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# SPHERICAL POSETS AND HOMOLOGY STABILITY FOR $0_{n,n}$

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## §0. INTRODUCTION

IN THIS PAPER we prove the following theorem: Let $F$ be a field, $F \neq \mathbf{Z}_2$, and let $0_{n,n}(F)$ be the orthogonal group of the quadratic form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then the map $i_*$: $H_k(0_{n,n}(F)) \to H_k(0_{n+1,n+1}(F))$ is onto for $n \geq 3k-1$ and an isomorphism for $n \geq 3k$, i.e. the $k$-th homology of the group $0_{n,n}(F)$ stabilizes at $n = 3k$.

Stability theorems for the homology of the general linear group of various classes of rings have been proved by Bass [1, p. 240], Quillen [2], Wagoner [3] and Charney [4].† The method used here for $0_{n,n}$ is based on Quillen's proof. We construct a simplicial complex $X_n$ with a natural $0_{n,n}$ action, and show that $X_n$ is $(n-1)$-connected. $X_n$ yields an acyclic chain complex $C_*$, and tensoring with a $\mathbf{Z}[0_{n,n}]$-free resolution $E_*$ of $\mathbf{Z}$ gives a double complex $E_* \otimes_{0_{n,n}} C_*$. The spectral sequence associated to this double complex converges to zero, and we examine this spectral sequence to obtain information about the homology of $0_{n,n}$.

It turns out that the information obtained involves a certain subgroup $S_{p,n}$ of $0_{n,n}$, and we must repeat the procedure, using a different complex $X^{p,n}$ for $S_{p,n}$ in order to learn enough about the homology of $S_{p,n}$ to prove the theorem.

In §1, we discuss the simplicial complexes $X_n$ and $X^{p,n}$; in §2 we construct the spectral sequences and prove the theorem.

## §1. THE SIMPLICIAL COMPLEXES ASSOCIATED TO $0_{n,n}$

The simplicial complexes we will use come from partially ordered sets (posets) of subspaces of a $2n$-dimensional vector space. We first give some notation and definitions.

Given a poset $X$, we can form a simplicial complex called the *realization* of $X$, denoted $|X|$, as follows: the 0-simplices of $|X|$ are the elements $x \in X$, and the $k$-simplices of $|X|$ are $(k+1)$-tuples $(x_0, \ldots, x_k)$ of elements of $X$, with $x_0 < x_1 < \cdots < x_k$. The natural identifications make this into a simplicial complex.

*Definition.* A poset is *$n$-spherical* if its realization is $n$-dimensional and $(n-1)$-connected.

*Definition.* Let $x$ be an element of $X$. The *height* of $x$, $h(x)$, is the length of a maximal totally ordered chain of elements less than $x$.

*Notation.* Let $x$ and $x'$ be elements of $X$. Then

$$X_{>x} = \{y \in X \mid y > x\}$$
$$X_{<x} = \{y \in X \mid y < x\}$$
$$(x, x') = \{y \in X \mid x < y < x'\}.$$

†W. Van der Kallen (preprint, Utrecht, December 1979) has proved a very general stability theorem for the general linear group.

*Definition.* A poset $X$ is *Cohen–Macaulay of dimension n*, denoted $X \in CM^n$, if
(i) $X$ is $n$-spherical
(ii) $X_{>x}$ is $(n-1-h(x))$-spherical for all $x \in X$.
(iii) $X_{<x}$ is $(h(x)-1)$-spherical for all $x \in X$.
(iv) $(x, x')$ is $(h(x')-h(x)-2)$-spherical for all $x < x'$ in $X$.

An important example of a Cohen–Macaulay poset is the set of all proper subspaces of a vector space $V$, partially ordered by inclusion. The Solomon–Tits theorem [5] says the realization of this poset has the homotopy type of a wedge of $(\dim V-2)$-spheres, and it follows easily that the poset is in fact Cohen–Macaulay.

We will be interested in the following generalization of the above poset. Let $W$ and $U$ be subspaces of the vector space $V$, with $\dim W = k$, $\dim U = m$, $\dim V = n$ and $k \leq m \leq n$. Consider the set $T = {}^W T^{U,V}$ of proper subspaces $A$ of $V$ such that $A \cap W = 0$ and $A + U = V$, partially ordered by inclusion. It turns out that $T$ is always Cohen–Macaulay; we will prove this in some special cases. Note that if $W = 0$ and $U = V$, this is just the Solomon–Tits theorem. We now consider the case $W \neq 0$, $U = V$.

PROPOSITION 1.1. *If* $W \neq 0$, $T = {}^W T^{V,V}$ *is homotopy equivalent to a wedge of* $(n-k-1)$-*spheres.*

*Proof.* We will proceed by induction on $n-k$, the case $n-k=1$ being obvious. We first prove a lemma.

LEMMA 1.2. *If* $A \in T$, *then* $T_{>A}$ *is* $(n\text{-}k\text{-}\dim A - 1)$-*spherical.*

*Proof.* $T_{>A} = \{B \supsetneq A \mid B \cap W = 0\}$. Pick a complement $A'$ for $A$ so that $A' \supset W$. Then the correspondence $B \mapsto B \cap A'$ gives a poset isomorphism

$$\{B \supsetneq A \mid B \cap W = 0\} \xrightarrow{\cong} \{B' \subseteq A' \mid B' \cap W = 0\};$$

by induction, the latter poset is $(n\text{-}k\text{-}\dim A - 1)$-spherical. □

Now let $l$ be a minimal element of $T$, i.e. $l$ is a line with $l \not\subseteq W$. Then the realization of $Y_0 = \{A \in T \text{ such that } A + l \in T\}$ is contractible via the maps $A \mapsto A + l \mapsto l$.

Let $A$ be an element of $T - Y_0$; then $A \cap W = 0$ but $(A+l) \cap W \neq 0$, or equivalently, $A \cap (l + W) = v$, where $v$ is a line not equal to $l$. Now

$$\begin{aligned} 1kA \cap Y_0 &= \{B \subsetneq A \mid B + l \in T\} \\ &= \{B \subsetneq A \mid (B+l) \cap W = 0\} \\ &= \{B \subsetneq A \mid B \cap v = 0\}; \end{aligned}$$

$\dim A \leq n-k < n$ and $\dim v = 1$, so by induction, the realization $|1kA \cap Y_0|$ is homotopy equivalent to a wedge of $(\dim A - 2)$-spheres.

Now define $Y_{i+1} = Y_0 \cup \{A \in T \mid \dim A \geq n-k-i\}$.

*Claim.* $Y_i$ is homotopy equivalent to a wedge of $(n-k-1)$-spheres, for $i \geq 1$.

*Proof.* If $i = 1$, take $A \in Y_1 - Y_0$. Then we have seen that $1kA \cap Y_0 \simeq VS^{n-k-2}$. Since $Y_0$ is contractible,

$$Y_1 \simeq \bigvee_{A \in Y_1 - Y_0} \operatorname{susp}(VS^{n-k-2}) \simeq VS^{n-k-1}.$$

If $i > 1$, look at $A \in Y_i - Y_{i-1}$. Then

$$1kA \cap Y_{i-1} = \{1kA \cap Y_0\} \cup \{B \underset{\neq}{\supset} A | B \cap W = 0\}.$$

Since every element of the second subset contains every element of the first, the realization of their union is the join of their realizations; so using the lemma we have

$$|1kA \cap Y_{i-1}| \cong VS^{n-k-i-1} * VS^{i-2}$$
$$\cong VS^{n-k-2}.$$

Therefore we have

$$|Y_i| \simeq \Big(\bigvee_{A \in Y_i - Y_{i-1}} \operatorname{susp}(VS^{n-k-2})\Big) V |Y_{i-1}|$$
$$\simeq VS^{n-k-1}.$$

Since $Y_{n-k} = T$, we have proved the proposition. □

COROLLARY 1.3. *${}^0T^{U,V}$ is homotopy equivalent to a wedge of $(m-1)$-spheres, for $U \neq V$,* dim $U = m$.

*Proof.* Equip $V$ with a non-degenerate quadratic form. Then the map $A \mapsto A^\perp$ gives a poset isomorphism ${}^O T^{U,V} \to {}^{U^\perp}T^{V,V}$. □

PROPOSITION 1.4. *If $W \neq 0$ and* dim $U = n-1$, *then ${}^W T^{U,V}$ is homotopy equivalent to a wedge of $(n-k-1)$-spheres.*

*Proof.* The proof is identical to the proof of Proposition 1.1; note that the minimal element $l$ has the additional property that $l + U = V$, and that for $A$ in $T - Y_0$, we identify $1kA \cap Y_0$ inductively as

$$\{B \underset{\neq}{\subset} A | B \in T \text{ and } B + l \in T\} = \{B \underset{\neq}{\subset} A | B + U = V \text{ and } (B + l) \cap W = 0\}$$
$$= \{B \underset{\neq}{\subset} A | B + (U \cap A) = A \text{ and } B \cap v = 0\}.$$

Since $A$ is transverse to $U$, dim$(U \cap A) =$ dim $A - 1$, so the inductive hypotheses are satisfied. □

COROLLARY 1.5. *If $W \neq 0$ and* dim $U = n-1$, *then $T = {}^W T^{U,V}$ is Cohen–Macaulay of dimension $(n-k-1)$.*

*Proof.* By Proposition 1.4, $T$ is $(n-k-1)$-spherical. Let $A$, $A' \in T$. Then $T_{>A} =$

$\{B \underset{\neq}{\supset} A | B \cap W = 0\}$, which is $(n - k - \dim A - 1)$-spherical by Lemma 1.2. $T_{<A} =$ $\{B \underset{\neq}{\subset} A | B + U = V\}$, which is(dim $A - 2$)-spheircal by Corollary 1.3. And $(A, A') =$ $\{A \underset{\neq}{\subset} B \underset{\neq}{\subset} A'\}$, which is $(\dim A' - \dim A - 2)$-spheircal by the Solomon–Tits theorem. □

We now introduce the quadratic form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ on a $2n$-dimensional vector space $V$ with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. A vector space with such a form will be called a *hyperbolic* space. A subspace $A \subset V$ is *isotropic* if the inner product $v \cdot w$ of any two vectors $v, w \in A$ is zero. Let $X_n$ be the poset of nonzero isotropic subspaces of $V$, partially ordered by inclusion, and $X_{n,k} \subseteq X_n$ the poset of nonzero isotropic subspaces of dimension $\leq k$.

THEOREM 1.6. *$X_{n,k}$ is spherical of dimension $k - 1$.*

*Proof.* The proof will proceed by induction on $k$. If $k = 1$, the theorem is clear. Let

$$Y_0 = \{A \in X_{n,k} | A \cap (e_1{}^\perp) \neq 0 \quad \text{and} \quad \dim[(A \cap (e_1{}^\perp)) + e_1] \leq k\}.$$

Then the maps

$$A \mapsto A \cap (e_1{}^\perp) \mapsto [(A \cap e_1{}^\perp) + e_1] \mapsto e_1$$

give a contraction of the realization of $Y_0$ to the point $e_1$.

$X_{n,k} - Y_0$ consists of two types of subspaces, namely isotropic lines $a$ with $a \cdot e_1 \neq 0$ and $k$-dimensional isotropic subspaces $A$, with $A \subseteq e_1{}^\perp$ but $A \not\supseteq e_1$. In the latter case, $1kA \cap Y_0 = \{B \underset{\neq}{\subset} A | B \neq 0\}$, so $|1kA \cap Y_0|$ is homotopy equivalent to a wedge of $(k-2)$-spheres by the Solomon–Tits theorem. Thus if we let $Y_1 =$ $Y_0 \cup \{A \subset V | \dim A = k, A \cdot A = 0\}$, we have

$$\begin{aligned} |Y_1| &\simeq \underset{A \in Y_1 - Y_0}{V} \operatorname{susp} |1kA \cap Y_0| \\ &\simeq V \operatorname{susp}(VS^{k-2}) \simeq VS^{k-1}. \end{aligned}$$

Now let $a$ be an isotropic line with $a \cdot e_1 \neq 0$. Then $1ka \cap Y_1 = \{B \underset{\neq}{\supset} a | B \cdot B = 0\}$. The map $B \mapsto B \cap e_1{}^\perp$ gives a homotopy equivalence of this poset with $\{A \subset (e_1{}^\perp \cap a^\perp) | A \cdot A = 0,\ A \neq 0,\ \dim A \leq k - 1\}$. The subspace $(e_1{}^\perp \cap a^\perp)$ is hyperbolic of dimension $n - 1$, so by induction $|1ka \cap Y_1|$ is homotopy equivalent to a wedge of $(k-2)$-spheres. Therefore the realization of $X_{n,k}$ is homotopy equivalent to $|Y_1| \vee \underset{a \in X_{n,k} - Y_1}{V} \operatorname{susp} |1ka \cap Y_1| \simeq VS^{k-1} \vee VS^{k-1} \simeq VS^{k-1}$. □

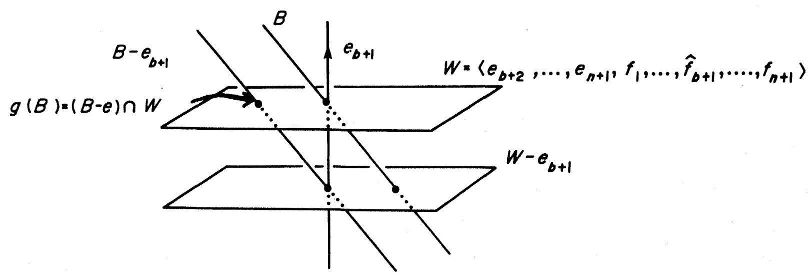
*Remark* 1.7. Note that in particular this shows that $X_n = \{$nonzero isotropic subspaces of $V\}$ is $(n-1)$-spherical. It is actually Cohen–Macaulay; if $A, A' \in X_n$, then $X_{n<A}$ and $(A, A')$ are $C - M$ by the Solomon–Tits theorem. To see that $X_{n>A}$ is $C - M$, choose a subspace $A'$, $\dim A' = \dim A$, such that $A \oplus A'$ is hyperbolic. Then $W = A^\perp \cap A'^\perp$ is a hyperbolic complement, and the map $B \mapsto B \cap W$ induces a homotopy equivalence $X_{n>A} \to \{B \subseteq W | B \cdot B = 0,\ B \neq 0\}$, which is $(n - \dim A - 1)$-spherical by Theorem 1.6.

There is one more poset which we will need to show is Cohen–Macaulay. Again let $V$ be a $2n$-dimensional vector space with quadratic form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Let $X^{k,n}$ denote the poset of all isotropic subspaces $A$ of $V$ such that $A + \langle e_{k+1}, \ldots, e_n, f_1, \ldots, f_n \rangle = V$. To prove that $X^{k,n}$ is Cohen–Macaulay, we will actually consider a more general class of posets and use a theorem of Qullen's to show that they are all Cohen–Macaulay. I am indebted to K. Igusa for most of the following argument.

Let $C^{b,a,n}$ be the poset of all isotropic subspaces of $W = \langle e_{b+1}, \ldots, e_n, f_1, \ldots, f_n \rangle$ such that $A + \langle e_{a+1}, \ldots, e_n, f_1, \ldots, f_n \rangle = W$ and $A \cap \langle f_1, \ldots, f_a \rangle = 0$. Note that $C^{0,0,n} = X_n$, and $C^{0,a,n} = X^{a,n}$. Let $\bar{C}^{b,a,n}$ be the poset of *affine* subspaces $X + \underline{v}$, where $X \in C^{b,a,n}$ (if $b = a = 0$, we allow $X = 0$) and $v \in \langle e_{b+1}, \ldots, e_n, f_1, \ldots, f_n \rangle$. Then we have the following relationship:

Proposition 1.8. *$\bar{C}^{b,a,n} \cong C^{b,a+1,n+1}$, for $a \geq b$.*

*Proof.* The map $g$: $C^{b,a+1,n+1} \to \bar{C}^{b,a,n}$ is given by $g(B) = \pi((B - e_{b+1}) \cap \langle e_{b+2}, \ldots, e_{n+1}, f_1, \ldots, f_{n+1} \rangle)$ where $\pi$ is projection along $f_{b+1}$. Then $g(B)$ is an affine subspace of $\langle e_{b+2}, \ldots, e_{n+1}, f_1, \ldots, \hat{f}_{b+1}, \ldots, f_{n+1} \rangle$ parallel to $\pi(B \cap \langle e_{b+2}, \ldots, e_{n+1}, f_1, \ldots, f_{n+1} \rangle)$, i.e. $g(B) \in \bar{C}^{b,a,n}$.




(The above picture is projected along $f_{b+1}$.) To define the inverse map, we first define maps $\psi$: $V \to V$ by $\psi(v) = v + e_{b+1} - \frac{1}{2}(v \cdot v) f_{b+1}$ and, for $v \in V$, define $\phi_v$: $V \to V$ by $\phi_v(u) = u - (v \cdot u) f_{b+1}$. Then the inverse map $f$: $\bar{C}^{b,a,n} \to C^{b,a+1,n+1}$ is given by $f(X + \underline{v}) = \langle \psi(v), \phi_v(X) \rangle$. Since $\phi_v$ is an orthogonal linear map with image contained in $\psi(v)^\perp$, we have $\langle \phi(v), \phi_v(X) \rangle$ is isotropic; transversality is guaranteed by the fact that $X$, and hence $\phi_v(X)$, is transverse to $\langle e_{b+1}, e_{a+2}, \ldots, e_{n+1}, f_1, \ldots, f_{n+1} \rangle$, and $\psi(v)$ has a nonzero $e_{b+1}$-component. Also, $\langle \psi(v), \phi_v(X) \rangle \cap \langle f_1, \ldots, f_a \rangle = 0$, and the map $f$ is independent of the choice of $v$, since if $X + \underline{v}_1 = X + \underline{v}_2$, then $v_1 - v_2 \in X$, so $\langle \psi(v_1), \phi_{v_1}(X) \rangle = \langle \psi(v_2), \phi_{v_2}(X) \rangle$.

It is clear that $g(f(X + \underline{v})) = X + \underline{v}$. To see that $f(g(B)) = B$, write $B = \langle e_{b+1} + v, X \rangle$, where $v$, $X \subset \langle e_{b+2}, \ldots, e_{n+1}, f_1, \ldots, f_{n+1} \rangle$. Then $g(B) = \pi v + \pi V$, and $f(g(B)) = \langle \psi(\pi v), \phi_{\pi v}(\pi X) \rangle = \langle e_{b+1} + \pi v - \frac{1}{2}(\pi v \cdot \pi v) f_{b+1}, \pi X - (\pi v, \pi X) f_{b+1} \rangle = \langle e_{b+1} + v, X \rangle = B$. □

To study the posets $\bar{C}^{b,a,n}$, we need the following definitions and theorem of Quillen's.

*Definition.* Let $X$ and $Y$ be posets. A map $f$: $X \to Y$ is a *poset map* if $x_1 > x_2$ implies $f(x_1) \geq f(x_2)$ for $x_1, x_2 \in X$.

*Notation.* Given a poset map $f$: $X \to Y$, then

$$f/y = \{x \in X \mid f(x) \leq y\}$$

$$f\backslash y = \{x \in X \mid f(x) \geq y\}.$$

*Definition.* A poset map $f\colon X \to Y$ is *strictly increasing* if $x_1 > x_2$ implies $f(x_1) > f(x_2)$ for $x_1, x_2 \in X$. We can now state Quillen's theorem [6, p. 120].

THEOREM. *Let $f\colon X \to Y$ be a poset map. Assume*
(i) *$Y$ is $n$-spherical.*
(ii) *$f/y$ is $h(y)$-spherical for all $y \in Y$.*
(iii) *$Y_{>y}$ is $(n - h(y) - 1)$-spherical for all $y \in Y$.*

Then $X$ is $n$-spherical.

COROLLARY 1.9. *Let $f\colon X \to Y$ be a strictly increasing poset map. Assume (i) $Y \in CM^n$.*
*(ii) $f/y \in CM^{h(y)}$ for all $y \in Y$. Then $X \in CM^n$.*

We want to apply this theorem to the posets $\bar{C}^{b,a,n}$. We first would like to (belatedly) introduce the notation $W_k^n = \langle e_{k+1}, \ldots, e_n, f_1, \ldots, f_n\rangle$. Now define a poset map $j\colon \bar{C}^{b,a,n} \to \bar{C}^{b+1,a,n}$ $(a > b)$ by the formula $j(X + \underline{v}) = (X + \underline{v}) \cap W_{b+1}^n$. We need to identify the "fibers" $j/Y$.

PROPOSITION 1.10. If $Y \in \bar{C}^{b+1,a,n}$, then $j\backslash Y \cong \bar{C}^{b,b,n-a+b-h(Y)} = \bar{C}^{b,b,n\text{-}\dim Y-1}$.

*Proof.* We may assume without loss of generality that $0 \in Y$. Then

$j\backslash Y = \{$subspaces $X \subseteq W_b^n$ such that $X \cdot X = 0$, $X + W_a^n = W_b^n$, $X \cap \langle f_1, \ldots, f_a\rangle = 0$ and $X \supseteq Y\}$.

It can easily be checked that the group of matrices in $0_{n,n}$ of the form

$$\begin{array}{l} n\left\{\begin{array}{l} a\left\{\begin{array}{l} b+1\{ \\ \\ \end{array}\right. \\ \\ \end{array}\right. \\ n\left\{\begin{array}{l} a\left\{\begin{array}{l} b+1\{ \\ \\ \end{array}\right. \\ \\ \end{array}\right. \end{array} \left(\begin{array}{ccc|ccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & * & A & 0 & 0 & B \\ \hline 0 & 0 & 0 & I & 0 & 0 \\ 0 & * & * & 0 & I & * \\ 0 & * & C & 0 & 0 & D \end{array}\right),$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in 0_{n-a,n-a}$ acts transitively on $k$-dimensional subspaces in $C^{b+1,a,n}$. Let $g$ be a matrix as above with $gY = \langle e_{b+2}, \ldots, e_a, \ldots, e_{a+h(Y)}\rangle$. Then

$$W_b^n \cap (gY)^\perp = \langle e_{b+1}, \ldots, e_n, f_1, \ldots, f_{b+1}, f_{a+h+1}, \ldots, f_n\rangle.$$

Let

$$\begin{aligned} W &= \langle e_{b+1}, e_{a+h+1}, \ldots, e_n, f_1, \ldots, f_b, f_{b+1}, f_{a+h+1}, \ldots, f_n\rangle \\ &\qquad \text{(which we think of as } (gY)^\perp/gY) \\ &\cong W_b^{n-d(Y)}; \end{aligned}$$

let

$$\begin{aligned} V &= \langle e_{a+h+1}, \ldots, e_n, f_1, \ldots, f_b, f_{b+1}, f_{a+h+1}, \ldots, f_n\rangle \\ &\quad (= W/\langle e_{b+1}\rangle) \cong W_{b+1}^{n-d(Y)}; \end{aligned}$$

and let

$$Z = \langle f_1, \ldots, f_b \rangle = gZ.$$

*Claim.* The map $C^{b,b+1,n-d(Y)} = C^{Z,V,M} \to j\backslash gY$ induced by the map $X \to X \oplus gY$ is an isomorphism.

*Proof.* To see this is well-defined, we must show $X \oplus gY$ is in $j\backslash gY$. $X \oplus gY$ clearly contains $gY$; since $X \subseteq gY^{\perp}$, we have $(X \oplus gY)\cdot(X \oplus gY) = 0$. Also, $gY$ projects onto $\langle e_{b+2}, \ldots, e_a \rangle$, and $X$ projects onto $\langle e_{b+1} \rangle$, so $X \oplus gY$ projects onto $\langle e_{b+1}, \ldots, e_a \rangle$, i.e. $(X \oplus gY) + W_a^n = W_b^n$. Finally, we must show that $(X \oplus gY) \cap \langle f_1, \ldots, f_a \rangle = 0$. But this is clear since

$$X \subseteq \langle e_{b+1}, e_{a+h+1}, \ldots, e_n, f_1, \ldots, f_{b+1}, f_{a+h+1}, \ldots, f_n \rangle$$
$$X \cap \langle f_1, \ldots, f_{b+1} \rangle = 0$$

and

$$gY = \langle e_{b+2}, \ldots, e_{a+h} \rangle.$$

It is also clear that the map is onto and injective, and a poset map, so gives a simplicial isomorphism of complexes.

Now to prove the assertion of the proposition, we need only observe that applying $g^{-1}$ to the above construction gives an isomorphism

$$C^{b,b+1,n-d}(Y) \cong C^{Z,g^{-1}V,g^{-1}W} \xrightarrow{\cong} j\backslash Y.$$

And by Proposition 1.8, $C^{b,b+1,n-d(Y)} \cong \bar{C}^{b,b,n-d(Y)-1}$. □

Let $p\colon \bar{C}^{a,a,n} \to \bar{C}^{0,0,n-a}$ be the map induced by projection along $\langle f, \ldots, f_a \rangle$, which we will also call $p$.

PROPOSITION 1.11. Let $Y \in \bar{C}^{0,0,n-a}$. Then $p/Y$ is Cohen–Macaulay of dimension $\dim Y - a$.

*Proof.* Without loss of generality, we may assume $Y$ contains 0. Recall that

$$\begin{aligned} p/Y &= \{X + \underline{v} \in \bar{C}^{a,a,n} \text{ s.t. } p(X + \underline{v}) \subseteq Y\} \\ &= \{X + \underline{v} \in \bar{C}^{a,a,n} \text{ s.t. } pX \subseteq Y \text{ and } pv \in Y\} \\ &= \{X + \underline{v} \in W_a^n \text{ s.t. } X \cdot X = 0,\ X \cap \langle f_1, \ldots, f_a \rangle = 0, \text{ and } pX, pv \subseteq Y\}. \end{aligned}$$

Since $pX \subseteq Y$, $pX$ is isotropic; therefore $pX \oplus \langle f_1, \ldots, f_a \rangle$ is isotropic. Since $X \subseteq p^{-1}(pX) = pX \oplus \langle f_1, \ldots, f_a \rangle$, $X$ itself is automatically isotropic. Therefore the above poset is equal to

$$\begin{aligned} &\{X + \underline{v} \subseteq W_a^n \text{ s.t. } X \cap \langle f_1, \ldots, f_a \rangle = 0 \text{ and } pX, pv \subseteq Y\} \\ &= \{X + \underline{v} \subseteq p^{-1}(Y) \text{ s.t. } X \cap \langle f_1, \ldots, f_a \rangle = 0\}. \end{aligned}$$

The map $(X + \underline{v}) \mapsto \langle X, (v, 1) \rangle$ now gives a poset isomorphism $p/Y \cong {}^{\langle f_1,\ldots,f_a \rangle}T^{p^{-1}Y, p^{-1}Y \oplus F}$. By Corollary 1.5, this is Cohen–Macaulay of dimension $\dim Y - a$. □

PROPOSITION 1.12. $\bar{C}^{0,0,n+1}$ is Cohen–Macaulay of dimension $n+1$.

*Proof.* The proof proceeds by induction on $n$. If $n=0$, $\bar{C}^{0,0,1}$ consists of cosets of $\langle e_1\rangle$, $\langle f_1\rangle$ and 0. The realization is clearly one-dimensional and connected, so is Cohen–Macaulay of dimension 1.

Now let

$$H = \langle e_2, \dots, e_{n+1}, f_1, \dots, f_{n+1}\rangle$$
$$H_\lambda = H + \lambda \underline{e_1}$$
$$P_\lambda = \{X \in \bar{C}^{0,0,n+1} \text{ such that } X \cap H_\lambda \neq \emptyset\}.$$

Then for any $\lambda \neq \mu$,

$$P_\lambda \cap P_\mu = \{\text{cosets of isotropic subspaces } A \text{ such that } A + H = V\} = \bar{C}^{0,1,n+1}.$$

The maps $\bar{C}^{0,1,n+1} \xrightarrow{j} \bar{C}^{1,1,n+1} \xrightarrow{p} \bar{C}^{0,0,n}$ are both strictly increasing. By Proposition 1.11, $p/Y$ is Cohen–Macaulay for all $Y \in \bar{C}^{0,0,n}$, and since $\bar{C}^{0,0,n}$ is Cohen–Macaulay by induction, Corollary 1.9 shows $\bar{C}^{1,1,n+1}$ is Cohen–Macaulay. By Proposition 1.11, $j\backslash Y \cong \bar{C}^{0,0,n\text{-dim}\,Y}$; this is Cohen–Macaulay by induction, so Corollary 1.9 again shows that $\bar{C}^{0,1,n+1}$ is Cohen–Macaulay.

*Claim.* $P_\lambda$ is homotopy equivalent to $\bar{C}^{0,0,n}$.

*Proof.* The map $X \mapsto X \cap H_\lambda$ gives a deformation retraction to cosets of isotropic subspaces in $H_\lambda$; this poset is isomorphic to the poset of cosets of isotropic subspaces of $H$. The maps $A \mapsto \langle A, f_1\rangle \mapsto \pi_{f_1}A$ then give a retraction of this poset to the poset of cosets of isotropic subspaces of $\langle e_2, \dots, e_{n+1}, f_2, \dots, f_{n+1}\rangle$ (here $\pi_{f_1}$ is projection along $f_1$), which is $\bar{C}^{0,0,n}$. □

By van Kampen's theorem, $\bar{C}^{0,0,n+1} = \bigcup_\lambda P_\lambda$ is simply connected if $n \geq 2$. If $n=1$, an edge-path calculation shows this is true [7, p. 39]. Therefore, in order to see that $\bar{C}^{0,0,n+1}$ is $(n+1)$-spherical, it suffices to show that $\bar{C}^{0,0,n+1} = \bigcup_\lambda P_\lambda$ has homology only in dimension $n+1$. We begin by considering $P_0 \cup P_1$; since $P_0$ and $P_1$ are each $n$-spherical, the Mayer–Vietoris sequence for $P_0 \cup P_1$ reduces to

$$0 \to H_{n+1}(P_0 \cup P_1) \to H_n(Q) \xrightarrow{f_0+f_1} H_nP_0 \oplus H_nP_1 \to H_n(P_0 \cup P_1) \to 0$$

where $Q = P_0 \cap P_1$ and $f_\lambda \colon Q \to P_\lambda$ is the inclusion map. Note that for *any* $\lambda \in F$, $f_\lambda$ is onto in homology, since we can easily construct a homotopy section $\phi_\lambda \colon P_\lambda \simeq \bar{C}^{0,0,n} \to \bar{C}^{0,1,n+1} = Q$; e.g. for $X + \underline{v}$ in $\bar{C}^{0,0,n}$, define $\phi_\lambda(X + \underline{v}) = \langle X, e_1\rangle + \underline{v}$.

*Claim.* $f_0 + f_1$ is onto, and hence $H_n(P_0 \cup P_1) = 0$.

*Proof.* Since $f_0$ and $f_1$ are each onto, it suffices to construct homotopy sections $s_0$ for $f_0$ and $s_1$ for $f_1$ such that $f_1 \circ s_0 \simeq 0$ and $f_0 \circ s_1 \simeq 0$. For $X + \underline{v}$ in $\bar{C}^{0,0,n}$, define

$$s_1(X + \underline{v}) = \text{subspace of } V \text{ spanned by the vectors } a - (a \cdot v)f_1 \text{ for } a \in X \text{ and the vector } e_1 - \tfrac{1}{2}(v \cdot v)f_1 + v$$

$$s_0(X+\underline{v}) = \underline{e}_1 + (\text{subspace spanned by the vectors } a+(a\cdot v)f_1,\ a\in X \text{ and } v-e_1+\tfrac{1}{2}(v\cdot v)f_1).$$

These sections are well-defined and inclusion-preserving; $f_0\circ s_1(X+\underline{v})$ is a *subspace* of $H_0 = H$, so can be retracted to the zero subspace, and $f_1\circ s_0$ is similarly homotopic to a point. □

Thus $H_n(P_0\cup P_1)=0$. Now we notice that adding any number of $P_\lambda$'s to $P_0\cup P_1$ does not add any $n$-dimensional homology, since each $f_\lambda$ is onto in homology. Since homology commutes with direct limits, we have $H_n(\bigcup_\lambda P_\lambda) = H_n(\bar{C}^{0,0,n+1}) = 0$.

To complete the proof that $\bar{C}^{0,0,n+1}$ is Cohen–Macaulay, we need to check the subcomplexes $\bar{C}^{0,0,n+1}_{<Y}$ and $\bar{C}^{0,0,n+1}_{>Y}$ for each $Y = X+\underline{v}$ in $\bar{C}^{0,0,n+1}$. In each case we may assume $Y\ni 0$. Then $\bar{C}^{0,0,n+1}_{>Y}$ is the set of isotropic subspaces containing $Y$, which is spherical by Remark 1.7. $\bar{C}^{0,0,n+1}_{<Y}$ is the set of all proper subspaces of $\langle a, X\rangle$ which are transverse to $\langle a\rangle$, via the map $B+\underline{w}\mapsto\langle B, w+a\rangle$. This is spherical by Corollary 1.3. □

We can now prove the theorem we were after, namely.

THEOREM 1.13. *$\bar{C}^{0,a,n}$ is Cohen–Macaulay of dimension $n-a$, for $a\geq 0$.*

*Proof.* For $a=0$, this is Proposition 1.12. If $a>0$, consider the map $p\colon \bar{C}^{a,a,n}\to \bar{C}^{0,0,n-a}$. By Proposition 1.11, $p/Y$ is Cohen–Macaulay for each $Y\in\bar{C}^{0,0,n-a}$; since $p$ is strictly increasing, Corollary 1.9 implies $\bar{C}^{a,a,n}$ is Cohen–Macaulay.

By induction and Proposition 1.10, each of the maps

$$\bar{C}^{0,a,n}\xrightarrow{j}\bar{C}^{1,a,n}\to\cdots\to\bar{C}^{a,a,n}$$

has Cohen–Macaulay fibers; therefore another application of Corollary 1.9 shows that $\bar{C}^{0,a,n}$ is Cohen–Macaulay. □

## §2. HOMOLOGY STABILITY FOR $0_{n,n}$

We will study the homology of $0_{n,n}$ by considering the action of $0_{n,n}$ on the simplicial complex $X_n$ = realization of $X_n$. Recall from §1 that $X_n$ = {non-zero isotropic subspaces of a $2n$-dimensional vector space}. We let $0_{n,n}$ act on $X_n$ on the left in the natural way; then the filtration of $X_n$ by the subcomplexes $X_{n,k}$ = realization of $\{A\in X_n \text{ s.t. } \dim A\leq k\}$ is equivariant. We have

$$\emptyset = X_{n,0}\subset X_{n,1}\subset\cdots\subset X_{n,n} = X_n.$$

By Theorem 1.6, $X_{n,k}$ is $(k-1)$-spherical; therefore the spectral sequence associated to this filtration, with $E^1_{p,q} = H_{p+q}(X_{n,p+1}, X_{n,p})$, collapses, giving an exact sequence

$$(*)\quad 0\to H_{n-1}(X_n)\to H_{n-1}(X_n, X_{n,n-1})\to H_{n-2}(X_{n,n-1}, X_{n,n-2})$$
$$\to\cdots\to H_1(X_{n,2}, X_{n,1})\to H_0(X_{n,1})\to \mathbf{Z}\to 0.$$

We can further identify these homology groups by noting that for $0\leq p\leq n-1$, we know $X_{n,p+1}$ is obtained from $X_{n,p}$ by attaching isotropic subspaces $A$ of dimension $p+1$. For each such $A$, $1kA\cap X_{n,p}$ = {all proper subspaces of $A$}; by the Solomon–

Tits theorem, the realization of $1kA \cap X_{n,p}$ is homotopy equivalent to a wedge of $(p-1)$-spheres. Therefore

$$\frac{X_{n,p+1}}{X_{n,p}} \simeq \bigvee_{\substack{\text{isotropic } A \\ \dim A = p+1}} \operatorname{susp}|1kA \cap X_{n,p}|,$$

so

$$H_p(X_{n,p+1}, X_{n,p}) \cong \bigoplus_{A^{p+1}} \tau(A^{p+1})$$

where $\tau(A^{p+1}) = H_{p-1}$ (Tits building for $A^{p+1}$). $0_{n,n}$ acts on this direct sum by permuting the $A^{p+1}$, as well as acting on each $\tau(A^{p+1})$.

Let $0 \to C_{n+1} \to C_n \to \cdots \to C_1 \to C_0 = \mathbf{Z} \to 0$ denote the exact sequence (*), and let $E0_{n,n*} = E_*$ be a free $\mathbf{Z}[0_{n,n}]$-resolution of $\mathbf{Z}$. Then we can form the double complex $E_* \otimes_{0_{n,n}} C_*$:

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & \\
\leftarrow & E_i \otimes C_{j+1} & \xleftarrow{\partial_E \otimes 1} & E_{i+1} \otimes C_{j+1} & \leftarrow \\
 & \downarrow{\scriptstyle 1\otimes(-1)^i\partial_c} & & \downarrow{\scriptstyle 1\otimes(-1)^{i+1}\partial_c} & \\
\leftarrow & E_i \otimes C_j & \xleftarrow[\partial_E \otimes 1]{} & E_{i+1} \otimes C_j & \leftarrow \\
 & \downarrow & & \downarrow &
\end{array}$$

The vertical filtration of this double complex gives a spectral sequence with $E^1_{p,q} = H_q(E_p \otimes C_*, (-1)^p \partial_c) = 0$ since $E_p$ is free and $C_*$ is exact. The horizontal filtration gives a spectral sequence with $E^1_{p,q} = H_q(E_* \otimes C_p, \partial_E) = H_q(0_{n,n}; C_p)$. If $0 < p < n+1$, we know $C_p = \bigoplus_{A^p} \tau(A^p)$, so $H_q(0_{n,n}; C_p) = H_q(0_{n,n}; \bigoplus_{A^p} \tau(A^p)) = H_q(0_{n,n}; Z[0_{n,n}] \otimes_{ZS_{p,n}} \tau_p)$, where $\tau_p = \tau(\langle e_1, \ldots, e_p \rangle)$ and $S_{p,n}$ is the stabilizer in $0_{n,n}$ of $\langle e_1, \ldots, e_p \rangle$. By Shapiro's lemma, this last homology group is isomorphic to $H_q(S_{p,n}; \tau_p)$. We have just proved

THEOREM 2.1. *There is a spectral sequence converging to zero with* $E^1_{p,q} = H_q(S_{p,n}; \tau_p)$ *for* $0 \le p \le n$. □

If we consider $0_{n,n}$ as acting on the left, it is easy to calculate that the stabilizer $S_{p,n}$ is the set of all matrices in $0_{n,n}$ of the form

$$\left\{\begin{matrix} \alpha & * & * & * \\ 0 & A & * & B \\ 0 & 0 & {}^t\alpha^{-1} & 0 \\ 0 & C & * & D \end{matrix}\right\},$$

where $\alpha \in GL_p$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in 0_{n-p,n-p}$. The subgroup $S_{p,n}$ acts transitively on the *right* on the set of $(p+i)$-dimensional isotropic subspaces $X \subseteq V$ with $X + \langle e_{p+1}, \ldots, e_n, f_1, \ldots, f_n \rangle = V$; i.e. $S_{p,n}$ acts "transitively" on $X^{p,n}$. In fact the subgroup

$F_{p,n}$ of $S_{p,n}$ which *fixes* $\langle e_1, \dots, e_p\rangle$ also acts transitively in this sense on $X^{p,n}$. Here $F_{p,n}$ consists of matrices in $S_{p,n}$ of the form

$$\begin{Bmatrix} I & * & * & * \\ 0 & A & * & B \\ 0 & 0 & I & 0 \\ 0 & C & * & D \end{Bmatrix}.$$

By Theorem 1.13, we know that $X^{p,n}$ is Cohen–Macaulay of dimension $n-p$. We can filter $X^{p,n}$ in the same way as $X_n$, by the subcomplexes $Z_i$ = realization of $\{A \in X^{p,n}$ such that $\dim A \le i+p\}$. Then

$$\emptyset \subset Z_0 \subset Z_1 \subset \cdots \subset Z_{n-p} = X^{p,n}.$$

For any vertex $A$ in $Z_i - Z_{i-1}$, $1kA \cap Z_{i-1} = |\{B \underset{\neq}{\subset} A$ such that $B + (A \cap W_p{}^n) = A\}|$, which is $(i-1)$-spherical by Corollary 1.3. Therefore

$$Z_i/Z_{i-1} \simeq \bigvee_{A^{i+p} \in X^{p,n}} \operatorname{susp} |1kA \cap Z_i|$$

$$\simeq \bigvee_{A^{i+p} \in X^{p,n}} (VS^i),$$

so

$$H_{i+j}(Z_i, Z_{i-1}) = \begin{cases} \bigoplus_{A^{i+p} \in X^{p,n}} \sigma(A^{i+p}), & j = 0 \\ 0 & j \neq 0 \end{cases}$$

where $\sigma(A^{i+p}) = H_{i-1}({}^0T^{(A \cap W_p{}^n), A})$.

Thus the spectral sequence of the filtration $\{Z_i\}$ collapses, giving an exact sequence

$$0 \to H_{n-p}(X^{p,n}) \to H_{n-p}(Z_{n-p}, Z_{n-p-1}) \to \cdots \to H_1(Z_1, Z_0) \to H_0(Z_0)$$
$$\to \mathbf{Z} \to 0.$$

We denote this complex by $D_*$; $D_*$ has a natural equivariant $F_{p,n}$-action. Let $EF_{p,n^*}$ be a free $\mathbf{Z}[F_{p,n}]$-resolution of $\mathbf{Z}$, and form the double complex $D_* \otimes_{F_{p,n}} EF_{p,n^*}$. As before, this gives a spectral sequence converging to zero with $E^1_{s,t} = H_t(F_{p,n}; D_s)$, and we use Shapiro's lemma to identify this term and obtain

THEOREM 2.2. *Let $R_{s,p,n}$ be the stabilizer in $F_{p,n}$ of the subspace $\langle e_1, \dots, e_{p+s}\rangle$ under the natural right action of $F_{p,n}$ on $X^{p,n}$ and let $\sigma_s = \sigma\langle e_1, \dots, e_{p+s}\rangle$. Then there is a spectral sequence converging to zero with $E^1_{0,t} = H_t(F_{p,n}; \mathbf{Z})$, and $E^1_{s,t} = H_t(R_{s-1,p,n}; \sigma_{s-1})$ for $0 < s < n-p+1$.*

The stability theorem we want says that $H_k(0_{n,n}) \to H_k(0_{n+1,n+1})$ is an isomorphism for $n$ sufficiently large with respect to $k$. We will prove this by showing that the relative groups $H_k(0_{n+1,n+1}, 0_{n,n})$ vanish for $n$ large. Therefore we actually want to consider relative versions of the spectral sequences 2.1 and 2.2. Let $G_n = 0_{n,n}$ or $S_{p,n}$, and $K_{n^*} = C_*$ or $D_*$ respectively. Then the inclusion $G_n \to G_{n+1}$ induces natural equivariant maps $K_{n*} \to K_{n+1*}$ and $EG_{n*} \to EG_{n+1*}$, and therefore a map from the spectral sequence for $G_n$ to the spectral sequence for $G_{n+1}$. If we take the mapping cone of this map, we get a "relative" spectral sequence [8], i.e.

THEOREM 2.3. *There is a spectral sequence converging to zero with* $E^1_{p,q} = H_q(S_{p,n+1}, S_{p,n}; \tau_p)$ *for* $0 \le p \le n$.

THEOREM 2.4. *There is a spectral sequence converging to zero with* $E^1_{0,t} = H_t(F_{p,n+1}, F_{p,n})$ *and* $E^1_{s,t} = H_t(R_{s-1,p,n+1}, R_{s-1,p,n}; \sigma_{s-1})$ *for* $0 < s < n+1$.

We are now ready to prove the main theorem.

THEOREM 2.5. *Let F be a field with more than two elements, and let* $0_n = 0_{n,n}(F)$. *Then* $H_k(0_{n+1}, 0_n) = 0$ *for* $n \ge 3k-1$.

*Proof.* We will prove the theorem by induction on $k$. Specifically, we will assume

$(a)_{k-1}$: $H_l(0_{n+1}, 0_n) = 0$ for $l \le k-1$ and $n \ge 3l-1$

$(b)_{k-1}$: $H_l(F_{p,n+1}, F_{p,n}) = 0$ for $n-p \ge 3l$ and $l \le k-1$.

Then to prove that $H_k(0_{n+1}, 0_n) = 0$, a diagram chase of the following diagram shows that it suffices to show that the map $j_*$: $H_k(0_{n-1}, 0_{n-2}) \to H_k(0_n, 0_{n-1})$ is onto:

$$\begin{array}{ccccccccc} & & H_k(0_{n-1}, 0_{n-2}) & \to & H_{k-1}(0_{n-2}) & & & & \\ & & \downarrow j_* & & \downarrow & & & & \\ H_k(0_n) & \to & H_k(0_n, 0_{n-1}) & \to & H_{k-1}(0_{n-1}) & \xrightarrow{i_*} & H_{k-1}(0_n) & \to & 0 \\ \downarrow & & \downarrow j_* & & & & & & \\ H_k(0_{n+1}) & \to & H_k(0_{n+1}, 0_n). & & & & & & \end{array}$$

The map $j_*$ is the composition of the maps

$$H_k(0_{n-1}, 0_{n-2}) \xrightarrow{d} H_k(F_{1,n}, F_{1,n-1}) \xrightarrow{i} H_k(S_{1,n}, S_{1,n-1}) \xrightarrow{d'} H_k(0_n, 0_{n-1})$$

where $i$ is induced by inclusion, $d$ is the $d_1$-map of the spectral sequence 2.4 and $d'$ is the $d_1$-map of the spectral sequence 2.3. We will use our induction hypotheses to show that each of the above maps is onto.

LEMMA 1. *Let*

$$d': H_k(S_{1,n}, S_{1,n-1}) \to H_k(0_n, 0_{n-1})$$

*be the* $d_1$*-map of the spectral sequence* 2.3. *Then for* $n \ge 3k-1$, $d'$ *is onto.*

*Proof.* Since we know the spectral sequence 2.3 converges to zero, we can show $d'$ is onto by showing that the terms $E^1_{s,k-s+1}$ are zero for $2 \le s \le k+1$ (so $d'$ is the only non-zero differential). If $k+1<n+1$, we have

$$E^1_{s,k-s+1}=H_{k-s+1}(S_{s,n},S_{s,n-1};\tau_s).$$

To show that this is zero, we consider the extensions

$$\begin{array}{ccccccc} 1\to F_{s,n-1} & \to & S_{s,n-1} & \to & GL_s & \to 1\\ \downarrow & & \downarrow & & \| & \\ 1\to F_{s,n} & \to & S_{s,n} & \to & GL_s & \to 1. \end{array}$$

The mapping cone spectral sequence of this diagram [3] has

$$E^2_{u,v}=H_u(GL_s;H_v(F_{s,n},F_{s,n-1};\tau_s)]\Rightarrow H_{u+v}(S_{s,n},S_{s,n-1};\tau_s).$$

Since $F_{s,n}$ acts trivially on $\tau_s$, our induction hypothesis $(b)_{k-1}$ shows $H_v(F_{s,n},F_{s,n-1};\tau_s)=0$ for $v<k$; thus all terms in the filtration of $H_{k-s+1}(S_{s,n},S_{s,n-1};\tau_s)$ are zero.

Lemma 2. *The map* $i$: $H_k(F_{1,n},F_{1,n-1})\to H_k(S_{1,n},S_{1,n-1})$ *induced by inclusion is onto for* $n\geq 3k-1$.

*Proof.* The diagram

$$\begin{array}{ccccccc} 1\to F_{1,n-1} & \to & S_{1,n-1} & \to & GL_1 & \to 1\\ \downarrow & & \downarrow & & \| & \\ 1\to F_{1,n} & \to & S_{1,n} & \to & GL_1 & \to 1 \end{array}$$

gives a spectral sequence

$$E^2_{u,v}=H_u(GL_1;H_v(F_{1,n},F_{1,n-1}))\Rightarrow H_{u+v}(S_{1,n},S_{1,n-1}).$$

All the terms of the filtration for $H_k(S_{1,n},S_{1,n-1})$ are zero by $(b)_{k-1}$ except $E^\infty_{0,k}$. Thus

$$\begin{aligned} H_k(F_{1,n},F_{1,n-1}) &\twoheadrightarrow H_0(GL_1;H_k(F_{1,n},F_{1,n-1}))=E^2_{0,k}\\ &\twoheadrightarrow E^\infty_{0,k}=H_k(S_{1,n},S_{1,n-1}). \end{aligned}$$

Lemma 3. *Let* $d$: $H_k(0_{n-1},0_{n-2})\to H_k(F_{1,n},F_{1,n-1})$ *be the* $d_1$*-map of the spectral sequence* 2.3. *Then for* $n\geq 3k-1$, $d$ *is onto.*

***Proof.*** As in Lemma 1, we need only show that $E^1_{s,k-s+1}=0$ for $2\leq s\leq k+1$. If $k+1<n+1$, we have $E^1_{s,k-s+1}=H_{k-s+1}(R_{s-1,1,n},R_{s-1,1,n-1};\sigma_{s-1})$. Recall that $R_{s,p,n}$ is the stabilizer in $S_{p,n}$ of $\langle e_1,\dots,e_{p+s}\rangle$, i.e. $R_{s,p,n}$ consists of matrices in $S_{p,n}$ of the form

$$n\left\{\begin{array}{l} p\{\\ s\{\\ \\ \end{array}\right.\begin{pmatrix} \alpha & * & 0 & 0 & 0 & 0\\ 0 & X & 0 & 0 & 0 & 0\\ 0 & * & A & 0 & 0 & B\\ 0 & 0 & 0 & {}^t\alpha^{-1} & 0 & 0\\ 0 & * & * & * & {}^tX^{-1} & *\\ 0 & * & C & 0 & 0 & D \end{pmatrix}$$

with $\alpha \in GL_p$, $X \in GL_x$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in 0_{n-p-s}$. The projection map

$$R_{s,p,n} \to G_{s,p} = \left\{ \begin{matrix} \alpha & * & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & {}^t\alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & * & {}^tX^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \right\}$$

has kernel $F_{s,n-p}$; the diagram

$$\begin{array}{ccccccccc} 1 \to & F_{s,n-p-1} & \to & R_{s,p,n-1} & \to & G_{s,p} & \to 1 \\ & \downarrow & & \downarrow & & \| & \\ 1 \to & F_{s,n-p} & \to & R_{s,p,n} & \to & G_{s,p} & \to 1 \end{array}$$

gives a spectral sequence with

$$E^2_{uv} = H_u(G_{s,p}; H_v(F_{s,n-p-1}; \sigma_s)) \Rightarrow H_{u+v}(R_{s,p,n}, R_{s,p,n-1}; \sigma).$$

Again $\sigma_s$ is a trivial $F_{s,n-p}$-module. For $p = 1$, $n > 3k - 1$ and $(b)_{k-1}$ guarantees that $E^2_{u,v} = 0$ if $u + v < k$, so all terms in the filtration of $H_{k-s+1}(R_{s-1,1,n}, R_{s-1,1,n-1}; \sigma_{s-1})$ are zero.

Lemmas 1–3 show that $j_*$ is onto, so $H_k(0_{n+1}, 0_n) = 0$. It remains to verify our induction hypotheses $(b)_k$: this is a consequance of the above statement together with Lemma 3. Q.E.D.

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