Finite length Solenoid potential and field

The surface current density is (Jackson, 1998):

$$\vec{K} = \frac{I}{L}\delta(\rho - a), \qquad z \in \left(-\frac{L}{2}, \frac{L}{2}\right)$$

The vector potential is:

$$\vec{A} = A_\phi \hat{\phi} = \hat{\phi}\frac{\mu_0}{4\pi}\frac{I}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^{2\pi}\int_0^{\infty}\frac{\delta(\rho' - a)\cos\phi'}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2a\rho\cos\phi'}}\rho' d\rho' \, d\phi' dz'$$

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^{\pi}\frac{\cos\phi'}{\sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi'}} d\phi' dz'$$

Simplify the form by setting $\zeta = (z - z')$ and the integration of ζ is a log function (Edmund E. Callaghan, 1960):

$$\int_0^{\pi}\cos\phi'\left[\ln\left(\zeta + \sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}\right)\right]_{\zeta_-}^{\zeta_+} d\phi' = \left[\int_0^{\pi}\cos\phi' \ln\big(\zeta + \alpha(\zeta)\big)\, d\phi'\right]_{\zeta_-}^{\zeta_+}$$

$$\alpha(\zeta) = \sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}, \qquad \zeta_\pm = z \mp \frac{L}{2}$$

Integration by path:

$$\int_0^{\pi}\cos\phi' \ln\big(\zeta + \alpha(\zeta)\big)\, d\phi' = \sin\phi' \ln\big(\zeta + \alpha(\zeta)\big)\Big]_0^{2\pi} - \int_0^{\pi}\sin\phi' \, d\Big(\ln\big(\zeta + \alpha(\zeta)\big)\Big)$$

The first term is zero, and the derivative of $\ln\big(\zeta + \alpha(\zeta)\big)$ is:

$$\frac{d\ln\big(\zeta + \alpha(\zeta)\big)}{d\phi'} = \frac{\rho a\sin\phi'}{(\alpha(\zeta) + \zeta)\alpha(\zeta)}$$

Multiple by $(\alpha(\zeta) + \zeta)/(\alpha(\zeta) - \zeta)$

$$= \frac{(\alpha(\zeta) - \zeta)\rho a\sin\phi'}{(\alpha^2(\zeta) + \zeta^2)\alpha(\zeta)} = \frac{\rho a\sin\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')} - \frac{\zeta\rho a\sin\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')\alpha(\zeta)}$$

The first term on the right side appeared on both $\alpha_\pm$, then,

$$\int_0^{\pi}\sin\phi' \, d\Big(\ln\big(\zeta + \alpha(\zeta)\big)\Big) = -\int_0^{\pi}\frac{\zeta\rho a\sin^2\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')\alpha(\zeta)} d\phi'$$

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia^2\rho}{L}\left[\zeta\int_0^{\pi}\frac{\sin^2\phi'}{(\rho^2 + a^2 - 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}} d\phi'\right]_{\zeta_-}^{\zeta_+}$$

By using the change of integration interval

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2(2\theta)}{((a+\rho)^2 - 4\rho a \sin^2\theta)\sqrt{\zeta^2 + (a+\rho)^2 - 4\rho a \sin^2\theta}} d\theta$$

$$= \frac{kh^2}{8\left(\sqrt{a\rho}\right)^3} \int_0^{\frac{\pi}{2}} \frac{\sin^2(2\theta)}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$= \frac{kh^2}{2\left(\sqrt{a\rho}\right)^3} \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta - \sin^4\theta}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$h^2 = \frac{4a\rho}{(a+\rho)^2}$$

$$k^2 = \frac{4a\rho}{(a+\rho)^2 + \zeta^2}$$

The integral can be spitted into 2 parts, the first part is (Milton Abramowitz, 1965) (NIST Digital Library of Mathematical Functions) :

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$= -\frac{1}{h^2} \int_0^{\frac{\pi}{2}} \frac{1 - h^2 \sin^2\theta - 1}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$= -\frac{1}{h^2} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}} - \frac{1}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$= \frac{1}{h^2}\left(\Pi(h^2, k^2) - K(k^2)\right)$$

Here

$$\Pi(n, m) = \int_0^{\frac{\pi}{2}} \frac{1}{(1 - n \sin^2\theta)\sqrt{1 - m \sin^2\theta}} d\theta$$

It is the elliptic integral of 3rd kind.

The 2nd part is:

$$\int_0^{\frac{\pi}{2}} \frac{-\sin^4\theta}{(1 - h^2 \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} d\theta$$

$$= \frac{1}{h^4}\int_0^{\frac{\pi}{2}} \frac{1 - h^4\sin^4\theta - 1}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}d\theta$$

$$= \frac{1}{h^4}\int_0^{\frac{\pi}{2}} \frac{1 + h^2\sin^4\theta}{\sqrt{1 - k^2\sin^2\theta}} - \frac{1}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}d\theta$$

$$= \frac{1}{h^4}\left(K(k^2) + \frac{h^2}{k^2}\big(K(k^2) - E(k^2)\big) - \Pi(h^2, k^2)\right)$$

Thus, combine it and we have:

$$A_\phi = \frac{\mu_0}{4\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\zeta k\left(\frac{k^2 + h^2 - h^2k^2}{h^2k^2}K(k^2) - \frac{1}{k^2}E(k^2) + \frac{h^2 - 1}{h^2}\Pi(h^2, k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

The magnetic field is the curl

$$B_\rho = [\nabla \times A_\phi]_\rho = -\frac{\partial}{\partial z}(A_\phi) = -\frac{\partial}{\partial \zeta}(A_\phi)$$

$$B_z = [\nabla \times A_\phi]_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) = \frac{1}{\rho}A_\phi + \frac{\partial A_\phi}{\partial \rho}$$

Using the derivative formulae for elliptic integral:

$$\frac{d}{dx}K(x) = -\frac{1}{2x}K(x) + \frac{1}{2x(1 - x)}E(k)$$

$$\frac{d}{dx}E(x) = -\frac{1}{2x}K(x) + \frac{1}{2k}E(k)$$

$$\frac{d}{dx}\Pi(\nu, x) = \frac{1}{2(x - 1)(\nu - x)}E(k) + \frac{1}{2(\nu - x)}\Pi(\nu, x)$$

Thus:

$$\frac{d}{dk}\left(k\left(\frac{k^2 + h^2 - h^2k^2}{h^2k^2}K(k^2) - \frac{1}{k^2}E(k^2) + \frac{h^2 - 1}{h^2}\Pi(h^2, k^2)\right)\right)$$

$$= -\frac{1}{k^2}K(k^2) + \frac{h^2}{k^2(h^2 - k^2)}E(k^2) + \frac{h^2 - 1}{(h^2 - k^2)}\Pi(h^2, k^2)$$

$$\frac{dk}{dz} = -\frac{k^3\left(z \pm \frac{L}{2}\right)}{4a\rho}, \qquad \frac{dk}{d\rho} = \frac{k}{2\rho} - \frac{k^3(a + \rho)}{4a\rho}$$

$$\frac{d}{dz}\big(zf(z)\big) = f(z) + z\frac{df(z)}{dz}, \qquad \frac{d}{d\rho}\left(\sqrt{\rho}f(\rho)\right) = \frac{f(\rho)}{2\sqrt{\rho}} + \sqrt{\rho}\frac{df(\rho)}{d\rho}$$

Then

$$B_\rho = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{2k}K(k^2)+\frac{1}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

Or, by using integration identity

$$\frac{d}{dx}\int_a^b f(x)dx = f(b)-f(a)$$

$$B_\rho = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{\cos\phi'}{\sqrt{\zeta^2+\rho^2+a^2-2\rho a\cos\phi'}}d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Since the angle integration is elliptic integral.

$$B_\rho = \frac{\mu_0}{2\pi}\frac{I}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{2k}K(k^2)+\frac{1}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}, \qquad k^2 = \frac{4a\rho}{(a+\rho)^2+\zeta^2}$$

Thus, the result is double verified. We should notices that the radial component of the magnetic field is just like 2 coil separated by distance L vertically.

The z-component is :

$$B_z = -\frac{\mu_0}{4\pi}\frac{I}{L2\sqrt{a\rho}}\left[\zeta k\left(K(k^2)+\frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

Or we can compute $\frac{\partial A_\phi}{\partial \rho}$, we use

$$A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \cos\phi' \ln\big(\zeta+\alpha(\zeta)\big)\,d\phi'\right]_{\zeta_-}^{\zeta_+}$$

By

$$\frac{\partial}{\partial\rho}\ln\big(\zeta+\alpha(\zeta)\big) = \frac{\rho-a\cos\phi'}{\alpha(\zeta)(\alpha(\zeta)+\zeta)}$$

Using the same trick

$$\frac{\rho-a\cos\phi'}{\alpha(\zeta)(\alpha(\zeta)+\zeta)} = \frac{(\rho-a\cos\phi')(\alpha(\zeta)-\zeta)}{\alpha(\zeta)(\alpha^2(\zeta)+\zeta^2)}$$

$$= \frac{(\rho-a\cos\phi')(\alpha(\zeta)-\zeta)}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')} = \frac{(\rho-a\cos\phi')}{(\rho^2+a^2-2\rho a\cos\phi')} - \frac{(\rho-a\cos\phi')\zeta}{\alpha(\zeta)(\rho^2+a^2-2\rho a\cos\phi')}$$

Therefore:

$$\frac{\partial A_\phi}{\partial \rho} = -\frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{\zeta\rho\cos\phi' - a\zeta\cos^2\phi'}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a\cos\phi')}d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Combined with

$$\frac{1}{\rho}A_\phi = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{a\zeta\sin^2\phi'}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a\cos\phi')}d\phi'\right]_{\zeta_-}^{\zeta_+}$$

Then the magnetic field is

$$B_z = \frac{\mu_0}{2\pi}\frac{Ia}{L}\left[\int_0^\pi \frac{\zeta(a - \rho\cos\phi')}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a\cos\phi')}d\phi'\right]_{\zeta_-}^{\zeta_+}$$

With change the interval

$$\int_0^\pi \frac{\zeta(a - \rho\cos\phi')}{(\rho^2 + a^2 - 2\rho a\cos\phi')\sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}}d\phi'$$

$$= \int_0^{\frac{\pi}{2}} \frac{a\zeta + \rho\zeta\cos(2\theta)}{((a+\rho)^2 - 4\rho a\sin^2\theta)\sqrt{\zeta^2 + (a+\rho)^2 - 4\rho a\sin^2\theta}}d\theta$$

$$= \frac{kh^2}{8\left(\sqrt{a\rho}\right)^3}\int_0^{\frac{\pi}{2}} \frac{(a+\rho)\zeta - 2\rho\zeta\sin^2\theta}{(1 - h^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}d\theta$$

$$= \frac{kh^2}{8\left(\sqrt{a\rho}\right)^3}\left((a+\rho)\zeta\,\Pi(h^2,k^2) - \frac{2\rho\zeta}{h^2}\left(\Pi(h^2,k^2) - K(k^2)\right)\right)$$

$$= \frac{k\zeta}{8\left(\sqrt{a\rho}\right)^3}\left((h^2(a+\rho) - 2\rho)\Pi(h^2,k^2) + 2\rho K(k^2)\right)$$

$$= -\frac{k\zeta}{8\left(\sqrt{a\rho}\right)^3}\left(2\rho K(k^2) + \frac{2\rho(a-\rho)}{a+\rho}\Pi(h^2,k^2)\right)$$

$$= -\frac{k\zeta}{4a\sqrt{a\rho}}\left(K(k^2) + \frac{(a-\rho)}{a+\rho}\Pi(h^2,k^2)\right)$$

Thus we get the same result.

$$B_z = -\frac{\mu_0}{4\pi}\frac{I}{L2\sqrt{a\rho}}\left[\zeta k\left(K(k^2) + \frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

In conclusion, the field is defined by:

$$A_\phi = \frac{\mu_0 I}{4\pi}\frac{1}{L}\sqrt{\frac{a}{\rho}}\left[\zeta k\left(\frac{k^2+h^2-h^2k^2}{h^2k^2}K(k^2)-\frac{1}{k^2}E(k^2)+\frac{h^2-1}{h^2}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

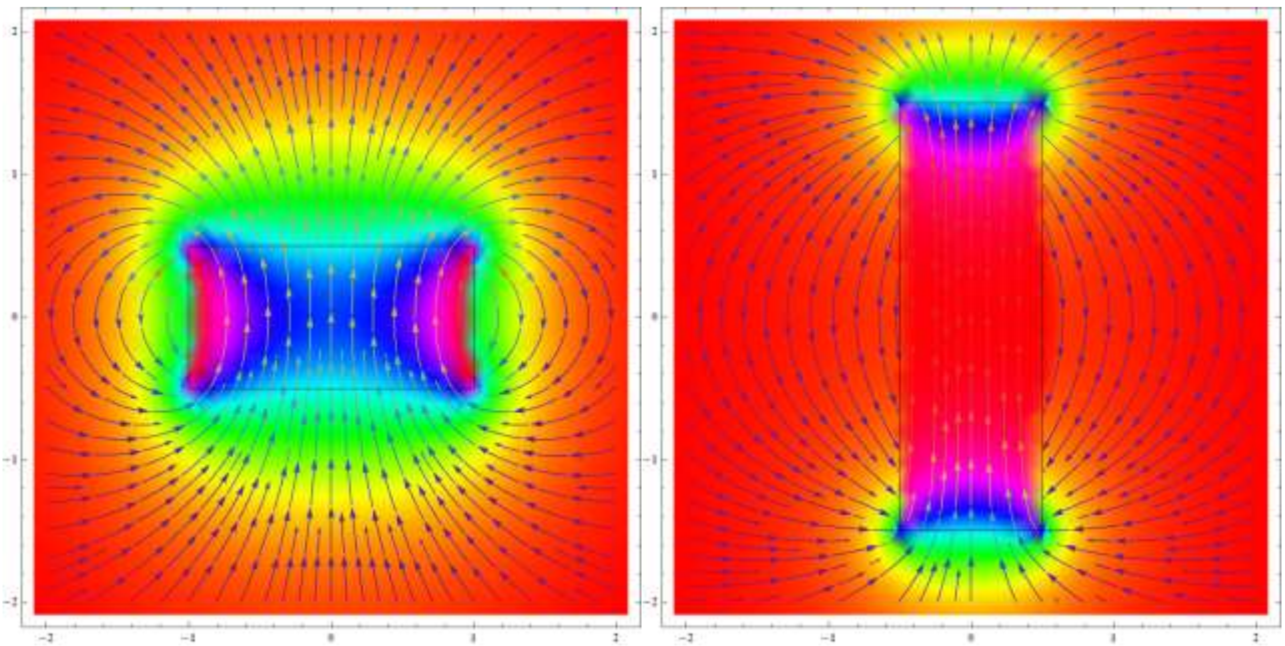
$$B_\rho = \frac{\mu_0 I}{4\pi}\frac{1}{L}\sqrt{\frac{a}{\rho}}\left[\left(\frac{k^2-2}{k}K(k^2)+\frac{2}{k}E(k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

$$B_z = -\frac{\mu_0 I}{4\pi}\frac{1}{2L\sqrt{a\rho}}\left[\zeta k\left(K(k^2)+\frac{a-\rho}{a+\rho}\Pi(h^2,k^2)\right)\right]_{\zeta_-}^{\zeta_+}$$

With

$$h^2 = \frac{4a\rho}{(a+\rho)^2}, \qquad k^2 = \frac{4a\rho}{(a+\rho)^2+\zeta^2}$$

Here is some plot



Works Cited

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