinct from the existing proof. This package is composed of a number of major subtheorems about $\mathcal{K}$-proper simple groups G, which will be described in detail in the next chapter. Within this $\mathcal{K}$-group context, they can be viewed as parallel results, which taken together yield the Classification Theorem.

Finally, many of the properties of $\mathcal{K}$-groups that we require are general results. On the other hand, as in the existing classification proof, our arguments require a great many detailed, specialized properties of individual quasisimple $\mathcal{K}$-groups or families of such groups, which for the most part have no intrinsic interest beyond the specific context in which they are needed. The seemingly excessive length of our $\mathcal{K}$-group "dictionary" is in part at least a consequence of strict adherence to the following principle: *A property of the known simple groups, or of $\mathcal{K}$-groups, can be used in the course of the proof only if it has been explicitly stated in a preliminary $\mathcal{K}$-group lemma.* This is in marked contrast to much of the existing literature, in which many such properties are invoked, often implicitly, without proof or reference.

We conclude with a comment only indirectly related to the $\mathcal{K}$-group hypothesis. This is the fact that we have endeavored to utilize methods of local group-theoretic analysis wherever possible. As a result, our treatments of simple groups with semidihedral or wreathed Sylow 2-subgroups and of groups with Sylow 2-subgroups of type $U_3(4)$ differ in significant ways from their existing classification. Indeed, the original proofs can be viewed as making maximum use of block-theoretic methods, whereas ours are designed to replace such arguments where feasible by ordinary character-theoretic or local arguments. Although this approach does not necessarily yield shorter proofs, nevertheless our taste is for internal arguments based on local subgroup structure.

30. The term $G \approx G^*$

As already described, the thrust of the classification proof is first to force the internal structure of the $\mathcal{K}$-proper simple group G under investigation to approximate that of some known simple group G^*, this conclusion designated by the symbol $G \approx G^*$, and then to turn this approximation into a presentation for G identical to one that holds in G^*, thereby proving that $G \cong G^*$. It is clearly critical for understanding the overall strategy to have a good picture of the various sets of internal conditions that characterize the term $G \approx G^*$.

As already stated, in the generic Lie type case the term $G \approx G^*$ is defined by the condition that G possess a vertical neighborhood generating a central extension of the generic target group G^*. Likewise for generic alternating groups (of degree at least 13) the term $G \approx G^*$ is expressed by the existence of a vertical neighborhood identical to that determined by a root four-subgroup of G^* and generating an alternating group. Analogous conditions define the term $G \approx A_n$, $9 \leq n \leq 12$.

On the other hand, in the remaining special type cases the corresponding definitions have four possible shapes, three of which have been indicated in the course of the preceding discussion. Historically, the oldest is that given by the involution fusion pattern together with the structure of the centralizers of involutions, and we take these conditions as the definition of the term $G \approx G^*$ when the target group G^* is either sporadic or alternating of degree 7. [As remarked earlier, in the case $G^* = M_{11}$, we must add the condition $|G| = |M_{11}|$ to distinguish G from the group $L_3(3)$, which has the same involution fusion pattern and centralizer of involution structure. Also $A_5 \cong L_2(4)$, $A_6 \cong L_2(9)$, and $A_8 \cong L_4(2)$ are best identified as groups of Lie type.]

Next for the groups of Lie type of Lie rank 1, we define the term $G \approx G^*$ by the conditions that G be doubly transitive with the structure of a point stabilizer closely approximating that of a Borel subgroup of G^*. [In particular, this covers A_5 and A_6.]

There remain (special) groups of Lie type which with a small number of exceptions are of Lie rank 2 and 3. Here definitions differ according as the target group G^* is of odd characteristic r or of characteristic 2. In the odd characteristic case, for most groups G^* of Lie rank 2, the term $G \approx G^*$ is defined by conditions on the involution fusion pattern, the structure of the centralizer of a 2-central involution, the structure of the "Weyl group" of G, and also the structure of a certain r-local subgroup containing a Sylow r-subgroup of G; for groups of larger Lie rank and $U_4(q)$ and $U_5(q)$, it is defined by the existence of an appropriate vertical neighborhood.

Finally, it is the quasithin type case that gives rise to most of the special groups of Lie type of characteristic 2. Since this case is to be treated by the amalgam method, the term $G \approx G^*$ is defined by conditions on the structure of maximal 2-local subgroups of G containing a given Sylow 2-subgroup. For the few groups of Lie type arising from the Klinger-Mason case, on the other hand, the term $G \approx G^*$ is defined by the existence of an appropriate vertical neighborhood.

31. Reading this series

Because of the extreme length of the proof of the Classification Theorem and hence of this series of monographs, it will be useful to give a few guidelines for reading the text.

First, the next chapter is a prerequisite for all that follows, for the detailed outline that it includes provides the context for the specific results to be established in each of the later chapters.

Furthermore, the introductory material of chapters [I_G], [I_A], [II_A], [II_P], [II_S], [II_G], etc., will include basic notation needed for the subsequent chapters, and so it too is required reading. On the other hand, although many of the results within these chapters are of independent interest, a large percentage, especially in the $\mathcal{K}$-group chapters [II_S], [III_S], etc., are highly technical and their hypotheses are difficult to grasp independently of the context in which they arise. Thus these chapters are best viewed primarily as reference material for the body of the text.

Beyond these caveats, we have attempted as much as possible to write each chapter of Parts II, III, IV, and V so that it can be read independently of any of the other chapters. However, occasionally there will be references to a result in a prior chapter — in most cases to the principal result of the earlier chapter.

PART I, CHAPTER 2

OUTLINE OF PROOF

Introduction

As stated in the preceding chapter, the purpose of this series of monographs is to provide a proof of the Classification Theorem, assuming the Background Results listed there. The theorem can be formulated equivalently as follows.

Classification Theorem. *Every $\mathcal{K}$-proper finite simple group is cyclic, a group of Lie type, an alternating group, or one of the 26 sporadic simple groups.*

Recall that a group G is $\mathcal{K}$-proper if and only if each of its proper subgroups is a $\mathcal{K}$-group; a group X is a $\mathcal{K}$-group if and only if every simple section of X (including X itself if X is simple) is isomorphic to a known simple group—i.e., to one of the groups in the conclusion of the Classification Theorem.

Throughout this series, we shall assume the Background Results, and the letter G will be reserved for a $\mathcal{K}$-proper noncyclic simple group.

Since both the Odd Order Theorem and Z^*-Theorem are included as Background Results, it follows in particular that G has even order and 2-rank at least 2.

As described in the brief sketch in sections 23–26 of Chapter 1, a good way to visualize the proof of the Classification Theorem is as a **grid** whose rows represent the major case divisions, as dictated by the distinct techniques needed for the analysis, and whose entries in each row represent the conclusions reached at successive critical stages of the argument in each of the individual cases. In fact, it is preferable to split this grid into two separate grids–the **uniqueness grid** and the **classification grid**. This is because the uniqueness results can be viewed as preparatory, ruling out the existence of certain special "uniqueness" configurations in G, while the results described in the classification grid constitute the main body of the classification proof, taking seven mutually exclusive and exhaustive possibilities for the structure of G and driving each case to the conclusion that G is a known simple group. Typically the results from the classification grid are proved by contradiction, with contradictions arising from appeal to uniqueness theorems.

The classification grid is itself divided into three major subgrids corresponding to the respective cases in which G is of **special odd type**, **special even type** and **generic type**. These terms were discussed in sections 23–26 of the preceding chapter and will be made precise below. Since the analysis in the generic case depends on certain results from the special cases, the logic of the total proof is clearest if the special cases appear in the classification grid prior to the generic case. On the other hand, because of the obvious importance of the determination of all "generic" simple groups and because of the extreme length of the analyses in the special cases, we shall present the generic case before the special cases, stating explicitly those special case results required for the proof.

Thus this series of monographs is divided into five parts as follows:

Part I: Preliminaries
Part II: Uniqueness results
Part III: Generic simple groups
Part IV: Special simple groups of odd type
Part V Special simple groups of even type

The uniqueness grid dictates the structure of Part II, and the classification grid dictates that of the subsequent three parts. Part I consists of the two chapters of this volume and two further chapters: one on general group theory, the other an introduction to the theory of $\mathcal{K}$-groups.

The uniqueness grid is divided into three parts: first, those uniqueness results that hold only for the prime 2, then those valid for all primes, and finally those that hold only for odd primes.

The classification grid is yet more elaborate. Part III corresponds to one row of the grid, while each of Parts IV and V corresponds to three rows. Thus the classification grid represents a total of seven broad cases—the generic one plus six special ones.

In organizing the grids, our aim has been to emphasize conceptual stages of the analysis of each case. Hence there is no necessary correlation among the lengths of the arguments required to pass from one stage to the next, either within the analysis of a given case or among those of different cases.

Although the solvability of groups of odd order is one of the Background Results, for conceptual completeness we are including the various stages of the analysis of such groups within the classification grid.

Note that the term $G \approx G^*$ for some known simple group G^* appears in both grids. As described in sections 24 and 30 of Chapter 1, this symbol is intended to incorporate the exact set of internal conditions on the group G from which a presentation for G is to be derived which yields the desired conclusion $G \cong G^*$. The meaning of $G \approx G^*$, which varies from case to case, will be discussed below.

After we present the grids in sections 2 and 3, the balance of the chapter will be devoted to an explanation of all terms appearing in the grids and a description of the successive stages of the proofs of each of the principal subsidiary theorems which taken together provide a proof of the Classification Theorem. This will then be followed by a brief discussion of the principal techniques used in the analysis.

A. The Grids

1. Some basic terminology

Some of the basic terms about to be defined were introduced in the preceding chapter, but we repeat the definitions for completeness. As is our convention throughout, G is a $\mathcal{K}$-proper simple group.

DEFINITION 1.1. Let X be an arbitrary finite group and p a prime. A **component of** X is a quasisimple subnormal subgroup of X. (A **quasisimple group** is a perfect central extension of a simple group.) Furthermore, we define

$$\mathcal{I}_p(X) = \{x \in X \mid x \text{ has order } p\}, \text{ and}$$
$$\mathcal{L}_p(X) = \{K \mid \text{for some } x \in \mathcal{I}_p(X),\ K \text{ is a component of } C_X(x)/O_{p'}(C_X(x))\}$$

As described in the preceding chapter, the structure of the sets $\mathcal{L}_p(G)$ in the $\mathcal{K}$-proper simple group G under investigation dominates the global logic of the classification proof.

DEFINITION 1.2. If $x \in \mathcal{I}_p(X)$, p a prime, and K is a component of $C_X(x)$ of order divisible by p, we say that the pair (x, K) is **terminal in** X if and only if K is a component of $C_X(y)$ for every $y \in \mathcal{I}_p(C_X(K))$.

When the element x or the prime p is clear from the context, we simply say that K **is terminal in** X, and call K a **terminal component**. This definition includes Definition 7.4 of Chapter 1 by the remarks made after that definition.

Recall that $\mathcal{K}$ is the set of quasisimple $\mathcal{K}$-groups, i.e., the set of perfect central extensions of simple groups of Lie type, alternating groups, and sporadic groups. We have

$$\mathcal{K} = \mathcal{C}hev \cup \mathcal{A}lt \cup \mathcal{S}por, \quad \text{and} \quad \mathcal{C}hev = \cup_p \mathcal{C}hev(p)$$

where $\mathcal{C}hev(p)$ (resp. $\mathcal{A}lt$ or $\mathcal{S}por$) is the set of all $K \in \mathcal{K}$ such that $K/Z(K)$ is simple of Lie type of characteristic p (resp. alternating or sporadic). Moreover, for a given prime p, $\mathcal{K}_p$ is the set of quasisimple $\mathcal{K}$-groups K for which $O_{p'}(K) = 1$. In section 12 below we shall precisely define a partition of $\mathcal{K}_p$ into two sets: $\mathcal{C}_p$, consisting of those groups which are "characteristic p-like", and $\mathcal{C}_{p'}$, the complementary set. The set $\mathcal{C}_2$ is required to define the key term "even type". We shall call a group in $\mathcal{C}_2$ (or $\mathcal{C}_p$, etc.) a $\mathcal{C}_2$**-group** (or $\mathcal{C}_p$-group, etc., respectively). We also use the standard notation $m_p(X)$ for the p-rank of X, the rank of the largest elementary abelian p-subgroups of X.

DEFINITION 1.3. We say that G is of **even type** if and only if the following conditions hold:

(1) Every element of $\mathcal{L}_2(G)$ is a $\mathcal{C}_2$-group;
(2) $O_{2'}(C_G(x)) = 1$ for every involution x of G; and
(3) $m_2(G) \geq 3$.

The simple groups of Lie type of characteristic 2 satisfy (1) and (2) of Definition 1.3 (with $\mathcal{L}_2(G)$ empty) and apart from $L_2(4)$, $U_3(4)$, $L_3(2)$, $Sp_4(2)'$, and $G_2(2)'$, they also satisfy (3).

We shall introduce in section 8 below a slightly stronger notion, that of "restricted" even type. By definition G is of restricted even type if and only if it is of even type and it satisfies some additional conditions related to specific small elements of $\mathcal{L}_2(G)$. We shall then say that G is of **odd type** if and only if it is not of restricted even type; and it is the dichotomy

restricted even type / odd type

which separates three of the special cases in the classification grid from the other three. However, for almost all purposes the reader need only keep in mind the simpler definition of even type.

DEFINITION 1.4. Suppose that G is of even type and let $S \in Syl_2(G)$. Then $\mathcal{M}(S) = \mathcal{M}(G; S)$ is the set of maximal 2-local subgroups of G containing $N_G(S)$. Also $\mathcal{M}_1(S) = \mathcal{M}_1(G; S)$ is the set of maximal 2-local subgroups M of G such that $|S : S \cap M| \leq 2$.

DEFINITION 1.5. Let G be of even type and let $S \in Syl_2(G)$. Then

$$\sigma(G) = \{p \mid p \text{ is an odd prime and } m_p(M) \geq 4 \text{ for some } M \in \mathcal{M}_1(S)\}.$$

It would be more elegant to use $\mathcal{M}(S)$ in the definition of $\sigma(G)$, but there are technical advantages to our definition.

DEFINITION 1.6. If G is of even type with $\sigma(G)$ empty, we say that G is **quasithin**[1] (or of **quasithin type**).

Next we define some notions related to uniqueness subgroups.

DEFINITION 1.7. Let p be a prime, P a p-subgroup of the group X, and k an integer such that $1 \leq k \leq m_p(P)$. Then

$$\Gamma_{P,k}(X) = \langle N_X(Q) \mid Q \leq P,\ m_p(Q) \geq k \rangle.$$

Moreover, $\Gamma^o_{P,2}(X)$ is the following large subgroup of $\Gamma_{P,2}(G)$:

$$\Gamma^o_{P,2}(X) = \langle N_X(Q) \mid Q \leq P,\ m_p(Q) \geq 2,\ m_p(QC_P(Q)) \geq 3 \rangle.$$

DEFINITION 1.8. Let p be a prime divisor of the order of the group X. A **strongly p-embedded subgroup of** X is a proper subgroup Y of X such that $\Gamma_{P,1}(X) \leq Y$ for some $P \in Syl_p(X)$.

The definition has several equivalent formulations. For example, a subgroup Y of X is strongly p-embedded in X if and only if $Y = N_X(Y)$ and in the permutation representation of X on the set Ω of X-conjugates of Y, every element of X of order p fixes exactly one point of Ω, but Ω itself has more than one point.

As is customary, when $p = 2$, we suppress p and say simply that Y is **strongly embedded** in X.

DEFINITION 1.9. A **2-uniqueness subgroup of** G is a proper subgroup Y of G such that for some $S \in Syl_2(G)$,

(a) If $m_2(S) \geq 3$, then $\Gamma^o_{S,2}(G) \leq Y$; and
(b) If $m_2(S) = 2$, then $\Gamma_{S,1}(G) \leq Y$.

As remarked earlier, we must have $m_2(S) \geq 2$ in any case.

DEFINITION 1.10. Suppose that $p \in \sigma(G)$. Then a **p-uniqueness subgroup of** G is a proper subgroup Y of G such that $\Gamma_{P,2}(G) \leq Y$ for some $P \in Syl_p(G)$.

In our analysis of groups of even type, when $\sigma(G) \neq \emptyset$ we endeavor to construct p-uniqueness subgroups for various $p \in \sigma(G)$ which satisfy some further similar conditions. Such subgroups will be called **strong p-uniqueness subgroups**; the definition of this term is not given until section 8 below, as it requires considerable technical detail. We then define

[1]This is a new usage of "quasithin"; see the Introduction to the Series.

DEFINITION 1.11. If G is of even type, then $\sigma_0(G)$ is the set of all $p \in \sigma(G)$ such that G does not possess a strong p-uniqueness subgroup.

On a first reading, the reader may well ignore the word "strong" and consider $\sigma_0(G)$ as the set of all primes $p \in \sigma(G)$ for which $\Gamma_{P,2}(G) = G$ for some (hence all) $P \in Syl_p(G)$.

Several additional important terms were mentioned in the preceding chapter without being defined precisely, and we prefer to defer their exact definition until later in the chapter: the sets $\mathcal{G}_p$ and $\mathcal{T}_p$ (section 13); vertical neighborhood (section 25); $G \approx G^*$ (sections 9, 19–23 and 26).

2. The uniqueness grid

As described in the preceding chapter, the uniqueness grid consists of a variety of results about the existence in G of strongly p-embedded subgroups or p-uniqueness subgroups for suitable primes p, or subgroups satisfying similar conditions. Some of these results determine the possibilities for G when such subgroups exist, and others establish sufficient conditions for G to contain such a subgroup.

A few comments concerning the uniqueness grid are in order, applicable to the classification grid as well. Each row in the grid represents a theorem about G. The first entry of the row indicates the hypothesis on G; the remaining entries, from left to right, are the results of the successive stages of the proof. At certain points, a bifurcation occurs–that is, the specified conclusions at a particular stage of the analysis consist of two distinct alternatives. Both are listed in the grid and the subsequent stages of the analysis of each are separately included. The listing of the stages of the proof of a uniqueness result can terminate in several ways: (a) G is shown to be isomorphic to some known simple group G^*, or to satisfy some other desired conclusion (the existence of a standard component or a strongly p-embedded subgroup, for example); (b) the conclusion at a given stage is identical to that reached at some stage of a previously listed uniqueness result, or (c) a contradiction is reached.

We note that the analysis of groups of even type with $|\mathcal{M}(S)| = 1$ uses results concerning 2-component preuniqueness subgroups. As a consequence, our p-component preuniqueness theorems will be established before those on groups with $|\mathcal{M}(S)| = 1$.

The precise statements of the theorems represented by the uniqueness grid will be given in sections 4–8 below, and the stages of their proofs will be discussed in sections 9–11.

3. The classification grid

In this section we present the seven-case classification grid, but again begin with a few preliminary comments. In the uniqueness grid, the points of bifurcation arise from exceptional configurations, the conclusion at such a stage including a "singular" solution. This phenomenon occurs in the classification grid, too, but most of the bifurcations occur for a more significant reason.

For example, when G is of $\mathcal{L}\mathcal{B}_2$-type (case 2 of the special odd type portion of the grid), the ultimate target groups G^* include (a) most of the groups of Lie type of odd characteristic and Lie rank 1, (b) most of the groups of Lie type of odd characteristic of Lie rank 2 (plus certain linear groups of Lie rank 3), and (c) A_7 and several sporadic groups. On the other hand, the definition of the term

UNIQUENESS GRID

Hypothesis	stage 1	stage 2	stage 3	stage 4
		$p = 2$		
G has strongly $\mathcal{Z}$-embedded subgroup M	G is doubly transitive on the cosets of M	M is strongly embedded	2-trans. of Suzuki type: $G \approx L_2(2^n)$, $Sz(2^n)$,$U_3(2^n)$	$G \cong L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$
G has a 2-uniqueness subgroup M	G has a proper 2-generated core	a) M is str. $\mathcal{Z}$-embedded b) $G \approx J_1$	b) $G \cong J_1$	
G even type, strongly closed abel. 2-subgrp	a) G has a strongly $\mathcal{Z}$-embedded subgroup b) $G \approx J_1$			
G even type,M weakly $\mathcal{Z}$-emb.	M is strongly $\mathcal{Z}$-embedded			
G of even type, $\lvert\mathcal{M}(S)\rvert = 1$; quasithin or uniqueness case	a) G has a near component 2-local uniqueness subgroup b) G has a 2-uniqueness subgroup	a) G has a standard near component	a) contradiction	
		p arbitrary; M, K satisfy p-component preuniqueness hypothesis (6.1)		
M controls rank 1 fusion	a) M is strongly p-embedded b) p odd, M is almost strongly p-embedded			
M controls rank 2 fusion	a) M controls rank 1 fusion b) p odd, M is almost strongly p-embedded			
M not standard	M controls rk 2 fusion			
K is terminal, $m_2(K) > 1$, $p = 2$	a) G has a strongly embedded subgroup b) K is standard	a) contradiction		
		p odd; G is of even type, $\sigma(G) \neq \emptyset$		
p-uniqueness subgroups $\forall p \in \sigma(G)$	$\{2, p\}$-uniqueness subgroups	a) 2-amalgam type, or b) $\lvert\mathcal{M}(S)\rvert = 1$	a) contradiction	

CLASSIFICATION GRID

Case	stage 1	stage 2	stage 3	stage 4	stage 5
Special Odd Type					
1. G has odd order	odd order uniqueness type	$\sigma^*(G)$-uniqueness type	$\{p, q\}$-parabolic type	contra-diction	
2. G has $\mathcal{LB}_2$-type	2-terminal G^*-type $G^* \in \mathcal{K}^{(2)*}$	a) $G \approx M_{12}$, Mc, Ly, $O'N$, $L_3(q)$, q odd b) 2-central G^*-type, $m_2(G^*) \geq 3$ c) 2-maximal G^*-type mod cores, $m_2(G^*) = 2$ $(G^* \in \mathcal{K}^{(2)*})$	$G \approx G^*$ $G^* \in \mathcal{K}^{(2)}$	$G \cong G^*$ $G^* \in \mathcal{K}^{(2)}$	
3. G has $\mathcal{LT}_2$-type	2-terminal $\mathcal{LT}_2$-type	$G \approx G^*$ $G^* \in \mathcal{K}^{(3)}$	$G \cong G^*$ $G^* \in \mathcal{K}^{(3)}$		
Special Even Type					
4. G has quasithin type	2-amalgam type	2-amalgam G^*-type, $G^* \in \mathcal{K}^{(4)}$	$G \approx G^*$ $G^* \in \mathcal{K}^{(4)}$	$G \cong G^*$ $G^* \in \mathcal{K}^{(4)}$	
5. G has $\mathcal{LC}_p$-type	wide $\mathcal{LC}_p$-type	quasisymplectic type $(p = 3)$	$G \approx G^*$ $G^* \in \mathcal{K}^{(5)}$	$G \cong G^*$ $G^* \in \mathcal{K}^{(5)}$	
6. G has $\mathcal{LT}_p$-type	p-terminal $\mathcal{LT}_p$-type	contradiction			
Generic Type					
7. G has generic type	$\frac{3}{2}$-balanced type	semisimple type	proper semisimple type	$G \approx G^*$ $G^* \in \mathcal{K}^{(7)}$	$G \cong G^*$ $G^* \in \mathcal{K}^{(7)}$

$G \approx G^*$ differs sharply according as G^* is one of the groups in (a), (b), or (c). As a consequence, the paths leading to the identification of G inevitably diverge into three subcases at some stage of the analysis.

Next, as remarked earlier, the conclusions derived at the various stages are based on the earlier uniqueness theorems. For instance, in the case that G is of odd type, the critical such result is that G has no 2-uniqueness subgroup. When G has even type, on the other hand, the corresponding result—that G has no strong p-uniqueness subgroup for the odd prime p on which we focus—is arranged in advance by our choosing $p \in \sigma_0(G)$ if possible. When $\sigma_0(G) = \emptyset$, then Theorem U(σ) (row 10 of the uniqueness grid) implies that $\sigma(G) = \emptyset$, so that G is quasithin.

The seven-case division is intended to split the classification proof into essentially disjoint parts, so that each known simple group—except for the groups $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$ and J_1, which possess 2-uniqueness subgroups—occurs exactly once as a target group for G.

The term "essentially disjoint" requires some clarification. For compactness of exposition, it is convenient to have a designation for the sets of simple groups that occur as solutions for G in each of the seven cases. We introduce in Definition 3.1 the important partition of the set of simple $\mathcal{K}$-groups into the sets $\mathcal{K}^{(1)}, \ldots, \mathcal{K}^{(7)}$ of target groups of the seven cases of the classification grid, and one additional set $\mathcal{K}^{(0)}$ consisting of the simple groups arising from the uniqueness grid.

The separation of cases is almost but not quite absolute. In a few cases (i.e., rows of the grid), the full hypothesis of the row is needed only at a particular stage of the analysis, so that subsequent results established in that case will be more generally valid. For example, the special odd type assumption when G is of $\mathcal{LB}_2$-type of 2-rank ≥ 3 (case 2 of the grid) is used only in stage 1. Hence, once it is shown that G is of 2-terminal $\mathcal{K}^{(2)}$-type, the subsequent analysis applies to an arbitrary $\mathcal{K}$-proper simple group—in particular, even if G is of generic type. This means that results about groups of 2-terminal $\mathcal{K}^{(2)}$-type will be applicable to the analysis of the generic case. It is for this reason that the clearest flow chart of the proof proceeds from the special to the generic cases.

Likewise the solutions for G in cases 2 and 3 include a few groups of even type, as will be seen below from the definition of the sets $\mathcal{K}^{(2)}$ and $\mathcal{K}^{(3)}$, despite the stated assumption that G is special of odd type. Again this arises as a matter of convenience in the proof, and is codified in the overall logic by the formulation of a sharper but ad hoc notion of "restricted even type" (see section 8). These extra groups of even type are *not* of restricted even type. However, for simplicity, we omit all such refinements from the grid and confine them to the definitions given in the text.

Definition 3.1. We partition the set of simple $\mathcal{K}$-groups into the following eight disjoint subsets $\mathcal{K}^{(i)}$, $0 \leq i \leq 7$ of target groups:

$\mathcal{K}^{(0)} = \{L_2(2^n), U_3(2^n), Sz(2^n) \mid n \geq 2\} \cup \{J_1\}$.

$\mathcal{K}^{(1)} = \emptyset$.

$\mathcal{K}^{(2)} = \{L_2(q)\,(q > 5), L_3(q), U_3(q), PSp_4(q)(q > 3), L_4(q)(q \not\equiv 1 \mod 8, q > 3), U_4(q)(q \not\equiv 7 \mod 8, q > 3), G_2(q)(q > 3), {}^2G_2(q)(q = 3^{2n+1}, q > 3), {}^3D_4(q)$, with q odd throughout; $A_7, M_{11}, M_{12}, Mc, Ly, ON\}$.

$\mathcal{K}^{(3)} = \{A_9, A_{10}, A_{11}\}$.

$\mathcal{K}^{(4)} = \{L_3(2^n)(n > 1), L_4(2^n)(n \geq 1), U_4(2^n)(n \geq 1), U_5(2^n)(n \geq 1), PSp_4(2^n)(n > 1), PSp_6(2^n)(n \geq 1), G_2(2^n)(n > 1), {}^2F_4(2^n)'(n \geq 1), {}^2D_4(2^n)(n \geq 1), {}^3D_4(2^n)(n \geq 1), L_5(2), L_6(2), L_7(2), U_6(2), Sp_8(2), D_4(2), F_4(2), L_4(3), U_4(3), G_2(3), A_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, He, Ru, Suz, Co_3, Co_2, F_5, F_3\}$.

$\mathcal{K}^{(5)} = \{P\Omega_7(3), P\Omega_8^{\pm}(3), {}^2D_5(2), U_7(2), {}^2E_6(2), Fi_{22}, Fi_{23}, Fi'_{24}, Co_1, F_2, F_1\}$.

$\mathcal{K}^{(6)} = \emptyset$.

$\mathcal{K}^{(7)} = \{\text{All groups of Lie type not already listed}\} \cup \{A_n \mid n \geq 13\}$.

Of the twenty-six sporadic groups, note that J_1 arises in a uniqueness theorem, and the other twenty-five arise in only one special odd and two special even cases:

$$\begin{aligned} \mathcal{L}\mathcal{B}_2\text{-type}: &\ M_{11}, M_{12}, Mc, Ly, ON \\ \text{Quasithin type}: &\ M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, He, Ru, Suz, Co_3, Co_2, F_5, F_3 \\ \mathcal{L}\mathcal{C}_3\text{-type}: &\ Fi_{22}, Fi_{23}, Fi_{24}', Co_1, F_2, F_1. \end{aligned}$$

The classification grid represents the sequence of seven theorems:

THEOREM $\mathcal{C}_i$ $(i = 1, \dots, 7)$. *Let G be a $\mathcal{K}$-proper simple group satisfying the assumptions of Case i, $1 \le i \le 7$. Then $G \cong G^*$ for some $G^* \in \mathcal{K}^{(i)}$.*

A more exact statement of each of the seven Theorems $\mathcal{C}_i$, $1 \le i \le 7$, is given below in sections 14–16. Of course, Theorem $\mathcal{C}_1$ is a Background Result.

This series of monographs is taken up with the proofs of Theorems $\mathcal{C}_i$, $2 \le i \le 7$, as well as of the various supporting theorems listed in the uniqueness grid. Together, these results and the Background Results will constitute a proof of the Classification Theorem.

A few comments about our notational conventions may help to clarify the rather formidable array of terms appearing in the grids. First, the major cases into which the analysis is divided are denoted by specifying that the given $\mathcal{K}$-proper simple group G is of a particular **type**. Thus we have the terms "uniqueness type", "even type", "quasithin type", "generic type", and "special type". Moreover, script characters are used to specify various sets of quasisimple $\mathcal{K}$-groups, some of which may occur as proper sections of G, such as $\mathcal{C}_{p'}$-groups, $\mathcal{G}_p$-groups, and $\mathcal{T}_p$-groups (arbitrary p) and $\mathcal{B}_2$-groups ($p = 2$); others of which are target groups (the $\mathcal{K}^{(i)}$). Script letters are also used to denote subcases; for example, among the special type cases, we have $\mathcal{L}\mathcal{T}_p$-type (arbitrary p), $\mathcal{L}\mathcal{B}_2$-type ($p = 2$), and $\mathcal{L}\mathcal{C}_p$-type (p odd), indicating that certain elements of $\mathcal{L}_p(G)$ lie in the appropriate set $\mathcal{T}_p$, $\mathcal{B}_2$, etc.

B. THE UNIQUENESS GRID

4. 2-uniqueness subgroups

In the next eight sections we give a more detailed description of the uniqueness grid. We begin with the statements of the principal theorems to be proved, covering the $p = 2$ portion of the grid in this section.

The notion of a strongly $\mathcal{Z}$-embedded subgroup, first considered by Aschbacher in his study of groups having a proper 2-generated core [**A3**], represents a natural generalization of the notion of a strongly embedded subgroup. Underlying this extension is the following basic proposition concerning groups with a strongly embedded subgroup (cf. [**G1,** §9; **Be3**]). We limit its statement to our group G.

Recall first that if $X \le G$ and T is a p-subgroup of X, p a prime, then X is said to **control the G-fusion** (resp. **strong G-fusion**) of the subset A of T if and only if for every $g \in G$ such that $A^g \subseteq T$, there is $x \in X$ such that $A^g = A^x$ (resp. $a^g = a^x$ for all $a \in A$); and X is said to **control (strong) G-fusion in T** if and only if X controls the (strong) G-fusion of every subset of T. Thus X controls G-fusion in T if and only if any two subsets of T are conjugate in G if and only if they are conjugate in X.

PROPOSITION 4.1. *If M is a strongly embedded subgroup of G, then the following conditions hold:*

(i) *$C_G(x) \leq M$ for every $x \in \mathcal{I}_2(M)$;*
(ii) *M controls G-fusion in a Sylow 2-subgroup of M;*
(iii) *G and M each have only one conjugacy class of involutions; and*
(iv) *If $H < G$ with $|H \cap M|$ even and $H \not\leq M$, and if we set $H_1 = \langle \mathcal{I}_2(H) \rangle$, then $H_1 \not\leq M$ and $H_1 \cap M$ is strongly embedded in H_1.*

In particular, (iii) implies that every involution of M is **2-central in** G, that is, lies in the center of a Sylow 2-subgroup of G.

We note also that if M is a proper subgroup of G of even order satisfying conditions (i) and (ii), then it is easily shown that M is strongly embedded in G.

Strong $\mathcal{Z}$-embedding is defined by analogous conditions on a subgroup M of G, relative to a single conjugacy class $\mathcal{Z}$ of 2-central involutions of G. For any subset X of G, we define

$$\mathcal{Z}_X = \mathcal{Z} \cap X.$$

In particular, if X is a subgroup of G, then $\langle \mathcal{Z}_X \rangle \lhd X$.

DEFINITION 4.2. Given a conjugacy class $\mathcal{Z}$ of 2-central involutions of G, we say that a proper subgroup M of G is **strongly $\mathcal{Z}$-embedded in** G if and only if the following conditions hold:

(1) $\mathcal{Z}_M \neq \emptyset$;
(2) $C_G(z) \leq M$ for every $z \in \mathcal{Z}_M$;
(3) M controls G-fusion in a Sylow 2-subgroup of M; and
(4) If H is a proper subgroup of G with $\mathcal{Z}_{H \cap M}$ nonempty and $H \not\leq M$, and if we set $H_1 = \langle \mathcal{Z}_H \rangle$, then $H_1 \not\leq M$ and $H_1 \cap M$ is strongly embedded in H_1.

By the proposition, if M is strongly embedded in G, then M is strongly $\mathcal{Z}$-embedded with $\mathcal{Z} = \mathcal{I}_2(G)$.

The first row of the uniqueness grid gives the flow chart for the fundamental Aschbacher-Bender-Suzuki theorem classifying groups with a strongly $\mathcal{Z}$-embedded subgroup. [As with our other results, it will be proved only under our assumption that G is a $\mathcal{K}$-proper simple group.]

Because of the importance of the strongly embedded subcase, we split the statement of the principal result into two separate theorems. Proper subgroups of G containing a strongly ($\mathcal{Z}$-)embedded subgroup are easily seen to be strongly ($\mathcal{Z}$-)embedded, so there is no loss in taking M maximal.

THEOREM SZ. (Uniqueness grid, row 1, stages 1 and 2) *Let M be a maximal subgroup of G. If M is strongly $\mathcal{Z}$-embedded in G for some conjugacy class $\mathcal{Z}$ of 2-central involutions of G, then the following conditions hold:*

(i) *M is strongly embedded in G; and*
(ii) *G acts doubly transitively by conjugation on the set of G-conjugates of M.*

This result covers the first two stages of the analysis. In particular, it shows that if G contains a strongly embedded subgroup M with M a maximal subgroup of G, then G is necessarily doubly transitive on the set of G-conjugates of M.

The second result deals with the strongly embedded case and thus covers the remaining stages of the analysis of groups with a strongly $\mathcal{Z}$-embedded subgroup.

THEOREM SE. (Uniqueness grid, row 1, stages 3 and 4) *Let M be a maximal subgroup of G. If M is strongly embedded in G, then for some $n \geq 2$, $G \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$, and M is a Borel subgroup of G.*

Next is an extension of Aschbacher's proper 2-generated core theorem [**A3**].

THEOREM U(2). (Uniqueness grid, row 2) *Let M be a maximal subgroup of G. If M is a 2-uniqueness subgroup, then one of the following holds:*
- (i) *M is strongly $\mathcal{Z}$-embedded in G for some conjugacy class $\mathcal{Z}$ of 2-central involutions of G (and hence $G \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$); or*
- (ii) *$G \cong J_1$.*

The first part of the proof shows that the hypothesis $\Gamma^o_{S,2}(G) \leq M$ leads to the stronger condition $\Gamma_{S,2}(G) \leq M$ if $m_2(S) \geq 3$. This stage of the analysis was originally investigated by Harada [**H2**]. Now since $\Gamma_{S,2}(G)$ is a proper subgroup of G, G has a proper 2-generated core by definition, and like Aschbacher we verify the hypotheses of Theorem SZ for G.

Finally, with the aid of Theorem SZ, we establish two additional important results needed for the classification of groups of even type. The first of these is Goldschmidt's strongly closed abelian 2-group theorem [**Go5**], limited here to groups of even type. Recall that a subgroup A of $S \in Syl_2(G)$ is said to be **strongly closed in S with respect to G** if and only if whenever $a \in A$, $g \in G$ and $a^g \in S$, then $a^g \in A$.

THEOREM SA. (Uniqueness grid, row 3) *Assume that G is of even type and let $S \in Syl_2(G)$. If S contains a nontrivial abelian subgroup that is strongly closed in S with respect to G, then one of the following holds:*
- (i) *G contains a strongly $\mathcal{Z}$-embedded maximal subgroup M for some conjugacy class $\mathcal{Z}$ of 2-central involutions of G (whence again $G \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$); or*
- (ii) *$G \cong J_1$.*

The second result is Holt's theorem in the particular case of groups of even type [**Ho1**].

THEOREM SF. (Uniqueness grid, row 4) *Assume that G is of even type, $S \in Syl_2(G)$, and M is a proper subgroup of G containing S with the following properties:*
- (a) *M controls G-fusion in S; and*
- (b) *$C_G(z) \leq M$ for some involution z of $Z(S)$.*

Then one of the following holds:
- (i) *M is strongly $\mathcal{Z}$-embedded with respect to the conjugacy class $\mathcal{Z}$ of involutions containing z (whence $G \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$); or*
- (ii) *$G \cong A_9$.*

Because G is assumed to be of even type, the possibilities $G \cong A_n$, n odd, $n > 9$, of Holt's theorem are excluded here. In the particular circumstances when Theorem SF is applied, the A_9 conclusion will be clearly impossible. Hence we do not include A_9 in $\mathcal{K}^{(0)}$. Accordingly, the term "G has a weakly $\mathcal{Z}$-embedded subgroup M" in the uniqueness grid means by definition that the hypotheses of Theorem SF hold, but $G \ncong A_9$.

5. Groups of restricted even type with $|\mathcal{M}(S)| = 1$

Application of the amalgam method in a group G of even type requires a Sylow 2-subgroup S of G to be contained in at least two maximal 2-local subgroups. Thus before using amalgams, one must first dispose of the case in which $|\mathcal{M}(S)| = 1$. In this case, if we let M be the unique maximal 2-local subgroup of G containing $N_G(S)$, it follows that

$$N_G(T) \leq M \text{ for every } 1 \neq T \leq S \text{ with } T \lhd N_G(S). \tag{5.1}$$

In particular, it follows that

$$\begin{aligned} &C(G,S) \leq M, \text{ where} \\ &C(G,S) = \langle N_G(T) \,|\, 1 \neq T \text{ char } S \rangle \end{aligned} \tag{5.2}$$

is the "**characteristic 2-core**".

Thus the condition $|\mathcal{M}(S)| = 1$ implies that the corresponding maximal 2-local subgroup M satisfies (5.2), a weak type of 2-uniqueness. In the original classification proof this situation was analyzed for arbitrary groups of characteristic 2 type. Fortunately, it turns out that the analysis is required for us only under certain conditions on the p-structure of G for odd primes p, namely when either $\sigma(G)$ is empty, or there are odd primes p for which $m_p(M) \geq 4$ and for each such p, M is a p-uniqueness subgroup. [By definition any such p lies in $\sigma(G)$.] The objective is again to prove that M must be strongly embedded in G, so that G is determined from Theorem SE.

We can therefore state the principal result to be proved in this connection as follows.

THEOREM $\mathcal{M}(S)$. (Uniqueness grid, row 5) *Assume that G is of restricted even type with $|\mathcal{M}(S)| = 1$ for $S \in Syl_2(G)$, and let M be the unique element of $\mathcal{M}(S)$. Suppose either that $\sigma(G) = \emptyset$ or that there exist odd primes p for which $m_p(M) \geq 4$, and for every such p, M is a p-uniqueness subgroup. Then M is strongly embedded in G (whence $G \cong L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$ for some $n \geq 2$).*

[The definition of restricted even type will be given in section 8.]

6. Component preuniqueness subgroups

We next describe the portion of the grid that deals with p-component preuniqueness subgroups, a notion we proceed to define.

DEFINITION 6.1. Let M be a proper subgroup of G, p a prime, and set $\overline{M} = M/O_{p'}(M)$. We say that M is a **p-component preuniqueness subgroup of G** (with respect to K) if and only if M contains a p-component K such that $C_G(x) \leq M$ for every $x \in \mathcal{I}_p(C_M(\overline{K}))$.

[The p-component K will usually not have to be mentioned as its identity will be clear from the context. Note that if K is a component of M, then $\mathcal{I}_p(C_M(\overline{K})) = \mathcal{I}_p(C_M(K))$.]

An important example of such a preuniqueness subgroup arises in connection with a terminal component K in G (Definition 1.2). Indeed, let Ω be the set of G-conjugates of K and define a relation $\sim$ on Ω by setting $I \sim J$ for I, $J \in \Omega$ if and only if either $[I, J] = 1$ or $I = J$. Using the condition of terminality, it is easy to show that $\sim$ is an equivalence relation on Ω. We denote by $[K]$ the product of all elements in the equivalence class containing K, so that $[K]$ is a semisimple group—i.e., a central product of quasisimple groups. Moreover, K is a component of $[K]$. Again using the terminality of K, one directly obtains the following result.

PROPOSITION. *If we set $M = N_G([K])$, then K is a component of M and $C_G(x) \leq M$ for every $x \in \mathcal{I}_p(C_M(K))$, so that M is a p-component preuniqueness subgroup. Moreover, M permutes the components of $[K]$ transitively by conjugation.*

We consider various conditions on a p-component preuniqueness subgroup M.

DEFINITION 6.2. Let p, $\overline{M}$ and $\overline{K}$ be as in Definition 6.1, let Q be a Sylow p-subgroup of $C_M(\overline{K})$, and expand Q to $P \in Syl_p(M)$. We say that M **controls rank 1 fusion (with respect to K)** if and only if M controls the strong G-fusion in P of every element of $\mathcal{I}_p(Q)$, and that M **controls rank 2 fusion (with respect to K)** if and only if M controls the strong G-fusion in P of every E_{p^2}-subgroup of Q. In addition, we say that M **is standard (with respect to K)** if K is normal in M.

Use of the term "standard" here is justified by the fact that if K is terminal, then K is standard in G if and only if $K = [K]$ and hence if and only if $K \lhd N_G([K])$.

As the terminology suggests, we shall aim to prove that a p-component preuniqueness subgroup is actually a p-uniqueness subgroup (Definition 1.10), indeed that it is strongly p-embedded in G. However, since we wish our p-component preuniqueness results to be valid for arbitrary p, we prefer not to impose at this time the additional assumption that G is of even type when p is odd (which is the only situation in which the odd prime results are to be applied). Unfortunately, without this assumption, it turns out not to be possible to establish strong p-embedding in every case. As a consequence, for odd p we are forced to introduce the notion of an **almost strongly p-embedded subgroup** M, which incorporates the exceptional configurations.

The precise definition is rather technical, and will be given in Definition 8.4. It should be remarked that part (4) of that definition arises not from the preuniqueness results we are discussing, but in the study of generic groups. Hence we only discuss the remaining alternatives (1)–(3) of the definition here. In addition to the condition that M be a p-uniqueness subgroup, the definition has essentially three critical parts in the exceptional configurations. The first restricts the isomorphism type of $\overline{K}$: if $m(Q) \geq 2$, the only case of interest at the moment, then $m_p(\overline{K}) = 1$. The second gives the structure of $F^*(\overline{M})$ and $\overline{K}^* = \langle \overline{K}^{\overline{M}} \rangle$, while the third gives the approximation to strong p-embedding that can be proved in these residual cases, asserting at least that $\Gamma_{R,1}(G) \leq M$ for some explicit large p-subgroup R of M.

To state our p-component preuniqueness results, we assume the following conditions, which we term the **p-component preuniqueness hypothesis**:

(6.1)

(1) M is a p-component preuniqueness subgroup of G with respect to K;

(2) M is a maximal subgroup of G;

(3) $m_p(C_M(\overline{K})) \geq 2$, where $\overline{M} = M/O_{p'}(M)$; and

(4) If p is odd, then $m_p(M) \geq 4$.

Under these assumptions we shall prove

THEOREM PU_1. (Uniqueness grid, row 6) *Assume the p-component preuniqueness hypothesis. If M controls rank 1 fusion with respect to K, then one of the following holds:*
(i) *M is strongly p-embedded in G; or*
(ii) *p is odd and M is almost strongly p-embedded in G.*

THEOREM PU_2. (Uniqueness grid, row 7) *Assume the p-component preuniqueness hypothesis. Further, if $p = 2$, assume that $m_2(K) \geq 2$. If M controls rank 2 fusion with respect to K, then M controls rank 1 fusion with respect to K (and so the conclusion of Theorem PU_1 holds).*

THEOREM PU_3. (Uniqueness grid, row 8) *Assume the p-component preuniqueness hypothesis. Further, if $p = 2$, assume that $m_2(K) \geq 2$. Then one of the following holds:*
(i) *M is standard; or*
(ii) *M controls rank 2 fusion with respect to K.*

Together, these three results have the following fundamental consequence.

THEOREM PS. *Assume the p-component preuniqueness hypothesis. Further, if $p = 2$, assume that $m_2(K) \geq 2$. Then one of the following holds:*
(i) *M is standard;*
(ii) *M is strongly p-embedded in G; or*
(iii) *p is odd and M is almost strongly p-embedded in G.*

Finally, by combining Theorems PS and SZ and using the fact that in the groups $L_2(2^n)$, $Sz(2^n)$ and $U_3(2^n)$ all centralizers of involutions are solvable, we obtain the basic Aschbacher-Gilman theorem [**A4, Gi1**] that terminal components of 2-rank at least 2 are necessarily standard for $p = 2$.

THEOREM TS. (Uniqueness grid, row 9) *Let $x \in \mathcal{I}_2(G)$ and let K be a component of $C_G(x)$ with K terminal in G. If $m_2(K) \geq 2$, then K is standard in G.*

7. The odd uniqueness theorem

The definition of **strong p-uniqueness subgroup** is limited to the case in which G is of even type with $\sigma(G) \neq \emptyset$ and $p \in \sigma(G)$, so that in particular it follows then from the definition of $\sigma(G)$ that $m_p(G) \geq 4$. As the definition is technical, requiring that a p-uniqueness subgroup satisfy some extra conditions, we defer it until the next section and simply state the main theorem of the odd uniqueness case.

THEOREM $U(\sigma)$. (Uniqueness grid, row 10) *Assume that G is of even type with $\sigma(G) \neq \emptyset$. If G contains a strong p-uniqueness subgroup for each $p \in \sigma(G)$, then $|\mathcal{M}(S)| = 1$ for any $S \in Syl_2(G)$.*

Since $\sigma(G)$ is empty when $G \cong L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$, Theorem $\mathcal{M}(S)$ then implies that the hypothesis of Theorem $U(\sigma)$ is never satisfied. Thus as a corollary we obtain

COROLLARY $U(\sigma)$. *If G is of even type with $\sigma(G) \neq \emptyset$, then $\sigma_0(G) \neq \emptyset$. That is, for some $p \in \sigma(G)$, G does not contain a strong p-uniqueness subgroup.*

Theorem U(σ) is analogous to Aschbacher's uniqueness theorem [**A16**] but clearly there are significant differences in the hypotheses.

In the even type case of the classification proof, except in the quasithin case, we work with primes $p \in \sigma_0(G)$ and attempt to construct strong p-uniqueness subgroups to reach a contradiction. The preuniqueness results stated in the previous section are useful for this purpose.

8. Some technical definitions

For completeness we include here the full definitions of the terms "almost strongly p-embedded", "strong p-uniqueness subgroup", and "restricted even type". This section can be omitted on a first reading.

As indicated earlier, of the three exceptional sets of conditions defining almost strong p-embedding, two represent the best conclusions short of strong p-embedding that we can prove about p-component preuniqueness subgroups in Theorem PU_1; the third represents a configuration reached in the analysis of G in the generic case when G has a strongly closed subgroup of order p.

In the next four definitions, we assume that M is a p-component preuniqueness subgroup with respect to K, and set $\overline{M} = M/O_{p'}(M)$ and let $Q \in Syl_p(C_M(\overline{K}))$. We let $P \in Syl_p(M)$ with $Q \leq P$.

Definition 8.1. M is **wreathed** if and only if the following conditions hold:
(1) $p \geq 5$;
(2) $m_p(K) = 1$;
(3) $K^* = \langle K^M \rangle$ has exactly p p-components;
(4) $\overline{K}^* = F^*(\overline{M})$;
(5) $|M : K^*|_p = p$;
(6) If we set $R = P \cap K^*$, then $\Gamma_{R,1}(G) \leq M$; and
(7) $\Gamma_{P,2}(G) \leq M$.

Definition 8.2. M is **almost p-constrained** if and only if the following conditions hold:
(1) p is odd;
(2) $m_p(K) = 1$;
(3) $F^*(\overline{M}) = \overline{K} \times \overline{Q}$;
(4) $P = T \times Q$, where $T = P \cap K$;
(5) $C_G(T)$ has a p-component J such that $Q \in Syl_p(J)$ and $J/O_{p'}(J) \cong L_2(p^n)$ $(n \geq 3)$, $U_3(p^n)$ $(n \geq 2)$, or ${}^2G_2(3^n)$ $(n \geq 3,\ p = 3)$; and
(6) $\Gamma_{P,2}(G) \leq M$ and $\Gamma_{Q,1}(G) \leq M$.

Definition 8.3. M is of **strongly closed type** if and only if the following conditions hold:
(1) p is odd;
(2) $K \in \mathcal{C}hev(2)$;
(3) $m_p(Q) = 1$;
(4) $\Omega_1(Q)$ is strongly closed in P with respect to G;
(5) If $X \leq P$ has order p and $m_p(C_P(X)) \geq 3$, then $N_G(X) \leq M$; and
(6) $\Gamma_{P,2}(G) \leq M$.

It is a quick consequence of the above definitions and simple properties of $\mathcal{K}$-groups that if M is of one of the three above types, then K is determined up to conjugacy in M as the p-component of M of largest p-rank. Hence the terms need not include the phrase "with respect to K".

Now we can define almost strong p-embedding.

DEFINITION 8.4. Let M be a p-component preuniqueness subgroup of G. Then M is **almost strongly p-embedded in** G if and only if one of the following holds:

(1) M is strongly p-embedded in G;
(2) M is wreathed;
(3) M is almost p-constrained; or
(4) M is of strongly closed type.

Now we turn to the notion of **strong p-uniqueness subgroup**, which incorporates those conclusions which we are able to prove at various points of our analysis of G by signalizer functor theory and p-local analysis, but—in the case G is of even type and $p \in \sigma(G)$—before any elaborate analysis of 2-local subgroups of G.

First, when p is odd and $\mathcal{L}_p(G) \subseteq \mathcal{C}_p$ (this is part of the $\mathcal{LC}_p$-case, the condition being an analogue for odd p of one of the conditions defining even type), a p-uniqueness subgroup is obtained not as a p-component preuniqueness subgroup, but purely from signalizer functors. We now formulate its key properties. First, we define:

DEFINITION 8.5. For any group X, prime p, and positive integer n,

$$\mathcal{S}_n^p(X) = \{E \leq X \mid E \cong E_{p^n} \text{ and } m_p(C_X(E)) = n\}, \text{ and}$$
$$\mathcal{S}^p(X) = \cup_{n=1}^{\infty} \mathcal{S}_n^p(X).$$

Now we can define:

DEFINITION 8.6. Let M be a proper subgroup of G, p a prime, $P \in Syl_p(M)$, and assume that $m_p(P) \geq 4$. Then M is an **$\mathcal{LC}_p$-uniqueness subgroup of** G if and only if the following conditions hold:

(1) $\Gamma_{P,2}(G) \leq M$;
(2) $O_{p'}(M)$ has even order, and if $T \in Syl_2(O_{p'}(M))$, then $C_M(T)$ is a p'-group; and
(3) For any $A \in \mathcal{S}^p(M)$, every A-invariant p'-subgroup of M lies in $O_{p'}(M)$.

Finally, we can define the term "strong p-uniqueness subgroup".

DEFINITION 8.7. Let p be an odd prime and M a proper subgroup of G with $m_p(G) \geq 4$. Then M is a **strong p-uniqueness subgroup** if and only if (1) and (2) hold:

(1) Either M is a p-component preuniqueness subgroup which is almost strongly p-embedded in G, or M is a $\mathcal{LC}_p$-uniqueness subgroup of G; and
(2) Either (a) or (b) holds:
    (a) $O_{p'}(M) \neq 1$, but $O_{p'}(X) = 1$ for all $X \leq G$ with $M < X$;
    (b) $M = N_G(K)$ for some $K \leq G$ with $K \in \mathcal{C}hev(2)$, $m_p(C_M(K)) \leq 1$, and M is almost strongly p-embedded in G.

In condition (2b), the almost strong p-embedding condition is actually redundant, i.e., M cannot be a $\mathcal{LC}_p$-uniqueness group. For it follows easily from the condition $m_p(M) \geq 4$ that $P \cap K \neq 1$. If M were an $\mathcal{LC}_p$-uniqueness group, however, then $[P \cap K, O_{p'}(M)]$ would have even order by the definition, which would contradict the quasisimplicity of K. A further observation in case (2b) is that if $m_p(C_M(K)) = 1$, then M must be of strongly closed type.

Now we turn to the third technical definition, that of restricted even type. It is used to record the fact that certain configurations in which G has even type have already been ruled out by arguments from the special odd type case. For example, as we shall see in section 12, $\mathcal{C}_2$ contains some Chevalley groups in odd characteristic, namely $L_2(q)$ with $q \in \mathcal{FM}9$ (that is, q is a Fermat prime, Mersenne prime or 9), whereas other $L_2(q)$, q odd, lie in $\mathcal{C}_{2'}$. Thus, as might be expected, our arguments in the case that G is of special odd type with some element of $\mathcal{L}_2(G)$ in $\mathcal{C}_{2'}$ actually are strong enough to rule out the possibility, under certain conditions, that G is of even type but $\mathcal{L}_2(G)$ has an element isomorphic to $L_2(q)$ for some $q \in \mathcal{FM}9$. In particular a number of configurations involving $L_2(5)$ and $L_2(9)$ can be ruled out which would be awkward to handle from the point of view of a group of even type. There are also some configurations involving $L_3(4)$ which can similarly be ruled out. The definition of restricted even type incorporates the assumption that these configurations have already been handled.

Definition 8.8. We say that G is of **restricted even type** if and only if the following conditions hold:

(1) G is of even type;
(2) Suppose that $x \in \mathcal{I}_2(G)$ and K is a component of $C_G(x)$ such that $K \cong L_2(q)$ with $q \in \mathcal{FM}9$. Then we have $q = 5$, 7, 9 or 17, and $m_2(G) \geq 4$;
(3) Suppose that $x \in \mathcal{I}_2(G)$ and K is a *terminal* component of $C_G(x)$ such that $K \cong L_2(q)$ with $q \in \mathcal{FM}9$. Then
  (a) $m_2(C_G(x)) \geq 4$;
  (b) x is not 2-central in G; and
  (c) Either $q = 5$ and $N_G(K)$ has Sylow 2-subgroups isomorphic to E_{16}, or $q = 9$ and $m_2(C_G(K)) = 1$; and
(4) Suppose that x and y are commuting involutions of G, K is a terminal component of $C_G(x)$, $K/Z(K) \cong L_3(4)$, $Z(K) \cong 1$, Z_2 or E_4, and y induces an automorphism of unitary type on K (that is, $C_{K/Z(K)}(y) \cong U_3(2)$). Then $|G : N_G(K)|_2 \geq 4$.

9. Groups with a strongly embedded subgroup

At this point we have stated the principal results to be proved in the uniqueness grid. We next discuss the stages of proof and explain the as yet undefined terms appearing in the grid.

In row 1, the term "doubly transitive of Suzuki type" is also the definition of $G \approx G^*$ for $G^* \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$, $n \geq 2$. It incorporates the conclusion of the first stage of the proof of Theorem SE (equivalently, the third stage of the classification of groups with a strongly $\mathcal{Z}$-embedded subgroup).

Let G then satisfy the assumptions of Theorem SE. Thus G is a $\mathcal{K}$-proper simple group containing a strongly embedded subgroup M with M a maximal subgroup of G. In view of Theorem SZ, we may assume that G is doubly transitive on the set Ω of G-conjugates of M.

Definition 9.1. We say that G **is doubly transitive of Suzuki type (on Ω)** if and only if a two-point stabilizer D fixing M (i.e., $D \leq M$) has a normal complement X in M (i.e., $M = DX$ with $X \lhd M$ and $D \cap X = 1$) satisfying the following conditions:

(1) X is nilpotent;
(2) X contains a Sylow 2-subgroup of M (and hence of G); and
(3) X acts regularly on $\Omega - \{M\}$.

Bender's theorem asserts the following:

THEOREM SE: STAGE 1. *G is doubly transitive of Suzuki type on Ω.*

In particular, G is a split (B, N)-pair of rank 1 (with $B = M$) and odd degree, since the regularity condition implies that $|\Omega| = |X| + 1$.

This is the starting point for Suzuki's work on doubly transitive permutation groups, which we have included as a Background Result with the proof as revised by Peterfalvi. With this result, the proof of Theorem SE is completed, and with it the classification of groups with a strongly $\mathcal{Z}$-embedded subgroup.

THEOREM SE: STAGE 2. *If G is doubly transitive of Suzuki type, then $G \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$ for some $n \geq 2$.*

10. Theorem $\mathcal{M}(S)$

We next discuss the stages of the proof of Theorem $\mathcal{M}(S)$ (row 4 of the uniqueness grid). In particular, we assume that G is of even type.

The theorem being a variation of the global $C(G, S)$-theorem, its proof also follows the Aschbacher χ-block method. However, because of possible confusion of terms between Aschbacher's blocks and Brauer's blocks of characters, we shall adopt alternate terminology here: the term "near component" in place of "block", and "linear or alternating near component" in place of "χ-block".

DEFINITION 10.1. Let X be a group, set $P = O_2(X)$, and assume that

(a) $C_X(P) \leq P$ (equivalently $F^*(X) = P$);
(b) X/P is quasisimple or simple (including the case of prime order); and
(c) $X = O^2(X)$.

We call X a **near component** provided the following additional conditions hold:

(1) $U = [P, X] \leq \Omega_1(Z(P))$; and
(2) X acts nontrivially and irreducibly on $V = U/C_U(X)$.

We can regard V as a faithful irreducible $\boldsymbol{F}_2[X/P]$-module and refer to V as the **associated module** of X. In addition, a subgroup X of a group Y is called a **near component of** Y if X is a near component and X is subnormal in Y.

The justification for this terminology is that near components of a group behave very much like components of a group; in particular, they satisfy an analogue [**Foo1, Foo2**] of $L_{2'}$-balance, which is a basic property of components.

DEFINITION 10.2. A near component X with associated module V will be called **linear** (respectively **alternating**) if and only if $X/O_2(X) \cong L_2(2^n)$ with $n \geq 2$ (respectively A_n with $n = 3$ or $n \geq 5$) and V is a natural 2-dimensional $\boldsymbol{F}_{2^n}$-module for $X/O_2(X) \cong L_2(2^n)$, considered as an $\boldsymbol{F}_2[X/O_2(X)]$-module (respectively the nontrivial irreducible constituent of the standard $\boldsymbol{F}_2$ permutation module for $X/O_2(X) \cong A_n$).

[The notion of near component arises in the local $C(G, S)$-theorem [**A11, A12**], which asserts that a group Y such that $F^*(Y) = O_2(Y)$ is the product of all its linear and alternating near components and $C(Y, S) = \langle N_Y(R) \mid 1 \neq R \text{ char } S\rangle$, where $S \in Syl_2(Y)$. The notion of a near component can also be extended to arbitrary primes p. However, for $p = 3$ and $X/P \cong SL_2(3^n)$, $P = O_3(X)$, in order for an analogue of the local $C(G, S)$-theorem to be true, it is necessary to allow X to have a structure approximating that of a maximal parabolic subgroup of $G_2(3^n)$. In particular, $[X, P]/\Phi([X, P])$ is the sum of two natural X/P-modules and X acts nontrivially on $\Omega_1(Z(P))$. Such "$G_2(3^n)$-type" near components arise in the course of the classification of groups with semidihedral or wreathed Sylow 2-subgroups.]

Assume now that G satisfies the assumptions of Theorem $\mathcal{M}(S)$, so that G is of even type, $|\mathcal{M}(S)| = 1$ for $S \in Syl_2(G)$, and if M is the unique element of $\mathcal{M}(S)$, then either (1) $\sigma(G) = \emptyset$ or (2) $m_p(M) \geq 4$ for some odd prime p, and for any such p, M is a p-uniqueness subgroup. [This last assumption is not required for Stage 1.] The first stage of the proof of the theorem proceeds as follows.

By Theorem U(2), we may assume that G does not contain a 2-uniqueness subgroup. Theorem SA therefore implies that S does not contain a nontrivial strongly closed abelian subgroup. It follows that M does not control G-fusion in S, otherwise $Z(O_2(M))$ would be such a nontrivial strongly closed abelian subgroup of S. (By definition of $\mathcal{M}(S)$, M is a 2-local subgroup, so $O_2(M) \neq 1$.) This observation constitutes the first step in the analysis.

As a consequence, one can apply the Alperin-Goldschmidt conjugation theorem [**Go1**] to conclude that there is $1 \neq D \leq S$ such that $H = N_G(D)$ has the following properties:

(10.1)
(1) $H \not\leq M$;
(2) $F^*(H) = O_2(H)$; and
(3) D contains every involution of $C_G(D)$.

We consider the set $\mathcal{N}(M)$ of all maximal 2-local subgroups N of G such that there exists a 2-subgroup $D \neq 1$ of $N \cap M$ such that if we set $H = N_N(D)$, then D, H satisfy the conditions of (10.1). [By the discussion preceding (10.1), $\mathcal{N}(M)$ is nonempty.]

We define a preordering $\succeq$ on the set $\mathcal{N}(M)$ as follows: if N, $N_1 \in \mathcal{N}(M)$ and $T \in Syl_2(N \cap M)$, $T_1 \in Syl_2(N_1 \cap M)$, we write $N \succeq N_1$ if and only if $m_2(T) \geq m_2(T_1)$, with $|T| \geq |T_1|$ in case $m_2(T) = m_2(T_1)$. We denote the set of maximal elements of $\mathcal{N}(M)$ under $\succeq$ by $\mathcal{N}^*(M)$. Finally, for $N \in \mathcal{N}^*(M)$ and $T \in Syl_2(N)$, set

$$C(N, T) = \langle N_N(T_0) \mid 1 \neq T_0 \text{ char } T\rangle.$$

The goal of Stage 1 is the following.

THEOREM $\mathcal{M}(S)$: STAGE 1. *Under the given assumptions on G, either G has a 2-uniqueness subgroup, or $\mathcal{N}^*(M) \neq \emptyset$ and the following conditions hold for any $N \in \mathcal{N}^*(M)$:*

(i) *$F^*(N) = O_2(N)$;*
(ii) *M contains a Sylow 2-subgroup T of N;*
(iii) *$C(N, T) \leq M$; and*
(iv) *N contains a linear or alternating near component K with $K \not\leq M$.*

We call M a **near component 2-local uniqueness subgroup**. Condition (iv) follows directly from (i), (ii), (iii) and Aschbacher's local $C(G, S)$ theorem, as remarked above. The difficulty in establishing the critical condition (i) arises from the presence of components in 2-local subgroups[2]. The following subsidiary result, using Theorem TS, underlies the proof. Here N and T are as above.

THEOREM. *Let $x \in \mathcal{I}_2(G)$, let L be a component of $C_G(x)$, let P be a Sylow 2-subgroup of $C_G(x)$, and assume*
(a) *$L \not\leq M$; and*
(b) *$P \leq M$ and $P \succeq T$.*
Then $L \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$ for some $n \geq 2$, and L is standard in G.

Next, let $N \in \mathcal{N}^*(M)$ and K be as in Theorem $\mathcal{M}(S)$: stage 1. Our assumptions on $\sigma(G)$ are now used to restrict both the possibilities for K and the number r of near components of N isomorphic to K: namely,

(1) If $K/O_2(K) \cong A_m$, then $m = 3$, 5, or 9; and
(2) If K is nonsolvable, then $r \leq 3$.

[When $K/O_2(K) \cong A_m$, Aschbacher's theorem implies that, in fact, $m = 2^n+1$ for some integer n, which explains why only the values $m = 3, 5$, or 9 occur here.]

As noted in Chapter 1, the analysis of this situation has been carried out by Richard Foote. Underlying it is the following "standardness" theorem for the K obtained in stage 1.

THEOREM $\mathcal{M}(S)$: STAGE 2. *Assume the hypotheses and conclusions of Theorem $\mathcal{M}(S)$: stage 1. Let T_1 be a subgroup of T with $|T : T_1| \leq 2$ and $O_2(N) \leq T_1$, and assume that if K is nonsolvable, then T_1 normalizes K. If N_0 is a maximal 2-local subgroup of G such that $\langle K, T_1\rangle \leq N_0$ and $N_0 \succeq N$, then K is a near component of N_0.*

We say that K is a **standard** near component. With this result Foote then argues to a contradiction.

THEOREM $\mathcal{M}(S)$: STAGE 3. *G cannot satisfy the assumptions and conclusions of Theorem $\mathcal{M}(S)$: stage 1.*

11. Theorem U(σ)

Finally we describe the stages of the proof of Theorem U(σ). We assume throughout that G is of restricted even type with $\sigma(G) \neq \emptyset$ and that for each $p \in \sigma(G)$, G contains a strong p-uniqueness subgroup M_p.

The goals of the first stage of the analysis are to reduce as much as possible to the 2-local case $F^*(M_p) = O_2(M_p)$, and to strengthen both the p-uniqueness and the 2-uniqueness properties of M_p. To describe the last of these, we need the following terminology. Define $\mathcal{Q}(M_p)$ to be the set of all p-subgroups Q of M_p such that either $m_p(Q) \geq 3$ or for some E_{p^2} subgroup A of Q, $\Gamma_{A,1}(G) \leq M_p$.

DEFINITION 11.1. We say that M_p is a **$\{2, p\}$-uniqueness subgroup of** G if and only if every 2-local subgroup of G containing an element of $\mathcal{Q}(M_p)$ lies in M_p.

THEOREM U(σ): STAGE 1. *The following conditions hold for each $p \in \sigma(G)$:*
(i) *Structure of M_p:*
 (a) *$O_{2'}(M_p) = 1$; and*

[2]This difficulty arises from our assuming that G is of even type rather than of characteristic 2 type, as was assumed in the corresponding result in the original classification proof.

(b) *Either* $F^*(M_p) = O_2(M_p)$, *or* $E(M_p) \in \mathcal{C}hev(2)$ *with* $C_{M_p}(E(M_p))$ *having p-rank at most 1;*

(ii) *2-uniqueness properties:*

(a) M_p *is a* $\{2,p\}$*-uniqueness subgroup of* G*; and*

(b) M *contains* $N_G(S)$ *for some* $S \in Syl_2(G)$*;*

(iii) *Additional p-uniqueness properties (by definition* M_p *is a strong p-uniqueness subgroup):*

(a) *If* $p = 3$*, then* M_p *is strongly p-embedded in* G*; and*

(b) *If* $p > 3$*,* $x \in \mathcal{I}_p(M_p)$*, and* $C_G(x) \not\leq M_p$*, then* $L_{p'}(C_G(x)) = K$ *satisfies the conditions* $K/O_{p'}(K) \cong L_2(p^n)$*,* $n \geq 3$*,* $\langle x \rangle$ *is strongly closed in a Sylow p-subgroup* P *of* M *with respect to* G*,* $P \in Syl_p(C_G(x))$*, and*

$$P = P_0 \times (P \cap K)$$

with P_0 *cyclic.*

In the original classification proof similar results were obtained by Aschbacher, Gorenstein and Lyons [**AGL1, GL1**] and here our proof is similar. However we achieve some simplification in the verification of the $\{2,p\}$-uniqueness property because of the definition of $\sigma(G)$. The focus is on the elimination of 2-local subgroups H such that $H \cap M_p$ contains both a Sylow 2-subgroup of H and an E_{p^2}-subgroup. For such a subgroup the p-uniqueness properties of M place severe restrictions on the structure of H. The case $m_p(H \cap M_p) \geq 4$ is then relatively easy to rule out, and once that has been done, the condition $\mathcal{M}_1(S) \neq \emptyset$ forces M_p itself to contain a subgroup of index at most 2 in a Sylow 2-subgroup of G. This in turn eases the analysis of the remaining possibility $m_p(H \cap M_p) \leq 3$.

Once the preparatory uniqueness properties have been established, the proof relies on the analysis of 2-amalgams and is the work of Gernot Stroth.

THEOREM U(σ): STAGES 2 AND 3. *If* G *satisfies the hypotheses and conclusions of Theorem U(*σ*): stage 1, then* $|\mathcal{M}(S)| = 1$ *for some* $S \in Syl_2(G)$*.*

If the conclusion of this theorem fails, Stroth makes a careful choice of $p \in \sigma(G)$, takes $S \in Syl_2(M_p)$, and chooses a second element $M \in \mathcal{M}(S)$. He then selects certain subgroups N and N_p of M and M_p, respectively, containing S but with $O_2(\langle N, N_p \rangle) = 1$, and argues to a contradiction by analysis of the amalgam of N and N_p. The $\{2,p\}$-uniqueness property is heavily used.

The conclusion of this theorem contradicts Theorem $\mathcal{M}(S)$, and so this completes the proof of Theorem U(σ).

C. THE CLASSIFICATION GRID: GENERIC AND SPECIAL SIMPLE GROUPS

12. $\mathcal{C}_p$-groups

We now give an analogous description of the classification grid. We begin with the definitions of the notions of the "generic" and "special" cases for G, which underlie the basic three-part division of the grid. We then give the precise conditions defining the seven rows of the grid and give the statements of Theorems $\mathcal{C}_i$, $1 \leq i \leq 7$.

Fundamental to the case division are partitions—one for each prime p—of the set of quasisimple $\mathcal{K}$-groups into three subsets. More precisely, for any p, recall that $\mathcal{K}_p$ is the set of all quasisimple $\mathcal{K}$-groups K such that $O_{p'}(K) = 1$; we shall define a partition

$$\mathcal{K}_p = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{G}_p. \tag{12.1}$$

We begin with the definition of $\mathcal{C}_p$. In view of Definition 1.3, this will complete the precise definition of "even type." To specify which elements of $\mathcal{K}_p$ are to be in $\mathcal{C}_p$, we first do this for simple K and then consider the nonsimple case.

As remarked in the preceding chapter, $\mathcal{C}_p$-groups are intended to include those quasisimple $\mathcal{K}$-groups considered to be "characteristic p-like." These include, of course, all the simple groups in $\mathcal{C}hev(p)$—the groups of Lie type of characteristic p. However, we are able to achieve certain economies by including some non-$\mathcal{C}hev(p)$ groups as well.

Unfortunately, the known quasisimple groups are too perverse to allow for a neat conceptual definition that separates the elements of $\mathcal{C}_p$ (i.e., the $\mathcal{C}_p$-groups) from the other quasisimple groups. We shall therefore make the definition by simply listing the $\mathcal{C}_p$-groups for each prime p. This will be followed by a conceptual description which yields roughly the same set of groups.

Definition 12.1. Let p be a prime. A simple $\mathcal{K}$-group K is a **$\mathcal{C}_p$-group** if and only if $K \in \mathcal{C}hev(p)$, or $K \cong A_p$, A_{2p}, or A_{3p}, or one of the following holds:

(1) $p = 2$ and either
  (a) $K \cong L_2(q)$, $q \in \mathcal{FM}9$ (i.e., q a Fermat or Mersenne prime or 9);
  (b) $K \cong L_3(3)$, $L_4(3)$, $U_4(3)$, or $G_2(3)$; or
  (c) $K \cong M_{11}$, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, Suz, Ru, Co_1, Co_2, Fi_{22}, Fi_{23}, Fi'_{24}, F_3, F_2, or F_1.
(2) $p = 3$ and either
  (a) $K \cong U_5(2)$, $U_6(2)$, $Sp_6(2)$, $D_4(2)$, ${}^3D_4(2)$, $F_4(2)$, ${}^2F_4(2)'$, or $PSp_4(8)$; or
  (b) $K \cong M_{11}$, J_3, Co_1, Co_2, Co_3, Mc, Ly, Suz, ON, Fi_{22}, Fi_{23}, Fi'_{24}, F_5, F_3, F_2, or F_1.
(3) $p = 5$ and either
  (a) $K \cong {}^2F_4(2)'$, ${}^2F_4(32)$, or $Sz(32)$; or
  (b) $K \cong J_2$, Co_1, Co_2, Co_3, HS, Mc, Ly, Ru, He, F_5, F_3, F_2, or F_1.
(4) $p = 7$ and $K \cong Co_1$, He, ON, Fi'_{24}, F_3, or F_1.
(5) $p = 11$ and $K \cong J_4$.

Furthermore, a quasisimple $\mathcal{K}$-group $K \in \mathcal{K}_p$ is a $\mathcal{C}_p$-group if and only if $K/O_p(K)$ is a $\mathcal{C}_p$-group, with the following exceptions:

For $p = 2$: The groups $SL_2(q)$, $q \in \mathcal{FM}9$, $2A_8$, $SL_4(3)$, $SU_4(3)$, $Sp_4(3)$, and $[X]L_3(4)$, $X = 4$, 4×2, or 4×4, are not included among the $\mathcal{C}_2$-groups.

For $p = 3$: The groups $3A_6$ and $3ON$ are not included among the $\mathcal{C}_3$-groups.

Finally, a quasisimple $\mathcal{K}$-group K with $O_{p'}(K) = 1$ is a **$\mathcal{C}_{p'}$-group** if K is not a $\mathcal{C}_p$-group; that is, $\mathcal{C}_{p'} = \mathcal{K}_p - \mathcal{C}_p$.

In the above definition, we have not necessarily listed Chevalley groups of ambiguous characteristic, since automatically all simple groups in $\mathcal{C}hev(p)$ lie in

$\mathcal{C}_p$. But we note that $\mathcal{C}_2$ contains $U_3(3)$, $PSp_4(3)$, and ${}^2G_2(3)'$, since these are isomorphic to $G_2(2)'$, $U_4(2)$, and $L_2(8)$, respectively, so lie in $\mathcal{C}hev(2)$.

We shall not explicitly list the nonsimple $\mathcal{C}_p$-groups at this time, but shall introduce the notation to be used in connection with covering groups.

If K is a quasisimple group and $Z \le Z(K)$ with Z cyclic, we shall write

$$K \cong A\overline{K}, \text{ where } \quad A = |Z| \text{ and } \overline{K} = K/Z.$$

Thus, for example, $SL_4(3) \cong 2L_4(3)$ and $SU_4(3) \cong 4U_4(3)$ are not $\mathcal{C}_2$-groups, but the group $2U_4(3) \cong SU_4(3)/Z_2$ is a $\mathcal{C}_2$-group.

On the other hand, if $Z \cong Z_1 \times Z_2$ is noncyclic with Z_1, Z_2 cyclic, we shall write

$$K = [A_1 \times A_2]\overline{K}, \text{ where } \quad A_i = |Z_i|,\ i = 1, 2.$$

Thus, for example, $2\Omega_8^+(3) = [2 \times 2]P\Omega_8^+(3)$; moreover, there exists a quasisimple group of the form $[4 \times 4]L_3(4)$.

It turns out, after the calculation of the Schur multiplier of each simple $\mathcal{K}$-group K and the action of $Aut(K)$ on it, that every quasisimple group K has one of the forms

$$A\overline{K} \qquad \text{or} \qquad [A_1 \times A_2]\overline{K}$$

where $\overline{K} = K/Z(K)$, and this notation determines K up to isomorphism with the following exceptions:

$$2D_n(q),\ q \text{ odd},\ n \text{ even},\ n > 4,\ 4L_3(4), \text{ and } \quad 3U_4(3),$$

for each of which there are exactly two isomorphism classes of groups.

We also use parentheses to indicate that we allow more than one possibility for the central subgroup Z—that Z may be trivial or of the indicated order. Thus, for example, $K \cong (2)A_n$ means that either $K \cong A_n$ or $K \cong 2A_n$. (The symbols $(S)L_n(q)$, $(P)\Omega_n(q)$, etc., are similarly ambiguous, representing a group which is isomorphic either to $SL_n(q)$ or $L_n(q)$, etc.) By contrast, a symbol inside a **square** bracket denotes the actual structure of Z: $K \cong [X]L_3(4)$, $X = 4$ or 4×2, means that $Z(K) \cong Z_4$ or $Z_4 \times Z_2$.

Finally, we give a roughly equivalent conceptual description of the notion of $\mathcal{C}_p$-group. We need some preliminary terminology. Fix a prime p.

First, an element $x \in \mathcal{I}_p(H)$ is ***p*-central (in *H*)** if x is in the center of a Sylow p-subgroup of H.

Next define

$$\mathcal{CH}_p = \{K \in \mathcal{K}_p \mid F^*(C_K(x)) = O_p(C_K(x)) \text{ for every } p\text{-central } x \in \mathcal{I}_p(K)\}.$$

It is immediate that every $K \in \mathcal{CH}_p$ is actually simple. Now given any set $\mathcal{H}_p$ of known simple groups, each of order divisible by p, define

$$\hat{\mathcal{H}}_p = \{K \in \mathcal{K}_p \mid K/Z(K) \in \mathcal{H}_p\},$$

and let $\mathcal{H}_p^*$ be the set of all simple groups $K \in \mathcal{K}$ such that either $K \in \mathcal{H}_p$, or $m_p(K) > 1$ and $\mathcal{L}_p(K) \subseteq \hat{\mathcal{H}}_p$. Thus $\mathcal{H}_p^*$ consists of $\mathcal{H}_p$ plus certain additional groups K, the components[3] of whose centralizers of elements of order p are required to be covering groups of $\mathcal{H}_p$-groups. In particular, $K \in \mathcal{H}_p^*$ if K is of characteristic p-type, or more generally if $\mathcal{L}_p(K)$ is empty and $m_p(K) > 1$.

[3]By the B_p property, the p-components of such centralizers are actually components.

Finally we define

$$\mathcal{J}_o(p) = \{J \mid J \text{ is simple and } J \in \mathcal{C}hev(p) \cup \{A_p\}\};$$
$$\mathcal{J}_1(p) = \mathcal{J}_o(p)^* \cap \mathcal{CH}_p; \text{ and}$$
$$\mathcal{J}_2(p) = \mathcal{J}_1(p)^* \cap \mathcal{CH}_p.$$

With this terminology, the set

$$\tilde{\mathcal{C}}_p = \hat{\mathcal{J}}_o(p) \cup \hat{\mathcal{J}}_1(p) \cup \hat{\mathcal{J}}_2(p)$$

is a reasonable approximation[4] to $\mathcal{C}_p$. Thus the $\tilde{\mathcal{C}}_p$-groups consist of the groups in $\mathcal{C}hev(p)$ and A_p, together with certain of their pumpups and "double" pumpups. An even closer approximation can be obtained by using the above definitions but expanding $\mathcal{CH}_p$ to include all groups $K \in \mathcal{K}_p$ with the following property: for every p-central $x \in \mathcal{I}_p(K)$, $O_{p'}(C_K(x)) = 1$ and each component of $E(C_K(x))$ lies in $\hat{\mathcal{J}}_0(p)$ but not in any $\hat{\mathcal{J}}_0(r)$ for any $r \neq p$.

The notion of a $\tilde{\mathcal{C}}_p$-group has been introduced solely for expository reasons. It is the $\mathcal{C}_p$-groups with which we shall work.

In particular, the definition of "even type" (Definition 1.3) is now complete.

13. $\mathcal{G}_p$-groups and $\mathcal{T}_p$-groups

Unfortunately the existence of an element K in $\mathcal{L}_p(G)$ with K a $\mathcal{C}_{p'}$-group for some prime p is not in itself sufficient to guarantee that the target group G^* is a "generic" simple group. On the contrary, there exist certain "in-between" situations whose analysis requires special methods.

To accommodate this case separation, it is necessary to divide the set $\mathcal{C}_{p'}$ into two subsets, the larger called $\mathcal{G}_p$, corresponding to the generic case, with the residual "intermediate" set called $\mathcal{T}_p$, $\mathcal{T}$ for "thin", as these groups all have low p-rank. We first list the elements of the sets $\mathcal{T}_p$ for each prime p, and take $\mathcal{G}_p$ as the set of $\mathcal{C}_{p'}$-groups left over.

DEFINITION 13.1. Let p be a prime. Then $\mathcal{T}_p$ is the set of all $\mathcal{C}_{p'}$-groups K such that one of the following holds:

(1) $p = 2$ and $K \cong L_2(q)$, q odd, $q \notin \mathcal{FM}9$, A_7, $SL_2(q)$, q odd, $2A_n$, $7 \leq n \leq 11$, or $[X]L_3(4)$, $X = 4$, 4×2, or 4×4;
(2) $p = 3$ and $K/Z(K) \cong L_3^\epsilon(q)$, $\epsilon = \pm 1$, $q \equiv \epsilon (\text{mod } 3)$, A_7, M_{12}, M_{22}, J_2 or $G_2(8)$, or $K \cong 3A_6$;
(3) $p = 5$ and $K \cong Fi_{22}$; or
(4) p is odd and $m_p(K) = 1$.

Furthermore,

$$\mathcal{G}_p = \mathcal{K}_p - \mathcal{C}_p - \mathcal{T}_p.$$

Since $\mathcal{T}_2$ by definition contains all groups in $\mathcal{K}_2$ with quaternion Sylow 2-subgroups, it follows that $\mathcal{T}_p$ contains, for each prime p, all elements of $\mathcal{K}_p$ of p-rank 1, with the exception of the $\mathcal{C}_p$-groups $L_2(p)$ and A_p, and (for $p = 5$) $Sz(32)$.

[4]For example, to consider simple groups for illustration, the only simple elements of $\tilde{\mathcal{C}}_p - \mathcal{C}_p$ are $D_4(3)$, He, F_5, A_9 and A_{10} (for $p = 2$) and $L_3(4)$ (for $p = 3$). The only simple elements of $\mathcal{C}_p - \tilde{\mathcal{C}}_p$ are A_{2p} and A_{3p} (for $p \geq 5$), Co_1 and F_3 (for $p = 7$), ${}^2F_4(32)$, $Sz(32)$ and He (for $p = 5$), $PSp_4(8)$, $G_2(8)$, $U_5(2)$, $D_4(2)$, Co_1, Suz, ON and Ly (for $p = 3$), and Fi_{23} (for $p = 2$). These elements fail to be in $\tilde{\mathcal{C}}_p$ generally only because they contain a p-central element x such that $E(C_K(x)) \neq 1$.

Note also that for $p = 2$ the groups A_n, $n = 9, 10, 11$, are $\mathcal{G}_2$-groups according to the above definition, while $A_8(\cong L_4(2))$ is a $\mathcal{C}_2$-group and A_7 is a $\mathcal{T}_2$-group; but their 2-fold covers $2A_n$ $(7 \le n \le 11)$ are all $\mathcal{T}_2$-groups.

We note also that if K is a sporadic $\mathcal{G}_p$-group, then one of the following holds:

(1) $p = 2$ and $K \cong J_1$, Mc, Ly, ON, He, Co_3, or F_5;
(2) $p = 3$ and $K \cong M_{23}$, M_{24}, J_4, Ru, HS, He, or $3ON$;
(3) $p = 5$ and $K \cong Suz$, Fi_{23}, or Fi'_{24};
(4) $p = 7$ and $K \cong F_2$; or
(5) $p = 11$ or 13, and $K \cong F_1$.

In particular, if p is odd, then in every case $m_p(K) = 2 + m_p(Z(K))$.

The sets $\mathcal{C}_p$, $\mathcal{T}_p$ and $\mathcal{G}_p$ have been constructed in such a way as to retain a property of "closure" under taking p-components of centralizers of elements of order p–indeed of any automorphisms of order p. Namely, it is verified without difficulty that

$$\text{If } K \in \mathcal{C}_p \text{ (resp. } \mathcal{T}_p\text{), then for any } x \in Aut(K) \text{ of order } p,\\ \text{all elements of } \mathcal{L}_p(K) \text{ lie in } \mathcal{C}_p \text{ (resp. } \mathcal{C}_p \cup \mathcal{T}_p\text{).} \tag{13.1}$$

Moreover,

$$\text{If } K \in \mathcal{C}_p \text{ (resp. } \mathcal{T}_p\text{),}\\ \text{then all nonidentity factor groups of } K \text{ lie in } \mathcal{C}_p \text{ (resp. } \mathcal{C}_p \cup \mathcal{T}_p\text{),}\\ \text{except if } p = 2 \text{ and } K \cong 2A_n,\ 9 \le n \le 11. \tag{13.2}$$

In some cases, these properties control whether pumpups are $\mathcal{C}_p$, $\mathcal{T}_p$, or $\mathcal{G}_p$-groups. For instance, suppose that K and L are p-components of $C_G(x)$ and $C_G(y)$, respectively, where x and y are of order p and L is obtained from K by a chain of pumpups. Because of (13.1) and (13.2), it follows that if $K/O_{p'}(K) \in \mathcal{G}_p$, then also $L/O_{p'}(L) \in \mathcal{G}_p$ unless $p = 2$ and $K/O(K) \cong A_n$, $9 \le n \le 11$.

14. Groups of special odd type

We can now define the major case divisions of the classification grid.

In the previous chapter we introduced the following refinements of $\mathcal{I}_p(G)$ and $\mathcal{L}_p(G)$.

Definition 14.1. Let p be a prime. Then

$$\mathcal{I}_p^o(G) = \{x \in \mathcal{I}_p(G) \,|\, m_p(C_G(x)) \ge 4\} \text{ if } p \text{ is odd,}$$
$$\mathcal{I}_2^o(G) = \{x \in \mathcal{I}_2(G) \,|\, m_2(C_G(x)) \ge 3\}, \text{ and}$$
$$\mathcal{L}_p^o(G) = \{L \,|\, L \text{ is a component of } C_G(x)/O_{p'}(C_G(x)) \text{ for some } x \in \mathcal{I}_p^o(G)\}.$$

In this section we consider the cases where G is not generic for the prime 2. These cases are incorporated into the following definition.

Definition 14.2. We say that G is of $\mathcal{L}_2$**-special type** if and only if every element of $\mathcal{L}_2^o(G)$ is either a $\mathcal{C}_2$-group or a $\mathcal{T}_2$-group (equivalently, no element of $\mathcal{L}_2^o(G)$ is a $\mathcal{G}_2$-group). Moreover, G is of **special odd type** if and only if G is of $\mathcal{L}_2$-special type but not of restricted even type.

It is clear from the definition that if $m_2(G) = 2$, then G is of special odd type. Moreover, as noted in the previous chapter, if $m_2(G) \geq 3$, then $\mathcal{L}_2^o(G) = \mathcal{L}_2(G)$ by an argument using the Thompson transfer lemma.

The analysis of groups of special odd type itself splits into three cases, the first of which is of course the solvability of groups of odd order, included as a Background Result. This accounts for case 1, the first row of the classification grid.

When G is of even order, two cases are drawn according to the principal techniques to be used in the analysis:

Case 2. The Bender method together with 2-fusion analysis and character theory.

Case 3. The signalizer functor method together with 2-fusion analysis.

A brief discussion of the relevant methods appears in sections 28–31 and 35–36 below.

The two cases are distinguished by the nature of the elements of $\mathcal{L}_2(G)$. The first arises primarily, but not exclusively, when some element of $\mathcal{L}_2(G)$ lies in the set $\mathcal{B}_2$, whose definition we repeat.

Definition 14.3. $\mathcal{B}_2$ consists of the groups $SL_2(q)$, q odd, $q > 3$; $[X]L_3(4)$, $X = 4$, 4×2 or 4×4; and $2A_n$, $7 \leq n \leq 11$.

Thus $\mathcal{B}_2 \subseteq \mathcal{T}_2$.

The following definition incorporates the configurations to be treated by the Bender method. It includes the case in which some element of $\mathcal{L}_2(G)$ is a $\mathcal{B}_2$-group, and some cases of small 2-rank.

Definition 14.4. If G is of special odd type, we say that G is of $\mathcal{L}\mathcal{B}_2$**-type** if and only if one of the following holds:
(1) Some element of $\mathcal{L}_2(G)$ is a $\mathcal{B}_2$-group; or
(2) $m_2(G) \leq 3$.

The complementary set defines the cases to be treated by the signalizer functor method.

Definition 14.5. We say that G is of $\mathcal{L}\mathcal{T}_2$**-type** if and only if G is of even order, G is of special odd type, but G is not of $\mathcal{L}\mathcal{B}_2$-type.

Now we can state Theorems $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$.

Theorem $\mathcal{C}_1$. *G is of even order.*

Theorem $\mathcal{C}_2$. *If G is of $\mathcal{L}\mathcal{B}_2$-type, but contains no 2-uniqueness subgroup, then $G \cong G^*$ for some $G^* \in \mathcal{K}^{(2)}$.*

Theorem $\mathcal{C}_3$. *If G is of $\mathcal{L}\mathcal{T}_2$-type, then $G \cong G^*$ for some $G^* \in \mathcal{K}^{(3)}$.*

These three theorems together with Theorem U(2) show that if G is of special odd type, then $G \cong G^*$ for some $K \in \mathcal{K}^{(0)} \cup \mathcal{K}^{(2)} \cup \mathcal{K}^{(3)}$.

[The hypothesis in Theorem $\mathcal{C}_2$ that there is no 2-uniqueness subgroup could be deleted, at the expense of having the groups $L_2(4)$, $U_3(4)$ and J_1 added to the conclusion. We have given the above statement of the theorem simply to preserve the device of keeping the target sets $\mathcal{K}^{(i)}$ pairwise disjoint.]

15. Groups of special even type

For odd primes, generic and special type groups are defined only when G is of restricted even type, so we make this assumption throughout. Moreover, when $\sigma(G)$ is nonempty, the definitions are limited to primes $p \in \sigma(G)$ (Definition 1.5). For such p, $m_p(G) \geq 4$ by definition, and so $\mathcal{I}_p^o(G)$ is nonempty.

DEFINITION 15.1. The group G of restricted even type is of $\mathcal{L}_p$**-special type** for $p \in \sigma(G)$ if and only if

$$\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p \cup \mathcal{T}_p,$$

or equivalently, no element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group. Furthermore, we say that G is of **special even type** if and only if G is of restricted even type, and either $\sigma_0(G) = \emptyset$ or G is of $\mathcal{L}_p$-special type for some $p \in \sigma_0(G)$.

The analysis of groups of special even type likewise divides into three major cases, each again determined by the principal techniques used in their classification:

CASE 4. The Goldschmidt amalgam method

CASE 5. The Klinger-Mason method

CASE 6. The signalizer functor method

These methods will be briefly described in Sections 22, 23 and 29–34.

We now give the precise conditions defining each of these cases. The condition for Case 4 is that G be quasithin (Definition 1.6). In this case, Theorem $\mathcal{M}(S)$ from the uniqueness grid implies that $|\mathcal{M}(S)| \geq 2$. It is this case whose analysis will be based on the Goldschmidt amalgam method. When G is not of quasithin type, then Theorem U(σ) from the uniqueness grid implies that $\sigma_0(G) \neq \emptyset$. This case is further split into two subcases (cases 5 and 6) according to the nature of the elements of $\mathcal{L}_p^o(G)$ for $p \in \sigma_0(G)$.

DEFINITION 15.2. Let $p \in \sigma_0(G)$. Then G is of $\mathcal{LC}_p$**-type** if and only if G is of restricted even type and $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$; and G is of $\mathcal{LT}_p$**-type** if and only if G is of restricted even type, $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p \cup \mathcal{T}_p$, but $\mathcal{L}_p^o(G) \not\subseteq \mathcal{C}_p$.

Thus for $p \in \sigma_0(G)$, G is of $\mathcal{L}_p$-special type if and only if it is either of $\mathcal{LC}_p$-type or of $\mathcal{LT}_p$-type.

The analysis of groups of $\mathcal{LC}_p$-type depends on the Klinger-Mason method; that of groups of $\mathcal{LT}_p$-type on the signalizer functor method.

Now we can state Theorems $\mathcal{C}_4, \mathcal{C}_5$ and $\mathcal{C}_6$. Throughout, *the hypotheses imply in particular that G is assumed to be of restricted even type.* In addition we assume, as we may in view of Theorem U(2) of the uniqueness grid, that G does not contain a 2-uniqueness subgroup.

THEOREM $\mathcal{C}_4$. *If G is of quasithin type but contains no 2-uniqueness subgroup, then $G \cong G^*$ for some $G^* \in \mathcal{K}^{(4)}$.*

THEOREM $\mathcal{C}_5$. *If G is of $\mathcal{LC}_p$-type for some $p \in \sigma_0(G)$, then $p = 3$ and $G \cong G^*$ for some $G^* \in \mathcal{K}^{(5)}$.*

THEOREM $\mathcal{C}_6$. *G is not of $\mathcal{LT}_p$-type for any $p \in \sigma_0(G)$.*

In both Theorems $\mathcal{C}_5$ and $\mathcal{C}_6$ the proofs reach contradictions by the construction of a strong p-uniqueness subgroup.

In view of Theorem $\mathrm{U}(\sigma)$, these three theorems completely classify G if G has special even type. As with Theorem $\mathcal{C}_2$, we could delete the hypothesis about 2-uniqueness subgroups from Theorem $\mathcal{C}_4$ at the expense of including $L_2(2^n)$, $Sz(2^n)$ and $U_3(2^n)$ $(n \geq 3)$ in the conclusion.

16. Groups of generic type

Finally we consider the generic situation.

DEFINITION 16.1. We say that G is of **generic odd type** if and only if some element of $\mathcal{L}_2^o(G)$ is a $\mathcal{G}_2$-group. We say that G is of **generic even type** if and only if G is of restricted even type, G is not of $\mathcal{L}_p$-special type for any $p \in \sigma_0(G)$, but $\sigma_0(G)$ is nonempty. (Thus, taking $p \in \sigma_0(G)$, it follows that some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group.)

Finally, G is of **generic type** if and only if it is either of generic odd type or generic even type.

We remark that groups of Lie type of characteristic 2 and large Lie rank are of generic even type, and those of odd characteristic are of generic odd type. The primes p mentioned in Definition 16.1 are never the characteristic; thus p is even when G is of odd generic type, and vice-versa.

In addition, the condition that G be of even type in the above definition forces the odd and even generic cases to be mutually exclusive, since $\mathcal{L}_2(G) \subseteq \mathcal{C}_2$ for groups of even type.

THEOREM $\mathcal{C}_7$. *If G is of generic type, then $G \cong G^*$ for some $G^* \in \mathcal{K}^{(7)}$.*

Together, Theorems $\mathcal{C}_i$, $1 \leq i \leq 7$, combined with results from the uniqueness grid, establish the Classification Theorem.

For clarity we restate the main logic. It is immediate from Definitions 14.2, 15.1, and 16.1 that a $\mathcal{K}$-proper simple group G is of special odd type, special even type or generic type. If G has a 2-uniqueness subgroup, then Theorem U(2) applies. Otherwise, if G is of special odd type, then Definition 14.5 shows that one of Theorems $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_3$ applies; if G is of special even type, then one of Theorems $\mathcal{C}_4$, $\mathcal{C}_5$ or $\mathcal{C}_6$ applies, as noted after Theorem $\mathcal{C}_6$; and if G is of generic type, then Theorem $\mathcal{C}_7$ applies. In every case, $G \in \mathcal{K}$.

D. The Classification Grid: The Stages of the Proof

17. Theorem $\mathcal{C}_1$

As with the uniqueness grid, we shall now describe the successive stages of the proofs of Theorem $\mathcal{C}_i$, $1 \leq i \leq 7$.

Although Theorem $\mathcal{C}_1$ is a Background Result, we are including a brief discussion of it because of parallels with other cases, particularly Theorem $\mathrm{U}(\sigma)$.

Assume then that G has odd order. Since G is $\mathcal{K}$-proper, all proper simple sections are of prime order and hence all proper subgroups of G are solvable. First, define

$$\sigma^*(G) = \{p \,|\, p \text{ is a prime and } m_p(G) \geq 3\}.$$

A simple transfer argument shows that $\sigma^*(G) \neq \emptyset$. The objective of the first stage of the analysis is the following:

THEOREM $\mathcal{C}_1$: STAGE 1. *For each $p \in \sigma^*(G)$ and each subgroup $E \cong E_{p^3}$ of G, E lies in a unique maximal subgroup of G.*

In the classification grid, we abbreviate the conclusion of this theorem by saying that G has odd order uniqueness type. This is analogous in a broad sense to the hypothesis of Theorem U(σ), and shows that the classification of groups of odd order reduces in some sense to a uniqueness-type problem. However, this is only partially true, for the subsequent stages of the analysis require precise information about the structure and embedding of p-local subgroups of G for primes p for which $m_p(G) \leq 2$, so that there are also features of the analysis that have a resemblance to the quasithin group problem.

These are included in the second stage of the analysis, the goal of which is to pin down the exact structure and embedding of the maximal subgroups of G to the extent possible by local group-theoretic methods. We shall not attempt to describe the possibilities in detail here. An expository account of their structure and embedding (as well as of the subsequent stages of the Feit-Thompson analysis) appears in [**G4,** Chapter 1]. Roughly speaking, a maximal subgroup M of G is shown to "resemble" a Frobenius group (in some cases, it is a Frobenius group) and its "Frobenius kernel" is shown to be approximately disjoint from its conjugates in G. Let us then abbreviate all these conditions, once the exact possibilities for the structure and embedding of the maximal subgroups of G have been determined to the extent possible by local group-theoretic methods, by saying that G is of $\sigma^*(G)$**-uniqueness type**.

Thus one next proves

THEOREM $\mathcal{C}_1$: STAGE 2. *G is of $\sigma^*(G)$-uniqueness type.*

This is analogous to Theorem U(σ): stage 1.

The third stage of the analysis involves exceptional character theory. In this stage, all but a single possibility for the configuration of maximal subgroups are eliminated. This residual case is expressed in terms of the structure and embedding of a maximal p-local subgroup of G for a suitable prime p. In particular, it is shown that some maximal subgroup of G is not a Frobenius group.

The following definition incorporates the goal of this analysis. It represents the set of conditions which Peterfalvi uses to derive a final contradiction in his simplification of Feit and Thompson's original generator-relation analysis.

DEFINITION 17.1. The group G is of $\{p, q\}$**-parabolic type** for the odd primes p and q if and only if

(1) $((p^q - 1)/p - 1), (p - 1)) = 1$ and $p > q$;
(2) G contains a Frobenius group $H = AP$ with kernel $P \cong E_{p^q}$ and cyclic complement $A \cong Z_k$, where $k = (p^q - 1)/(p - 1)$; and
(3) There exists a subgroup P_0 of P of order p and a P_0-invariant abelian q-subgroup Q of G such that P_0^y normalizes A for some $y \in Q$.

Thus Feit and Thompson next prove, using character theory:

THEOREM $\mathcal{C}_1$: STAGE 3. *G is of $\{p, q\}$-parabolic type for suitable primes p, q.*

Finally, this conclusion is shown to be impossible [**P3**].

THEOREM $\mathcal{C}_1$: STAGE 4. *G is not of $\{p, q\}$-parabolic type for any primes p, q.*

The last two stages are somewhat analogous to the last stages of Theorem $\mathcal{C}_5$ in their focus on the interplay between two primes, and also to Stroth's strategy in Theorem U(σ): stage 2, namely, to construct a pair of subgroups and derive a contradiction by amalgam-type analysis.

18. p-terminal p-components

The analysis of most of the remaining theorems involves the notion of a p-terminal p-component in the centralizer of an element of $\mathcal{I}_p(G)$, p a prime. The notion extends that of a terminal component (Definition 1.2) in the presence of p'-core obstruction, and was defined in section 7 of the preceding chapter. We briefly recall the definition and basic properties.

First, by definition, the p-components of a group X are the minimal subnormal subgroups K of X such that the image of K in $\overline{X} = X/O_{p'}(X)$ is a component of $\overline{X}$. Each component of $\overline{X}$ is then the image of a unique p-component of X, and the product of the p-components of X is called $L_{p'}(X)$, the p-layer.

Thus the set $\mathcal{L}_p(G)$ (resp. $\mathcal{L}_p^o(G)$) consists of the groups $K/O_{p'}(K)$ as K ranges over the set of p-components of $C_G(x)$ and x ranges over $\mathcal{I}_p(G)$ (resp. $\mathcal{I}_p^o(G)$).

It follows quickly from the definition that a p-component K of X is a component of X if and only if K centralizes $O_{p'}(X)$, or equivalently if and only if K is quasisimple. Moreover, if K is a p-component of the group X, and we set $\overline{X} = X/O_{p'}(X)$, then for any $y \in \mathcal{I}_p(C_X(\overline{K}))$, $L_{p'}(C_K(y))$ maps onto $\overline{K}$. Here $C_X(\overline{K})$ is the subgroup of $N_X(K)$ acting trivially on $\overline{K}$ by conjugation.

The fundamental definition is the following:

DEFINITION 18.1. Let K be a p-component of $C_G(x)$ for $x \in \mathcal{I}_p(G)$. We say that K, or (x, K), is **p-terminal in G** if and only if the following conditions hold:

(i) For any element $y \in \mathcal{I}_p(C_{C_G(x)}(K/O_{p'}(K)))$, $L_{p'}(C_K(y))$ is contained in a p-component J of $C_G(y)$ such that $J/O_{p'}(J) \cong K/O_{p'}(K)$; and

(ii) For any $Q \in Syl_p(C_{C_G(x)}(K/O_{p'}(K)))$ and $y \in \mathcal{I}_p(Z(Q))$, if J is determined as in (i), then $Q \in Syl_p(C_{C_G(y)}(J/O_{p'}(J)))$.

Note that if $m_p(C_G(K/O_{p'}(K))) = 1$, then it is automatic that K is p-terminal in G.

We shall abuse terminology slightly by calling such a K a **p-terminal p-component (in G)**. Of course such a subgroup is a p-component not of G, but of $C_G(x)$ for some x of order p. In addition, for brevity, if K is a p-terminal p-component in G and $\overline{K} = K/O_{p'}(K)$, we shall say that $\overline{K}$ itself is p-terminal in G.

Observe that if a p-component K of some $C_G(x)$ is quasisimple, then $C_G(K) = C_G(K/O_{p'}(K))$, by an elementary property of quasisimple groups. Thus $K = L_{p'}(C_K(y))$ for each $y \in \mathcal{I}_p(C_G(K))$. We therefore have the following criterion for a p-terminal p-component to be terminal in G: If K is a p-component of $C_G(x)$ for some $x \in \mathcal{I}_p(G)$, with K p-terminal in G, then K is terminal in G if and only if K centralizes $O_{p'}(C_G(y))$ for every $y \in \mathcal{I}_p(C_G(K))$. The condition includes the case $y = x$.

As observed in the previous chapter, if we start with an element $x \in \mathcal{I}_p(G)$ and a p-component K of $C_G(x)$, it is possible by a sequence of pumpups to arrive at a p-terminal p-component K^* of $C_G(x^*)$ for some $x^* \in \mathcal{I}_p(G)$. This process is the first step of the analysis in several rows of the classification grid. Moreover, x^* and K^* may be taken to inherit certain properties of x and K. For example, because of $L_{p'}$-balance, the properties (13.1) and (13.2) imply that if $K/O_{p'}(K)$ lies in $\mathcal{G}_p$ or $\mathcal{G}_p \cup \mathcal{T}_p$, then the same holds for K^* unless $p = 2$ and $K/O(K) \cong A_n$, $9 \leq n \leq 11$. Likewise it is important to be able to assert that if $x \in \mathcal{I}_p^o(G)$, then the same holds for x^*. This is a triviality for $p = 2$, and is straightforward if p is odd and $K/O_{p'}(K) \in \mathcal{G}_p$, since then K contains an E_{p^2} subgroup disjoint from $O_{p'p}(K)$. With some more difficulty it can also be established for suitable K in the case of groups of $\mathcal{LT}_p$-type, p odd.

Finally, we note that it is useful for us to extend the notion of p-terminality in a natural way to so-called solvable p-components[5] for $p = 2$ and 3. We state the definition of these objects here, but do not give the extension. A subgroup I of $C_G(x)$ is called a **solvable p-component** if x has order $p = 2$ or 3, $I \geq O_{p'}(C_G(x))$, and in $\overline{C_G(x)} = C_G(x)/O_{p'}(C_G(x))$, $\overline{I} \cong SL_2(3)$ or $SU_3(2)$, respectively, and the normal closure of $\overline{I}$ in $\overline{C_G(x)}$ is the product of elementwise commuting $SL_2(3)$- or $SU_3(2)$-subgroups of $\overline{C_G(x)}$, one of which is $\overline{I}$. I is a **solvable component** if and only if the normal closure of I in $C_G(x)$ is the product of elementwise commuting $SL_2(3)$- or $SU_3(2)$-subgroups of $C_G(x)$, one of which is I.

19. Centralizer of involution patterns

As indicated in the preceding chapter, the term $G \approx G^*$ when G^* is a sporadic group or a low degree alternating group is expressed entirely in terms of the structure of centralizers of involutions. We repeat the definition.

Definition 19.1. The groups G and G^* have the **same involution fusion pattern** if and only if there is an isomorphism $x \mapsto x^*$ from a Sylow 2-subgroup S of G onto a Sylow 2-subgroup of G^* such that for any two involutions u, $v \in S$, u and v are conjugate in G if and only u^* and v^* are conjugate in G^*. Moreover, if this is the case, we then say that G and G^* have the **same centralizer of involution pattern** if the isomorphism can be chosen so that in addition $C_G(u) \cong C_{G^*}(u^*)$ for every involution $u \in S$.

Definition 19.2. If G^* is sporadic or $G^* \cong A_n$, $7 \leq n \leq 12$, $n \neq 8$, we write $G \approx G^*$ if and only if G and G^* have the same centralizer of involution pattern, and in the cases $G^* = A_7$ and M_{11} if and only if in addition $|G| = |G^*|$.

The additional condition in the M_{11} case is required to distinguish it from the group $L_3(3)$, which has the same centralizer of involution pattern as M_{11}. On the other hand, in the A_7 case the required order for G is in fact obtained in the process of determining the structure of the centralizer of an involution. Finally, $A_8 \cong L_4(2)$ is more easily identified as the group $L_4(2)$ of Lie type.

Although the term $G \approx G^*$ differs from the above for the remaining known simple groups, nevertheless much of the analysis of Theorem $\mathcal{C}_2$, to be described in the next section, is concerned with showing that the centralizer of involution

[5] This was done by Aschbacher for $p = 2$ [**A5**], and the case $p = 3$ is similar.

pattern in G approximates that of one of the target groups G^*, this approximation becoming sharper as one moves from stage to stage.

20. Theorem $\mathcal{C}_2$

The classification of groups of $\mathcal{LB}_2$-type (Definition 14.4) is the most elaborate of any of the results to be derived in either grid. For one thing, this case involves the most complex bifurcation pattern; for another, the conclusions to be reached at various stages of the analysis have by far the most technical statements. For simplicity, in this introductory chapter we shall give only an approximate description of the precise results to be established. This will be entirely sufficient to convey the nature of the overall proof. [A discussion of the $SL_2(q)$ subcase is given in section 25 of the preceding chapter.]

First, it is necessary to enlarge the set $\mathcal{K}^{(2)}$ slightly. We set

$$\mathcal{K}^{(2)*} = \mathcal{K}^{(2)} \cup \{L_2(4), U_3(4), J_1\}.$$

Our assumption that G is not of 2-uniqueness type eventually excludes these three added groups, but this does not occur until a late stage of the analysis.

The first stage of the analysis uses almost exclusively the techniques of 2-fusion and 2-transfer. The conclusions of this stage are incorporated into the following definition.

Definition 20.1. If G is of $\mathcal{LB}_2$-type, we say that G is of **2-terminal G^*-type** for $G^* \in \mathcal{K}^{(2)*}$ if and only if the following conditions hold:

(i) G and G^* have the same involution fusion pattern (in particular, G and G^* have isomorphic Sylow 2-subgroups); and

(ii) If z is a 2-central involution of G, then the structure of $C_G(z)/O_{2'}(C_G(z))$ closely approximates that of $C_{G^*}(z^*)/O_{2'}(C_{G^*}(z^*))$, where z^* is a 2-central involution of G^*.

The precise meaning of condition (ii) will be given in Part IV. We confine ourselves to the following remarks for now. Every $G^* \in \mathcal{K}^{(2)*}$ has only one conjugacy class of 2-central involutions, and most such G^* have only one class of involutions altogether. Furthermore, with the exception of the groups $G^* = L_2(q)$, q odd, $U_3(4)$ and M_{12}, $C_{G^*}(z^*)$ contains a component or solvable component K^* for each $G^* \in \mathcal{K}^{(2)*}$. In all cases where such a K^* exists, the precise meaning of (ii) includes the assertion that $C_G(z)$ possesses a 2-component or solvable 2-component K with $K/O_{2'}(K)$ isomorphic to K^* or at least closely approximating K^*. In these cases we shall say that G is of **2-terminal G^*-type** (z, K).

Theorem $\mathcal{C}_2$: stage 1. *If G is of $\mathcal{LB}_2$-type, then G is of 2-terminal G^*-type for some $G^* \in \mathcal{K}^{(2)*}$.*

The second stage of the analysis involves the application of the Bender method to study $O_{2'}(C_G(z))$. The sharpest results are achieved when $m_2(G) \geq 3$. In fact, when G^* is sporadic and $m_2(G) \geq 3$, namely when $G^* = M_{12}$, Mc, Ly, ON or J_1, one already reaches the conditions defining $G \approx G^*$. In particular, the case $G^* = J_1$ is eliminated at this stage of the analysis since the conditions $G \approx J_1$ immediately imply that G contains a 2-uniqueness subgroup.

The conclusions reached in the non-sporadic cases of 2-rank at least 3 are incorporated in the following definition.

DEFINITION 20.2. Assume that G is of 2-terminal G^*-type (z, K) for $G^* \in \mathcal{K}^{(2)}$ of Lie type with $m_2(G^*) \geq 3$. We say that G is of **2-central G^*-type** if and only if the following conditions hold:
(1) $L_{2'}(C_G(z)) = E(C_G(z))$; and
(2) Either $O_{2'}(C_G(z)) = 1$, or $G^* = L_4(q)$ or $U_4(q)$ and $O_{2'}(C_G(z))$ is cyclic of order dividing $q \pm 1$, respectively.

We remark that (1) follows from (2), but it is included for emphasis. Again we write **2-central G^*-type** (z, K) if we wish to specify z and K.

When G^* has 2-rank 2, the corresponding definitions focus on a maximal subgroup M of G containing $C_G(z)$, $z \in \mathcal{I}_2(G)$. By far the most elaborate analysis occurs in the $L_3(q)$ case and here too the final conclusion is by far the sharpest.

DEFINITION 20.3. If G is of 2-terminal $L_3(q)$-type, $q = p^n$, p an odd prime, we write $G \approx L_3(q)$ if and only if the following conditions hold for some maximal subgroup M of G containing $C_G(z)$:
(1) $F^*(M) = O_p(M) \cong E_{q^2}$ with $O_p(M)$ inverted by z;
(2) $C_G(z) \cong M/O_p(M)$ and $M/O_p(M)$ is isomorphic to a subgroup of $\Gamma L_2(q)$ containing $SL_2(q)$; and
(3) M contains a Sylow p-subgroup of G.

Note that M is approximately isomorphic to a maximal parabolic subgroup of G^*. From this detailed information it is easy to reach the split (B, N)-pair structure of G.

In the remaining cases, the information derived in this second stage places rather weak restrictions on the structure of $O_{2'}(M)$. We emphasize this fact by our choice of terminology. The precise definitions are technical and subdivided into several cases. For simplicity we state only the principal conclusions. Recall that M is a maximal subgroup of G containing $C_G(z)$.

DEFINITION 20.4. Assume that G is of 2-terminal G^*-type with $m_2(G^*) = 2$ and $G^* \not\cong L_3(q)$. We say that G is of **2-maximal G^*-type mod cores** if and only if the following conditions hold:
(1) $O_2(M) \neq 1$; and
(2) $L_{2'}(C_G(z)) = E(C_G(z))$.

Condition (1) easily implies that $M = C_G(z)$ unless $G^* \cong L_2(7)$, $L_2(9)$ or A_7 and $M/O_{2'}(M) \cong \Sigma_4$. For the cases $G^* = U_3(3)$ or M_{11}, condition (2) is replaced by the condition that $C_G(z)$ contain a normal Q_8-subgroup. Now at last we can state the goal of stage 2.

THEOREM $\mathcal{C}_2$: STAGE 2. *If G is of 2-terminal G^*-type for $G^* \in \mathcal{K}^{(2)*}$, then one of the following holds:*
(i) *$G^* = M_{12}$, Mc, Ly, ON or $L_3(q)$, q odd, and $G \approx G^*$;*
(ii) *$G^* \in \mathcal{C}hev$, $m_2(G^*) \geq 3$, and G is of 2-central G^*-type; or*
(iii) *$m_2(G^*) = 2$, $G^* \not\cong L_3(q)$ for any q, and G is of 2-maximal G^*-type mod cores.*

In conclusion (ii), the structure of $C_G(z)$ is essentially completely specified. On the other hand, in conclusion (iii), we are given no information about the structure and embedding in G of $O_{2'}(M)$ for the maximal subgroup M of G containing $C_G(z)$. However, detailed information of this nature is required to prove that $G \approx G^*$ in this case.

Achieving this objective involves a very long and difficult analysis involving local methods, character theory and counting arguments. Moreover, although there are many analogies and similarities among the arguments employed in the various cases, the details are sufficiently different that it is best at this point to separate the proof into cases according as (1) $G^* = L_2(q)$ or A_7; (2) $G^* = U_3(4)$; or (3) $G^* = M_{11}$ or $U_3(q)$, q odd. In all cases one determines the order of G and identifies G as a doubly transitive permutation group with a specified point stabilizer. It is in the latter phase of this analysis that the $U_3(4)$ case is eliminated by showing that G contains a 2-uniqueness subgroup.

Note also that the doubly transitive target groups in $\mathcal{K}^{(2)}$ include the groups ${}^2G_2(q)$ of 2-rank 3. Moreover, the passage from 2-central ${}^2G_2(q)$-type to double transitivity again utilizes character-theoretic methods. Because of this, it is preferable to treat the ${}^2G_2(q)$-case along with the 2-rank 2 cases. Hence at this point in the analysis, we split the groups in $\mathcal{K}^{(2)}$ of Lie type according as they have Lie rank 1 or Lie rank at least 2.

This shows the complexity of the succeeding stages of the classification of groups of $\mathcal{LB}_2$-type. The precise conclusions one reaches from the local group-theoretic analysis are extremely technical. We limit ourselves therefore to describing only the one portion of its conclusions that is uniform for all G^*, and to making a few additional comments.

We assume then that G is of 2-maximal G^*-type mod cores for some $G^* \in \mathcal{K}^{(2)*}$ of 2-rank 2 and with $G^* \not\cong L_3(q)$ for any q. With z and M as in the definition, set $Q = O_{2'}(M)$, let $t \in \mathcal{I}_2(M) - O_2(M)$ and put $B = [Q, t]$.

Then one goal of the local analysis is to verify the following conditions:

(1) B is a Hall subgroup of $F(Q)$;

(2) B is abelian and inverted by t;

(3) $N_G(B_0) \leq M$ for all $1 \neq B_0 \leq B$.

[The goal of the subsequent character-theoretic and counting analysis is then to force $B = 1$ except when the target group G^* is A_7, in which case it is to force $B \cong Z_3$.]

One of the principal alternative outcomes of the local analysis is that Q has essentially the structure of a Frobenius group with kernel B and complement A, with additional information about the embedding in G of $N_G(A_0)$ for $1 \neq A_0 \leq A$. A second alternative is that $N_G(X) \not\leq M$ for some $1 \neq X \leq F(Q)$, with specific structural information concerning $N_G(X)$.

Further analysis proceeds by means of exceptional character theory, modular character theory and counting arguments. We now describe the precise nature of the double transitivity we reach in the third stage of the analysis when G^* is of Lie rank 1. It is incorporated into the following definition.

Definition 20.5. Assume that G is either of 2-maximal G^*-type mod cores for $G^* = L_2(p^n)$, p odd, $p^n > 5$, or $U_3(p^n)$, p odd, or of 2-central G^*-type for $G^* = {}^2G_2(p^n)$, $p = 3$, $n > 1$. We say that G is of **doubly transitive G^*-type** and write $G \approx G^*$ if and only if the following conditions hold:

(1) G acts doubly transitively on a set Ω;
(2) If G_a is the stabilizer of the point $a \in \Omega$, then $G_a = N_G(P)$ for some $P \in Syl_p(G)$, and G_a has the following properties:
 (a) $|P| = p^n$, p^{3n} or p^{3n}, respectively;

(b) P acts regularly on $\Omega - \{a\}$ (whence $|\Omega| = |P| + 1$);
(c) $G_a = PY$, where Y is cyclic of the respective order $(p^n - 1)/2$, $(p^{2n} - 1)/d$, d odd, $d|(p^n + 1)$, or $p^n - 1$ with $p = 3$ and n odd, $n > 1$; and
(d) Correspondingly G_a, $G_a/\Phi(P)$ or $O^2(G_a)$ is a Frobenius group with kernel P, $P/\Phi(P)$ or P, respectively; and

(3) The following conditions hold:
(a) If $G^* = L_2(p^n)$, then Y is inverted by an involution t of G and $C_G(y) \leq Y\langle t\rangle$ for every $y \in Y^\#$;
(b) If $G^* = U_3(p^n)$, then for any involution z of G, $C_G(z) \cong GU_2(p^n)/D$, where D is a cyclic normal subgroup of order d. Moreover, for a suitable choice of z, $O_{2'}(C_G(z)) \leq Y$ and $C_G(y) \leq C_G(z)$ for every $y \in O_{2'}(C_G(z))^\#$; and
(c) If $G^* = {}^2G_2(p^n)$, then for any involution z of G, $C_G(z) \cong Z_2 \times L_2(p^n)$. Moreover, for a suitable choice of z, $z \in Y$, Y is inverted by an involution of $C_G(z)$, and $C_G(y) = Y$ for every $y \in O_{2'}(Y)^\#$.

For the groups of Lie rank ≥ 2, the definition of $G \approx G^*$ is extremely detailed, depending on the isomorphism type of G^*. We have already defined $G \approx L_3(q)$, so we just consider the remaining cases. For all the cases, $G \approx G^*$ includes a condition that G have subgroups H and N with $H \lhd N$, resembling a Cartan subgroup and monomial subgroup H^* and N^* of G^*. For the cases $G^* = L_3(q)$, $PSp_4(q)$, $G_2(q)$ and ${}^3D_4(q)$, it includes a condition that G have a p-local subgroup M resembling a certain maximal parabolic subgroup M^* of G^*, namely the normalizer of a long root group. For the cases $G^* = L_4^\pm(q)$, it includes a condition that for 2-central involutions z and z^* of G and G^*, respectively, $C_G(z)$ and $C_{G^*}(z^*)$ resemble each other. The conditions are too technical to state here, but we note that for $G^* = PSp_4(q)$, for example, the critical requirements on M are that $O_p(M)$ is special of order q^3 with center of order q; $M/O_p(M) \cong M^*/O_p(M^*)$; the central involution of $M/O_p(M)$ inverts $O_p(M)/Z$; and M contains a Sylow p-subgroup of G. We summarize all these requirements by saying that G has subgroups H, N, M and $C_G(z)$ of the same **complexion** as the corresponding subgroups of G^*.

DEFINITION 20.6. If G is of 2-central G^*-type for $G^* \in \mathcal{K}^{(2)}$ of Lie type of Lie rank at least 2 but $G^* \not\cong L_3(q)$, q odd, we set $G \approx G^*$ if and only if G has subgroups H and N of the same complexion as the corresponding subgroups of G^*; furthermore, according as $G^* = PSp_4(q)$, $G_2(q)$, ${}^3D_4(q)$ or $L_4^\pm(q)$, G has a subgroup M, M, M or $C_G(z)$, respectively, of the same complexion as the corresponding subgroup of G^*.

THEOREM $\mathcal{C}_2$: STAGE 3. *If G is of $\mathcal{LB}_2$-type, then $G \approx G^*$ for some $G^* \in \mathcal{K}^{(2)}$.*

We complete the classification of groups of $\mathcal{LB}_2$-type by turning this internal approximation into an isomorphism.

THEOREM $\mathcal{C}_2$: STAGE 4. *If G is of $\mathcal{LB}_2$-type, with $G \approx G^*$ for some $G^* \in \mathcal{K}^{(2)}$, then $G \cong G^*$.*

The sporadic and Lie rank 1 cases of the above theorem are Background Results. In the Lie rank ≥ 2 cases, one argues that the multiplication table of G is uniquely determined either because G is a split (B, N)-pair of rank 2 or by the Curtis-Tits theorem or a variation of it.

21. Theorem $\mathcal{C}_3$

Next we discuss the classification of groups of $\mathcal{LT}_2$-type (Definition 14.5). In the analysis of groups of $\mathcal{LB}_2$-type, the existence of an involution z in G whose centralizer contains a 2-component K with K 2-terminal in G was essentially a direct consequence of the conditions defining this case. In contrast, the existence of such an involution in the $\mathcal{LT}_2$-case is the entire objective of stage 1. The desired conclusion is incorporated into the following definition.

DEFINITION 21.1. If G is of $\mathcal{LT}_2$-type, we say that G is of **2-terminal $\mathcal{LT}_2$-type** if and only if for some $x \in \mathcal{I}_2(G)$, $C_G(x)$ has a 2-component K with K 2-terminal in G and $K/O_{2'}(K) \cong L_2(q)$, q odd, $q \geq 5$, or A_7.

THEOREM $\mathcal{C}_3$: STAGE 1. *If G is of $\mathcal{LT}_2$-type, then G is of 2-terminal $\mathcal{LT}_2$-type.*

In view of (13.1) and (13.2), as remarked in section 18, the difficulty in establishing this theorem centers around 2-components of centralizers of involutions that are isomorphic mod $2'$-cores to $L_2(5)$, $L_2(7)$ or $L_2(9)$, since such 2-components are also in $\mathcal{C}_2$ and hence may possess proper pumpups outside of the desired set, such as $L_2(16)$, $L_3(4)$ and $PSp_4(4)$, respectively.

The aim after stage 1 is to force K to be terminal in G and hence standard in view of Theorem TS. In the contrary case, we use signalizer functor theory to embed $C_G(x)$ in a 2-component preuniqueness subgroup M and then invoke our general results on such subgroups to show that M is a 2-uniqueness subgroup, a contradiction in view of our uniqueness theorems. Once K is standard, it then follows easily that the only possibilities for K are $L_2(5)$ ($\cong A_5$), $L_2(9)$ ($\cong A_6$), or A_7. With some additional work, we pin down the exact isomorphism type of a Sylow 2-subgroup of G and then argue that G has the same centralizer of involution pattern as A_9, A_{10}, or A_{11}, respectively, so that $G \approx G^*$ for some $G^* \in \mathcal{K}_3$, which is the goal of stage 2.

THEOREM $\mathcal{C}_3$: STAGE 2. *If G is of 2-terminal $\mathcal{LT}_2$-type, then $G \approx G^*$ for some $G^* \in \mathcal{K}^{(3)}$.*

Finally, from the Thompson order formula (Theorem 35.1 below), it follows directly that $|G| = |A_n|$, $n = 9$, 10 or 11, and we now have sufficient information to complete the identification of G.

THEOREM $\mathcal{C}_3$: STAGE 3. *If $G \approx G^*$ for some $G^* \in \mathcal{K}^{(3)}$, then $G \cong G^*$.*

This completes the classification of groups of special odd type.

22. Theorem $\mathcal{C}_4$

We now present a similar outline for groups of special even type, beginning with those of quasithin type. Because the analysis of groups of quasithin type has not been completed as of this writing, the discussion in this section must be viewed as only a tentative outline of their classification.

The goal of the first stage of the analysis is to focus on an amalgam in G, which exists by virtue of Theorem $\mathcal{M}(S)$.

DEFINITION 22.1. Let G be of quasithin type. Then we say that G is of **2-amalgam type** if and only if there are $S \in Syl_2(G)$ and 2-local subgroups M_1 and M_2 of G such that the following conditions hold:

(i) $S \leq M_1 \cap M_2$ and $N_{M_1}(S) = N_{M_2}(S)$; and

(ii) $O_2(\langle M_1, M_2\rangle) = 1$.
In this case we say that G is of **2-amalgam type** $(M_1, M_2; S)$.

THEOREM $\mathcal{C}_4$: STAGE 1. *If G is of quasithin type, then G is of 2-amalgam type.*

The second stage has two goals. The first is to reduce the number of amalgam problems that require resolution, that is, the number of possibilities for the subgroups $M_i/O_2(M_i)$, perhaps by making a different choice of the M_i. The second is to solve these amalgam problems, that is, use the amalgam method to analyze the structure of $O_2(M_i)$. In the end the M_i should have a structure closely approximating that of maximal 2-local subgroups of some $G^* \in \mathcal{K}^{(4)}$.

The precise objectives for this stage will depend on the group G^*. Let us say that G is of **2-amalgam G^*-type** for $G^* \in \mathcal{K}^{(4)}$ if and only if G is of 2-amalgam type $(M_1, M_2; S)$, and there are maximal 2-local subgroups M_1^*, M_2^* in G^* containing a common Sylow 2-subgroup S^* of G^* such that $O^{2'}(M_i)$ is approximately isomorphic to $O^{2'}(M_i^*)$, $i = 1$ and 2; and $O^{2'}(M_1 \cap M_2)$ is approximately isomorphic to $O^{2'}(M_1^* \cap M_2^*)$. The notion of "approximately" will depend on G^*.

At the minimum, this approximation asserts that chief factors of the groups in question are isomorphic, but in some cases an exact isomorphism is achieved at this stage. Moreover, when $G^* \in \mathcal{K}^{(4)}$ is alternating or sporadic, $O^{2'}(M_i) = M_i$, $i = 1, 2$, and $O^{2'}(M_1 \cap M_2) = M_1 \cap M_2$. On the other hand, when $G^* \in \mathcal{K}^{(4)}$ is of Lie type, one must, in general, allow for automorphisms of odd order acting on $O^{2'}(M_i)$ (so that, in effect, one is obtaining an approximation of G with some subgroup of $Aut(G^*)$), and it is for this reason that the conditions must be expressed in terms of the groups $O^{2'}(M_i)$.

THEOREM $\mathcal{C}_4$: STAGE 2. *If G is quasithin of 2-amalgam type, then G is of 2-amalgam G^*-type for some $G^* \in \mathcal{K}^{(4)}$.*

This condition defines the term $G \approx G^*$ when $G^* \in \mathcal{K}^{(4)}$ is of Lie type.

DEFINITION 22.2. If $G^* \in \mathcal{K}^{(4)}$ is of Lie type, then $G \approx G^*$ if and only if G is of 2-amalgam G^*-type.

When $G^* \in \mathcal{K}^{(4)}$ is alternating or sporadic, the goal is now to show that G and G^* have the same centralizer of involution pattern, whence $G \approx G^*$ in these cases as well.

THEOREM $\mathcal{C}_4$: STAGE 3. *If G is of 2-amalgam type for $G^* \in \mathcal{K}^{(4)}$, then $G \approx G^*$.*

It thus remains to turn these approximations into exact isomorphisms. In the Lie type case, this is achieved either by showing that G is a split (B, N)-pair, with a uniquely determined multiplication table, or by establishing the Curtis-Tits relations or a variant of them. [As usual, the argument depends on the fact that G does not contain a 2-uniqueness subgroup, in the sense that one constructs a subgroup $G_0 \cong G^*$, and the nonexistence of a 2-uniqueness subgroup is used to show that $G_0 = G$.] On the other hand, in the case $G^* = A_{12}$, one again has $|G| = |A_{12}|$ by the Thompson order formula and one then identifies G via the standard presentation for A_{12} in terms of generating involutions. Furthermore, in the $\mathcal{K}^{(4)}$ sporadic cases the desired isomorphism is a consequence of the Background Results.

Thus the final stage is the following.

THEOREM $\mathcal{C}_4$: STAGE 4. *If $G \approx G^*$ for $G^* \in \mathcal{K}^{(4)}$, then $G \cong G^*$.*

23. Theorem $\mathcal{C}_5$

Next we assume that G is of $\mathcal{LC}_p$-type for some $p \in \sigma_0(G)$. Then according to Definition 15.2, G is of restricted even type and every element of $\mathcal{L}_p^o(G)$ is a $\mathcal{C}_p$-group. Since $p \in \sigma_0(G)$, G does not contain a strong p-uniqueness subgroup, and $p \in \sigma(G)$.

The analysis in this case has been carried out completely in most, but not all, subcases. Thus the steps outlined here are strictly speaking provisional, though they are accurate for all the subcases which have been completely analyzed.

To state the objective of the first stage of the analysis, we need a preliminary definition.

DEFINITION 23.1. $\mathcal{B}_*^p(G)$ is the set of elementary abelian p-subgroups B of G lying in a 2-local subgroup of G and of maximal rank subject to this condition.

Since $p \in \sigma(G)$, the definition of this set of primes implies that $m_p(B) \geq 4$ for any $B \in \mathcal{B}_*^p(G)$. On the other hand, 2-locals containing a subgroup of index at most 2 in a Sylow 2-subgroup of G do not necessarily contain an element of $\mathcal{B}_*^p(G)$.

Now we can define the key term.

DEFINITION 23.2. We say that G is of **wide $\mathcal{LC}_p$-type** if and only if it is of $\mathcal{LC}_p$-type and

$$m_p(C_G(B)) > m_p(B) \text{ for every } B \in \mathcal{B}_*^p(G).$$

THEOREM $\mathcal{C}_5$: STAGE 1. *If G is of $\mathcal{LC}_p$-type, then G is of wide $\mathcal{LC}_p$-type.*

In the contrary case, if one takes $B \in \mathcal{B}_*^p(G)$ with $m_p(C_G(B)) = m_p(B)$, then as p is odd it follows rather quickly with the aid of the signalizer lemma of Bender and Thompson [**Be1; GL1,** I-20.1] that $\Theta(C_G(b)) = O_{p'}(C_G(b))$ for $b \in B^\#$ defines a nontrivial B-signalizer functor Θ on G whose closure is of even order. This implies that $\Gamma_{B,2}(G)$ is contained in a proper subgroup M of G, and now using signalizer functor theory, one argues that M is a strong p-uniqueness subgroup, contrary to assumption.

This is the point from which Klinger and Mason [**KMa1**] began their analysis of groups G of both characteristic 2- and characteristic p-type. [Thus $F^*(N)$ is a 2-group for every 2-local subgroup N of G and a p-group for every p-local subgroup N of G.] Our hypothesis that G is of $\mathcal{LC}_p$-type includes the possibility that G is of characteristic p-type, and our analysis parallels and generalizes theirs. It was shown in [**GL1**] that their arguments could be extended when the characteristic p-type condition was relaxed to allow centralizers of p-elements to have components in $\mathcal{C}hev(p)$; and now we further relax the hypotheses, expanding $\mathcal{C}hev(p)$ to $\mathcal{C}_p$ and allowing centralizers of involutions to have components in $\mathcal{C}_2$. These weaker hypotheses lead us to some target groups, whereas Klinger and Mason simply reach a contradiction, at least if some 2-local subgroup of G has p-rank at least 4, as is the case here.

The conclusion of the first phase of their analysis is that a maximal B-invariant 2-subgroup T of G is acted on faithfully by B, and is of **symplectic type**: a central product of an extra-special 2-group and a cyclic, quaternion, dihedral or semidihedral 2-group. Under the present weaker assumption that G is of even type, there are additional possibilities, and so we make the following definition.

DEFINITION 23.3. A **symplectic pair in** G is a pair (B, T) such that
(i) $B \in \mathcal{B}_*^p(G)$;
(ii) T is a maximal B-invariant 2-subgroup of G; and
(iii) Either T is of symplectic type or T has order 2.
Furthermore, if T has order 2, we call the symplectic pair (B, T) **trivial**, and if $C_B(T) = 1$, we call the pair **faithful**.

In addition to having to cope with trivial symplectic pairs we unavoidably encounter one other configuration which is mixed, and arises in the target group $G^* = \Omega_8^-(3)$. We say that a symplectic pair (B, T) is of $\Omega_8^-(3)$**-type** if and only if $p = 3$, $B \cong E_{3^4}$, $BT/C_B(BT)$ is the central product of two copies of $SL_2(3)$, and $C_G(Z(T))$ has a component $K \cong L_2(9)$.

The following definition then incorporates the goal of Stage 2.

DEFINITION 23.4. If G is of wide $\mathcal{LC}_p$-type, then we say that G is of **quasisymplectic type** if and only if $p = 3$, G has symplectic pairs, and every symplectic pair in G is trivial, faithful or of $\Omega_8^-(3)$-type.

THEOREM $\mathcal{C}_5$: STAGE 2. *If G is of wide $\mathcal{LC}_p$-type for $p \in \sigma(G)$, then G is of quasisymplectic type.*

The second phase of the Klinger-Mason analysis (Theorem D of [**KMa1**]) shows that there is no simple group of characteristic 2 type and characteristic p type which is of quasisymplectic type. On the other hand, under the present weaker even type and $\mathcal{LC}_p$-type hypothesis, there turn out to be twelve groups of quasisymplectic $\mathcal{LC}_p$-type: six of Lie type ($\Omega_7(3)$, $P\Omega_8^-(3)$, $P\Omega_8^+(3)$, $P\Omega_{10}^-(2)$, $U_7(2)$, ${}^2E_6(2)$) and six sporadic (Fi_{22}, Fi_{23}, Fi_{24}', Co_1, F_2, F_1).

For the six groups of Lie type, the next stage of the analysis consists in showing that G satisfies conditions somewhat similar to those holding in $\mathcal{L}_p$-generic groups of Lie type, the latter expressed by the condition that G contain a "neighborhood" from which the Steinberg relations for G can then be deduced. We shall use the same term for the groups in $\mathcal{K}^{(5)}$ of Lie type (see section 25). As in the generic case, we shall take the existence of such a neighborhood as definition of the term $G \approx G^*$ for $G^* \in \mathcal{K}^{(5)}$ of Lie type.

In the case of the six sporadic groups of $\mathcal{K}^{(5)}$, the following definition incorporates the objective of the bulk of the third stage of the analysis. We have chosen the term "neighborhood" here solely because of the suggestive parallel with the generic situation.

DEFINITION 23.5. Let $G^* \in \mathcal{K}^{(5)}$. If G is of quasisymplectic type, we call the configuration $(U, C_G(z), C_G(y), C_G(t))$ a $Z_6 \times Z_2$**-neighborhood of type** G^* if and only if the following conditions hold:
(1) z, y and t have orders 2, 3 and 2, respectively, and $U = \langle zy \rangle \times \langle t \rangle \cong Z_6 \times Z_2$;
(2) There exist commuting elements z^*, y^* and t^* of G^* and isomorphisms $C_G(z) \cong C_{G^*}(z^*)$, $C_G(y) \cong C_{G^*}(y^*)$ and $C_G(t) \cong C_{G^*}(t^*)$, all carrying z to z^*, y to y^* and t to t^*;
(3) z is 2-central in G with $F^*(C_G(z))$ a 2-group;
(4) $C_G(y)$ has a component K with $\langle y \rangle \in Syl_3(C_G(K))$; and
(5) $t \in O_2(C_G(z))$ and $E(C_G(t)) \neq 1$.

The existence of a $Z_6 \times Z_2$-neighborhood of type $G^* \in \mathcal{K}^{(5)}$ with $G^* \in \mathcal{S}por$ eventually yields that G and G^* have the same centralizer of involution pattern and hence that $G \approx G^*$.

Thus the full objective of stage 3 is the following.

THEOREM $\mathcal{C}_5$: STAGE 3. *If G is of quasisymplectic type, then $G \approx G^*$ for some $G^* \in \mathcal{K}^{(5)}$.*

The final stage recognizes G by means of the Steinberg relations or the Background Results.

THEOREM $\mathcal{C}_5$: STAGE 4. *If $G \approx G^*$ for some $G^* \in \mathcal{K}^{(5)}$, then $G \cong G^*$.*

24. Theorem $\mathcal{C}_6$

There is a considerable parallel between the proofs of Theorems $\mathcal{C}_3$ and $\mathcal{C}_6$. The principal distinction is that as G is now of even type, the analysis leads to a contradiction: if $p \in \sigma_0(G)$ and G is of $\mathcal{LT}_p$-type, it is shown that G has a strong p-uniqueness subgroup, contrary to the definition of $\sigma_0(G)$. We let $\sigma_{\mathcal{T}}(G)$ be the set of primes $p \in \sigma_0(G)$ such that G is of $\mathcal{LT}_p$-type. Thus for any $p \in \sigma_{\mathcal{T}}(G)$ and any $x \in \mathcal{I}_p^o(G)$, every component of $C_G(x)/O_{p'}(C_G(x))$ is in $\mathcal{C}_p$ or $\mathcal{T}_p$; and there is some x for which some such component is in $\mathcal{T}_p$. (The list of $\mathcal{T}_p$-groups is given in Definition 13.1.)

DEFINITION 24.1. We say that G is of **p-terminal $\mathcal{LT}_p$-type** if and only if there is $x \in \mathcal{I}_p^o(G)$ and a p-component K of $C_G(x)$ such that K is p-terminal in G and $K/O_{p'}(K) \in \mathcal{T}_p$.

THEOREM $\mathcal{C}_6$: STAGE 1. *If G is of restricted even type and $p \in \sigma_{\mathcal{T}}(G)$, then G is of p-terminal $\mathcal{LT}_p$-type.*

This result has been mentioned already in section 18. If $\mathcal{I}_p(G)$ were equal to $\mathcal{I}_p^o(G)$, it would be a consequence of general properties of p-components and the fact that pumpups of elements of $\mathcal{T}_p$ cannot lie in $\mathcal{C}_p$. The difficulty is due to the set $\mathcal{I}_p(G) - \mathcal{I}_p^o(G)$.

Next, using signalizer functor theory and the fact that groups in $\mathcal{T}_p$ have low p-rank, we are able to apply our p-component preuniqueness theorems to prove

THEOREM $\mathcal{C}_6$: STAGE 2. *If G is of restricted even type, then for any $p \in \sigma_{\mathcal{T}}(G)$, G contains a strong p-uniqueness subgroup.*

However, by definition $\sigma_{\mathcal{T}}(G) \subseteq \sigma_0(G)$, that is, such uniqueness subgroups do not exist for any $p \in \sigma_{\mathcal{T}}(G)$. Therefore $\sigma_{\mathcal{T}}(G) = \emptyset$, proving Theorem $\mathcal{C}_6$.

25. Vertical neighborhoods

To describe the goal of the analysis in the generic case, we begin by stating a property of the groups of Lie type in $\mathcal{K}^{(7)}$.

First, if K is any quasisimple group of Lie type, it will be convenient to call inner-diagonal-graph automorphisms of K **algebraic automorphisms**. The term is natural since these are the automorphisms arising from automorphisms of the algebraic group overlying K.

PROPOSITION 25.1. *Let $G^* \in \mathcal{K}^{(7)}$ with $G^* = {}^d\mathcal{L}(q)$ a simple group of Lie type. Then for some prime p dividing $q \pm 1$, and specifically for $p = 2$ if q is odd, G^* contains an E_{p^2}-subgroup U^* with the following properties:*

(1) *For some $x^* \in U^{*\#}$, $C_{G^*}(x^*)$ contains a component K^* with K^* a $\mathcal{G}_p$-group and K^* terminal in G^*;*
(2) *U^* leaves K^* invariant and elements of $U^* - \langle x^* \rangle$ induce algebraic automorphisms on K^*;*
(3) *$C_{K^*}(U^*)$ contains a component I^* such that the image $\overline{I}^*$ of I^* in $\overline{K^*U^*} = K^*U^*/C_{K^*U^*}(K^*)$ is terminal in $\overline{K^*U^*}$;*
(4) *For each $u^* \in U^{*\#}$, I^* is contained in a component I_{u^*} of $C_{G^*}(u^*)$; and*
(5) *For some $u^* \in U^* - \langle x^* \rangle, I^* < I_{u^*}$.*

Thus $I_{x^*} = K^*$. We call such a configuration $(U^*, I^*, \{I_{u^*} \mid u^* \in (U^*)^\#\})$ a **vertical neighborhood in** G^*. By condition (4), each pumpup I_{u^*} of I^* is vertical. The subgroup U^* is called the **base** of the neighborhood. Also each of the groups I_{u^*} is called a **neighbor** of (x^*, K^*).

[The base of a vertical neighborhood consists of semisimple elements, and usually but not always lies in a torus in the algebraic group overlying G. Nor is it true conversely that a subgroup isomorphic to E_{p^2} with $p|(q \pm 1)$, and lying in a torus, necessarily is the base of some vertical neighborhood; the terminality condition may easily fail.]

We emphasize that we require $p = 2$ if q is odd. Thus according to the definition, only groups of Lie type of characteristic 2 possess vertical neighborhoods for odd p. In fact, groups of Lie type over $\boldsymbol{F}_q$, q odd, in general also contain E_{p^2}-subgroups U^* satisfying the above conditions for suitable odd p dividing $q \pm 1$. However, as it is our intent to identify each group in $\mathcal{K}^{(7)}$ only once in the course of the analysis, we enforce this limitation on p.

Furthermore, G^* may possess more than one conjugacy class of vertical neighborhoods and the Steinberg relations for G^* may be more easily constructed from one such vertical neighborhood than from another. It is in stage 3 below that we find a preferred neighborhood.

A slight modification of the above definition allows us to include the six groups of Lie type in $\mathcal{K}^{(5)}$ as groups of Lie type possessing a vertical neighborhood. Indeed, if G^* is one of these groups, we let $p = 2$ or 3 according as the characteristic of G^* is 3 or 2, respectively. Then G^* contains a subgroup $U^* \cong E_{p^2}$ satisfying all the above conditions *except that* $K^* \notin \mathcal{G}_p$; indeed the components I_{u^*} for $u^* \in (U^*)^\#$ are $\mathcal{C}_p$-groups. Namely, for $G^* = \Omega_7(3)$, $P\Omega_8^+(3)$ or $\Omega_8^-(3)$, two I_{u^*} are $2U_4(3)$ and the third is $PSp_4(3)$, $2U_4(3)$ or $L_4(3)$, respectively; for $G^* = {}^2D_5(2)$ or ${}^2E_6(2)$, half the I_{u^*} are $D_4(2)$ and the rest are $U_4(2)$ or $U_6(2)$, respectively; and for $G^* = U_7(2)$, half the I_{u^*} are $SU_6(2)$ and the rest are $U_5(2)$.

An entirely analogous result holds for the alternating groups in $\mathcal{K}^{(7)}$. If $G^* = A_n$, $n \geq 13$, taking a four subgroup U^* generated by two disjoint products of two transpositions, we obtain three components I_{u^*} isomorphic to A_{n-4}, A_{n-4}, and A_{n-8}. Again we call the corresponding configuration a **vertical neighborhood**. Likewise the subgroup of order 4 generated by a pair of disjoint transpositions yields a vertical neighborhood in the symmetric groups Σ_n, $n \geq 9$.

The definition of a vertical neighborhood can be transferred to G in the natural way. Let G^* be a group of Lie type in $\mathcal{K}^{(5)} \cup \mathcal{K}^{(7)}$, or A_n, $n \geq 13$, or Σ_n, $n \geq 11$. Let

$(U^*, I^*, \{I_{u^*} \mid u^* \in (U^*)^\#\})$ be a vertical neighborhood in G^*. Let U be a subgroup of G such that the following conditions hold:

(1) There is an isomorphism $\alpha : U \to U^*$;
(2) $C_G(U)$ contains a component I such that $m_p(C_G(I)) = 2$;
(3) For each $u \in U^\#$, I is contained in a component I_u of $C_G(u)$; and
(4) There are isomorphisms $\pi_u : I_u U \to I_{\alpha(u)} U^*$, for each $u \in U^\#$, which coincide on I and extend α.

Under these circumstances, we say that the configuration $(U, I, I_y,\, y \in U^\#)$ is a **vertical neighborhood (of type G^*)**.

In practice, it is often the case that the relations determined by the I_{u^*} (and their intersections) are already determined by a proper subset of the I_{u^*}. Indeed, this is obviously true if one of the I_{u^*} is I^* itself. In such cases, we may suppress superfluous I_{u^*} in describing the given vertical neighborhood.

26. Theorem $\mathcal{C}_7$

In sections 22, 26 and 27 of the preceding chapter, we have described the principal moves in the classification of generic groups. Here we formalize the successive stages of the analysis.

Thus we assume that G is of generic odd type or generic even type. Moreover, in the latter case if we let $\sigma_{\mathcal{G}}(G)$ be the set of all primes $p \in \sigma(G)$ such that some element of $\mathcal{L}_p^o(G)$ is a $\mathcal{G}_p$-group, then $\sigma_{\mathcal{G}}(G) \neq \emptyset$ and G contains a q-uniqueness subgroup for every $q \in \sigma(G) - \sigma_{\mathcal{G}}(G)$, i.e., $\sigma_0(G) \subseteq \sigma_{\mathcal{G}}(G)$.

If G has generic even type we fix a prime $p \in \sigma_0(G)$, so that $p \in \sigma_{\mathcal{G}}(G)$ and G does not contain a strong p-uniqueness subgroup. If G is of generic odd type we put $p = 2$, so that by Theorem U(2), G does not contain a 2-uniqueness subgroup.

Choosing a natural ordering on the elements of $\mathcal{L}_p(G)$ that are $\mathcal{G}_p$-groups, one easily produces an element $x \in \mathcal{I}_p^o(G)$ such that $C_G(x)$ has a p-component K with $K/O_{p'}(K)$ a $\mathcal{G}_p$-group and K p-terminal in G. [The exceptions A_n, $9 \le n \le 11$, noted after (13.2) could lead to terminal $2A_n$ components, $9 \le n \le 11$, but such configurations have already been analyzed as a side problem in conjunction with the $\mathcal{LB}_2$-type case, a more natural context.]

The case in which $p = 2$ and $K/O_{2'}(K) \cong 2A_n$ ($n \ge 12$ as $K/O_{2'}(K)$ is a $\mathcal{G}_p$-group) is exceptional and is eliminated by a fusion analysis. In all the other cases, we use signalizer functor theory to eliminate p'-core obstruction, which is the aim of the first two stages of the analysis.

Indeed, in the first stage we verify that G is 3/2-balanced with respect to a suitable E_{p^3}-subgroup A of $C_G(x)$ suitably embedded in $K\langle x\rangle$ (see section 29 for the definition). Since G does not contain a (strong) p-uniqueness subgroup, it follows with the aid of our uniqueness results that $C_G(x)$ cannot be embedded in a p-component preuniqueness subgroup. This is shown in the second stage to imply that the associated functor $\Theta_{3/2}$ on E_{p^2}-subgroups of A is trivial (see section 29), which in turn leads to the elimination of p'-core obstruction.

DEFINITION 26.1. We say that G is of **3/2-balanced type with respect to x and K** if and only if

(1) $x \in \mathcal{I}_p^o(G)$ and K is a p-component of $C_G(x)$;
(2) If $p = 2$, then $K/O_{2'}(K) \not\cong 2A_n$, $n \ge 12$; and
(3) G is 3/2-balanced with respect to a suitably chosen E_{p^3}-subgroup A of $K\langle x\rangle$.

THEOREM $\mathcal{C}_7$: STAGE 1. *Let G be of generic type. Then G is of $3/2$-balanced type with respect to a suitable $x \in \mathcal{I}_p^o(G)$ and p-component K of $C_G(x)$ such that K is p-terminal in G and $K/O_{p'}(K)$ is a $\mathcal{G}_p$-group.*

The next definition describes the aim of stage 2. For uniformity, in the case that $p = 2$ and $K/O_{2'}(K) \cong A_n$ we shall call any involution of Σ_n **algebraic**. Note that $n \geq 9$ as $K/O_{2'}(K)$ is a $\mathcal{G}_p$-group.

DEFINITION 26.2. We say that G is of **semisimple type (with respect to x, y and K)** if and only if the following conditions hold:

(1) $x \in \mathcal{I}_p^o(G)$ and K is a component of $C_G(x)$;
(2) K is terminal in G;
(3) $y \in \mathcal{I}_p(C_G(x))$, y induces a nontrivial algebraic automorphism on K and $C_K(y)$ contains a component I (for $p = 2$ or 3, possibly a solvable $SL_2(3)$ or $SU_3(2)$ component) such that the image $\overline{I}$ of I in $\overline{K}\langle\overline{y}\rangle = K\langle y\rangle/C_{K\langle y\rangle}(K)$ is terminal in $\overline{K}\langle\overline{y}\rangle$; moreover, for every $u \in \langle x, y\rangle^{\#}$, $I \leq E(C_G(u))$ (with an appropriate modification in the $SL_2(3)$ and $SU_3(2)$ cases).

THEOREM $\mathcal{C}_7$: STAGE 2. *If G is of $3/2$-balanced type with respect to x and K, then for a suitable y, G is of semisimple type with respect to x, y and K.*

As described in Section 26 of the preceding chapter, Stage 3 has a number of conclusions, involving the isomorphism type of K as well as properties of a subterminal (x, K)-pair (y, I) and its associated neighborhood $\mathcal{N} = (\langle x, y\rangle, I, \{I_u \mid u \in \langle x, y\rangle^{\#}\})$, where I_u denotes the pumpup of I in $C_G(u)$. These properties are expressed in terms of the definition of vertical neighborhood and, when K is of Lie type, of the definitions of both a level neighborhood and a splitting prime. Moreover, when G is of even type (whence necessarily K is of Lie type of characteristic 2), a choice of the prime $p \in \sigma_0(G)$ is necessary to achieve splitting.

We shall not repeat the rather detailed conclusions of Stage 3, but for brevity shall refer to them by saying that G is of **proper semisimple type** (with respect to $\mathcal{N}$). Thus in Stage 3, we prove

THEOREM $\mathcal{C}_7$: STAGE 3. *There exists a prime $p \in \sigma(G)$ and a p-terminal $\mathcal{G}_p$-pair (x, K) possessing a subterminal pair (y, I) and associated neighborhood $\mathcal{N}$ such that G is of proper semisimple type with respect to $\mathcal{N}$.*

We define the **span of** $\mathcal{N}$ to be the subgroup

$$G_o(\mathcal{N}) = \langle I_u \mid u \in \langle x, y\rangle^{\#}\rangle.$$

Likewise as described in the preceding chapter, the goal of Stage 4 is to prove

THEOREM $\mathcal{C}_7$: STAGE 4. *If G is of proper semisimple type with respect to the neighborhood $\mathcal{N}$, then $G_o(\mathcal{N}) \cong G^*$ for some $G^* \in \mathcal{K}^{(7)}$.*

We take the conclusion of Stage 4 as the definition of the term $G \approx G^*$.

Finally, using 2-uniqueness results, we prove

THEOREM $\mathcal{C}_7$: STAGE 5. *If $G \approx G^*$ for some $G^* \in \mathcal{K}^{(7)}$, then $G \cong G^*$.*

E. Principal Techniques of the Proof

27. Fusion

We briefly describe the principal methods underlying the proof of the Classification Theorem. We make no attempt to be comprehensive, but rather single out some of the key techniques that are basic for the analysis. Since the purpose here is solely to illuminate the proof, we shall omit the references needed to justify our assertions. Throughout, as always, G is a $\mathcal{K}$-proper simple group.

We begin with fusion. Fusion analysis is most critical for the prime 2 in the study of groups of special odd type, but certain general consequences of the Alperin-Goldschmidt conjugation theorem are critical for other primes and in other cases.

The first two results concern 2-fusion. Fix $S \in Syl_2(G)$.

Theorem 27.1. (Glauberman's Z^*-theorem) *If $z \in \mathcal{I}_2(S)$, then $z^g \in S - \langle z \rangle$ for some $g \in G$.*

Theorem 27.2. (Extremal conjugation) *Suppose $T \lhd S$ with S/T cyclic. If $y \in \mathcal{I}_2(S - T)$, then the following conditions hold for some $g \in G$:*
(i) *$y^g \in T$;*
(ii) *$C_S(y^g) \in Syl_2(C_G(y^g))$; and*
(iii) *$C_S(y)^g \leq C_S(y^g)$.*

This is a version of the Thompson transfer lemma. Involutions x of S for which $C_S(x) \in Syl_2(C_G(x))$ are said to be **extremal in** S (with respect to G).

Next we state the Alperin-Goldschmidt conjugation theorem.

Theorem 27.3. *Let $P \in Syl_p(G)$, and let A be a subset of P. If $g \in G$ and $A^g \leq P$, then there exist subgroups D_i of P and elements $g_i \in N_i = N_G(D_i)$, $1 \leq i \leq n$, for some positive integer n, satisfying*
(i) *$C_G(D_i) \leq O_{p'p}(N_i)$, $1 \leq i \leq n$;*
(ii) *$O_{p'p}(N_i) = O_{p'}(N_i) \times D_i$, $1 \leq i \leq n$;*
(iii) *$N_i \cap P \in Syl_p(N_i)$, $1 \leq i \leq n$;*
(iv) *$A \subseteq D_1$ and $A^{g_1 g_2 \cdots g_j} \subseteq D_{j+1}$, $1 \leq j \leq n-1$; and*
(v) *$A^g = A^{g_1 g_2 \cdots g_n}$, indeed $g = cg_1 \cdots g_n$ for some $c \in C_G(A)$.*

The theorem asserts that all G-fusion in P can be "factored" as a product of conjugations within p-local subgroups of restricted types. Furthermore, as $Z(P) \leq C_G(D_i)$, (i) and (ii) imply that $Z(P) \leq D_i$ for each i.

It is this last fact that enables one to establish the following corollary of the theorem.

Theorem 27.4. *Let M be a subgroup of G containing P and z an element of order p in $Z(P)$. If $C_G(z) \leq M$ and M controls the G-fusion of z in P, then the following conditions hold:*
(i) *If Q is a p-subgroup of G containing z, then $N_G(Q) \leq M$; and*
(ii) *M controls G-fusion in P.*

[The definitions of control of fusion appear in Section 4.]

28. The Bender method

Underlying the Bender method is the Bender uniqueness theorem, a basic result which is critical in several parts of the classification proof, including the analysis of groups of $\mathcal{LB}_2$-type and the proofs of the solvable signalizer functor theorem and the Odd Order Theorem. Recall that for any group X, the generalized Fitting subgroup $F^*(X)$ is the product of the layer $E(X)$ and the Fitting subgroup $F(X)$ of X, that is, the product of all components and normal p-subgroups of X. Furthermore, for any set of primes π, $O_\pi(X)$ is the unique largest normal π-subgroup of X, and π' is the set of all primes not in π. We again limit the discussion to G.

DEFINITION 28.1. Let M and N be maximal subgroups of G. Then $M \rightsquigarrow N$ if and only if there is $1 \neq Q \leq F(M)$ such that $N_{F^*(M)}(Q) \leq N$.

THEOREM 28.2. (Bender) *Let M and N be maximal subgroups of G such that $M \rightsquigarrow N$, and let π be the set of primes dividing $|F(M)|$. Then*
(i) *If $|\pi| > 1$, then $O_\pi(F(N)) \leq M$;*
(ii) *$O_{\pi'}(F(N)) \cap M = 1$; and*
(iii) *If also $N \rightsquigarrow M$, then either $M = N$ or both $F^*(M)$ and $F^*(N)$ are p-groups for the same prime p.*

To illustrate the Bender method, we consider the analysis of the $\mathcal{LB}_2$-case. We fix an involution z and consider maximal subgroups M of G containing $C_G(z)$. The analysis divides naturally into three cases:
(1) For some such M, $F^*(M)$ has even order;
(2) For any such M, $F^*(M)$ has odd order but not prime power order;
(3) For some such M, $F^*(M)$ is a p-group for some odd prime p.

In case (1), the fact that G is $\mathcal{K}$-proper together with the Bender-Suzuki theorem yields in almost all cases that $M = C_G(z)$. This in turn leads easily to the B_2-property for $C_G(z)$, i.e., $L_{2'}(C_G(z)) = E(C_G(z))$.

The specific analysis in case (2) depends on the nature of a Sylow 2-subgroup S of G; for simplicity let us assume here that S is dihedral, a case analyzed by Bender. An examination of $\mathcal{K}$-groups shows that if N is any maximal subgroup of G containing z, then N has the following critical property: for every $C_N(z)$-invariant subgroup W of N of odd order, $[z, W] \leq O(N)$. Bender chooses a suitable M and, using this property, makes a sequence of applications of the uniqueness theorem. Eventually it applies to M and M^g, where $g \in G$ is chosen so that $s = z^g$ and sz both centralize z and are G-conjugate but not M-conjugate to z. (The existence of g follows from fusion analysis and the Bender-Suzuki theorem.) The uniqueness theorem yields that $M = M^g$ so that $g \in M$, a contradiction since z and s are not M-conjugate.

Thus, as is typical of the method, we are reduced to the case that a suitably chosen maximal subgroup M of G has the property that $F^*(M) = O_p(M)$ for some odd prime p. Unfortunately, in the presence of quadratic modules, this case has been amenable to successful analysis only under very restricted assumptions. The definition of a group of $\mathcal{LB}_2$-type in fact incorporates the extent to which we have been able to push through this final case, whose elaborate resolution accounts in part for the length of the analysis of groups of $\mathcal{LB}_2$-type. It is from this analysis that the parabolic structure of G emerges leading to the case $G \approx L_3(p^n)$, p odd. Fortunately, the signalizer functor method for the prime 2 is applicable when G is not of $\mathcal{LB}_2$-type.

29. Signalizer functors and k-balanced groups

The signalizer functor method is built on two theories. One consists of the signalizer functor theorems, from the basic solvable signalizer functor theorem to McBride's extension to the nonsolvable case (Theorem 29.2 below). The other, developed by Gorenstein and Walter, concerns the existence of k-balanced signalizer functors for various values of k. The signalizer functor method is utilized for all primes p in both the generic and special cases.

We fix a prime p and a positive integer k. First denote by $\mathcal{E}^p(X)$ the set of elementary abelian p-subgroups of the group X and by $\mathcal{E}^p_k(X)$ the subset of $\mathcal{E}^p(X)$ consisting of those elements of rank $\geq k$.

Definition 29.1. If $A \in \mathcal{E}^p(G)$, an **A-signalizer functor on G** is a correspondence Θ associating to each $a \in A^\#$ an A-invariant p'-subgroup $\Theta(C_G(a))$ of $C_G(a)$ such that

$$\Theta(C_G(a)) \cap C_G(a') = \Theta(C_G(a')) \cap C_G(a) \qquad \text{for all } a,\, a' \in A^\#.$$

Furthermore Θ is said to be **solvable** if each $\Theta(C_G(a))$ is solvable and to be **trivial** if each $\Theta(C_G(a)) = 1$. In addition, the **closure** $\Theta(G;A)$ of Θ in G is defined to be

$$\Theta(G;A) = \langle \Theta(C_G(a)) \,|\, a \in A^\# \rangle.$$

The signalizer functor Θ is said to be **closed** if and only if its closure is a p'-group; it is said to be **complete** if and only if it is closed and in addition, $\Theta(C_G(a)) = C_{\Theta(G;A)}(a)$ for all $a \in A^\#$.

Note that for $p = 2$, each $\Theta(C_G(a))$, being a $2'$-group, is of odd order and so is solvable. Hence in this case all A-signalizer functors on G are necessarily solvable.

Theorem 29.2. (Signalizer functor theorem) *If $A \in \mathcal{E}^p_3(G)$ and Θ is an A-signalizer functor on G, then $\Theta(G;A)$ is a p'-group, and moreover Θ is complete.*

Remark. In the solvable case, the theorem in this generality is due to Glauberman. In the general case, the only known proof uses the hypothesis that G is a $\mathcal{K}$-proper group in case Θ is not solvable, the case dealt with by McBride. The reason is that the proof requires certain properties for automorphisms of order p of simple sections of G of order prime to p; in fact these necessary properties are enjoyed by all simple $\mathcal{K}$-groups of order prime to p.

Now we turn to the second pillar of the method, the k-balanced signalizer functors. For any group X and any $B \in \mathcal{E}^p(X)$, define $\Delta_X(B)$ by

$$\Delta_X(B) = \bigcap_{b \in B^\#} O_{p'}(C_X(b)).$$

Definition 29.3. If $A \in \mathcal{E}^p_{k+1}(G)$, we say that G is **k-balanced with respect to A** if and only if for each $B \in \mathcal{E}^p_k(A)$ and each $a \in A^\#$

$$\Delta_G(B) \cap C_G(a) \leq O_{p'}(C_G(a));$$

and we say that G is **weakly k-balanced with respect to A** if and only if for each such B and a

$$[\Delta_G(B) \cap C_G(a), A] \leq O_{p'}(C_G(a)).$$

The following result shows that the existence of A's with either of these properties is sufficient for most applications to yield signalizer functors on G.

THEOREM 29.4. *Let $A \in \mathcal{E}^p_{k+2}(G)$, k a positive integer, and assume that G is either k-balanced or weakly k-balanced with respect to A. Correspondingly, for each $a \in A^\#$, define $\Theta(C_G(a))$ as follows:*

$$\Theta(C_G(a)) = \langle \Delta_G(B) \cap C_G(a) \mid B \in \mathcal{E}^p_k(A)\rangle;\ \textit{or}$$
$$\Theta(C_G(a)) = \langle \Delta_G(B) \cap C_G(a) \mid B \in \mathcal{E}^p_k(A)\rangle O_{p'}(C_G(A)).$$

Then Θ is an A-signalizer functor on G.

We refer to Θ as **the k-balanced** or **weakly k-balanced A-signalizer functor on G**.

Thus in either case the signalizer functor theorem implies that Θ is complete, and in particular that the closure $\Theta(G; A)$ of A is a p'-group.

In the k-balanced case, one can easily establish that

$$\Theta(G; A) = \langle \Delta_G(B) \mid B \in \mathcal{E}_k(A)\rangle.$$

Similarly, for any $E \in \mathcal{E}_{k+1}(A)$,

$$\Theta(G; A) = \langle \Delta_G(B) \mid B \in \mathcal{E}_k(E)\rangle = \Theta(G; E). \tag{29.1}$$

However, $N_G(E)$ obviously normalizes the right side of this equation, so we have

THEOREM 29.5. *Let $A \in \mathcal{E}^p_{k+2}(G)$, k a positive integer, and assume that G is k-balanced with respect to A. Let Θ be the k-balanced A-signalizer functor on G. Then*

$$\Gamma_{A,k+1}(G) \le N_G(\Theta(G; A)). \tag{29.2}$$

On the other hand, because of the presence of the factor $O_{p'}(C_G(A))$ in each $\Theta(C_G(a))$ in the weakly k-balanced case, it is not possible to verify (29.1) in that case and thus obtain an analogue of Theorem 29.5. However, there exists an effective substitute for Theorem 29.5 under slightly stronger conditions, developed by Goldschmidt and Aschbacher. The conditions are incorporated into the following definition.

DEFINITION 29.6. If $A \in \mathcal{E}^p_{k+1}(G)$, we say that G is **$(k + \frac{1}{2})$-balanced with respect to** A if and only if G is both weakly k-balanced and $(k+1)$-balanced with respect to A.

Moreover, suppose that $A \in \mathcal{E}^p_{k+2}(G)$ and G is $(k + \frac{1}{2})$-balanced with respect to A. For each $E \in \mathcal{E}^p_{k+1}(A)$, set

$$\Theta_{k+\frac{1}{2}}(G; E) = \langle \Delta_G(E), [\Delta_G(D), E] \mid D \in \mathcal{E}^p_k(E)\rangle. \tag{29.3}$$

We refer to $\Theta_{k+\frac{1}{2}}$ as **the associated $(k + \frac{1}{2})$-balanced functor on** G.

There is no signalizer functor $\Theta_{k+\frac{1}{2}}$, but we use the notation $\Theta_{k+\frac{1}{2}}(G; E)$ because of the following analogy with (29.1), which is proved by a direct generalization of an argument made originally by Aschbacher in the case $p = 2$, $k = 1$.

$$\Theta_{k+\frac{1}{2}}(G; A) = \Theta_{k+\frac{1}{2}}(G; E) \text{ for any } E \in \mathcal{E}^p_{k+1}(A). \tag{29.4}$$

However, it is immediate from the definition of $\Theta_{k+\frac{1}{2}}(G;E)$ that it is invariant under $N_G(E)$. Thus we have

THEOREM 29.7. *If $A \in \mathcal{E}^p_{k+2}(G)$, k a positive integer, and G is $(k+\frac{1}{2})$-balanced with respect to A, then*

$$\Gamma_{A,k+1}(G) \leq N_G(\Theta_{k+\frac{1}{2}}(G;A)).$$

Though $\Theta_{k+\frac{1}{2}}(G;A)$ does not arise from an A-signalizer functor on G, we have by its definition and from weak k-balance the condition

$$\Theta_{k+\frac{1}{2}}(G;A) \leq \Theta(G;A), \tag{29.4}$$

where Θ denotes, as above, the weakly k-balanced A-signalizer functor on G. As Θ is an A-signalizer functor on G, it follows from Theorem 29.2 and (29.4) that $\Theta_{k+\frac{1}{2}}(G;A)$ is a p'-group, which is the critical use made of the signalizer functor theorem. Thus Theorem 29.7 is indeed an effective substitute for Theorem 29.5, allowing one to extend the signalizer functor method to the case of $(k+\frac{1}{2})$-balance. The value of this is that for the applications, $(k+\frac{1}{2})$-balance is a significantly weaker condition to verify than k-balance, for a given value of k.

It is thus critical to be able to verify either k-balance or $(k+\frac{1}{2})$-balance with respect to A. As shown by Gorenstein and Walter, this verification can be reduced to checking local properties of certain elements of $\mathcal{L}_p(G)$, namely the components of $C_G(a)/O_{p'}(C_G(a))$ as a ranges over $A^{\#}$. For convenience, denote this subset of $\mathcal{L}_p(G)$ by $\mathcal{L}_p(G;A)$.

DEFINITION 29.8. Let K be a quasisimple $\mathcal{K}$-group. We say that K is **locally k-balanced** or **weakly locally k-balanced** if and only if for each H such that $Inn(K) \leq H \leq Aut(K)$, each $E \in \mathcal{E}^p_k(H)$ and each $x \in \mathcal{I}_p(C_H(E))$, we have respectively

$$\Delta_H(E) = 1 \text{ or}$$
$$[\Delta_H(E), x] = 1$$

Furthermore, we say that K is **locally $(k+\frac{1}{2})$-balanced** if K is both locally $(k+1)$-balanced and weakly locally k-balanced.

The following result shows the connection between local balance and balance.

THEOREM 29.9. *Let $A \in \mathcal{E}^p_{k+2}(G)$, k a positive integer. If each element of $\mathcal{L}_p(G;A)$ is locally k-balanced, then G is k-balanced with respect to A. If each element of $\mathcal{L}_p(G;A)$ is locally $(k+\frac{1}{2})$-balanced and A-invariant, then G is $(k+\frac{1}{2})$-balanced with respect to A.*

REMARKS. A more practical version of the theorem holds, with the hypothesis that the elements of $\mathcal{L}_p(G;A)$ satisfy only a weaker form of local k-balance or $(k+\frac{1}{2})$-balance, relativized to A. The theorem also indicates that verification of $(k+\frac{1}{2})$-balance requires additional analysis in the presence of components not left invariant by A.

30. $L_{p'}$-balance, the B_p-property and pumpups

These topics were introduced in sections 6 and 7 of the previous chapter. We make some further remarks here, following the work of Gorenstein and Walter. Throughout this section, p is a fixed prime. Combining the fact that G is $\mathcal{K}$-proper with $L_{p'}$-balance and the B_p-property, we obtain:

THEOREM 30.1. *Let K be a p-component of $C_G(x)$, $x \in \mathcal{I}_p(G)$, $y \in \mathcal{I}_p(C_G(x))$, $L = \langle K^{\langle y \rangle} \rangle$, and let I be a p-component*[6] *of $C_L(y)$. Then the following conditions hold:*

(i) *$I \le L_{p'}(C_G(y))$;*

(ii) *Set $\overline{C_G(y)} = C_G(y)/O_{p'}(C_G(y))$ and let J be the normal closure of I in $L_{p'}(C_G(y))$. Then $\overline{I}$ is a component of $C_{\overline{J}}(\overline{x})$, and one of the following holds:*

  (a) *$\overline{I} = \overline{J}$;*

  (b) *J is a single x-invariant p-component of $C_G(y)$ with $\overline{I} < \overline{J}$; or*

  (c) *J is the product of p isomorphic p-components $J_1, \ldots, J_p$ of $C_G(y)$ cycled by x, with $I/O_{p'}(I)$ a homomorphic image of each $J_i/O_{p'}(J_i)$.*

In (ii) one can say "component" rather than "p-component" by virtue of the B_p-property. The pumpup J of I is trivial, vertical and proper, or diagonal according as (a), (b), or (c) holds. When I covers $K/O_{p'}(K)$, i.e., $y \in C_{C_G(x)}(K/O_{p'}(K))$, then J is also called a pumpup of K itself.

The crucial significance of this result is that one is able to determine the possible isomorphism types of $\overline{J}$ from that of $\overline{I}$. Namely, if $\overline{J}$ is quasisimple, then one may use the known tables of components of the centralizers of automorphisms of order p of quasisimple $\mathcal{K}$-groups; if $\overline{J}$ is not quasisimple, then each component of $\overline{J}$ is isomorphic to a covering group of $\overline{I}$. This fact underlies the passage to p-terminal components described in section 18 as well as innumerable specific calculations.

Of course, if $J \le E(C_G(y))$, then J itself is a product of components of $C_G(y)$ and the B_p-property implies that I is quasisimple and a component of $C_J(x)$.

There exists an important related result, known as $L^*_{p'}$-balance, that describes the embedding of $O_{p'}(C_G(x)) \cap C_G(y)$ in $C_G(y)$. To state it, we need some further notation.

First, observe that if X is any group with $O_{p'}(X) = 1$, p a prime, then clearly $F^*(X) = E(X)O_p(X)$ and $E(X) = L_{p'}(X)$.

DEFINITION 30.2. Let X be a group with $O_{p'}(X) = 1$, and let P be a Sylow p-subgroup of $E(X)$. We define $L^*_{p'}(X)$ to be $E(X)O_{p'}(C_X(P))$. More generally, if $O_{p'}(X) \ne 1$, we define $L^*_{p'}(X)$ to be the full preimage in X of $L^*_{p'}(X/O_{p'}(X))$.

Note that it is immediate from the definition that $L^*_{p'}(X)$ normalizes each p-component of X. Moreover, by a Frattini argument, the definition is independent of the choice of P.

THEOREM 30.3. ($L^*_{p'}$-balance) *Let X be a $\mathcal{K}$-group and N a p-local subgroup of X. Then $L^*_{p'}(N) \le L^*_{p'}(X)$.*

[6]Thus either $L = K$ (when y leaves K invariant) or L is the product of p isomorphic p-components of $C_G(x)$ cycled by y; and in the latter case, I is a "diagonal" of L, so that $I/O_{p'p}(I)$ projects isomorphically onto each component of $L/O_{p'p}(L)$, and $I/O_{p'}(I)$ is isomorphic to a homomorphic image of $K/O_{p'}(K)$.

Like $L_{p'}$-balance, this has consequences for pairs of commuting elements of G of order p, an important example of which is the following.

THEOREM 30.4. *If x and y are commuting elements of G of order p and we set $D = O_{p'}(C_G(x)) \cap C_G(y)$ and $\overline{C_G(y)} = C_G(y)/O_{p'}(C_G(y))$, then the following conditions hold:*
(i) *$D \leq L^*_{p'}(C_G(y))$;*
(ii) *D normalizes every p-component of $C_G(y)$;*
(iii) *$\overline{D}$ centralizes every component of $\overline{C_G(y)}$ that is either centralized by $\overline{x}$ or not normalized by $\overline{x}$; and*
(iv) *$\overline{D}$ centralizes $O_p(\overline{C_G(y)})$.*

If $\overline{D} \neq 1$, then by the F^*-theorem there is a component $\overline{K}$ of $\overline{C_G(y)}$ not centralized by $\overline{D}$. By the theorem $\overline{N} = \overline{KD\langle x\rangle}$ is a group with $\overline{K} \lhd \overline{N}$ and $\overline{x}$ not centralizing $\overline{K}$. Hence if we set $N^* = \overline{N}/C_{\overline{N}}(\overline{K})$, then K^* is simple, $N^* \leq Aut(K^*)$, $D^* \neq 1$, and $x^* \neq 1$. Moreover, as $D \leq O_{p'}(C_{C_G(y)}(x))$, one can show that $D^* \leq O_{p'}(C_{N^*}(x^*))$ and consequently according to the definition K^* is not locally 1-balanced with respect to $\langle x^*\rangle$. Thus K^* is limited to the simple $\mathcal{K}$-groups that are not locally 1-balanced for the prime p, and the structure of D^* is limited by that of $O_{p'}(C_{N^*}(x^*))$, which is always small in some sense of the word, because K is a $\mathcal{K}$-group.

31. The signalizer functor method

We now explain the objectives of the signalizer functor method as it relates to the $\mathcal{K}$-proper simple group G. In brief, *trivial* signalizer functors lead to the elimination of p'-core obstruction, while *nontrivial* signalizer functors lead to the existence of p-uniqueness subgroups.

This principle is most evident when $m_p(G) \geq 3$ and every element of $\mathcal{L}_p(G)$ is locally balanced (i.e., locally 1-balanced) for the prime p—for instance, when $\mathcal{L}_p(G)$ is empty. In that case G is balanced with respect to every $A \in \mathcal{E}^p_3(G)$. Using the signalizer functor theorem, one obtains the following result rather easily:

THEOREM 31.1. *Let p be a prime such that $m_p(G) \geq 3$. If every element of $\mathcal{L}_p(G)$ is locally balanced for p and A is any element of $\mathcal{E}^p_3(G)$, then according as the 1-balanced A-signalizer functor Θ on G is trivial or nontrivial, we have:*
(i) *$\Theta(G; A) = 1$ and $O_{p'}(C_G(x)) = 1$ for all $x \in \mathcal{I}_p(G)$ with $m_p(C_G(x)) \geq 3$; or*
(ii) *$\Theta(G; A) \neq 1$, $M = N_G(\Theta(G; A))$ is a proper subgroup of G, and for any $P \in Syl_p(G)$ containing A, $\Gamma^o_{P,2}(G) \leq M$.*

We remark once again that when $p = 2$ and $m_2(G) \geq 3$, the Thompson transfer lemma implies directly that $m_2(C_G(x)) \geq 3$ for every $x \in \mathcal{I}_2(G)$, so that if (i) holds, then $O(C_G(x)) = 1$ for every involution x of G. In particular, under the hypotheses of the theorem, we conclude that either every 2-component of $C_G(x)$ is quasisimple for every $x \in \mathcal{I}_2(G)$ or M is a 2-uniqueness subgroup. For p odd the conclusions are still very strong as well.

Application of the signalizer functor method is more elaborate in the presence of elements of $\mathcal{L}_p(G)$ that are not locally balanced for p. In the first place, the analysis becomes "relativized", centered around a suitably chosen element $x \in \mathcal{I}_p(G)$ and p-component K of $C_G(x)$ with K p-terminal in G. The pair x and K are found via the pumping up process, in practice K being chosen so that $K/O_{p'}(K)$ is maximal in a suitable ordering of the elements of $\mathcal{L}_p(G)$. Usually this ordering yields in particular that $K/O_{p'}(K)$ is "well-generated" with respect to elementary abelian p-subgroups of $C_G(x)$ acting on it.

One then argues that there is $A \in \mathcal{E}^p(C_G(x))$ with $m_p(A) \geq k+2$ for some positive integer k for which G is either k-balanced or $(k+\frac{1}{2})$-balanced with respect to A. Moreover, one chooses A so that k is as small as possible (the best one can hope for in general is $k=1$ with G being $\frac{3}{2}$-balanced with respect to A). Then

$$\Gamma_{A,k+1}(G) \leq N_G(\Phi(G;A)), \tag{31.1}$$

where Φ is either the k-balanced A-signalizer functor Θ or the associated $(k+\frac{1}{2})$-balanced functor $\Theta_{k+\frac{1}{2}}$ on G, respectively.

The goal of the analysis is entirely similar to that in the 1-balanced case above, relativized to x and K: If $\Phi(G;A)=1$, the aim is to show that K and certain of its neighbors are semisimple, while if $\Phi(G;A) \neq 1$, the aim is to show that $M = N_G(\Phi(G;A))$ is a p-component preuniqueness subgroup containing $C_G(x)$. The value of k is important here, for a crucial step in verifying that $C_G(x) \leq M$ is the inclusion $K \leq M$, which will follow from (31.1) provided $K/O_{p'}(K)$ is well enough generated with respect to A, namely,

$$K \leq \Gamma_{A,k+1}(G).$$

The smaller the value of k, the fewer "non-generated" counterexamples K to this desired condition there will be. On the other hand, the smaller the value of k, the less likely it is for k-balance or $k+\frac{1}{2}$-balance to hold; the choice of k is therefore a compromise dictated by the specific nature of the elements of $\mathcal{L}_p(G;A)$. The problem with many of the groups in $\mathcal{T}_p$ is that there is no satisfactory value of k.

32. Near components, pushing up, and failure of factorization

In the next three sections we describe some general techniques that are central to the 2-local analysis of groups of even type. The results of this and the next section are crucial in the study of the quasithin case as well as in certain uniqueness theorems: Theorem U(σ) and Theorem $\mathcal{M}(S)$.

We refer the reader to section 10 for the definitions connected with near components. We first describe the general context that leads to the existence of a near component. It is the existence of a subgroup M of G (again we limit the discussion to our $\mathcal{K}$-proper simple group G) which "controls" many 2-local subgroups of G and a second subgroup X of G which represents an obstruction to the control of further 2-local data by M. The conditions on X are the following:

(1) $X \not\leq M$;
(2) $F^*(X) = O_2(X)$;
(3) There is $T \in Syl_2(X)$ with $T \leq M$;
(4) $N_G(T_o) \leq M$ for all $1 \neq T_o$ char T; and
(5) $C(X,T) < X$.

Here $C(X,T)$ is defined by $C(X,T) = \langle N_X(T_0) \mid 1 \neq T_0 \text{ char } T\rangle$, and condition (5) is an obvious consequence of the other conditions.

Aschbacher has established the following general result [**A11, A12**], known as the **local $C(X,T)$-theorem**, about groups X satisfying conditions (2) and (5). For us, as G is $\mathcal{K}$-proper, his result is only required under the additional assumption that X is a $\mathcal{K}$-group.

THEOREM 32.1. *Assume that X is a $\mathcal{K}$-group with $F^*(X) = O_2(X)$. Then X is the product of $C(X,T)$ and all the linear and alternating near components K of X such that $K/O_2(K) \cong L_2(2^n)$, $n \geq 2$, or A_{2^n+1}, $n \geq 1$. Consequently, if $C(X,T) < X$, then X possesses such a near component.*

The proof of Theorem 32.1 involves two techniques that are fundamental for the analysis of groups of even type. The first step of the proof concerns failure of Thompson factorization and quadratic modules, introduced in section 8 of the previous chapter.

Let

$$\mathcal{A}(T) = \{A \mid A \leq T,\ A \cong E_{2^n},\ n = m_2(T)\} \text{ and}$$
$$J(T) = \langle A \mid A \in \mathcal{A}(T)\rangle.$$

$J(T)$ is called the Thompson subgroup of T. Also set $Z_1(T) = \Omega_1(Z(T))$. Then $Z_1(T)$ and $J(T)$ are each characteristic in T and so by our assumption on $C(X,T)$, certainly

$$C_X(Z_1(T))N_X(J(T)) < X. \tag{32.1}$$

If we set $V = \langle Z_1(T)^X\rangle$, then the condition $F^*(X) = O_2(X)$ implies that V is an elementary abelian normal 2-subgroup of X. Furthermore, by (32.1), if we set $C = C_X(V)$, it follows by a Frattini argument that $J(T) \not\leq C$. Indeed if we put $\overline{X} = X/C$, the following conditions quickly follow for some $A \in \mathcal{A}(T)$:

(1) V is a faithful $\overline{X}$-module;
(2) $O_2(\overline{X}) = 1$;
(3) $\overline{A} \neq 1$ and $\overline{A}$ is elementary abelian; and
(4) $|\overline{A}||C_V(\overline{A})| \geq |V|$,

with (2) following from the definition of V, and (4) following because $A \in \mathcal{A}(T)$ has the consequences $|A| \geq |(A \cap C)V|$ and $V \cap A = C_V(A) = C_V(\overline{A})$. Therefore V is a failure of factorization $\overline{X}$-module, by definition. Furthermore, an application of the Thompson Replacement Theorem shows that for a suitable choice of A, the following condition also holds:

$$[V, \overline{A}, \overline{A}] = 1,$$

so that V is quadratic with respect to $\overline{A}$. This is the situation that is first analyzed in the proof of Theorem 32.1.

When $F^*(\overline{X})$ is solvable, Thompson's original analysis, refined by Glauberman, shows that $SL_2(2) \cong \Sigma_3$ is involved in $\overline{X}$ and describes precisely how it is involved. When $E(\overline{X}) \neq 1$, the following basic result originally obtained by Aschbacher is used for a first reduction in the structure of $\overline{X}$. [The proof can now take advantage of the recent analysis of quadratic modules for $\mathcal{K}$-groups by Meierfrankenfeld and Stroth [**MeSt1**].]

THEOREM 32.2. *Let Y be a group with $F^*(Y) \in \mathcal{K}$. If Y possesses a failure of factorization module W, then $F^*(Y) \in \mathcal{C}hev(2) \cup \mathcal{A}lt$.*

Failure of factorization modules as well as quadratic modules have been extensively studied, and the possibilities for W have been determined for various choices of the $\mathcal{K}$-group $F^*(Y)$. Such information is needed in the analysis of all the theorems mentioned at the beginning of this section.

When $E(\overline{X}) \neq 1$, one can arrange to apply Theorem 32.2 to suitable components of $\overline{X} = X/C_X(V)$. We consider here only the case that there is such a component in $\mathcal{C}hev(2)$. Then one can use the generation of the groups in $\mathcal{C}hev(2)$ of Lie rank at least 2 by their parabolic subgroups and a further reduction argument of Baumann to reduce the proof of Theorem 32.1 in the Lie type characteristic 2 case to the following minimal case:

$$X/O_2(X) \cong L_2(2^n), \quad n \geq 2.$$

In this case, $N_X(T)$ is, in fact, the unique maximal subgroup of X containing T, whence the condition $C(X,T) < X$ is equivalent to the assertion:

No nontrivial characteristic subgroup of T is normal in X.

The Baumann-Glauberman-Niles theorem covers this special case of the local $C(X,T)$ theorem, namely:

THEOREM 32.3. *Let X be a group such that $F^*(X) = O_2(X)$ and $X/O_2(X) \cong L_2(2^n)$, $n \geq 1$. If no nontrivial characteristic subgroup of a Sylow 2-subgroup T of X is normal in X, then $O^2(X)$ is a near component.*

Theorem 32.3 is an example of a **pushing up** situation, characterized in general by replacing the condition $C(X,T) < X$ by the condition that for some subgroup A of $Aut(T)$, no nontrivial A-invariant subgroup of T is normal in X. In practice, A arises within $N_G(T)$. Such pushing up problems with other possibilities for $F^*(X/O_2(X))$ besides $L_2(2^n)$ have also been extensively studied.

33. The amalgam method

The amalgam method, introduced by Goldschmidt [**Go6**], is a general technique for studying groups generated by a pair of subgroups without a common normal subgroup. The most important applications occur in the following situation:

(33.1)
(1) X is generated by two subgroups M_1 and M_2;
(2) No nonidentity subgroup of $M_1 \cap M_2$ is normal in both M_1 and M_2;
(3) M_1 and M_2 share a Sylow p-subgroup S for some prime p; and
(4) $F^*(M_i) = O_p(M_i)$, $i = 1, 2$.

The amalgam method involves a combination of local group theory and geometry, with large portions of the argument not requiring X to be finite. The purpose is to understand the amalgam consisting of M_1, M_2, and the inclusion mappings of $M_1 \cap M_2$ into these subgroups, rather than X, which is a "completion" of this amalgam. Indeed, one typically replaces X by the free amalgamated product of M_1 and M_2 amalgamated over the subgroup $M_1 \cap M_2$, and the main goal of the method is to force the structure of the "parabolic" subgroups M_1 and M_2, given the structure of the factor groups $M_i/O_p(M_i)$. Short of this, one aims at least to bound the number of chief factors of each M_i contained within $O_p(M_i)$, which in particular will bound $|O_p(M_i)|$, $i = 1, 2$. In practice, this method applies with $p = 2$ in the analysis both of quasithin groups and of Theorem U(σ). In both cases, the q-structure of $M_i/O_p(M_i)$ for odd primes q is restricted severely by the hypotheses of the respective cases, viz., $\sigma(G) = \emptyset$ and $\sigma_0(G) = \emptyset$.

One associates a graph Γ with the given configuration: the vertices of Γ are the right cosets of M_1 and M_2 in X, with two distinct vertices defined to be adjacent if they have a nontrivial intersection.

Obviously distinct cosets of M_i have trivial intersection for each $i = 1, 2$, so if two distinct vertices $M_j x, M_k y$ for $x, y \in G$ are adjacent, then necessarily $j \neq k$, so by definition Γ is bipartite. Furthermore, X acts by right multiplication on Γ, and this action is faithful because of (2) above, so that $X \leq Aut(\Gamma)$.

One easily establishes the following elementary properties of this set-up.

PROPOSITION 33.1. *The following conditions hold:*

(i) Γ *is connected;*
(ii) X *operates edge-transitively but not vertex-transitively on* Γ*;*
(iii) *The stabilizer in* X *of a vertex of* Γ *is an* X*-conjugate of* M_1 *or* M_2*;*
(iv) *The stabilizer in* X *of an edge of* Γ *is an* X*-conjugate of* $M_1 \cap M_2$*; and*
(v) *If* X *is the free amalgamated product of* M_1 *and* M_2 *over* $M_1 \cap M_2$*, then* Γ *is a tree.*

For simplicity, one denotes the vertices of Γ by letters α, β, γ, etc., with X_α, X_β, X_γ, etc., their respective stabilizers in X. For a vertex α of Γ, let $\Gamma(\alpha)$ be the set of vertices adjacent to α. Then the following facts are also part of the set-up.

PROPOSITION 33.2. X_α *is transitive on* $\Gamma(\alpha)$ *for any* $\alpha \in \Gamma$.

PROPOSITION 33.3. *Let* α, β *be adjacent vertices in* Γ *and let* $T \leq X_\alpha \cap X_\beta$ *. If* $(N_G(T))_\gamma$ *is transitive on* $\Gamma(\gamma)$ *for both* $\gamma = \alpha$ *and* $\gamma = \beta$*, then* $T = 1$.

In the ensuing discussion, we shall assume that $p = 2$ and shall describe briefly the key ingredients and objectives of the amalgam method. For any $\alpha \in \Gamma$, let Q_α be the kernel of the action of X_α on $\Gamma(\alpha)$. By (33.1)(3) and Proposition 33.2, $Q_\alpha \geq O_2(X_\alpha)$, and for simplicity we shall also assume that $Q_\alpha = O_2(X_\alpha)$ for each α.

Also, let $S_\alpha \in Syl_2(X_\alpha)$ and set $A_\alpha = \Omega_1(Z(S_\alpha))$. Since $C_{X_\alpha}(Q_\alpha) \leq Q_\alpha$ by our hypothesis (33.1)(4) on M_1 and M_2, we have $A_\alpha \leq \Omega_1(Z(Q_\alpha))$. Hence if we set

$$Z_\alpha = \langle A_\alpha^{X_\alpha} \rangle,$$

then $Z_\alpha \leq \Omega_1(Z(Q_\alpha))$ and $Z_\alpha \lhd X_\alpha$.

Finally, if we set $\overline{X}_\alpha = X_\alpha / C_{X_\alpha}(Z_\alpha)$, it quickly follows that

$$O_2(\overline{X}_\alpha) = 1, \text{ and}$$
$$Z_\alpha \text{ is a faithful } \overline{X}_\alpha\text{-module.}$$

The analysis focuses on the action of the subgroups Z_α on the graph Γ, more specifically on connected paths

$$\alpha = \alpha_1 \;\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\; \alpha_2 \;-\!\!\!-\; \cdots \;-\!\!\!-\; \alpha_{b-1} \;\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\; \alpha_b = \beta$$

of **minimal length** b with the following property: Z_α fixes each vertex of $\Gamma(\alpha_i)$ for all $1 \leq i < b$, but does not fix all the vertices of $\Gamma(\alpha_b) = \Gamma(\beta)$.

Group-theoretically, this is equivalent to the assertion that

$$Z_\alpha \leq Q_{\alpha_i} \text{ for all } 1 \leq i < b,\ Z_\alpha \leq X_\beta, \text{ but } Z_\alpha \not\leq Q_\beta.$$

The goal of the analysis is to force the integer b to be small, and use this to bound the number of chief factors of X contained within Q_α.

Now since $Z_\alpha \le X_\beta$, Z_α acts on Z_β. Furthermore, because of the minimal choice of b, it follows that likewise $Z_\beta \le X_\alpha$ and so Z_β acts on Z_α. Therefore $[Z_\alpha, Z_\beta] \le Z_\alpha \cap Z_\beta$, so $[Z_\alpha, Z_\beta, Z_\beta] = [Z_\beta, Z_\alpha, Z_\alpha] = 1$. Thus the action of $\overline{Z}_\beta$ on Z_α is quadratic as is the action of $\overline{Z}_\alpha$ on Z_β.

The analysis divides into two major cases according as

$$[Z_\alpha, Z_\beta] \ne 1 \text{ or } [Z_\alpha, Z_\beta] = 1.$$

In the first case, $Z_\beta \not\le Q_\alpha$ and so there is symmetry between α and β. Defining $C_\alpha = C_{Z_\alpha}(Z_\beta)$ and $C_\beta = C_{Z_\beta}(Z_\alpha)$, we can therefore choose notation so that

$$|Z_\alpha|/|C_\alpha| \ge |Z_\beta|/|C_\beta|.$$

However, this immediately yields that

$$|\overline{Z}_\alpha||C_{Z_\beta}(Z_\alpha)| \ge |Z_\beta|,$$

and hence that Z_β is a failure of factorization module for $\overline{X}_\beta$.

Thus the analysis in the first case leads directly to the study of failure of factorization modules for the group $\overline{X}_\beta$ in its action on Z_β . It is here that the possible structures of the groups $M_i/O_2(M_i)$ enter critically. Because of the restrictions on these groups in the applications, as noted above, the possible structures of X_β are severely restricted.

The analysis in the second case is similar to the first, but somewhat more complicated. We mention here only one subcase, related to the local $C(X,T)$-theorem. It involves α and an adjacent vertex γ. Then $Q_\alpha \le X_\gamma$. The assumption in this subcase is that

(33.2) $$Q_\alpha \in Syl_2(H_\gamma), \text{ where } H_\gamma = \langle Q_\alpha^{X_\gamma} \rangle.$$

Assume also that

(33.3) $$O_2(\langle X_\alpha, N \rangle) = 1 \text{ for all } N \le X_\gamma \text{ such that } N \not\le X_\alpha.$$

This condition can frequently be arranged by an appropriate initial choice of M_1 and M_2, and holds for instance if X_α is a maximal 2-local subgroup of G. Now (33.3) implies that for any $1 \ne Q_0$ char Q_α, we have $N_{X_\gamma}(Q_0) \le X_\alpha \cap X_\gamma$. Moreover, by (33.2) and a Frattini argument, $N_{X_\gamma}(Q_\alpha)$ and hence $X_\alpha \cap X_\gamma$ cover X_γ/H_γ. But $X_\gamma \not\le X_\alpha$ and so $H_\gamma \cap X_\alpha < H_\gamma$. Thus

$$C(H_\gamma; Q_\alpha) = \langle N_{H_\gamma}(Q_o) \,|\, 1 \ne Q_o \text{ char } Q_\alpha \rangle \le H_\gamma \cap X_\alpha < H_\gamma.$$

In view of (33.1)(4) and the local $C(X,T)$-theorem, H_γ therefore necessarily contains a near component.

34. Local analysis at two primes

To the extent possible, our analysis deals with the p-local structure of G for a fixed prime p. The choice of p depends on the case being analyzed. However, at several critical places, there is an interplay between p- and q-local structures for distinct primes p and q. Such interplay, for example, is intrinsic to the Bender method and central to the proof of the Feit-Thompson theorem.

We mention here three important examples in the analysis of groups of even type where p-local and 2-local structures interact. First, in the generic case, once the existence of a terminal component K has been established in $C_G(x)$ for some $x \in \mathcal{I}_p^o(G)$, $p \in \sigma_0(G)$, the proof that $K \in \mathcal{C}hev(2)$ relies on an analysis of centralizers of involutions $y \in K$, using the fact that $C_K(y) \lhd C_{C_G(y)}(x)$ and the restrictions imposed on $C_G(y)$ by the condition that G have even type.

Second, in the analysis of groups of $\mathcal{LC}_p$-type, the argument in the case where G is of wide $\mathcal{LC}_p$-type moves freely between p-local and 2-local subgroups, and indeed the goal of the argument, that G has a $Z_6 \times Z_2$-neighborhood, clearly involves both 2- and 3-local structure.

Third, the initial stage of the proof of Theorem U(σ) establishes 2-uniqueness properties of a subgroup M_p which was constructed from p-local analysis as having certain p-uniqueness properties.

Unfortunately, there is no direct analogue of $L_{p'}$-balance (Theorem 30.2) to apply to a pair of commuting elements x, $y \in G$ of **distinct** prime orders p and q. Indeed, if N is a q-local subgroup of the $\mathcal{K}$-group X, then $L_{p'}(N)$ need not lie in $L_{p'}(X)$.[7] Nevertheless the obstruction to such an inclusion can sometimes be pinpointed, as the following elementary proposition shows.

PROPOSITION 34.1. *Let p and q be distinct primes, and let X be a $\mathcal{K}$-group such that $O_{p'}(X) = 1$. Let Q be a q-subgroup of X and K a component of $E(C_X(Q))$. Then one of the following holds:*

(i) *$K \le E(X)$, and Q permutes transitively the set of components of X not centralized by K; or*

(ii) *There exists a QK-invariant p-subgroup R of X such that $[K, R] \neq 1$.*

This proposition, like $L_{p'}$-balance, can be applied when x and y are commuting elements of orders p and q, respectively, $O_{p'}(C_G(x)) = 1$, and K is a component of $C_{E(C_G(y))}(x))$, to give the embedding of K in $X = C_G(x)$. When the first alternative holds, we have the analogue of $L_{p'}$-balance: $K \le E(C_G(x))$, and indeed K is a component of $E(C_{C_G(x)}(y))$, so if the structure of K is known, then the structure of $E(C_G(x))$ is limited.

On the other hand, the second alternative can sometimes be avoided by choosing x and y appropriately. For example, the Thompson "dihedral lemma", originally used in the N-group paper for a similar purpose, is used in this way in the analysis of groups of wide $\mathcal{LC}_p$-type. The lemma is the following.

PROPOSITION 34.2. *If the elementary abelian 2-group A of rank n acts faithfully on the p-group R, p odd, then the semidirect product AR contains the direct product of n dihedral groups of order $2p$. Moreover, $m_{2,p}(AR) \ge n - 1$.*

---

[7]For example, take X to be the holomorph of $E = E_{p^n}$ and let q be a prime divisor of $p-1$. Then a complement H to E in X is a q-local subgroup of X, and $L_{p'}(H) = H'$, but $L_{p'}(X) = 1$.

Here the **2-local p-rank** of a group X is defined by

$$m_{2,p}(X) = \max\{m_p(N) \mid N \leq X,\ O_2(N) \neq 1\}.$$

We then can draw the following useful conclusion.

PROPOSITION 34.3. *Suppose that p is an odd prime, $x \in \mathcal{I}_p^o(G)$, $O_{p'}(C_G(x)) = 1$, y is an involution of $C_G(x)$, and K is a component of $E(C_{C_G(y)}(x))$. Suppose that $m_2(K) - m_2(Z(K)) > m_{2,p}(G) + 1$. Then $K \leq E(C_G(x))$.*

This proposition is one reason why it is advantageous to study elements x of order p lying in an element $B \in \mathcal{B}_*^p(G)$, since by definition $m_p(B) = m_{2,p}(G)$. When G is of even type and $E(C_G(y))$ has a component L, then $L \in \mathcal{C}_2$ and so typically the 2-rank of L is much larger than its p-rank. As long as $m_p(C_B(K))$ is not too large, Proposition 34.3 finds application.

We conclude by mentioning another elementary result which is at the heart of the analysis of groups of wide $\mathcal{LC}_p$-type and gives rise to 2-groups of symplectic type in G.

PROPOSITION 34.4. *Let p be an odd prime and let $B \in \mathcal{B}_*^p(G)$, so that $B \cong E_{p^n}$, $n = m_{2,p}(G)$, and B normalizes a nontrivial 2-subgroup T of G. Suppose that $n \geq 2$ and that for every hyperplane A of B, $F^*(C_G(A))$ is a p-group. Further, if $p = 3$, assume that $C_T(B) \neq 1$. Then every characteristic abelian subgroup of T is cyclic, i.e., T is of symplectic type or cyclic.*

To illustrate this, suppose that G is of $\mathcal{LC}_p$-type and $O_{p'}(C_G(x)) = 1$ for all $x \in \mathcal{I}_p^o(G)$. Then the components of $C_G(x)$ for $x \in B^\#$ lie in $\mathcal{C}_p$. In the simplest case, these components actually lie in $\mathcal{C}hev(p)$ and each one admits a nontrivial inner automorphism induced by some element of a hyperplane A of B. By $L_{p'}$-balance and the fact that groups in $\mathcal{C}hev(p)$ are of characteristic p-type, this leads to the desired condition that $F^*(C_G(A))$ is a p-group. The condition that $C_T(B) \neq 1$ can usually be achieved by a suitable choice of B and T.

35. Character theory and group order formulas

The only general result that we use that depends on character theory is Glauberman's Z^*-theorem, which generalizes the Brauer-Suzuki theorem on groups with quaternion Sylow 2-subgroups. Beyond this, character theory is used in the classification proof only in connection with groups of special odd type (including groups of odd order) and in connection with certain properties of the sporadic groups and a few other $\mathcal{K}$-groups.

The primary use of ordinary character theory is through the Brauer-Suzuki theory of exceptional characters and related isometries and coherence between the character ring of G and character rings of suitable subgroups of G. Such character calculations play a key role in the analysis both of groups of odd order and of groups with dihedral, rank 2 homocyclic abelian or type $U_3(4)$ Sylow 2-subgroups.

The theory of blocks of characters, in particular Brauer's first and second main theorems, is essential for the analysis of groups with Sylow 2-subgroups which are semidihedral, wreathed, or elementary abelian of order 8, but it is also used in the rank 2 homocyclic abelian and $U_3(4)$ type cases.

One of the principal uses of character theory is to derive formulas for the order of the group G under investigation in terms of the structure of the centralizer of an involution. We give such an example below. However, when G has more than one conjugacy class of involutions, Thompson has derived such a formula by entirely elementary, non character-theoretic calculations. Frequent application of this formula is made in the course of the classification of groups of special type, so we shall state it here.

Assume then that G has more than one conjugacy class of involutions and fix involutions u, v of G with u not conjugate to v in G. For any involution w of G, define $r(u, v, w)$ to be the number of ordered pairs (u_1, v_1) such that u_1 is G-conjugate to u, v_1 is G-conjugate to v, and $w \in \langle u_1 v_1 \rangle$. (Since u_1 and v_1 are not conjugate, their product must have even order.)

With this notation, Thompson counts the ordered pairs consisting of a conjugate of u and a conjugate of v in two different ways to arrive at his formula:

THEOREM 35.1. (Thompson Order Formula) *The order of G is given by the following expression:*

$$|G| = |C_G(u)||C_G(v)| \sum_w \frac{r(u, v, w)}{|C_G(w)|},$$

the sum taken over a system of representatives w of the conjugacy classes of involutions of G.

In the particular case in which there are just two conjugacy classes of involutions in G, the formula becomes

$$|G| = r(u, v, v)|C_G(u)| + r(u, v, u)|C_G(v)|.$$

It is for the case in which G has only one conjugacy class of involutions that character theory is needed to derive analogous formulas for $|G|$. We shall state Brauer's result in the semidihedral/wreathed case (under the assumption that G is what Brauer calls "regular"). His formula depends on the prior determination of the degrees of the characters in the principal 2-block of G in terms of the structure of the centralizer of an involution.

Assume then that G has a semidihedral or wreathed Sylow 2-subgroup S and let z be the unique involution of $Z(S)$. An elementary fusion analysis implies that $C_G(z)$ does not have a normal 2-complement. But now as G is $\mathcal{K}$-proper, it follows that $\overline{C_G(z)} = C_G(z)/O_{2'}(C_G(z))$ contains a normal subgroup $\overline{L} \cong SL_2(q)$ for some odd prime power q. Since G has only one class of involutions, q depends only on G; we refer to it as the **characteristic power** of G, and it is a key entry in the order formula for G. We write $q = p^n$, p an odd prime.

Note that the target groups here are the groups $L_3(q)$, $U_3(q)$, q odd, and M_{11}; correspondingly their characteristic powers are in fact q, q and 3.

Now let U be a four-subgroup of S with $U \lhd S$ if S is wreathed. Since $m_2(S) = 2$, $z \in U$, so $C_G(U) \leq C_G(z)$.

DEFINITION 35.2. We say that G is **regular** if and only if the p-part of $|G|$ is at least as large as the p-part of $|C_G(z)|^3/|C_G(U)|^2$.

Brauer has shown that G is necessarily regular when S is wreathed. However, when S is semidihedral, G need not be regular. Indeed, if $G = M_{11}$, then $C_G(z) \cong GL_2(3)$, whence the 3-part of $|C_G(z)|^3/|C_G(U)|^2$ is 27, but $|M_{11}| = 8 \cdot 9 \cdot 10 \cdot 11$ is not divisible by 27.

Here then is Brauer's formula.

THEOREM 35.3. *Assume that G has a semidihedral or wreathed Sylow 2-subgroup and is regular of characteristic power q. Then for a suitable value of $\epsilon = \pm 1$, we have*

$$|G| = \frac{|C_G(z)|^3}{|C_G(U)|^2} \cdot \frac{q^2 - \epsilon q + 1}{(q-\epsilon)^2}.$$

The two possibilities for ϵ correspond to the two target groups $G^* = L_3(q)$ or $U_3(q)$.

36. Identification of the groups of Lie type

In sections 11, 26 and 27 of the preceding chapter we have described the Steinberg presentation for the groups of Lie type (of Lie rank ≥ 2) and have discussed several alternative ways of verifying these relations—via Tits's classification of (B,N)-pairs, the Curtis-Tits theorem (Chapter 1, Theorem 27.3) and variations of it, or the Gilman-Griess theorem for the groups of characteristic 2 and Lie rank at least 4 (Chapter 1, Theorem 27.6). The last two approaches are most effective for groups of Lie rank at least 3. [Gilman and Griess have established a modification of Theorem 27.6, applicable in the Lie rank 3 cases.]

The Curtis-Tits and Gilman-Griess theorems do not apply to groups of Lie rank at most 2, on the other hand. Variants of the Curtis-Tits theorem can be used if the untwisted Lie rank is at least 3, for instance to identify $U_n(q)$ for $n = 4$ or 5 by means of subgroups stabilizing nonsingular subspaces on the natural n-dimensional module over $\boldsymbol{F}_{q^2}$. Other groups of Lie rank at most 2 are best identified from their split (B,N)-pair structure, which we now describe in some detail. We begin with the Lie rank 2 case, where the required conditions are more easily described. As always, G is a $\mathcal{K}$-proper simple group.

Let $G(q) = G^*$ be a simple target group of Lie rank 2. Thus $G^* = B^*N^*B^*$ with $H^* = B^* \cap N^* \lhd N^*$, H^* an abelian p'-group, $B^* = H^*U^*$, where $U^* \in Syl_p(G^*)$, U^* is the product of H^*-root subgroups, the Weyl group $W^* = N^*/H^*$ is a dihedral group generated by two fundamental reflections, and the Bruhat lemma (Definition 11.1(iv)) describes the products of double cosets $B^*x^*B^* \cdot B^*y^*B^*$, for x^*, $y^* \in N^*$. Again this description of G^* determines G^* up to isomorphism. Indeed if G is a simple split (B,N)-pair of rank 2, and we set $H = B \cap N$ and let U be a nilpotent normal complement to H in B, then one can define H-root subgroups in U and the following result holds [**Ri1,Ti5**]:

THEOREM 36.1. *In the situation just described, if there are isomorphisms $W \to W^*$ and $B \to B^*$ carrying H to H^* and H-root subgroups of U to corresponding H^*-root subgroups of U^*, then $G \cong G^*$.*

Indeed, Tits shows that under these hypotheses, the corresponding buildings for G and G^* are isomorphic. We also remark that an analogous result holds for split (B,N)-pairs of rank ≥ 3; however, this is not in general the most efficient approach for establishing the Steinberg relations.

The condition in the above result that $B \cong B^*$ can in fact be weakened to conditions which are more appropriate to the results of amalgam analysis (corresponding to a target group $G(q)$, q even) or to the results of analysis of centralizers of involutions (for the case q odd). If we let P_i^*, $i = 1$, 2, be the maximal parabolic subgroups of $G(q)$ containing B^*, then the P_i^* have Levi decompositions $P_i^* = K_i^*U_i^*$ with $U_i^* \cap K_i^* = 1$, U_i^* a p-group and K_i^* the product of H^* and a normal rank 1 subgroup of Lie type. If one assumes that $W \cong W^*$ and G has subgroups P_i, $i = 1, 2$, such that $P_i \geq B$, $P_i = K_i U_i$ with $K_i \cong K_i^*$, $|U_i| = |U_i^*|$, and such that certain sections of U_i are isomorphic to corresponding sections of U_i^*, even as K_i-modules, then it turns out that the condition $B \cong B^*$ can be derived without repeating the complete classification of split (B, N)-pairs of rank 2 of Fong and Seitz (see [**GLS1, FoSe1**]).

The identification problem in the Lie rank 1 case is much more difficult. In this case, it is necessary to delve into portions of the multiplication table of G outside proper subgroups to show how the table is determined from the given rank 1 split (B, N)-pair data.

We shall explain this briefly. In this case $W = N/H \cong Z_2$, and hence if we fix $w \in N - H$, then

$$G = BNB = B \cup BwB, \tag{36.1}$$

whence G is a doubly transitive subgroup of Σ_Ω, where Ω is the set of right cosets of B. Now the prior analysis reduces one to the case that $B = HU$ with $U \lhd B$ and $U \in Syl_p(G)$. In particular, it follows from (36.1) that

$$G = B \cup BwU, \tag{36.2}$$

and the (B, N)-pair axioms imply that each element $y \in BwU$ has a unique representation of the form

$$y = huwv \text{ with } \quad h \in H \text{ and } \quad u, v \in U. \tag{36.3}$$

Hence for each $t \in U^\#$, the element wtw can be uniquely expressed in the form

$$wtw = h(t)g(t)wf(t), \tag{36.4}$$

where $h(t) \in H$ and $f(t)$, $g(t) \in U$.

It is clear from (36.3) and (36.4) that the multiplication table of G is completely determined by the structures of B and $N = H\langle w \rangle$ and the functions f, g and h. But much less is sufficient. Namely, since G is simple, it is easy to see that G is generated by U and w. Moreover, because of (36.2) and (36.3), Ω can be parametrized by U and a single additional symbol. The action of U on Ω is then determined by the structure of U, and the action of w on Ω is completely determined by the single function $f(t)$.

Thus one obtains:

PROPOSITION 36.2. *The multiplication table of G is completely determined by the structure of U and the function f from $U^\#$ to $U^\#$.*

Peterfalvi uses this last result in his revision and extension of O'Nan's earlier characterization of $U_3(q)$, q odd, as a split (B, N)-pair of rank 1.

37. Properties of $\mathcal{K}$-groups

As the preceding discussion makes amply clear, almost all of the above techniques depend for their validity and usefulness on the fact that the critical subgroups of G to which they are to be applied are $\mathcal{K}$-groups. Verification of the necessary $\mathcal{K}$-group properties reduces ultimately — frequently by Bender's F^*-theorem (Theorem 4.8(i) of Chapter 1) — to a combination of specific properties of simple and almost simple $\mathcal{K}$-groups, and properties of solvable groups.

We have also indicated the nature of many of these simple $\mathcal{K}$-group properties. We shall not present any of these results here, but instead shall simply list the various categories into which they fall:

1. Automorphisms
2. Schur multipliers
3. Structure of local subgroups, especially the p-components of centralizers of elements of order p, but also other local subgroups
4. Generation, particularly by p-local subgroups
5. Local balance
6. Sylow p-structure and fusion
7. p-rank and embedding of elementary abelian p-subgroups
8. Other subgroup structures
9. $\boldsymbol{F}_p$-representations for the groups in $\mathcal{C}hev(p)$

It seems that the smaller the $\mathcal{K}$-groups one encounters in an analysis, the longer and more detailed is the list of their properties which one needs. Thus the analysis of simple groups of special type seems to require digging deeply into the minutiae of their subgroup structures. In contrast, the generic case analysis utilizes only a shorter list of comparatively general properties of simple $\mathcal{K}$-groups, and even within the set of generic groups, the ones of larger Lie rank are amenable to more general arguments.

F. Notational conventions

In general, our notation will be standard. However, our notation for the simple groups is that of Table I of Chapter 1; this differs in some respects from the *Atlas* notation [**CPNNW1**].

For the sake of compactness, we shall adopt some abbreviated conventions, such as the "bar convention" for homomorphic images: if $\overline{X}, \tilde{X}, X^*$, etc. is a homomorphic image of the group X, then the image of any subgroup or subset Y of X will be denoted by $\overline{Y}, \tilde{Y}, Y^*$, etc., respectively.

For any subset Y of the ambient $\mathcal{K}$-proper simple group G under investigation, we shall write C_Y and N_Y for $C_G(Y)$ and $N_G(Y)$, respectively. [This notation will be used only relative to G itself.]

In addition, as the prime p will be fixed in each chapter, we shall write for any group X, $m(X)$, $L(X)$, $\mathcal{I}(X)$, $\mathcal{E}_k(X)$, $\mathcal{L}(X)$, etc., for $m_p(X)$, $L_{p'}(X)$, $\mathcal{I}_p(X)$, $\mathcal{E}_k^p(X)$, $\mathcal{L}_p(X)$, etc., respectively.

However, this abbreviated notation will be used only in the body of the text, the introduction to each chapter being written with the standard, fuller notation.

Finally, at the end of each volume, we shall include a glossary of terms and symbols introduced in that volume.

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GLOSSARY

PAGE	SYMBOL
90	$\sim$
97	$\succeq$
4–5	$[\mathrm{I}_1]$, $[\mathrm{I}_2]$, $[\mathrm{I}_A]$, $[\mathrm{I}_G]$, $[\mathrm{II}_A]$, $[\mathrm{II}_G]$, $[\mathrm{II}_P]$, $[\mathrm{II}_S]$
60, 101	$2A_n$, $(2)A_n$
101	$2D_n(q)$
101	$3U_4(3)$
101	$4L_3(4)$
101	$[A_1 \times A_2]\overline{K}$ (central extension)
101	$A\overline{K}$ (central extension)
81	$\mathcal{A}lt$
6, 8	A_n
7, 8	$A_n(q)$, ${}^2A_n(q)$, $A_n^+(q)$, $A_n^-(q)$
130	$\mathcal{A}(T)$
60, 104	$\mathcal{B}_2$, $\mathcal{B}_2$-group
7, 8, 10	$B_n(q)$, ${}^2B_2(2^n)$
116	$\mathcal{B}_*^p(G)$
24	$B_{p'}(X)$
7, 8	$C_n(q)$
9, 11	Co_1, Co_2, Co_3
100	$\mathcal{C}_p$, $\mathcal{C}_p$-group
102	$\tilde{\mathcal{C}}_p$
100	$\mathcal{C}_{p'}$, $\mathcal{C}_{p'}$-group
81	$\mathcal{C}hev$, $\mathcal{C}hev(p)$
90	$C(G,S)$
101	$\mathcal{C}\mathcal{H}_p$
22	$C(K,x)$
63	C_x
22	$C_X(A/B)$
14	$C_X(V)$
139	C_Y
64, 124	$\Delta_X(B)$
7, 8, 10	$D_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, $D_n^+(q)$, $D_n^-(q)$
7, 8, 10	$E_n(q)$, ${}^2E_6(q)$, $E_6^+(q)$, $E_6^-(q)$
14	E_{p^n}
124	$\mathcal{E}^p(X)$, $\mathcal{E}_k^p(X)$
139	$\mathcal{E}(X)$, $\mathcal{E}_k(X)$
17	$E(X)$

18	$\Phi(X)$
9, 11	F_1, F_2, F_3, F_5
7, 8, 10	$F_4(q)$, ${}^2F_4(2^n)$, ${}^2F_4(2)'$
9, 11	Fi_{22}, Fi_{23}, Fi'_{24}
95	$\mathcal{FM}9$
6	$\boldsymbol{F}_q$
16	$F(X)$
17	$F^*(X)$
66	$\gamma(G)$
132	Γ, $\Gamma(\alpha)$
82	$\Gamma^o_{P,2}(X)$
82	$\Gamma_{P,k}(X)$
27	G^*
109–121	$G \approx G^*$
66, 121	$G_0(\mathcal{N})$
7, 8, 10	$G_2(q)$, ${}^2G_2(3^n)$
71	G_α, G^*_α
6	$GL_n(q)$
102	$\mathcal{G}_p$
32–33	$G(q)$, $\hat{G}(q)$
7	$GU_n(q)$
9, 11	He
101	$\mathcal{H}_p$, $\hat{\mathcal{H}}_p$, $\mathcal{H}^*_p$
9, 11	HS
55, 81	$\mathcal{I}_p(G)$
55, 103	$\mathcal{I}^o_p(G)$, $\mathcal{I}^o_2(G)$
13	$Inn(X)$
14	Int, $Int(x)$
139	$\mathcal{I}(X)$
102	$\mathcal{J}_0(p)$, $\mathcal{J}_1(p)$, $\mathcal{J}_2(p)$
9, 11	J_1, J_2, J_3, J_4
26, 130	$J(P)$
90	$[K]$
53, 81	$\mathcal{K}$
110	$\mathcal{K}^{(2)*}$
86	$\mathcal{K}^{(i)}$, $i = 0, \ldots, 7$
53, 81	$\mathcal{K}_p$
12	$\mathcal{K}$-group
12	$\mathcal{K}$-proper
6, 8	$L_n(q) = L^+_n(q)$
8	$L^-_n(q)$
55, 103	$\mathcal{L}^o_p(G)$
53, 81	$\mathcal{L}_p(G)$
126	$\mathcal{L}_p(G; A)$
20	$L_{p'}(X)$
21	$\hat{L}_{p'}(X)$
127	$L^*_{p'}(X)$
7	$\mathcal{L}(q)$

139	$\mathcal{L}(X)$
139	$L(X)$
9, 11	Ly
135	$m_{2,p}(G)$
9, 11	$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
9, 11	Mc
123	$M \rightsquigarrow N$
30, 81	$m_p(A)$
58, 60, 82	$\mathcal{M}(S)$, $\mathcal{M}_1(S)$
139	$m(X)$
65	$\mathcal{N}$
97	$\mathcal{N}(M)$, $\mathcal{N}^*(M)$
19	$N_X(Q)$
139	N_Y
26	$\Omega_1(B)$
117	$\Omega_8^-(3)$-type
9, 11	$O'N$
24	$O_{p'p}(X)$
19	$O^p(X)$, $O^{p'}(X)$, $O^\pi(X)$
29	$Out(X)$
19	$O(X)$, $O_p(X)$, $O_{p'}(X)$, $O_\pi(X)$
19	π, π'
32	Π
7, 8, 101	$P\Omega_n^\pm(q)$, $P\Omega_n(q)$, $(P)\Omega_n(q)$
6, 8	$PSL_n(q)$
7, 8	$PSp_n(q)$
7, 8	$PSU_n(q)$
98	$\mathcal{Q}(M_p)$
9,11	Ru
37, 82	$\sigma(G)$
107	$\sigma^*(G)$
58, 83	$\sigma_0(G)$
120	$\sigma_{\mathcal{G}}(G)$
118	$\sigma_{\mathcal{T}}(G)$
32	Σ, Σ^+, Σ^-
6	Σ_n
94	$\mathcal{S}^p(X)$, $\mathcal{S}^p_n(X)$
29	$S(G)$
6	$SL_n(q)$
101	$(S)L_n(q)$
81	$\mathcal{S}por$
7	$SU_n(q)$
9, 11	Suz
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30, 124–125	Θ, $\Theta(C_G(a))$, $\Theta(G;A)$, $\Theta_{k+\frac{1}{2}}$, $\Theta_{k+\frac{1}{2}}(G;A)$
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8	$U_n(q)$
139	$\overline{X}$, $\tilde{X}$, X^*

101	$[X]L_3(4)$
26	$Z(J(P))$
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115	2-amalgam G^*-type
114–115	2-amalgam type, 2-amalgam type $(M_1, M_2; S)$
111	2-central G^*-type
88	2-central involution
135	2-local p-rank
111	2-maximal G^*-type mod cores
98	$\{2, p\}$-uniqueness subgroup
110	2-terminal G^*-type
114	2-terminal $\mathcal{LT}_2$-type
82	2-uniqueness subgroup
64	3/2-balance
120	3/2-balanced type
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44–50	Background Results, Background References
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34	(B, N)-pair, split (B, N)-pair
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13	composition factor, length, series
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91	control of rank 1 (or rank 2) fusion
16	covering group
112	doubly transitive G^*-type
95–96	doubly transitive of Suzuki type
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16	Fitting subgroup
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18	Frattini subgroup
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106	generic odd type
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46, 109	involution fusion pattern
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125	k-balanced signalizer functor, weakly k-balanced signalizer functor
125	$(k+\frac{1}{2})$-balance
17	layer
104	$\mathcal{L}\mathcal{B}_2$-type
105	$\mathcal{L}\mathcal{C}_p$-type
94	$\mathcal{L}\mathcal{C}_p$-uniqueness subgroup
32	Lie rank (twisted or untwisted)
128	local balance
126	local $(k+\frac{1}{2})$-balance
126	local k-balance, weak local k-balance
19	local subgroup
21	$L_{p'}$-balance
127	$L^*_{p'}$-balance
57–58	$\mathcal{L}_p$-generic type
58, 103	$\mathcal{L}_p$-special type, $\mathcal{L}_2$-special type
104	$\mathcal{L}\mathcal{T}_2$-type
105	$\mathcal{L}\mathcal{T}_p$-type
34	monomial subgroup
96, 98	near component (of Y): linear, alternating, standard
97	near component 2-local uniqueness subgroup
64, 119	neighbor, (y, I)-neighbor
65, 119, 120	neighborhood: level, split, vertical
8, 9	notation for simple groups
58–59, 81	odd type
101	p-central element of order p
20	p-component
91	p-component preuniqueness hypothesis
90	p-component preuniqueness subgroup
20	p-constrained group
19	p'-core
20	p-layer
19	p-local subgroup
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121	proper semisimple type
24	p-solvable group
64	p-source

23, 108	p-terminal p-component
63	p-terminal $\mathcal{G}_p$-pair
118	p-terminal $\mathcal{LT}_p$-type
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82	p-uniqueness subgroup
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117	quasisymplectic type
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