
POLYNÔMES
ALAIN LASCOUX

POLYNOMIALS

Alain Lascoux

September 16, 2013

Contents

Abstract.

We give eight[1] *linear bases of the ring of polynomials in n indeterminates :* Schubert polynomials, Grothendieck polynomials, flag elementary/complete functions, Demazure characters (key polynomials) for types A, B, C, D, Macdonald polynomials.

All these bases are triangular in the basis of monomials, with respect to appropriate orders. We introduce different scalar products and compute the adjoint bases of the previous polynomials.

*We provide recursions (*transition formulas*) which allow to cut these polynomials into smaller ones of the same family.*

We recover the multiplicative structure of the ring of polynomials by describing the multiplication by a single variable.

In type A we lift the Schubert polynomials and Demazure characters to the free algebra.

We recover by symmetrisation Schur functions and symmetric Macdonald polynomials in type A, and symplectic and orthogonal Schur functions in types B, C, D.

[1] In fact, counting adjoint bases and deformations, many more, but the next lucky number, 88, seems out of reach for the moment.

Introduction

Polynomials appeared since the beginnings of algebra, and it may seem that there is not much to say, nowadays, about the space of polynomials as a vector space. In the case of a single variable x, many linear bases of $\mathfrak{Pol}(x)$ other than the powers of x have been described, starting with the Newton's interpolation polynomials. The theory of orthogonal polynomials flourished during the whole XIX^e century, providing many more bases.

In the case of symmetric polynomials, Newton, again, gave a basis of products of elementary functions. The transition matrices between these functions and the monomial functions were already considered in the $XVIII^e$ century by Vandermonde in particular. Later, the chevalier Faa de Bruno, Cayley, Kostka spent much energy computing different other transition matrices. It happens in fact that there is a fundamental basis, the basis of Schur functions. A great majority of the classical problems in the theory of symmetric functions involve this basis, and leads to a combinatorics of diagrams of partitions and Young tableaux.

The picture is not so bright when one relaxes the condition of symmetry and consider $\mathfrak{Pol}(x_1, \ldots, x_n)$ in full generality. In fact, computer algebra systems like Maple or Mathematica do not know the ring of polynomials in several variables with coefficients in $\mathbb{Z}$, but only the ring $\mathbb{Z}[x_1] \otimes \mathbb{Z}[x_2] \otimes \cdots \otimes \mathbb{Z}[x_n]$. Since 40 years, geometry and representation theory provided a new incentive for describing linear bases of polynomials. The cohomology theory and the K-theory flag manifolds lead to different bases related to Schubert varieties: *Demazure characters, Schubert polynomials, Grothendieck polynomials*. Independently, the theory of orthogonal polynomials, in conjunction with root systems, developed in the direction of several variables, with the work of Koornwinder, Macdonald and many others.

In these notes, we shall mostly restrict to *Schubert polynomials, Grothendieck polynomials, Demazure characters (key polynomials), Macdonald polynomials.* These objects will be obtained using simple operators such as *Newton's divided differences* and their deformations. Such operators act on two consecutive variables at a time, say x_i, x_{+1}, and commute with multiplication with symmetric functions in x_i, x_{i+1}. Therefore, they are characterized by their action on $1, x_{i+1}$ (which is a basis of $\mathfrak{Pol}(x_i, x_{i+1})$ as a free $\mathfrak{Sym}(x_i, x_{i+1})$-module). In type A, computations will not require more than the rules figuring in the following tableau, which expresses the images of $1, x_{i+1}$ under different operators, and indicates the related polynomials.

operator	$s_i+\partial_i$	∂_i	π_i	$\widehat{\pi}_i$	$(1-x_{i+1})\partial_i$	T_i
1	1	0	1	0	1	t
x_{i+1}	x_i-1	-1	0	$-x_{i+1}$	$x_i+x_{i+1}-1$	x_i
polynoms	*Jack*	*Schubert*	*Demazure Grothendieck*	*Demazure*	$\widetilde{G}$ *Grothendieck*	*Macdonald Hall−Littlewood*

To be complete, we have to add to this list the operators π_n^B, π_n^C and π_i^D in the case of key polynomials for types B, C, D, and the translation $f(x_1, \ldots, x_n) \to f(x_n/q, x_1, \ldots, x_{n-1})(x_n-1)$ in the case of Macdonald polynomials, but this does not change the picture: it is remarkable that such simple rules suffice to generate interesting families of polynomials. As a matter of fact, one also needs initial polynomials. In the case of Demazure characters, one starts with dominant monomials $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For Schubert polynomials, one introduces another set of variables, and one takes $Y_\lambda := \prod_{i=1..n, j=1..\lambda_i}(x_i - y_j)$. For Grothendieck polynomials, one takes $G_\lambda := \prod_{i=1..n, j=1..\lambda_i}(1 - y_j x_i^{-1})$, still with the requirement that $\lambda_1 \geq \cdots \geq \lambda_n$. In the case of Macdonald polynomials, one needs only one starting point, which is 1, because the translation operator increases degree and allows to generate polynomials of any degree.

Schubert and Macdonald polynomials can also be defined by interpolation properties. Indeed, to each $v \in \mathbb{N}^n$, one associates a *spectral vector* $\langle v \rangle^y$ (which is a permutation of $y_1, y_2, \ldots$), and another spectral vector $\langle v \rangle^{tq}$ (with components which are monomials in t, q). Now the Schubert polynomial Y_v and the Macdonald polynomial M_v are the only polynomials, up to normalization, of degree $d = |v| = v_1 + \ldots + v_n$, such that

$$Y_v\big(\langle u \rangle^y\big) = 0 \qquad \& \qquad M_v\big(\langle u \rangle^{tq}\big) = 0 \ \forall u : \ |u| \leq d, \ u \neq v \, .$$

it is easy to check that the vanishing conditions imply a recursion on polynomials, the image of a Schubert polynomial under ∂_i being another Schubert polynomial (when it is not 0), and the image of a Macdonald polynomial under $T_i + c$ being another Macdonald polynomial (when choosing appropriately the constant c).

Divided differences are discrete analogues of derivatives. One can thus expect a discrete analogue of the *multivariate Taylor formula*. In the case of functions of a single variable, this discrete analogue is the *Newton interpolation formula*. In the multivariate case, the universal coefficients appearing as coefficients of products of divided differences are precisely the Schubert polynomials, and this is a direct consequence of their vanishing properties.

In these notes, we have put the emphasis on Grothendieck polynomials, because the literature on this subject is rather scanty , apart from the *Graßmannian* case, which is the case where the polynomials are symmetric and can be treated as deformations of Schur functions. We do not touch the subject of Schubert polynomials for types B, C, D (see [10, 47, 49, 41, 132, 133]). They require introducing the operation $x_n \to -x_n$, while, for Demazure characters and K-theory, one must use $x_n \to x_n^{-1}$. In type A on the contrary, cohomology and K-theory can be mixed, operators like $\pi_i + \partial_i$ make sense.

Schubert, Grothendieck polynomials and Demazure characters are directly associated to the basis $\{\partial_\sigma : \sigma \in \mathfrak{S}_n\}$ of the Nil Hecke algebra, and to the basis $\{\pi_\sigma : \sigma \in \mathfrak{S}_n\}$ of the 0-Hecke algebra. We give two more bases, and their adjoint, of $\mathfrak{Pol}(x_1, \ldots, x_n)$, corresponding to the basis $\{\nabla_\sigma : \sigma \in \mathfrak{S}_n\}$, and to the Kazhdan-Lusztig basis $\{C_\sigma : \sigma \in \mathfrak{S}_n\}$ of the Hecke algebra.

Linear algebra is not enough, the ring $\mathfrak{Pol}(x_1, \ldots, x_n)$ has also a multiplicative structure that one needs to describe. We mostly restrict to multiplication by a single variable, which is enough to determine the multiplicative structure in each of the bases that we consider. Already this simple case involves fine properties of the Ehresmann-Bruhat order on the symmetric group (or on the affine symmetric group in the case of Macdonald polynomials). It is clear, however, that more work should be invested in that direction, the product of two general Schubert polynomials or two Grothendieck polynomials having, for example, many geometrical consequences . Fomin and Kirillov [40] have introduced an quadratic algebra to explain the connections between the Ehresmann-Bruhat order and Schubert calculus.

Having different bases, one may look for the relations between them. We consider the relations between Schubert and Grothendieck, Schubert and Demazure, Macdonald and key polynomials, but this subject is far from being exhausted.

Polynomials can be written uniquely as linear combination of *flag elementary functions*) (products of the type $\ldots e_i(x_1, x_2, x_3) e_j(x_1, x_2) e_k(x_1)$). Since the natural way to lift an elementary function of degree k in the free algebra is to take the sum of all strictly decreasing words of degree k, one has therefore a natural embedding, as a $\mathbb{Z}$-module, of $\mathfrak{Pol}(x_1, \ldots, x_n)$ in the free algebra on n letters. We shall rather use a distinguished quotient of the free algebra, the *plactic algebra* $\mathfrak{Plac}(n)$, quotient by the relations

$$cab \equiv acb,\ bac \equiv bca,\ baa \equiv aba,\ bab \equiv bba,\ a < b < c\,.$$

The lift of $\mathfrak{Sym}(x_1 \ldots, x_n)$ in $\mathfrak{Plac}(n)$ has now recovered its multiplicative structure, compared to the lift in the free algebra where one must have recourse to operations like shuffle instead of concatanation of words. In others words, one has an embedding of $\mathfrak{Sym}(x_1 \ldots, x_n)$ into a non-commutative algebra, and therefore any identity on symmetric polynomials translates automatically into a statement in the non-commutative world. Combinatorists will have no difficulty in going one step further in the translation and use Young tableaux, Dyck paths or non-intersecting paths instead of mere words. In short, the diagram

$$\begin{tikzcd} & \mathfrak{Plac} \arrow[dr, "Ev"] & \\ \mathfrak{Sym} \arrow[ur, "s_\lambda \to \sum T"] \arrow[rr] & & \mathfrak{Pol} \end{tikzcd}$$

where the left arrow sends a Schur function s_λ onto the sum of all tableaux of shape λ in the alphabet $\{1,\ldots,n\}$, and Ev is the commutative evaluation, allows to pass from algebraic identities on symmetric functions to statements about words and tableaux.

Simple transpositions can be lifted to the free algebra, inducing an action of the symmetric group on the free algebra. The isobaric divided differences π_i can also be lifted to the free algebra, but they do not satisfy the braid relations any more. This does not prevent using them on the lifts of Schubert polynomials and of Demazure characters. In particular, this is the most sensible way of understanding the decomposition of Schubert polynomials as a positive sum of key polynomials. One still has a commutative diagram, identifying the Demazure characters $\{K_v : v \in \mathbb{N}^n\}$ with the "free" Demazure characters $\{K_v^{\mathcal{F}} : v \in \mathbb{N}^n\}$. However, one has lost multiplication, $\mathfrak{Pol}(x_1,\ldots,x_n)$ is considered as the free module with basis the Demazure characters.

$$\begin{tikzcd} \mathfrak{Schub} = \langle K_v^{\mathcal{F}} \rangle \arrow[rr] \arrow[dr, "Ev"'] & & \mathfrak{Free} \arrow[dl, "Ev"] \\ & \mathfrak{Pol} & \end{tikzcd}$$

We use two structures on the ring of polynomials in $x_1,\ldots,x_n$, with coefficients in $\mathbf{y}$: as a module over $\mathbb{Z}[\mathbf{y}]$ with basis the infinite family of Schubert polynomials $\{Y_v(\mathbf{x}_n,\mathbf{y}) : v \in \mathbb{N}^n\}$, or as a free module of dimension $n!$ over $\mathbb{Z}[\mathbf{y}] \otimes \mathfrak{Sym}(\mathbf{x}_n)$, with basis $\{Y_v(\mathbf{x}_n,\mathbf{y}) : v \le \rho = [n{-}1,\ldots,0]\}$. We show in the appendix how to extend this finite Schubert basis in types C, D so as to obtain a pair of adjoint bases for $\mathfrak{Pol}(x_1^\pm,\ldots,x_n^\pm)$ as a free-module under the invariants of the Weyl group.

Chapter 1

Operators on polynomials

1.1 A, B, C, D

What are the simplest operations on vectors ?

- add
- concatanate
- transpose two consecutive components
- multiply a component by -1

Thus, acting on vectors $v \in \mathbb{Z}^n$ one has the following operators (denoted on the right) corresponding to the root systems of type A, B, C, D :

$$\begin{aligned} v\,s_i &= [\ldots, v_{i+1}, v_i, \ldots]\ ,\ 1 \le i < n, \\ v\,s_i^B = v\,s_i^C &= [\ldots, -v_i, \ldots],\ 1 \le i \le n, \\ v\,s_i^D &= [\ldots, -v_i, -v_{i-1}, \ldots],\ 2 \le i \le n\,. \end{aligned}$$

The groups generated by $s_1, \ldots, s_{n-1}$ (resp. $s_1, \ldots, s_{n-1}, s_n^B$, resp. $s_1, \ldots,$ s_{n-1}, s_n^D) are the Weyl groups of type A, BC, D. We shall distinguish between B and C later, when acting on polynomials.

The orbit of the vector $[1, 2, \ldots, n]$ consists of all *permutations* of $1, \ldots, n$ for type A, all *signed permutations* for type B, C, and all signed permutations with an even number of "-" in type D. The elements of the different groups can be denoted by these objects.

The generators satisfy the *braid relations* (or *Coxeter relations*)

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \& \quad s_i s_j = s_j s_i\,,\ |i-j| \neq 1\,, \tag{1.1.1}$$

$$s_{n-1}s_n^B s_{n-1}s_n^B = s_n^B s_{n-1}s_n^B s_{n-1} \quad \& \quad s_i s_n^B = s_n^B s_i,\ i \le n-2\,, \tag{1.1.2}$$

$$s_{n-2}s_n^D s_{n-2} = s_n^D s_{n-2}s_n^D \quad \& \quad s_i s_n^B = s_n^B s_i,\ i \ne n-2\,. \tag{1.1.3}$$

An expression of an element w of the group as a product of generators is called a *decomposition*, and when this product is of minimal length, it is called a *reduced decomposition*, the length being called the *length* of w and denoted $\ell(w)$.

By recursion on n, it is easy to write reduced decompositions of the maximal element w_0 of the group for type A_{n-1}, B_n, C_n, D_n. Write $1, \ldots, n$ for $s_1, \ldots, s_{n-1}$ and s_n^B or s_n^D. Then w_0 admits the following reduced decompositions (that we have cut into self-explanitory blocks; read blocks from left to right)

- type A $\quad \emptyset \quad \boxed{n{-}1} \quad \boxed{n{-}2}\boxed{n{-}1} \quad \cdots \quad \boxed{1}\boxed{2}\boxed{\cdots}\boxed{n{-}1}$

- type BC $\quad \boxed{n} \quad \begin{array}{|c|c|}\hline n{-}1 & n \\ \hline \multicolumn{1}{c|}{} & n{-}1 \\ \cline{2-2}\end{array} \quad \cdots \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline \multicolumn{3}{c|}{} & \vdots \\ \cline{4-4} \multicolumn{3}{c|}{} & 2 \\ \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \cline{4-4}\end{array}$

- type D $\quad \boxed{\binom{n-1}{n}}\ \boxed{n{-}2\binom{n-1}{n}n{-}2} \ \cdots \ \boxed{1\,2\cdots n{-}2\binom{n-1}{n}n{-}2\cdots 2\,1}$

In the case of type D we have written $\binom{n-1}{n}$ for the commutative product $s_{n-1}s_n^D$.

Erase in each block a right factor[1]. The resulting decomposition is still reduced, and the group elements are in bijection with these decompositions. Therefore, the sequence of lengths of the remaining left factors *codes* the elements for type A and B. In type D, one has to use an extra symbol to distinguish between a factor $s_k \cdots s_{n-2}s_{n-1}$ and a factor $s_k \cdots s_{n-2}s_n$.

Many combinatorial properties of permutations are more easily seen by taking, in type A, another decomposition. Instead of reading the successive rows of

[1]In type D_3, for example, the right factors of the block $1\binom{2}{3}1$ are $\emptyset$, 1, 21, 31, $\binom{2}{3}1$, $1\binom{2}{3}1$.

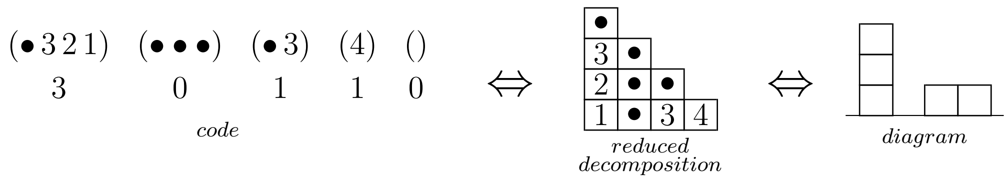
$$\begin{array}{|c|c|c|c|}\cline{1-1} n{-}1 \\ \cline{1-2} n{-}2 & n{-}1 \\ \cline{1-3} \cdots & \cdots & \cdots \\ \hline 1 & 2 & \cdots & n{-}1 \\ \hline \end{array}$$

one takes the successive columns, and thus chooses the decomposition

$$(n{-}1,\ldots,1)(n{-}1,\ldots,2)\ldots(n{-}1) \;\leftrightarrow\; \begin{array}{|c|}\hline n{-}1 \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \end{array}\;\; \begin{array}{|c|}\hline n{-}1 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}\;\cdots\;\boxed{n{-}1}\,.$$

It is easy to check that the decompositions obtained by taking arbitrary right factors of the successive blocks (= bottom parts of the columns) are reduced and in bijection with permutations.

For example, for $n = 5$,



is a reduced decomposition, that we shall call *canonical reduced decomposition*, of the permutation $s_3s_2s_1s_3s_4 = [4, 1, 3, 5, 2]$, and the sequence $[3, 0, 1, 1, 0]$ of lengths of the right factors is called the *code* of the permutation (one can represent the code by a *diagram* of boxes piled on the ground).

Given σ in the symmetric group $\mathfrak{S}_n$, its code $\mathfrak{c}(\sigma)$ can also be described as the vector v of components $v_i := \#\{j : j > i \;\&\; \sigma_i > \sigma_j\}$, which describes the inversions of σ. The sum $|v| = v_1 + \cdots + v_N$ is therefore the length $\ell(\sigma)$ of σ.

Having groups, one has also *group algebras*. Instead of enumerating the elements of the group W, together with their lengths one can now write a generating series which is called the *Poincaré polynomial*

$$\sum_{w\in W} q^{\ell(w)}\,.$$

From the preceding canonical decompositions, denoting by $[i]$ the q-integer $(q^i - 1)/(q - 1)$, one obtains the following Poincaré polynomials :

- type A $\quad [1]\,[2]\cdots[n]\quad$,
- type BC $\quad [2]\,[4]\cdots[2n]\quad$,

- type D $\quad [2]\,[4]\,\cdots\,[2n-2]\,[n]\quad$.

One can embed a Weyl group of type B_n, C_n, D_n into $\mathfrak{S}_{2n}$, as a subgroup, by sending s_i to $s_i s_{2n-i}$, $1 \le i \le n{-}1$, s_n^B and s_n^C to s_n, and s_n^D to $s_n s_{n+1} s_{n-1} s_n$. This amounts transforming a signed permutation v by $v_i \to \sigma_i = v_i$ if $v_i > 0$, and $v_i \to \sigma_i = 2n{+}1{+}v_i$ if $v_i < 0$, $i = 1, \ldots, n$, and completing by symmetry: $\sigma_{2n-i} = 2n{+}1 - \sigma_i$, thus obtaining a permutation in $\mathfrak{S}_{2n}$.

An *inversion* of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) such that $i < j$ and $\sigma_i > \sigma_j$. One inherits from the embedding into $\mathfrak{S}_{2n}$, taking into account symmetries, inversions for type B, C, D. If w is sent to σ, then an inversion is a pair $i, j : 1 \le i < j \le n$ such that $\sigma_i > \sigma_j$ or such that $\sigma_i > \sigma_{2n+1-j}$. In type B, C, the indices $i : 1 \le i \le n$ such that $w_i < 0$ (equivalently, $\sigma_i > \sigma_{2n+1-i}$) are also inversions. It is easy to see by recursion that the length coincides with the number of inversions.

1.2 Reduced decompositions in type A

In type A, we shall use graphical displays to handle more easily the braid relations. A *column* is defined to be a strictly decreasing sequence of integers. Any two-dimensional display of integers must be read columnwise, from left to right, each integer i being interpreted as s_i (or some other operators indexed by integers, depending on the context). A display is *reduced* if the corresponding product of s_i's is reduced. For example, $\begin{smallmatrix}1&3&\\&2&3\\&1&2\end{smallmatrix}$ must be read (1)(321)(32) and interpreted as $s_1 s_3 s_2 s_1 s_3 s_2$ (which happens to be a reduced decomposition of the permutation $[4, 3, 2, 1]$). With these conventions, the braid relation $s_1 s_2 s_1 = s_2 s_1 s_2$ becomes $\begin{smallmatrix}1&2\\&1\end{smallmatrix} = \begin{smallmatrix}2&\\1&2\end{smallmatrix}$. More generally, one has the following commutation lemma.

Lemma 1.2.1. *Let u, v be two columns such that uv is reduced and each letter of u also occurs in v. Then $uv = vu^+$, where u^+ is obtained from u by increasing each letter of u by* 1.

Proof. By induction on the size of u, the statement reduces to the case where $u = i$ is a single letter. Because iv is reduced, v must be of the type $v = v'\, i{+}1\, i\, v''$, with all the letters of v' bigger or equal to $i{+}2$, and all the letters of v'' less or equal to $i{-}1$. In that case,

$$iv = v'\, i\, i{+}1\, i\, v'' = v'\, i{+}1\, i\, i{+}1\, v'' = v'\, i{+}1\, i\, v''\, i{+}1\,,$$

as wanted. QED

For example, starting from the canonical reduced decomposition of $\omega = [5, 4, 3, 2, 1]$, one obtains the decompositions

$$\begin{smallmatrix}4&&&\\3&4&&\\2&3&4&\\1&2&3&4\end{smallmatrix} = \begin{smallmatrix}3&4&&\\2&3&&\\1&2&4&\\&1&3&4\end{smallmatrix} = \begin{smallmatrix}3&&&\\2&3&4&\\1&2&3&\\&&2&\\&&1&4\end{smallmatrix} = \begin{smallmatrix}2&3&4&\\1&2&3&\\&1&2&\\&&1&4\end{smallmatrix} = \begin{smallmatrix}2&3&&\\1&2&&\\&1&3&4\\&&&3\\&&&2\\&&&1\end{smallmatrix} = \begin{smallmatrix}2&&&\\1&2&3&4\\&&2&3\\&&1&2\\&&&1\end{smallmatrix} = \begin{smallmatrix}1&2&3&4\\&1&2&3\\&&1&2\\&&&1\end{smallmatrix}.$$

(these are 7 among the $2^8 \times 3$ reduced decompositions of ω).

1.3 Acting on polynomials with the symmetric group

Of course, considering vectors as exponents of monomials: $x^v = x_1^{v_1} x_2^{v_2} \cdots$, we get operators on polynomials: $v \to vs_i$ induces the simple transposition of x_i, x_{i+1} : $x^v \to x^{vs_i}$, and similarly for types B, D. No need to point out that addition of exponents corresponds to product of monomials, and that concatenation corresponds to a shifted product that we shall use when considering non-commutative symmetric functions:

$$u \in \mathbb{Z}^n, v \in \mathbb{Z}^m \ \to \ x^{u,v} = x_1^{u_1} \cdots x_n^{u_n} x_{n+1}^{v_1} \cdots x_{n+m}^{v_m} .$$

If v is such that $v_1 \geq \cdots \geq v_n$, then v is called *dominant* (we also say that v is a *partition*, terminal zeros being allowed). When $v_1 \leq \cdots \leq v_n$, then v is *antidominant*. The reversed vector $[v_n, \ldots, v_1]$ is denoted $v\omega$. Reordering v increasingly (resp. decreasingly) is denoted $v\uparrow$ (resp. $v\downarrow$).

Instead of vectors in $\mathbb{N}^n$, one may use permutations. We have just to reverse the correspondence seen above between permutations and *codes*[2]. One identifies $\sigma \in \mathfrak{S}_N$ and $[\sigma, N{+}1, N{+}2, \ldots]$; this corresponds to concatenating 0's to the right of the code of σ. For example, one identifies the two permutations $[2, 4, 1, 5, 3]$ and $[2, 4, 1, 5, 3, 6, 7, \ldots]$, as well as their codes $[1, 2, 0, 1, 0]$ and $[1, 2, 0, 1, 0, 0, 0, \ldots]$.

Let us consider in more details the space $\mathfrak{Pol}(x_1, x_2)$ of polynomials in $x_1^{\pm}, x_2^{\pm}$, with the simple transposition s of x_1, x_2. One remarks that s commutes with multiplication with symmetric functions in x_1, x_2 (whose space is denoted $\mathfrak{Sym}(x_1, x_2)$).

Every $f \in \mathfrak{Pol}(x_1, x_2)$ can be written

$$f = \frac{f + f^s}{2} + \frac{f - f^s}{2} = \frac{f + f^s}{2} + (x_1 - x_2) \left(\frac{f - f^s}{2(x_1{-}x_2)} \right) .$$

This means that every polynomial in $\mathfrak{Pol}(x_1, x_2)$ can be written uniquely as a linear combination of the polynomials 1 and $(x_1{-}x_2)$, with coefficients in $\mathfrak{Sym}(x_1, x_2)$. In other words $\mathfrak{Pol}(x_1, x_2)$ is a free $\mathfrak{Sym}(x_1, x_2)$-module of rank 2, and one can choose as natural bases $\{1, x_1{-}x_2\}$, $\{1, x_2\}$ or $\{1, x_1\}$.

The last choice corresponds to writing f as

$$f = x_1 \left(\frac{f - f^s}{x_1{-}x_2} \right) + \left(\frac{x_1 f^s - x_2 f}{x_1{-}x_2} \right) ,$$

the action of s being determined by

$$\{1, x_1\} \longrightarrow \{1, x_2 = -x_1 + (x_1{+}x_2)\}$$

[2]This correspondence is in fact due to Rothe (1800), who defined a planar diagram representing the inversions of a permutation.

and represented by the matrix

$$\begin{bmatrix} 1 & x_1+x_2 \\ 0 & -1 \end{bmatrix}.$$

Since a 2×2 matrix has 4 entries, this is not a big step to consider more general actions, such as

$$\{1, x_1\} \longrightarrow \{0, 1\},$$

which, for a general polynomial f, translate into

$$f \longrightarrow (f - f^s)\frac{1}{x_1 - x_2} := f\,\partial_1\,,$$

and is called *Newton divided difference.*

Similarly

$$\{1, x_2\} \to \{1, 0\} \quad \text{induces} \quad f \to (x_1 f - x_2 f^s)\frac{1}{x_1 - x_2} := f\,\pi_1\,,$$

$$\{1, x_1\} \to \{0, x_2\} \quad \text{induces} \quad f \to (f - f^s)\frac{x_2}{x_1 - x_2} := f\,\widehat{\pi}_1\,,$$

$$\{1, x_2\} \to \{t, x_1\} \quad \text{induces} \quad f \to f\pi_1(t-1) + f^s := f\,T_1\,,$$

$$\{1, x_1\} \to \{1, tx_2\} \quad \text{induces} \quad f \to f\widehat{\pi}_1(t-1) + f^s := f\,\widehat{T}_1\,,$$

which are, respectively, two kinds of *isobaric divided differences*, and two choices of a generator of the *Hecke algebra* $\mathcal{H}_2$ of the symmetric group $\mathfrak{S}_2$.

Of course, for every pair of consecutive variables x_i, x_{i+1}, one defines similar operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i, \hat{T}_i$. The following table summarizes their action on the basis $\{1, x_{i+1}\}$ of $\mathfrak{Pol}(x_i, x_{i+1})$ as a free $\mathfrak{Sym}(x_i, x_{i+1})$-module :

operator	s_i	∂_i	π_i	$\widehat{\pi}_i$	T_i	$\widehat{T}_i$
equivalent form		$(1-s_i)\frac{1}{x_i-x_{i+1}}$	$x_i\partial_i$	$\partial_i x_{i+1}$	$\pi_i(t-1)+s_i$	$\widehat{\pi}_i(t-1)+s_i$
1	1	0	1	0	t	1
x_{i+1}	x_i	-1	0	$-x_{i+1}$	x_i	$x_i+x_{i+1}-tx_{i+1}$

Equivalently, these different operators are represented, in the basis $\{1, x_{i+1}\}$ of the free module $\mathfrak{Pol}(x_i, x_{i+1})$, by the matrices

$$s_i = \begin{bmatrix} 1 & x_i+x_{i+1} \\ 0 & -1 \end{bmatrix},\ \partial_i = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},\ \pi_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\widehat{\pi}_i = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\ T_i = \begin{bmatrix} t & x_i+x_{i+1} \\ 0 & -1 \end{bmatrix},\ \widehat{T}_i = \begin{bmatrix} 1 & x_i+x_{i+1} \\ 0 & -t \end{bmatrix}.$$

All these operators are of the type

$$D_i = \mathbf{1}\, P(x_i, x_{i+1}) + s_i\, Q(x_i, x_{i+1})\,, \tag{1.3.1}$$

with P, Q rational functions, that is to say, they are linear combination of the identity operator and a simple transposition with rational coefficients. The operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i, \widehat{T}_i$ all satisfy the type A-*braid relations*

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \quad \& \quad D_i D_j = D_j D_i\,,\ |i-j| \neq 1\,.$$

One discovers that these operators also satisfy a *Hecke relation*

$$s_i s_i = 1,\ \partial_i\partial_i = 0,\ \pi_i\pi_i = \pi_i,\ \widehat{\pi}_i\widehat{\pi}_i = -\widehat{\pi}_i,\ (T_i - t)(T_i + 1) = 0,\ (\widehat{T}_i + t)(\widehat{T}_i - 1) = 0.$$

Let us check for example the relation $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$. These two operators commute with symmetric functions in x_1, x_2, x_3, and decrease degree by 3. We can take as a basis of $\mathfrak{Pol}(\mathbf{x}_n)$ (as a free module over $\mathfrak{Sym}(\mathbf{x}_3)$) the 6 monomials $\{x^v : [0,0,0] \leq v \leq [2,1,0]\}$. The first five are sent to 0 by $\partial_1\partial_2\partial_1$ and $\partial_2\partial_1\partial_2$ for degree reason, there remains only to check that $x^{210}\partial_1\partial_2\partial_1 = x^{210}\partial_2\partial_1\partial_2 = 1$ to conclude that, indeed, $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.

As a consequence of the braid relations, there exists operators $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma, T_\sigma$, indexed by permutations σ, which are obtained by taking any reduced decomposition of σ and the corresponding product of operators D_i.

1.4 Commutation relations

Divided differences satisfy Leibnitz[3] formulas[4], as easily seen from the definition:

$$fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{s_i} = g\,(f\partial_i) + g\partial_i\, f^{s_i}\ . \tag{1.4.1}$$

Iterating, one obtains the image of fg under any product of divided differences :

$$fg\,\partial_i\partial_j \ldots \partial_h = \sum_{\epsilon_i,\ldots\epsilon_h \in \{0,1\}} \left(f\partial_i^{\epsilon_i}\partial_j^{\epsilon_j}\cdots\partial_h^{\epsilon_h}\right)\left(g s_i^{\epsilon_i}\partial_i^{1-\epsilon_i} s_j^{\epsilon_j}\partial_j^{1-\epsilon_j}\cdots s_h^{\epsilon_h}\partial_h^{1-\epsilon_h}\right). \tag{1.4.2}$$

It may be appropriate to use a tensor notation, the above formula being the expansion of

$$f \otimes g\,(\partial_i \otimes s_i + 1 \otimes \partial_i)(\partial_j \otimes s_j + 1 \otimes \partial_j)\ldots(\partial_h \otimes s_h + 1 \otimes \partial_h)\,.$$

[3]For fear of being called Leinisse, Leibnitz chosed the spelling "Leibnitz" in his letters to the Académie des Sciences. We shall respect his choice.

[4]Notice that formulas are disymmetrical in f, g, one has two expressions for the image of a product.

In particular, when $g = x_i$, relations (1.4.1) may be seen as commutation relations :

$$x_i\partial_i = \partial_i x_{i+1} + 1 \quad \& \quad x_i\pi_i = \pi_i x_{i+1} + x_i \quad \& \quad x_i\widehat{\pi}_i = \widehat{\pi}_i x_{i+1} + x_{i+1}\,, \tag{1.4.3}$$

the relations $x_iT_i = T_ix_{i+1}+(t-1)x_i$ together with the trivial commutations $x_jT_i = T_ix_j$, when $|j-i| \neq 1$, being taken as axioms of the *affine Hecke algebra*[5].

Since $\widehat{\pi}_i = \partial_i x_{i+1}$, one has also $\widehat{\pi}_i x_i = \partial_i x_{i+1}x_i = x_{i+1}x_i\partial_i = x_{i+1}\pi_i$, and by iteration, reading the objects by successive columns,

$$\boxed{\begin{array}{c}\widehat{\pi}_n\\ \widehat{\pi}_{n-1}\\ \vdots\\ \widehat{\pi}_i\end{array}}\; x_i = x_{n+1}\boxed{\begin{array}{c}\pi_n\\ \pi_{n-1}\\ \vdots\\ \pi_i\end{array}}\,, \qquad \boxed{\begin{array}{cc}\widehat{\pi}_j & \cdots\widehat{\pi}_n\\ \vdots & \vdots\\ \widehat{\pi}_{i+j-n} & \cdots\widehat{\pi}_i\end{array}}\; x_i = x_{j+1}\boxed{\begin{array}{cc}\pi_j & \cdots\pi_n\\ \vdots & \vdots\\ \pi_{i+j-n} & \cdots\pi_i\end{array}}$$

We shall need some more commutation rules. For example,

$$\pi_1\pi_2\pi_3x_1x_2x_3 = x_2x_3x_4\pi_1\pi_2\pi_3 + x_1x_2x_3x_4\pi_1\pi_2 + x_1x_2x_4\pi_1\pi_3 + x_1x_3x_4\pi_2\pi_3$$

and to iterate such relations, we prefer to represent them graphically as

$$\underset{x_1}{\boxed{1}}\;\underset{x_2}{\boxed{2}}\;\underset{x_3}{\boxed{3}} = \overset{x_2}{\boxed{1}}\;\overset{x_3}{\boxed{2}}\;\overset{x_4}{\boxed{3}} + x_1\;\overset{x_2}{\boxed{1}}\;\overset{x_3}{\boxed{2}}\;\boxed{\bullet}$$
$$+\, x_1\;\overset{x_2}{\boxed{1}}\;\boxed{\bullet}\;\overset{x_4}{\boxed{3}} + x_1\;\boxed{\bullet}\;\overset{x_3}{\boxed{2}}\;\overset{x_4}{\boxed{3}}$$

In general, given an antidominant $v \in \mathbb{N}^k$, the *v-diagram* $\mathcal{V}$ is the array with columns of length $v_1, \ldots, v_n$ filled by decreasing integers as follows :

$$\mathcal{V} = \begin{array}{|c|c|c|c|}\cline{4-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & u_k\\ \cline{2-4} \multicolumn{1}{c}{} & u_2 & \cdots & \vdots\\ \cline{2-4} \multicolumn{1}{c}{} & \vdots & \cdots & \vdots\\ \hline u_1 & \vdots & \cdots & \vdots\\ \hline \vdots & \vdots & \cdots & \vdots\\ \hline 1 & 2 & \cdots & k\\ \hline\end{array}\,,$$

where $u = v + [0, 1, \ldots, k{-}1]$, and $\pi^{\mathcal{V}}$, $\widehat{\pi}^{\mathcal{V}}$, are the columnwise-reading of $\mathcal{V}$, interpreting i as π_i or $\widehat{\pi}_i$ respectively.

Iterating the preceding commutation rules, one obtains the following lemma.

[5] For the *double affine Hecke algebra* for the type A, omnipresent in the work of Cherednik, one needs also to define T_0 or an affine operation.

Lemma 1.4.1. *Let $v \in \mathbb{N}^k$ be antidominant, $\mathcal{V}$ its associated diagram, n be an integer such $n > v_k{+}k$. Then*

$$\pi^{\mathcal{V}} \frac{1}{x_1 \cdots x_k} = \frac{1}{x_{v_1+1} \cdots x_{v_k+k}} \, \widehat{\pi}^{\mathcal{V}} .$$

Equivalently, multiplying by the factor $x_1 \ldots x_n$ which commutes with $\pi_i, \widehat{\pi}_i$ for $i < n$, one has

$$\pi^{\mathcal{V}} x_{k+1} \ldots x_n = \left(\frac{x_1 \ldots, x_n}{x_{v_1+1} \cdots x_{v_k+k}} \right) \widehat{\pi}^{\mathcal{V}} . \tag{1.4.4}$$

A *punched v-diagram* $\mathcal{U}$ is what results after punching holes in a v-diagram, in such a way that there are no two holes in the same row or same column, and such that no two holes occupy the South-West and North-East corner of a rectangle contained in the diagram. We forbid $\begin{array}{|c|c|c|}\hline \bullet & & \bullet \\ \hline\end{array}$, $\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \bullet \\ \hline\end{array}$, $\begin{array}{|c|c|c|c|}\hline & & & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline\end{array}$.

Label the rows of a v-diagram by the first entry of each row, and the columns by $v_1{+}1, \ldots, v_n{+}n$. The *weight* of a punched v-diagram $\mathcal{U}$, that we denote $x^{\mathcal{U}}$, is the product $\prod_{rows} x_i \prod_{columns} x_j$, keeping the indices of punched rows, and of full columns. By $\pi^{\mathcal{U}}$ we mean the reading of $\mathcal{U}$ columnwise, from left to right, interpreting each i as π_i and ignoring the holes.

Let us give an example of a punched diagram for $v = [2, 2, 4, 4, 4]$.

$$\begin{array}{ccccc|l}
x_3 & x_4 & x_7 & x_8 & x_9 & \\ \cline{3-5}
 & & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & 8 & x_6 \\ \cline{3-5}
 & & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & 7 & x_5 \\ \hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} & 6 & x_2 \\ \hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & 5 & x_1 \\ \hline
\end{array}
\qquad\qquad
\begin{array}{ccccc|l}
x_3 & & & x_8 & & \\ \cline{3-5}
 & & \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{7} & 8 & x_6 \\ \cline{3-5}
 & & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & 7 & \\ \hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\bullet} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} & 6 & x_2 \\ \hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \bullet & x_1 \\ \hline
\end{array}$$

coordinates and filling weight of a punched diagram

The punched 133-diagrams with two holes, together with their weights, are

$$\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & \bullet & 5 \\ \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline\end{array} \\ x_1 x_4 x_6
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & 4 & 5 \\ \cline{2-3} & \bullet & 4 \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline\end{array} \\ x_1 x_3 x_6
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & 4 & \bullet \\ \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline\end{array} \\ x_1 x_5 x_4
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & 4 & 5 \\ \cline{2-3} & 3 & \bullet \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline\end{array} \\ x_1 x_5 x_3
\end{array}$$

$$\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & \bullet & 5 \\ \cline{2-3} & 3 & \bullet \\ \hline \multicolumn{1}{|c|}{1} & 2 & 3 \\ \hline\end{array} \\ x_2 x_4 x_3
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & \bullet & 5 \\ \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & \bullet \\ \hline\end{array} \\ x_2 x_4 x_1
\end{array}
\quad
\begin{array}{c}
\begin{array}{c|c|c|}\cline{2-3} & 4 & 5 \\ \cline{2-3} & \bullet & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & \bullet \\ \hline\end{array} \\ x_2 x_3 x_1
\end{array} .$$

We shall need more commutation relations.

Lemma 1.4.2. *For any positive integer n, one has*

$$\frac{1}{x_1\cdots x_{n+1}}\pi_1\cdots\pi_n\, x_1\cdots x_n = \frac{1}{x_1}\pi_1\cdots\pi_n + \sum_{i=1}^{n}\frac{1}{x_{i+1}}\pi_1\cdots\pi_{i-1}\pi_{i+1}\cdots\pi_n\,, \tag{1.4.5}$$

$$\pi_1\cdots\pi_n\, x_2\cdots x_n\, \pi_1\cdots\pi_{n-1} = x_3\cdots x_{n+1}\, \pi_1\cdots\pi_n\pi_1\cdots\pi_{n-1}\,. \tag{1.4.6}$$

Given $v\in\mathbb{N}^n$ antidominant, $\mathcal{V}$ its associated diagram, then

$$\pi^{\mathcal{V}}\, x_1\cdots x_n = \sum_{\mathcal{U}} x^{\mathcal{U}}\,\pi^{\mathcal{U}}\,, \tag{1.4.7}$$

sum over all the punched v-diagrams.

Proof. The first two assertions are obtained by iterating the relation $\pi_i x_i = x_{i+1}\pi_i + x_i$. Let us check the last one by recursion, adding a top row to the diagram $\mathcal{V}$.

One therefore has to evaluate a product of the type $\pi_r\cdots\pi_m x^{\mathcal{U}}\,\pi^{\mathcal{U}}$, where the restriction of $x^{\mathcal{U}}$ to $\{x_r,\ldots,x_{m+1}\}$ is a subword of $x_r\cdots x_m$ which points out full columns in $\mathcal{U}$.

Let us first examine the case where $x_r\notin\mathcal{U}$. Taking specific values to simplify the exposition, ignoring the left part figured by hearts, one has to evaluate

$$\begin{array}{cc|c|c|c|c|c|}
\multicolumn{2}{c}{} & \multicolumn{1}{c}{\pi_{15}} & \multicolumn{1}{c}{\pi_{16}} & \multicolumn{1}{c}{\pi_{17}} & \multicolumn{1}{c}{\pi_{18}} & \multicolumn{1}{c}{\pi_{19}}\\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{\cdot} & \multicolumn{1}{c}{x_{16}} & \multicolumn{1}{c}{x_{17}} & \multicolumn{1}{c}{\cdot} & \multicolumn{1}{c}{x_{19}}\\
\cline{3-7}
 & \heartsuit & 14 & 15 & 16 & 17 & 18\\ \cline{3-7}
\heartsuit & \heartsuit & \bullet & 14 & 15 & 16 & 17\\ \cline{3-7}
\heartsuit & \heartsuit & 12 & 13 & 14 & 15 & 16\\ \cline{3-7}
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15\\ \cline{3-7}
\end{array}$$

By commutation of the incomplete columns with the complete ones, one obtains

$$\begin{array}{cc|c|c|c|c|c|c|}
\multicolumn{2}{c}{} & \multicolumn{1}{c}{\pi_{15}} & \multicolumn{1}{c}{\pi_{16}} & \multicolumn{1}{c}{\pi_{17}} & \multicolumn{1}{c}{\pi_{18}} & \multicolumn{1}{c}{\pi_{19}} & \multicolumn{1}{c}{}\\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{\cdot} & \multicolumn{1}{c}{x_{16}} & \multicolumn{1}{c}{x_{17}} & \multicolumn{1}{c}{\cdot} & \multicolumn{1}{c}{x_{19}} & \multicolumn{1}{c}{}\\
\cline{3-7}
 & \heartsuit & \cdot & 15 & 16 & \cdot & 18 & \multicolumn{1}{c}{}\\ \cline{3-8}
\heartsuit & \heartsuit & \bullet & 14 & 15 & 16 & 17 & 18\\ \cline{3-8}
\heartsuit & \heartsuit & 12 & 13 & 14 & \cdot & 16 & 17\\ \cline{3-8}
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15 & 16\\ \cline{3-8}
\end{array}\;,$$

from which one extracts the left factor $(\pi_{15}\pi_{16}\pi_{17}\,x_{16}x_{17}\,\pi_{15}\pi_{16})(\pi_{18}\pi_{19}\,x_{19}\,\pi_{18})$, which, thanks to (1.4.6), is equal to $x_{17}x_{18}x_{20}\,(\pi_{15}\pi_{16}\pi_{15}\pi_{17}\pi_{16})(\pi_{18}\pi_{19}\pi_{18})$. We therefore have transformed $x^{\mathcal{U}}\,\pi^{\mathcal{U}}$ into $x^{\mathcal{U}^+}\,\pi^{\mathcal{U}^+}$, where $\mathcal{U}^+$ is obtained from $\mathcal{U}$ by adding a top row.

Let us consider now the case where $x_r \in x^{\mathcal{U}}$. Still with the same example, one has to evaluate

$$\begin{array}{cc|c|c|c|c|c|}
 & & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & \pi_{19} \\
 & & x_{15} & x_{16} & x_{17} & \cdot & x_{19} \\ \hline
 & \heartsuit & 14 & 15 & 16 & 17 & 18 \\ \hline
\heartsuit & \heartsuit & 13 & 14 & 15 & 16 & 17 \\ \hline
\heartsuit & \heartsuit & 12 & 13 & 14 & 15 & 16 \\ \hline
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15 \\ \hline
\end{array}$$

Thanks to (1.4.5), the factor $\pi_{15}\pi_{16}\pi_{17}\,(x_{15}x_{16}x_{17})$ is equal to the sum

$$\begin{array}{ccc} x_{16} & x_{17} & x_{18} \\ \pi_{15} & \pi_{16} & \pi_{17} \end{array} + \begin{array}{cccc} & & x_{17} & x_{18} \\ x_{15} & \bullet & \pi_{16} & \pi_{17} \end{array} + \begin{array}{cccc} & x_{16} & & x_{18} \\ x_{15} & \pi_{15} & \bullet & \pi_{17} \end{array} + \begin{array}{cccc} & x_{16} & x_{17} & \\ x_{15} & \pi_{15} & \pi_{16} & \bullet \end{array} .$$

Adding a top row to the diagram $\mathcal{V}$ has resulted in adding a top row to $\mathcal{U}$, or adding a row with only one hole, in all possible manners such that the new hole is left of the already existing holes in the last block of columns. This finishes the proof of the lemma. QED

For example, for $v = [1, 2, 2]$, one has

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}\, x_1x_2x_3 = x_2x_4x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$+\, x_1x_4x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array} + x_2x_3x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} + x_1x_2x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & \bullet & 3 \\ \hline \end{array}$$

$$+\, x_2x_3x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline 1 & 2 & 3 \\ \hline \end{array} + x_1x_2x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array} + x_1x_3x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array}$$

$$+\, x_1x_3x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline \bullet & 2 & 3 \\ \hline \end{array} + x_1x_2x_3 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array} .$$

Comparing the relations $\pi_1 x_2 = x_1\pi_1 - x_2$ and $x_1(-\widehat{\pi}_1) = (-\widehat{\pi}_1)x_2 - x_2$, one obtains a symmetry between commuting any π_σ with a polynomial f, and commuting f^ω and $\widehat{\pi}_{\omega\sigma^{-1}\omega}$:

Lemma 1.4.3. *Given n, $\sigma \in \mathfrak{S}_n$, and a polynomial $f(\mathbf{x}_n)$, suppose known the commutation*

$$\pi_\sigma f(\mathbf{x}_n) = \sum\nolimits_\zeta g_\zeta(\mathbf{x}_n)\,\pi_\zeta \,.$$

Then one has

$$f(\mathbf{x}_n^\omega)\,\widehat{\pi}_{\omega\sigma^{-1}\omega} = \sum\nolimits_\zeta (-1)^{\ell(\sigma)-\ell(\zeta)}\widehat{\pi}_{\omega\zeta^{-1}\omega}\, g_\zeta(\mathbf{x}_n^\omega)\,. \tag{1.4.8}$$

Similarly,

$$\widehat{\pi}_\sigma f(\mathbf{x}_n) = \sum\nolimits_\zeta g_\zeta(\mathbf{x}_n)\,\widehat{\pi}_\zeta$$

implies

$$f(\mathbf{x}_n^\omega)\,\pi_{\omega\sigma^{-1}\omega} = \sum\nolimits_\zeta (-1)^{\ell(\sigma)-\ell(\zeta)}\pi_{\omega\zeta^{-1}\omega}\, g_\zeta(\mathbf{x}_n^\omega)\,. \tag{1.4.9}$$

For example, for $n=3$, one has

$$\begin{array}{rcl} \pi_1\pi_2x_2 &=& x_3\pi_1\pi_2 + x_1\pi_1 - x_1\,, \\ x_2\widehat{\pi}_1\widehat{\pi}_2 &=& \widehat{\pi}_1\widehat{\pi}_2x_1 - \widehat{\pi}_2x_3 - x_3 \end{array}$$

and

$$\begin{array}{rcl} \widehat{\pi}_1\widehat{\pi}_2x_2^2 &=& x_3^2\widehat{\pi}_1\widehat{\pi}_2 + x_3(x_1{+}x_3)\widehat{\pi}_1 + x_3x_2\,, \\ x_2^2\pi_1\pi_2 &=& \pi_1\pi_2x_1^2 - \pi_2(x_1(x_1+x_3) - x_1x_2\,. \end{array}$$

Punched diagrams can also be used to describe the commutation of a product $\widehat{\pi}^{\mathcal{V}}$ with a monomial. For an antidominant $v \in \mathbb{N}^k$, $n = v_k{+}k$, $\mathcal{V}$ associated to v, let us take the monomial $x_{k+1}\dots x_n$. Transposing diagrams along the main diagonal, and introducing signs exchange the two cases. For example, for $v=[2,2]$, one has

$$\underset{x_1\;x_2}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}} = \overset{x_3\;x_4}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}} + \overset{x_2\;x_4}{\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&2\\\hline\end{array}} + \overset{x_3\;x_2}{\begin{array}{|c|c|}\hline 2&\bullet\\\hline 1&2\\\hline\end{array}} + \overset{x_1\;x_4}{\begin{array}{|c|c|}\hline 2&3\\\hline \bullet&2\\\hline\end{array}} + \overset{x_3\;x_1}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&\bullet\\\hline\end{array}} + \overset{x_2\;x_1}{\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&\bullet\\\hline\end{array}}\;,$$

that is,

$$\begin{aligned} \pi_2\pi_3\pi_1\pi_2x_1x_2 = x_3x_4\pi_2\pi_3\pi_1\pi_2 + x_2x_4\pi_3\pi_1\pi_2 + x_3x_2\pi_2\pi_1\pi_2 \\ + x_1x_4\pi_2\pi_3\pi_2 + x_3x_1\pi_2\pi_3\pi_1 + x_1x_2\pi_3\pi_1\,, \end{aligned}$$

while

$$\begin{aligned} \widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2x_3x_4 = x_2x_1\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2 - x_3x_1\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2 - x_4x_1\widehat{\pi}_2\widehat{\pi}_1\widehat{\pi}_2 \\ - x_2x_3\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_2 - x_2x_4\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1 + x_3x_4\widehat{\pi}_3\widehat{\pi}_1 \end{aligned}$$

can be displayed as

$$\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}\begin{array}{c}x_4\\x_3\end{array} = \begin{array}{c}x_2\\x_1\end{array}\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array} - \begin{array}{c}x_3\\x_1\end{array}\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&2\\\hline\end{array} - \begin{array}{c}x_4\\x_1\end{array}\begin{array}{|c|c|}\hline 2&\bullet\\\hline 1&2\\\hline\end{array} - \begin{array}{c}x_2\\x_3\end{array}\begin{array}{|c|c|}\hline 2&3\\\hline \bullet&2\\\hline\end{array} - \begin{array}{c}x_2\\x_4\end{array}\begin{array}{|c|c|}\hline 2&3\\\hline 1&\bullet\\\hline\end{array} + \begin{array}{c}x_3\\x_4\end{array}\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&\bullet\\\hline\end{array}\,.$$

The operators of the type (1.3.1) and preserving polynomials are characterized in [125]. They are essentially deformations of divided differences, though their explicit expression can look more frightening. For example, the operators (depending on the parameters $u_1,\dots,u_4,p,q,r$)

$$f \to f\,\frac{((qu_1+pu_3)x_i + (qu_2+pu_4))(u_3x_{i+1}+u_4)}{u_1u_4-u_2u_3}\,\partial_i + r f^{s_i} := f\,D_i$$

do satisfy the braid relations.

1.5 Maximal operators for type A

The operators associated to the maximal permutation $\omega = [n, \ldots, 1]$ play a proeminent role. In fact, they all come from the projector onto the alternating 1-dimensional representation of $\mathfrak{S}_n$, already used by Cauchy and Jacobi :

$$f \to \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)} f^\sigma \ .$$

Indeed, writing Δ for the Vandermonde $\det(x_i^{j-1})_{i,j=1}^n = \prod_{1\le i<j\le n}(x_i - x_j)$, with $\rho = [n-1, \ldots, 0]$, and thus $x^\rho = x_1^{n-1} x_2^{n-2} \cdots x_n^0$, one has the following proposition.

Proposition 1.5.1. *Given* $\mathbf{x}$ *of cardinality* n*, the divided differences* ∂_ω*,* π_ω *and* $\widehat{\pi}_\omega$ *verify :*

$$\partial_\omega = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \, \frac{1}{\Delta} \ , \tag{1.5.1}$$

$$\pi_\omega = x^\rho \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \, \frac{1}{\Delta} \ , \tag{1.5.2}$$

$$\widehat{\pi}_\omega = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \, \frac{(x^\rho)^\omega}{\Delta} \ . \tag{1.5.3}$$

Proof. As in the case $n = 2$, we prefer to characterize operators by their action on a basis. The monomials $x^u : u \le \rho$ are a basis of $\mathfrak{Pol}(n)$ as a free $\mathfrak{Sym}(n)$-module. They all are sent to 0 by ∂_ω as well as by $\sum \pm\sigma\Delta^{-1}$ for degree reasons, except x^ρ which is sent to 1 (this is the only computation to perform) by both operators. This proof can be adapted for π_ω and $\widehat{\pi}_\omega$. QED

We have not mentioned T_ω in the proposition, because this is not a symmetrizer, since, for $n = 2$ for example, $x_2 T_1 = x_1$. However, $x_2(T_1 + 1) = x_1 + x_2$ and $1(T_1 + 1) = t + 1$. This indicates that one has to take the Yang-Baxter deformation of T_ω for $v = [1, t, \ldots, t^{n-1}]$ if one wants a symmetrizer. Indeed one has, as we shall see in more details in (1.9.9), the following symmetrizer in the Hecke algebra (as shows the last expression):

$$(T_1 + 1)\left(T_2 + \frac{t-1}{t^2-1}\right)\left(T_3 + \frac{t-1}{t^3-1}\right) \cdots (T_1 + 1)\left(T_2 + \frac{t-1}{t^2-1}\right)(T_1 + 1)$$
$$= \sum_{\sigma\in\mathfrak{S}_n} T_\sigma = \prod_{1\le i<j\le n} (t x_i - x_j)\, \partial_\omega \, .$$

We shall frequently use the action of ∂_ω on a product $f_1(x_1)\cdots f_n(x_n)$ of functions of a single variable. In that case, the sum $\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)} \big(f_1(x_1)\cdots f_n(x_n)\big)^\sigma$ is equal to the determinant $\Big| f_i(x_j) \Big|$, and one may view

$$f_1(x_1)\cdots f_n(x_n)\, \partial_\omega = \Big| f_i(x_j) \Big|_{i,j=1\ldots n} \Delta^{-1} \tag{1.5.4}$$

as the *discrete Wronskian* of the functions $f_1, \ldots, f_n$. .

Schur functions correspond to the case where $f_1, \ldots, f_n$ are powers of a variable, factorial Schur functions arise when taking instead modified powers $x(x-1)\ldots(x-k)$, while q-factorial Schur functions stem from q-powers $(x-1)(x-q)\ldots(x-q^k)$. More precisely, for any $v \in \mathbb{N}^n$, the *Schur function* $s_v(\mathbf{x}_n)$ is equal to $x^{v+\rho}\,\partial_\omega$, the *factorial Schur function* of index v is equal to

$$\Big(x_1(x_1-1)\ldots(x_1-v_1-n+2)\Big)\ldots\Big(x_n(x_n-1)\ldots(x_n-v_n+1)\Big)\,\partial_\omega$$

and the *q-factorial Schur function* of index v is equal to

$$\Big((x_1-1)(x_1-q)\ldots(x_1-q^{v_1+n-1})\Big)\ldots\Big((x_n-1)(x_n-q)\ldots(x_n-q^{v_n})\Big)\,\partial_\omega\,.$$

For example, when $n=3$ and $v=[5,2,1]$, then the corresponding factorial Schur function is equal to

$$\begin{aligned}(x_1-1)\ldots(x_1-q^6)(x_2-1)(x_2-q)(x_2-q^2)(x_3-1)\partial_{321}\\ = \frac{1}{\Delta}\begin{vmatrix}(x_1-1)\ldots(x_1-q^6) & (x_2-1)\ldots(x_2-q^6) & (x_3-1)\ldots(x_3-q^6)\\ (x_1-1)\ldots(x_1-q^2) & (x_2-1)\ldots(x_2-q^2) & (x_3-1)\ldots(x_3-q^2)\\ x_1-1 & x_2-1 & x_3-1\end{vmatrix}\,.\end{aligned}$$

We shall interpret it later as the specialization $y_1=1, y_2=q, y_3=q^2, \ldots$ of the Graßmannian Schubert polynomial $Y_{125}(\mathbf{x},\mathbf{y})$.

Divided differences can be defined for any pair x_i, x_j, and not only consecutive variables :

$$\partial_{i,j} :\ f \to (f - f^{\tau_{ij}})(x_i - x_j)^{-1}\,,$$

τ_{ij} being the transposition of x_i, x_j. We shall need these differences to factorize ∂_ω.

Lemma 1.5.2. *Let* $n = 2m$, $\omega' = [m, \ldots, 1, 2m, \ldots, m{+}1]$, $\omega = [2m, \ldots, 1]$. *Then*

$$\partial_{\omega'}\, \partial_{1,m+1}\partial_{2,m+2} \ldots \partial_{m,2m}\, \partial_{\omega'} = (\text{-}1)^{\binom{m}{2}} m!\, \partial_\omega\,. \tag{1.5.5}$$

Proof. The left-hand side commutes with multiplication by elements of $\mathfrak{Sym}(\mathbf{x}_n)$, and decreases degree by $\binom{m}{2}$. It is therefore sufficient to test its action on x^ρ to characterize it. One has $x^\rho \partial_{\omega'} = x^{m^m, 0^m}$, $x^\rho \partial_{\omega'}\, \partial_{1,m+1} \ldots \partial_{m,2m} = \sum x^v$, sum over all $v \in \mathbb{N}^n$ such that $v_i + v_{m+i} = m\text{-}1$, $i = 1, \ldots, m$. Each such monomial has a non-zero image under $\partial_{\omega'}$ if and only if $v_1, \ldots, v_m$ is a permutation of $[m\text{-}1, \ldots, 0]$. There are $m!$ such monomials, which each contribute to $x^{m-1,\ldots,0,0,\ldots,m-1}\partial_{\omega'} = (\text{-}1)^{\binom{m}{2}}$ to the right-hand side. QED

For example, for $n = 4$, one has $\partial_{2143}\partial_{13}\partial_{24}\partial_{2143} = -2\partial_{4321}$. Many other decompositions are possible, e.g.

$$\partial_{12}\partial_{14}\partial_{34}\partial_{23}\partial_{13}\partial_{24} = \partial_{4321} = \partial_{14}\partial_{13}\partial_{24}\partial_{23}\partial_{24}\partial_{13} = \partial_{23}\partial_{13}\partial_{24}\partial_{14}\partial_{34}\partial_{12}\,.$$

1.6 Littlewood's formulas

One can combine the above operators with change of variables $x_i \to \varphi(x_i)$. The maximal divided difference ∂_ω becomes $\sum (\pm\sigma)\,\Delta(\varphi(\mathbf{x}))^{-1} = \partial_\omega \Delta(\mathbf{x})\Delta(\varphi(\mathbf{x}))^{-1}$, and it remains to find functions φ furnishing an interesting Vandermonde $\Delta(\varphi(\mathbf{x}))$.

Notice that if $\varphi(x_i) = g(x_i)/f(x_i)$, then

$$\left| f(x_i)^{n-1} \quad f(x_i)^{n-2} g(x_i) \quad \cdots \quad g(x_i)^{n-1} \right|_{i=1..n} = \prod_i (f(x_i)^{n-1}\, \Delta(\varphi(\mathbf{x}))\,.$$

Taking $f(x_i) = x_i$, $g(x_i) = 1 + x_i^k$, $k \geq 0$, and remarking that $(1+x_i^k)/x_i - (1+x_j^k)/x_j = (x_i^{-1} - x_j^{-1})\big(1 - x_i x_j s_{k-2}(x_i{+}x_j)\big)$, one obtains that

$$\begin{aligned}
\left| x_i^{n-1} \quad x_i^{n-2}(1{+}x_i^k) \quad \cdots \quad (1{+}x_i^k)^{n-1} \right|_{i=1\ldots n} & \Delta(\mathbf{x})^{-1} \\
&= (1+x_2^k)(1+x_3^k)^2 \ldots (1+x_n)^{n-1}\, \pi_\omega \\
&= \prod_{1\leq i<j\leq n} \big(1 - x_i x_j s_{k-2}(x_i{+}x_j)\big)\,, \qquad (1.6.1)
\end{aligned}$$

the first equality resulting from the definition of π_ω.

In the case $k = 2$, the preceding determinant can be transformed into

$$\left| x_i^{n-1} \quad x_i^{n-2}(1{+}x_i^2) \quad x_i^{n-2}(1{+}x_i^4) \quad \cdots \quad (1{+}x_i^{2n-2}) \right|_{i=1\ldots n}\,.$$

Since the operator π_ω sends x^v, $v \in \mathbb{N}^n$ onto the Schur function $s_{v(\mathbf{x})}$, the preceding identity, still in the case $k = 2$, can be written as

$$\begin{aligned}
\prod_{1\leq i<j\leq n} (1 - x_i x_j) &= (1+x_2^2)(1+x_3^2)^2 \ldots (1+x_n^2)^{n-1})\, \pi_\omega \\
&= (1+x_2^2)(1+x_3^4) \ldots (1+x_n^{2n-2})\, \pi_\omega \\
= \sum_{\epsilon=[\epsilon_1,\ldots,\epsilon_n]\in\{0,1\}^n} (\text{-}1)^{|\epsilon|} s_{[0\epsilon_1, 2\epsilon_2, \ldots, (2n-2)\epsilon_n]}(\mathbf{x}) &= 1 + s_{02}(\mathbf{x}) + s_{004}(\mathbf{x}) + s_{024}(\mathbf{x}) + \ldots \\
&= 1 - s_{11}(\mathbf{x}) + s_{211}(\mathbf{x}) - s_{222}(\mathbf{x}) + \ldots \\
&= 1 + \sum_{r,\alpha} (-1)^{|\alpha|} s_{(\alpha|\alpha+1^r)}(\mathbf{x})\,, \qquad (1.6.2)
\end{aligned}$$

sum over all r, all $\alpha = [\alpha_1, \ldots, \alpha_r]$, $\alpha_1 > \alpha_2 > \ldots \alpha_r \geq 0$, using the Frobenius notation[6] for partitions.

Similar identities, known to Littlewood [143], [146, p. 78], can be obtained as easily, the reordering of the indices of the Schur functions being translated into

6

properties of diagonal hooks.

$$\begin{aligned}\prod_i (1-x_i) \prod_{1\le i<j\le n} (1-x_ix_j) &= (1-x_1)(1-x_2^3)\dots(1-x_n^{2n-1})\,\pi_\omega \\ &= \sum_{\epsilon=[\epsilon_1,\dots,\epsilon_n]\in\{0,1\}^n} (-1)^{|\epsilon|} s_{[\epsilon_1,3\epsilon_2,\dots,(2n-1)\epsilon_n]}(\mathbf{x}) \\ &= 1 - s_1(\mathbf{x}) - s_{03}(\mathbf{x}) + s_{13}(\mathbf{x}) - s_{005}(\mathbf{x}) + s_{105}(\mathbf{x}) + s_{035}(\mathbf{x}) - s_{135}(\mathbf{x}) + \dots \\ &= 1 - s_1(\mathbf{x}) + s_{21}(\mathbf{x}) - s_{22}(\mathbf{x}) - s_{311}(\mathbf{x}) + s_{321}(\mathbf{x}) - s_{332}(\mathbf{x}) + s_{333}(\mathbf{x}) + \dots \\ &= 1 + \sum_\alpha (-1)^{|\alpha|} s_{(\alpha|\alpha)}(\mathbf{x})\,. \end{aligned} \tag{1.6.3}$$

$$\begin{aligned}\prod_{1\le i\le j\le n} (1-x_ix_j) &= (1-x_1^2)(1-x_2^4)\dots(1-x_n^{2n})\,\pi_\omega \\ &= \sum_{\epsilon=[\epsilon_1,\dots,\epsilon_n]\in\{0,1\}^n} (-1)^{|\epsilon|} s_{[2\epsilon_1,4\epsilon_2,\dots,2n\epsilon_n]}(\mathbf{x}) \\ &= 1 - s_2(\mathbf{x}) - s_{04}(\mathbf{x}) + s_{24}(\mathbf{x}) - s_{006}(\mathbf{x}) + s_{206}(\mathbf{x}) + s_{046}(\mathbf{x}) - s_{246}(\mathbf{x}) + \dots \\ &= 1 - s_2(\mathbf{x}) + s_{31}(\mathbf{x}) - s_{33}(\mathbf{x}) - s_{411}(\mathbf{x}) + s_{431}(\mathbf{x}) - s_{442}(\mathbf{x}) + s_{444}(\mathbf{x}) + \dots \\ &= 1 + \sum_{r,\beta} (-1)^{|\beta|} s_{(\beta+1^r|\beta)}(\mathbf{x})\,. \end{aligned} \tag{1.6.4}$$

$$\begin{aligned}\prod_{i=1}^n (1-x_i) \prod_{1\le i\le j\le n} (1-x_ix_j) &\\ = (1-x_1)(1-x_1^2)(1-x_2^2)(1-x_2^3)\dots(1-x_n^n)(1-x_n^{n+1})\,\pi_\omega &\\ = \big(1 - s_1(\mathbf{x}) + s_{11}(\mathbf{x}) - s_{111}(\mathbf{x}) + \dots\big) \sum_{\epsilon_i\in\{0,1\}} (-1)^{|\epsilon|} s_{[2\epsilon_1,4\epsilon_2,\dots,2n\epsilon_n]}(\mathbf{x})\,. & \end{aligned} \tag{1.6.5}$$

One can generalize these formulas by adding letters to the alphabet $\mathbf{x}$. For example, using $\mathbf{x}\cup\{1\}$ in (1.6.2), one obtains

$$\begin{vmatrix} x_1^n & x_1^{n-1}+x_1^{n+1} & \dots & 1+x_1^{2n} \\ \vdots & \vdots & & \vdots \\ x_n^n & x_n^{n-1}+x_n^{n+1} & \dots & 1+x_n^{2n} \\ 1 & 1 & \dots & 1 \end{vmatrix} \frac{1}{\Delta(\mathbf{x})} = \prod_{i=1}^n (1-x_i)^2 \prod_{1\le i<j\le n} (x_ix_j-1)\,, \tag{1.6.6}$$

the factor $\prod(1-x_i)^2$ being due to $s_{11}(\mathbf{x}+1) = s_{11}(\mathbf{x}) + s_1(\mathbf{x})$ and $\Delta(\mathbf{x}+1) = \Delta(\mathbf{x})\prod(1-x_i)$. More variations of this type can be found in [105].

All the preceding formulas can be interpreted, in terms of λ-rings, as describing the plethysms $\Lambda^i(S^2)$ or $\Lambda^i(\Lambda^2)$, and have counterparts describing $S^i(S^2)$ or $S^i(\Lambda^2)$. Let us show that the symmetrizer π_ω still allow to describe the generating function of this last plethysms.

Proposition 1.6.1. *For a given n, one has*

$$\prod_{i\leq j}(1-x_ix_j)^{-1} = \frac{1}{(1\text{-}x_1^2)(1\text{-}x_1^2x_2^2)\dots(1\text{-}x_1^2\dots x_n^2)}\,\pi_\omega \tag{1.6.7}$$

$$= \sum_{even\,rows} s_\lambda(\mathbf{x})$$

$$\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1\text{-}x_1x_2)(1\text{-}x_1\dots x_4)(1\text{-}x_1\dots x_6)\dots}\,\pi_\omega \tag{1.6.8}$$

$$= \sum_{even\,columns} s_\lambda(\mathbf{x})$$

$$\prod(1\text{-}x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1\text{-}x_1)(1\text{-}x_1x_2)\dots(1\text{-}x_1\dots x_n)}\,\pi_\omega \tag{1.6.9}$$

$$= \sum s_\lambda(\mathbf{x})$$

$$\prod(1\text{-}x_i)^{-2}\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1\text{-}x_1)^2(1\text{-}x_1x_2)^2\dots(1\text{-}x_1\dots x_n)^2}\,\pi_\omega \tag{1.6.10}$$

$$= \sum(\lambda_1\text{-}\lambda_2\text{+}1)(\lambda_2\text{-}\lambda_3\text{+}1)\dots(\lambda_n\text{+}1)s_\lambda(\mathbf{x}). \tag{1.6.11}$$

Proof. One can use induction on n, factorizing $\pi_\omega = \pi_{\omega'}\pi_\omega$, with $\omega' = [n\text{-}1,\dots,1]$. Thus one is left with computing the image under π_ω of the quotient of the two successive denominators appearing in the left-hand sides. For the first formula, it means computing

$$\begin{aligned}(1-x_1x_n)\dots(1-x_1x_{n-1})(1-x_n^2)(1-x_1\dots x_n)^{-1}\pi_\omega \\ = (1-x_ne_1+\dots+(\text{-}x_n)^ne_n))(1-x_1\dots x_n)^{-1}\pi_\omega\,,\end{aligned}$$

$e_1,\dots,e_n$ being the elementary symmetric functions in $\mathbf{x}_n$, and therefore commuting with π_ω. Since $x_n,\dots,x_n^{n-1}$ are sent to 0, and $(-x_n)^n\pi_\omega = -x_1\dots x_n$, the above expression is equal to 1, thus proving (1.6.7). The other formulas require no more pain. Moreover, the rational functions in the right-hand sides expanding as sums of dominant monomials, the expressions in terms of Schur functions follow immediately. QED

One should try expressions more general than products of factors $(1\pm u)^{\pm 1}$, with u monomial. I shall give a single example.

Lemma 1.6.2. *Given n, then*

$$\frac{1}{(1\text{-}x^1\text{-}x^2)(1\text{-}x^{22})(1\text{-}x^{111}\text{-}x^{222})(1\text{-}x^{2222})\dots}\,\pi_\omega = \prod_i \frac{1}{1-x_i-x_i^2}\prod_{i<j}\frac{1}{1-x_ix_j}\,. \tag{1.6.12}$$

Proof. Let G_n be the right-hand side. Using induction on n, one has to compute $G_{n-1}/G_n\pi_\omega$. This depends on parity, and taking $n=4,5$ will be generic enough to follow the proof.

$$G_3/G_4\pi_{4321} = (1\text{-}x_1x_4)(1\text{-}x_2x_4)(1\text{-}x_3x_4)(1-x_4-x_4^2)\,\pi_{4321}$$
$$= \prod_{i=1}^{4}(1-x_ix_4)\pi_{4321} - x_4(1\text{-}x_1x_4)(1\text{-}x_2x_4)(1\text{-}x_3x_4)\pi_{4321}\,.$$

One has already seen that $\prod(1-x_ix_4)\pi_{4321} = 1-x^{2222}$, and one checks that all the monomials appearing in $x_4(\dots)$ are sent to 0 under π_{4321}. In the case of $G_4/G_5\pi_{54321}$ on the contrary, the monomial $-x^{11003}$ is such that $-x^{11003}\pi_{54321} = -x^{11111}$, and thus, $G_4/G_5\pi_{54321} = 1-x^{22222}-x^{11111}$. In both cases, the resulting factor is what is required by the left-hand side of (1.6.12) to ensure equality. QED

The left-hand side of (1.6.12) expands as a positive sum of Schur functions, which multiplicities that are easily written in terms of the multiplicities of parts in the conjugate partitions.

1.7 Yang-Baxter relations

With a little more work, one can construct operators offering still more parameters.

The uniform shift $D_i \to D_i + 1$, $i = 1, \ldots, n{-}1$, destroys in general the braid relations[7]. For example,

$$(1+s_1)(1+s_2)(1+s_1) = 2 + 2s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1 \\ \neq (1+s_2)(1+s_1)(1+s_2) \ .$$

However

$$(1+s_1)(\frac{1}{2}+s_2)(1+s_1) = (1+s_2)(\frac{1}{2}+s_1)(1+s_2) \ ,$$

because both sides expand (in the group algebra of $\mathfrak{S}_3$) into the sum of all permutations.

Therefore, one abandons uniform shifts, but how to find compatible shifts like $1, 1/2, 1$?

The solution is due to Young [195], and called Yang-Baxter equation [194, 5] because Young-Yang-Baxter would be confusing.

One chooses an arbitrary vector of parameters $v = [v_1, \ldots, v_n]$ (called *spectral vector*), and each time one operates with D_i, $i = 1, \ldots, n{-}1$, one modifies accordingly the spectral vector by $v \to vs_i$.

Now, the shift to use depends only on the difference of the spectral values exchanged, with similar rules for the different varieties of operators D_i.

More precisely, given i, let $a = v_i$, $b = v_{i+1}$ the corresponding components of the spectral vector. Then, instead of $s_i, \partial_i, \pi_i, \widehat{\pi}_i, T_i$ respectively, one takes

$$s_i + \frac{1}{b-a},\ \partial_i + \frac{1}{b-a},\ \pi_i + \frac{1}{b/a-1},\ \widehat{\pi}_i + \frac{1}{b/a-1},\ T_i + \frac{t-1}{b/a-1}$$

(the careful reader adds "provided $b \neq a$").

For $n = 3$, the Yang-Baxter relations for s_i, ∂_i, π_i and T_i, and a spectral vector v are, writing $v_2{-}v_1 = a$, $v_3{-}v_2 = b$, $v_2/v_1 = \alpha$, $v_3/v_2 = \beta$,

[7] it only works for $\widehat{\pi}_i \to \widehat{\pi}_i + 1 = \pi_i$.

$$\begin{tikzcd}
& 123 \arrow[dl, "s_1+\frac{1}{a}"'] \arrow[dr, "s_2+\frac{1}{b}"] & \\
213 \arrow[d, "s_2+\frac{1}{a+b}"'] & & 132 \arrow[d, "s_1+\frac{1}{a+b}"] \\
231 \arrow[dr, "s_1+\frac{1}{b}"'] & & 312 \arrow[dl, "s_2+\frac{1}{a}"] \\
& 321 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 123 \arrow[dl, "\partial_1+\frac{1}{a}"'] \arrow[dr, "\partial_2+\frac{1}{b}"] & \\
213 \arrow[d, "\partial_2+\frac{1}{a+b}"'] & & 132 \arrow[d, "\partial_1+\frac{1}{a+b}"] \\
231 \arrow[dr, "\partial_1+\frac{1}{b}"'] & & 312 \arrow[dl, "\partial_2+\frac{1}{a}"] \\
& 321 &
\end{tikzcd}$$

$$\begin{tikzcd}
& 123 \arrow[dl, "\pi_1+\frac{1}{\alpha-1}"'] \arrow[dr, "\pi_2+\frac{1}{\beta-1}"] & \\
213 \arrow[d, "\pi_2+\frac{1}{\alpha\beta-1}"'] & & 132 \arrow[d, "\pi_1+\frac{1}{\alpha\beta-1}"] \\
231 \arrow[dr, "\pi_1+\frac{1}{\beta-1}"'] & & 312 \arrow[dl, "\pi_2+\frac{1}{\alpha-1}"] \\
& 321 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 123 \arrow[dl, "T_1+\frac{t-1}{\alpha-1}"'] \arrow[dr, "T_2+\frac{t-1}{\beta-1}"] & \\
213 \arrow[d, "T_2+\frac{t-1}{\alpha\beta-1}"'] & & 132 \arrow[d, "T_1+\frac{t_1-1}{\alpha\beta-1}"] \\
231 \arrow[dr, "T_1+\frac{t-1}{\beta-1}"'] & & 312 \arrow[dl, "T_2+\frac{t-1}{\alpha-1}"] \\
& 321 &
\end{tikzcd}$$

The fact that each hexagon closes means that the two paths from top to bottom give equal elements when evaluated as products of the labels on the edges.

Thanks to the Yang-Baxter relations, to each spectral vector v, is associated a family of operators $D^v_\sigma : \sigma \in \mathfrak{S}_n$, obtained by taking products corresponding to reduced decompositions.

For example, for $\mathfrak{S}_3$, and $v = [y_1, y_2, y_3]$, one has the operators

$$\partial^v_{123} = 1\,, \quad \partial^v_{213} = \partial_1 + \frac{1}{y_2 - y_1}\,, \quad \partial^v_{132} = \partial_2 + \frac{1}{y_3 - y_2}\,,$$

$$\partial^v_{231} = \partial_1\partial_2 + \partial_2\frac{1}{y_2 - y_1} + \partial_1\frac{1}{y_3 - y_1} + \frac{1}{(y_2 - y_1)(y_3 - y_1)}\,,$$

$$\partial^v_{312} = \partial_2\partial_1 + \partial_1\frac{1}{y_3 - y_2} + \partial_2\frac{1}{y_3 - y_1} + \frac{1}{(y_3 - y_2)(y_3 - y_1)}\,,$$

$$\partial^v_{321} = \partial_1\partial_2\partial_1 + \partial_1\partial_2\frac{1}{y_3 - y_2} + \partial_2\partial_1\frac{1}{y_2 - y_1} + \partial_1\frac{1}{(y_2 - y_1)(y_3 - y_1)} + \partial_2\frac{1}{(y_3 - y_2)(y_3 - y_1)} + \frac{1}{(y_2 - y_1)(y_3 - y_1)(y_3 - y_2)}\,.$$

One recognizes that the product $(1 + s_1)(2^{-1} + s_2)(1 + s_1)$ corresponds to the choice $D_i = s_i$, $\sigma = [3, 2, 1]$, $v = [1, 2, 3]$. The reader will guess, and prove, that for any n, the choice $D_i = s_i$, $\sigma = \omega :== [n, \ldots, 1]$, $v = [1, 2, \ldots, n]$ gives

$$(1 + s_1)\left((\frac{1}{2} + s_2)(1 + s_1)\right)\cdots\left((\frac{1}{n-1} + s_{n-1})\cdots(\frac{1}{2} + s_2)(1 + s_1)\right) = \sum_{\sigma\in\mathfrak{S}_n} \sigma\,.$$

One can also twist the action of the symmetric group, and use $D_i = \partial_i + s_i$. The operators D_i still satisfy the braid relations, together with the relations $D_i^2 = 1$. Therefore, the operators $D_1, \ldots, D_{n-1}$ provide a twisted action of the symmetric group on $\mathfrak{Pol}(\mathbf{x}_n)$. Since the Yang-Baxter shifts are the same for ∂_i and s_i, they can also be used for $\partial_i + s_i$. In particular, one can take the spectral vector $[1, 2, \ldots, n]$.

$$\begin{tikzcd}
& \boxed{123} \arrow[dl, "\partial_1 + s_1{+}1"'] \arrow[dr, "\partial_2 + s_2{+}1"] & \\
\boxed{213} \arrow[d, "\partial_2 + s_2{+}\frac{1}{2}"'] & & \boxed{132} \arrow[d, "\partial_1 + s_1{+}\frac{1}{2}"] \\
\boxed{231} \arrow[dr, "\partial_1 + s_1{+}1"'] & & \boxed{312} \arrow[dl, "\partial_2 + s_2{+}1"] \\
& \boxed{321} &
\end{tikzcd}$$

Let us show that the maximal Yang-Baxter element for this choice of spectral vector is still a symmetrizer. In the case $n = 2$, one has indeed

$$\partial_1 + s_1 + 1 = (1 + x_1 - x_2)\,\partial_1\,.$$

Lemma 1.7.1. *Given n, let*

$$\square_\omega = \Big((D_1{+}1)\dots(D_{n-1}+\frac{1}{n-1})\Big)\Big((D_1{+}1)\dots(D_{n-2}+\frac{1}{n-2})\Big)\dots\Big(D_1\Big).$$

Then

$$\square_\omega = \prod_{1\le i<j\le n}(1+x_i-x_j)\,\partial_\omega\,. \tag{1.7.1}$$

Proof. Both sides of (1.7.1) commute with multiplication with symmetric functions, it is therefore sufficient to test their action on a basis of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free $\mathfrak{Sym}(\mathbf{x}_n)$ module. But instead of the basis of monomials $\{x^v : v\le\rho\}$ used above, we shall use a basis of homogeneous polynomials $\{Y_v : v\le\rho\}$ in their linear span, such that each Y_v has a least one symmetry[8] in some x_i, x_{i+1}, except for $Y_{n-1,\dots,0} = x^\rho$. But using symmetric rational functions in $\mathbf{x}_n$ instead of elements of $\mathfrak{Sym}(\mathbf{x}_n)$, we can take the polynomials $Y_v\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)$ as a test cohort. All these elements, except in the case $v=\rho$, are sent to 0 by $\prod_{1\le i<j\le n}(1+x_i-x_j)\,\partial_\omega$ because the factor $\prod_{i\ne j}(1+x_i-x_j)$, being symmetrical, commutes with ∂_ω, and because $Y_v\partial_\omega = 0$ for degree reasons.

On the other hand, if Y_v has the symmetry x_i, x_{i+1}, then, by commutation,

$$\begin{aligned} Y_v\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)(\partial_i+s_i+1) &= Y_v\Bigg(\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)\Bigg)(1{+}x_i{-}x_{i+1})\,\partial_i \\ &= Y_v\partial_i\Bigg(\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)\Bigg)(1{+}x_i{-}x_{i+1}) = 0\,. \end{aligned}$$

Since, thanks to Yang-Baxter equation, one can factorize on the left of $\square_\omega$ any D_i+1, the image of $Y_v\prod(1+x_j-x_i)$ under $\square_\omega$ is 0 when $v\ne\rho$. Thus, both sides of (1.7.1) coincide up to multiplication by a rational symmetric function. To determine this constant, it is sufficient to see that

$$1(\partial_1{+}s_1{+}1)(\partial_1{+}s_1{+}2^{-1})\cdots = n! = \prod_{1\le i<j\le n}(1{+}x_i{-}x_j)\,\partial_\omega\,,$$

and this ensures the required equality. QED

The Yang-Baxter rules do not exhaust the realm of interesting factorized expressions. Let us take[9]

$$\begin{aligned}&((1-y_1\partial_1)(1-y_1\partial_2)\cdots(1-y_1\partial_{n-1}))\\ &\qquad\qquad((1-y_2\partial_1)(1-y_2\partial_2)\cdots(1-y_2\partial_{n-2}))\cdots((1-y_{n-1}\partial_1))\end{aligned}$$

[8] tTo show that such a basis exists is easy by induction on n, we shall see later that the Schubert polynomials $Y_v(\mathbf{x},\mathbf{0})$ satisfy such properties.

[9] This product of divided differences is the generating function of Schubert polynomials in the pair of alphabets $\mathbf{y},\mathbf{0}$, in the algebra of divided differences, also called the *Nil-Coxeter algebra* [39] see (8.3.2).

and show that this element can be used to transform the staircase monomial x^ρ, with $\rho = [n{-}1, \ldots, 0]$, into a product of factors of the type $x_i - y_j$.

Let us make the step-by-step computation for $n = 4$, displaying the factors of the polynomials planarly.

$$
\begin{array}{|c|c|c|}\hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1 \\ \hline \end{array}
\xrightarrow{1-y_1\partial_1}
\begin{array}{|c|c|c|}\hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
\xrightarrow{1-y_1\partial_2}
\begin{array}{|c|c|c|}\hline x_3 & & \\ \hline x_2 & x_2-y_1 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
$$

$$
\xrightarrow{1-y_1\partial_3}
\begin{array}{|c|c|c|}\hline x_3-y_1 & & \\ \hline x_2 & x_2-y_1 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
\quad \cdots \quad
\xrightarrow{1-y_3\partial_1}
\begin{array}{|c|c|c|}\hline x_3-y_1 & & \\ \hline x_2-y_2 & x_2-y_1 & \\ \hline x_1-y_3 & x_1-y_2 & x_1-y_1 \\ \hline \end{array} .
$$

Each step is of the type $f\, x_i(1 - y\partial_i) = f\,(x_i - y)$, with f symmetrical in x_i, x_{i+1}. In final, we have obtained the function $\prod_{i+j\le 4}(x_i - y_j)$ by using only that $1\partial_i = 0$, $x_i\partial_i = 1$. This function, together with the "staircase monomial" x^{3210}, will play a key role in all the sequel. This identity can be written more compactly, still reading the planar arrays by columns (reading by rows still works in the present case), as

$$
\begin{array}{|c|c|c|}\hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1 \\ \hline \end{array}
\;
\begin{array}{|c|c|c|}\hline 1-y_1\partial_1 & 1-y_1\partial_2 & 1-y_1\partial_3 \\ \hline & 1-y_2\partial_1 & 1-y_2\partial_2 \\ \hline & & 1-y_3\partial_1 \\ \hline \end{array}
=
\begin{array}{|c|c|c|}\hline x_3-y_1 & & \\ \hline x_2-y_2 & x_2-y_1 & \\ \hline x_1-y_3 & x_1-y_2 & x_1-y_1 \\ \hline \end{array} . \tag{1.7.2}
$$

1.8 Yang-Baxter bases and the Hecke algebra

The Yang-Baxter relations constitute a powerful tool to define linear bases with an explicit action of the Hecke algebra (or of the different algebras obtained by specialization, the first interesting one being the group algebra of the Weyl group).

In this section we shall change the conventions for the Hecke algebra, compared to the preceding section, to bring into prominence some symmetries.

The *Hecke algebra* $\mathcal{H}_n$ of the symmetric group $\mathfrak{S}_n$ is the algebra generated by $T_1, \ldots, T_{n-1}$ satisfying the braid relations together with the Hecke relations

$$(T_i - t_1)(T_i - t_2) = 0\ ,\ i = 1, \ldots, n{-}1\,,$$

for some fixed generic parameters t_1, t_2. For Macdonald polynomials, one takes $t_1 = t$, $t_2 = -1$. The 0-*Hecke algebra* is the specialisation $t_1 = 0$, $t_2 = -1$ of the Hecke algebra (that one can realize as the algebra generated by $\widehat{\pi}_1, \ldots, \widehat{\pi}_{n-1}$). The 00-*Hecke algebra*, also called *NilCoxeter algebra*, is the specialisation $t_1 = 0$, $t_2 = 0$. It can be realized as the algebra generated by $\partial_1, \ldots, \partial_{n-1}$. .

From the point of view of operators, the Hecke algebra is the algebra generated by operators T_i such that each T_i acts on x_i, x_{i+1} only, commutes with $\mathfrak{Sym}(x_i, x_{i+1})$, and acts on $\{1, x_{i+1}\}$ by

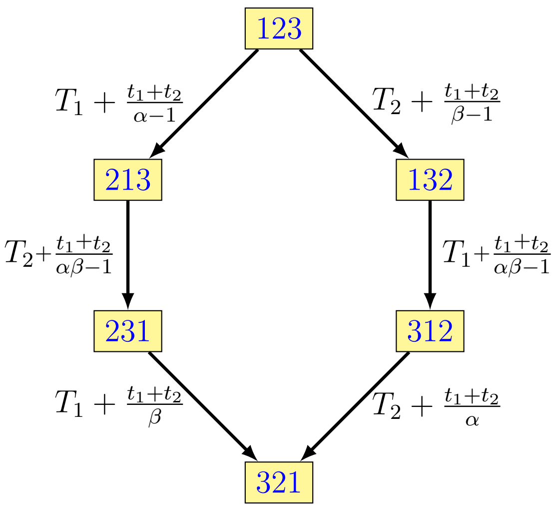
$$1\, T_i = t_1 \qquad \& \qquad x_{i+1}\, T_i = -t_2 x_i\,.$$

One has therefore $T_i = \pi_i(t_1 + t_2) - s_i t_2$.

The general *Yang-Baxter equation*[10] depends on two generic parameters α, β :

$$\left(T_1 + \frac{t_1{+}t_2}{\alpha{-}1}\right)\left(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1}\right)\left(T_1 + \frac{t_1{+}t_2}{\beta{-}1}\right) = \left(T_2 + \frac{t_1{+}t_2}{\beta{-}1}\right)\left(T_1 + \frac{t_1{+}t_2}{\alpha\beta{-}1}\right)\left(T_2 + \frac{t_1{+}t_2}{\alpha{-}1}\right)\,. \qquad (1.8.1)$$

Graphically, it reads



Given n, one takes an arbitrary *spectral vector* $[\gamma_1, \ldots, \gamma_n]$ of indeterminates. The *Yang-Baxter basis* $\{\mho_\sigma^\gamma : \sigma \in \mathfrak{S}_n\}$ *corresponding to* $[\gamma_1, \ldots, \gamma_n]$ is defined recursively, as follows, starting from $\mho_\sigma^\gamma = 1$ for the identity permutation:

$$\mho_{\sigma s_i}^\gamma = \mho_\sigma^\gamma \left(T_i + \frac{t_1 + t_2}{\gamma_{\sigma_{i+1}}/\gamma_{\sigma_i} - 1}\right) \text{ for } \sigma_i < \sigma_{i+1}\,. \qquad (1.8.2)$$

The consistency of the definition is insured by the Yang-Baxter equation (1.8.1). Notice that arrows are reversible in the generic case. Indeed, for any i, any

[10] The Yang-Baxter relations for the group algebra of $\mathfrak{S}_n$, for the algebra of divided differences, and for the algebra of isobaric divided differences are the limits $t_1 = 1, t_2 = -1$, $t_1 = 0, t_2 = 0$, $t_1 = 1, t_2 = 0$ of (1.8.1 respectively.

$\gamma \neq 0, 1$, one has

$$\left(T_i + \frac{t_1+t_2}{\gamma-1}\right)\left(T_i + \frac{t_1+t_2}{\gamma^{-1}-1}\right) = \left(t_1 + \frac{t_1+t_2}{\gamma-1}\right)\left(t_1 + \frac{t_1+t_2}{\gamma^{-1}-1}\right) = -\frac{(t_1\gamma+t_2)(t_1+t_2\gamma)}{(\gamma-1)^2}\,.$$

It is clear that the set $\{\mho_\sigma^\gamma : \sigma \in \mathfrak{S}_n\}$ constitute a linear basis of $\mathcal{H}_n$, because $\mho_\sigma = T_\sigma + \sum_{v:\ell(v)<\ell(\sigma)} c_\sigma^v T_v$. Since this basis is generated using the T_i's, it is immediate to write the matrices representing the Hecke algebra in this basis. The matrices representing each T_i are made of 2×2 blocks corresponding to the spaces $\langle \mho_\sigma^\gamma, \mho_{\sigma s_i}^\gamma \rangle$. They generalize the *semi-normal representation of the symmetric group* due to Young[11].

Indeed, for $\mathfrak{S}_2$, and the spectral vector $[1, \gamma]$, the Yang-Baxter basis is $\{1, T_1 + (t_1+t_2)(\gamma-1)^{-1}\}$, and the matrix representing T_1 is given on the left, while Young's matrix (which is the limit for $\gamma = (-t_1/t_2)^g, (-t_1/t_2) \to 1$) is given on the right [163] :

$$\begin{bmatrix} -(t_1+t_2)(\gamma-1)^{-1} & -(t_1\gamma+t_2)(t_1+\gamma t_2)(\gamma-1)^{-2} \\ 1 & (t_1+t_2)(\gamma^{-1}-1)^{-1} \end{bmatrix} , \begin{bmatrix} -g^{-1} & 1-g^{-2} \\ 1 & g^{-1} \end{bmatrix} \tag{1.8.3}$$

One could write the similar matrices for the other types B, C, D, once the Yang-Baxter relations have been written for these types.

Irreducible representations can be obtained by either degeneration of the spectral vector, or by making the Hecke algebra act on polynomials. For example, in the case of the symmetric group, a *Specht representation* is obtained by acting on a product of Vandermondes on consecutive variables. Similarly, acting on a product of t-t Vandermondes $\prod_{a\le i<j\le b}(x_i - tx_j)$ on blocks of consecutive variables produces an irreducible representation of the Hecke algebra.

Yang-Baxter bases possess many symmetries. Let $f \to \omega \star f \star \omega$ be the automorphism of $\mathcal{H}_n$ induced by $T_\sigma \to \omega \star T_\sigma \star \omega = T_{\omega\sigma\omega}$. Then one has

Lemma 1.8.1. *The Yang-Baxter bases associated to the spectral vectors* $[y_1, \dots, y_n]$ *and* $[y_n^{-1}, \dots, y_1^{-1}]$ *satisfy the relations*

$$\mho_\sigma^{y_n^{-1},\dots,y_1^{-1}} = \omega \star \mho_{\omega\sigma\omega}^{y_1,\dots,y_n} \star \omega\ , \quad \sigma \in \mathfrak{S}_n\,. \tag{1.8.4}$$

Proof. In the case $n = 2$, this is the identity

$$T_1 + \frac{t_1+t_2}{y_1^{-1}/y_2^{-1}-1} = \omega \star \left(T_1 + \frac{t_1+t_2}{y_2/y_1-1}\right) \star \omega = T_1 + \frac{t_1+t_2}{y_2/y_1-1}\,.$$

For a general σ and i such that $\ell(\sigma s_i) \ge \ell(\sigma)$, putting $\gamma = y_{n-i}^{-1}/y_{n+1-i}^{-1}$, one has

$$\begin{aligned} \mho_\sigma^{y_n^{-1},\dots,y_1^{-1}}\left(T_i + \frac{t_1+t_2}{\gamma-1}\right) &= (\omega \star \mho_{\omega\sigma\omega}^{y_1,\dots,y_n} \star \omega)\left(\omega \star \left(T_i + \frac{t_1+t_2}{\gamma-1}\right) \star \omega\right) \\ &= \omega \star \left(\mho_{\omega\sigma\omega}^{y_1,\dots,y_n}\left(T_i + \frac{t_1+t_2}{y_{n+1-i}/y_{n-i}-1}\right)\right) \star \omega\,, \end{aligned}$$

[11] We have taken generic parameters. To build general representations, one also needs blocks of size 1!.

and this proves the statement by induction on length. QED

We also need another involution $f \to \widehat{f}$ induced by

$$T_i \to \widehat{T_i} = T_i - (t_1{+}t_2)\,,\ t_1 \to -t_2\,,\ t_2 \to -t_1\,.$$

Notice that $\widehat{T}_1, \ldots, \widehat{T}_{n-1}$ satisfy the braid relations, together with the Hecke relations

$$\left(\widehat{T_i} + t_2\right)\left(\widehat{T_i} + t_1\right) = 0\,,$$

and that $T_i\widehat{T_i} = -t_1t_2$.

Let now $f \to f^\vee$ be the anti-automorphism induced by $(T_\sigma)^\vee = T_{\sigma^{-1}}$. Define a quadratic form $(\ ,\ )^{\mathcal{H}}$ on $\mathcal{H}_n$ by

$$(f\,,\,g)^{\mathcal{H}} = f\,g^\vee \cap T_\omega\,, \tag{1.8.5}$$

i.e. by taking the coefficient of T_ω in the product $f\,g^\vee$.

The basis $\{\widehat{T_\sigma}\}$ is clearly the adjoint of $\{T_{\omega\sigma}\}$, i.e. one has

$$\left(T_{\omega\sigma}\,,\,\widehat{T_\zeta}\right)^{\mathcal{H}} = \delta_{\sigma,\zeta}\,,\ \ \sigma,\zeta \in \mathfrak{S}_n\,.$$

Testing the statements on the pairs $T_\sigma, \widehat{T_\zeta}$, one checks :

$$(T_i\,f\,,\,g)^{\mathcal{H}} = (f\,,\,T_{n-i}g)^{\mathcal{H}} \quad \& \quad (fT_i\,,\,g)^{\mathcal{H}} = (f\,,\,gT_i)^{\mathcal{H}}\,. \tag{1.8.6}$$

The quadratic form can be restricted to two-dimensional spaces, for which one has the following property of a Yang-Baxter basis.

Lemma 1.8.2. *Let $f, g \in \mathcal{H}_n$, i, γ be such that*

$$\left(f\,,\,g\right)^{\mathcal{H}} = 0 \quad \& \quad \left(f(T_i + \frac{t_1+t_2}{\gamma-1})\,,\,g\right)^{\mathcal{H}} = 1\,.$$

Then

$$\left(f\,,\,g(T_i + \frac{t_1+t_2}{\gamma^{-1}-1})\right)^{\mathcal{H}} = 1 \quad \& \quad \left(f(T_i + \frac{t_1+t_2}{\gamma-1})\,,\,g(T_i + \frac{t_1+t_2}{\gamma^{-1}-1})\right)^{\mathcal{H}} = 0\,.$$

Proof. One transfers the factor $(T_i{+}\bullet)$ to the left, and uses that $(T_i{+}(t_1{+}t_2)(\gamma{-}1)^{-1})(T_i{+}(t_1{+}t_2)(\gamma^{-1}{-}1)^{-1})$ be a scalar. QED

In other words, the two Yang-Baxter bases associated with the spectral vectors $[1,\gamma]$ and $[1,\gamma^{-1}]$ are adjoint of each other with respect to $(\ ,\ )^{\mathcal{H}}$.

Combining the Yang-Baxter relations and the preceding lemma, one can evaluate scalar products of factorized elements. For example

$$\left((T_1 + \frac{t_1{+}t_2}{\alpha{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1})(T_1 + \frac{t_1{+}t_2}{\beta{-}1})\,,\,(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}{-}1})(T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}{-}1})\right)^{\mathcal{H}}$$

$$= \left((T_1 + \frac{t_1{+}t_2}{\alpha{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1})(T_1 + \frac{t_1{+}t_2}{\beta{-}1})(T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}{-}1})(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}{-}1})\right) \cap T_{321}$$

can be computed by reducing the length of the expression, replacing some factors $T_i + (t_1{+}t_2)(\gamma^{-1}{-}1)^{-1})$ by a sum of two terms $(T_i{+}c_1) + c_2$ to fit the parameters in the Yang-Baxter relations. But it is simpler to move the RHS of the scalar product to the left, obtaining

$$\left((T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}{-}1})(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}{-}1})(T_1 + \frac{t_1{+}t_2}{\alpha{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1})(T_1 + \frac{t_1{+}t_2}{\beta{-}1})\,,\, 1 \right)^{\mathcal{H}}$$

which reduces to a scalar multiple of $(T_1 + (t_1{+}t_2)(\beta{-}1)^{-1},\, ,\, 1)^{\mathcal{H}} = 0$.

This example is some instance of a general orthogonality of Yang-Baxter bases. Let us write $T_i(a,b) = T_i + (t_1{+}t_2)(y_b y_a^{-1} - 1)^{-1}$, $\mathbf{y}\omega = [y_n \ldots, y_1]$, and first settle the case of the maximal Yang-Baxter element.

Lemma 1.8.3. *The element $\mho^{\mathbf{y}}_{\omega}$ satisfies the $n!$ equations*

$$\left(\mho^{\mathbf{y}}_{\omega}\,,\, \mho^{\mathbf{y}\omega}_{\sigma}\right)^{\mathcal{H}} = \delta_{1,\sigma}\,. \tag{1.8.7}$$

Proof. One takes a reduced decomposition $s_{i_1}s_{i_2}\ldots s_{i_r}$ of σ. Then there exists integers such that

$$\mho^{\mathbf{y}\omega}_{\sigma} = T_{i_1}(a_1,b_1)T_{i_2}(a_2,b_2)\ldots T_{i_r}(a_r,b_r)\,.$$

One can factor $\omega = \sigma^{-1}(\sigma\omega)$, and correspondingly write the maximal element as

$$\mho^{\mathbf{y}}_{\omega} = T_{n-i_1}(b_1,a_1)T_{n-i_2}(b_2,a_2)\ldots T_{n-i_r}(b_r,a_r)\ \bullet\bullet\bullet\ .$$

Tanks to (1.8.6) ,

$$\left(\mho^{\mathbf{y}}_{\omega}\,,\, T_{i_1}(a_1,b_1)\ldots T_{i_r}(a_r,b_r)\right)^{\mathcal{H}}$$
$$= \left(T_{n-i_r}(a_r,b_r)\ldots T_{n-i_1}(a_1,b_1)\, T_{n-i_1}(b_1,a_1)\ldots T_{n-i_r}(b_r,a_r)\bullet\bullet\bullet\,,\, 1\right)^{\mathcal{H}}$$

is a scalar multiple of $\left(\bullet\bullet\bullet\,,\,1\right)^{\mathcal{H}}$, and therefore null if σ is not the identity permutation. QED

The following duality property of Yang-Baxter bases is given in [114, Th.5.1].

Theorem 1.8.4. *The Yang-Baxter bases associated to the spectral vectors $[y_1,\ldots,y_n]$ and $[y_n,\ldots,y_1]$ satisfy the relations*

$$\left(\mho^{\mathbf{y}}_{\sigma}\,,\, \mho^{\mathbf{y}\omega}_{\zeta}\right)^{\mathcal{H}} = \delta_{\sigma,\omega\zeta}\,, \tag{1.8.8}$$

that is, they are adjoint of each other.

Proof. When $\sigma = \omega$, this is property (1.8.7). One proves the general statement by decreasing induction on $\ell(\sigma)$, using Lemma 1.8.2. QED

Given any product

$$\big(T_i + \alpha(t_1{+}t_2)\big)\ldots\big(T_k + \gamma(t_1{+}t_2)\big) = \sum_\zeta c_\zeta T_\zeta\,,$$

then the product

$$\big(\widehat{T_i} - \alpha(t_1{+}t_2)\big)\ldots\big(\widehat{T_k} - \gamma(t_1{+}t_2)\big)$$

is equal to $\sum_\zeta \widehat{c_\zeta}\widehat{T_\zeta}$. This remark allows to rewrite the orthogonality relation (1.9.5). Define the coefficients c_σ^η by $\mho_\sigma^{\mathbf{y}} = \sum c_\sigma^\eta(\mathbf{y})T_\eta$, and recall that the involution $c \to \widehat{c}$ acts by $t_1 \to -t_2$, $t_2 \to -t_1$.

Corollary 1.8.5. *Let $\sigma, \zeta \in \mathfrak{S}_n$. Then*

$$\sum_{\eta\in\mathfrak{S}_n} \widehat{c_\sigma^\eta}(y_n^{-1},\ldots,y_1^{-1})\, c_\zeta^{\omega\eta}(\mathbf{y}) \;=\; \delta_{\omega\sigma,\zeta} \tag{1.8.9}$$

$$\sum_{\eta\in\mathfrak{S}_n} \widehat{c_\sigma^\eta}(\mathbf{y})\, c_\zeta^{\eta\omega}(\mathbf{y}) \;=\; \delta_{\sigma,\zeta\omega} \tag{1.8.10}$$

Proof. One uses that

$$\mho_\zeta^{\mathbf{y}\omega} = \sum_\eta \widehat{c_\zeta^\eta}(y_n^{-1},\ldots,y_1^{-1})\,\widehat{T}_\eta\,,$$

and that the symmetry (1.8.4) translates into $c_\sigma^\eta(y_n^{-1},\ldots,y_1^{-1}) = c_{\omega\sigma\omega}^{\omega\eta\omega}(y_1,\ldots,y_n)$. QED

Each of the relations (1.8.9) or (1.8.10) can be used to describe the inverse of the matrix of Yang-Baxter coefficients $\big[c_\sigma^\eta\big]$.

1.9 t_1t_2-Yang-Baxter bases

For $k > 1$, write

$$[k] = t_1^{k-1} - t_2 t_1^{k-2} + \cdots + (-t_2)^{k-1} \quad , \quad [-k] = t_2^{k-1} - t_1 t_2^{k-2} + \cdots + (-t_1)^{k-1} \,,$$

with the convention that $[0] = 0$, $[1] = 1 = [-1]$. Define, for all $k \in \mathbb{Z}$, $k \neq 0$,

$$T_i(k) = T_i + \frac{t_1 + t_2}{(-t_1/t_2)^k - 1} = \begin{cases} T_i - t_2^k\,[-k]^{-1} \,, & k > 0 \\ T_i - t_1^{-k}[k]^{-1} \,, & k < 0 \end{cases} \,, \tag{1.9.1}$$

adding $T_i(0) = T_i$.

Thus

$$T_i(1) = T_i - t_2,\ T_i(2) = T_i - \frac{t_2^2}{t_2 - t_1},\ T_i(3) = T_i - \frac{t_2^3}{t_2^2 - t_1t_2 + t_1^2} \,, \ldots \,,$$

$$T_i(-1) = T_i - t_1,\ T_i(-2) = T_i - \frac{t_1^2}{t_1 - t_2},\ T_i(-3) = T_i - \frac{t_1^3}{t_1^2 - t_1t_2 + t_2^2} \,, \ldots$$

We denote $\mathbb{U}_i = T_i(1)$ and $\nabla_i = T_i(-1)$ the two factors of the Hecke relation for T_i. Acting on $\{1, x_i\}$, one checks that

$$\nabla_i = \partial_i(t_2 x_i + t_1 x_{i+1}) \quad \& \quad \mathbb{U}_i = (t_1 x_i + t_2 x_{i+1})\partial_i \,. \tag{1.9.2}$$

Notice that for $k > 0$ one has

$$T_i(k)T_i(-k) = -t_1t_2 \frac{[k{-}1]\,[k{+}1]}{[k]^2} \,, \tag{1.9.3}$$

so that, for $k \neq \pm 1$, $T_i(k)$ and $T_i(-k)$ are inverse of each other up to a scalar. More generally, the Yang-Baxter equation (1.8.1) implies that, for any $i > 0$, any $k, r \in \mathbb{Z}$ such that $k, r, k{+}r \neq 0$, one has

$$T_i(k)T_{i+1}(k{+}r)T_i(r) = T_{i+1}(r)T_i(k{+}r)T_{i+1}(k) \tag{1.9.4}$$

Taking the spectral vectors $[t_1^{n-1}, -t_1^{n-2}t_2, \ldots, (-t_2)^{n-1}]$, and $[t_2^{n-1}, -t_1t_2^{n-2}, \ldots, (-t_1)^{n-1}]$, one obtains a pair of adjoint Yang-Baxter bases which are exchanged by the involution exchanging t_1, t_2. We shall denote these two bases $\{\nabla_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\mathbb{U}_\sigma : \sigma \in \mathfrak{S}_n\}$ respectively. Here is the basis associated to the spectral vector

$[t_2^2, -t_1t_2, t_1^2]$

$$\begin{tikzcd}[column sep=small]
& \mathbb{U}_{123} = 1 \arrow[dl, dash] \arrow[dr, equal] & \\
\mathbb{U}_{213} = T_1 - t_2 \arrow[d, equal] & & \mathbb{U}_{132} = T_2 - t_2 \arrow[d, dash] \\
\begin{array}{c} \mathbb{U}_{231} = T_1T_2 - t_2T_2 \\ -\frac{t_2^2}{t_2-t_1}T_1 + \frac{t_2^3}{t_2-t_1} \end{array} \arrow[dr, dash] & & \begin{array}{c} \mathbb{U}_{312} = T_2T_1 - t_2T_1 \\ -\frac{t_2^2}{t_2-t_1}T_2 + \frac{t_2^3}{t_2-t_1} \end{array} \arrow[dl, equal] \\
& \mathbb{U}_{321} = \begin{array}{c} T_1T_2T_1 - t_2T_2T_1 - t_2T_1T_2 \\ +t_2^2T_1 + t_2^2T_2 - t_2^3 \end{array} &
\end{tikzcd}$$

and the basis associated to the spectral vector $[t_1^2, -t_1t_2, t_2^2]$

$$\begin{tikzcd}[column sep=small]
& \nabla_{123} = 1 \arrow[dl, dash] \arrow[dr, equal] & \\
\nabla_{213} = T_1 - t_1 \arrow[d, equal] & & \nabla_{132} = T_2 - t_1 \arrow[d, dash] \\
\begin{array}{c} \nabla_{231} = T_1T_2 - t_1T_2 \\ -\frac{t_1^2}{t_1-t_2}T_1 + \frac{t_1^3}{t_1-t_2} \end{array} \arrow[dr, dash] & & \begin{array}{c} \nabla_{312} = T_2T_1 - t_1T_1 \\ -\frac{t_1^2}{t_1-t_2}T_2 + \frac{t_1^3}{t_1-t_2} \end{array} \arrow[dl, equal] \\
& \nabla_{321} = \begin{array}{c} T_1T_2T_1 - t_1T_2T_1 - t_1T_1T_2 \\ +t_1^2T_1 + t_1^2T_2 - t_1^3 \end{array} &
\end{tikzcd}$$

One notices that $\nabla_{213}, \nabla_{132}, \nabla_{321}$, as well as $\mathbb{U}_{213}, \mathbb{U}_{132}, \mathbb{U}_{321}$ are quasi-idempotents. This is due to the choice of the spectral vectors.

As a special case of (1.9.5), one has

Corollary 1.9.1. *The bases $\{\mathbb{U}_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\nabla_\sigma : \sigma \in \mathfrak{S}_n\}$ are adjoint of each other. Precisely, one has*

$$\Big(\mathbb{U}_\sigma\,,\,\nabla_\zeta\Big)^{\mathcal{H}} = \delta_{\sigma,\omega\zeta}\,. \tag{1.9.5}$$

The preceding corollary furnishes in particular the transition between $\{\mathbb{U}_\sigma\}$ and $\{\nabla_\sigma\}$:

$$\mathbb{U}_\sigma = \sum_{\zeta \le \sigma} \big(\mathbb{U}_\sigma, \mathbb{U}_{\omega\zeta}\big)^{\mathcal{H}}\, \nabla_\zeta\,.$$

The inverse of the transition matrix is obtained by conjugation with the diagonal matrix $[(-1)^{\ell(\sigma)},\ \sigma \in \mathfrak{S}_n]$. Non-zero entries correspond to pairs ζ, σ such that

$\zeta \le \sigma$ with respect to the Ehresmann-Bruhat order. Thus this matrix may be considered as "weighing" the order. We shall see later another weight given by the Kazhdan-Lusztig polynomials.

The case where σ is a Coxeter element is specially interesting since then the interval $[1, \sigma]$ is boolean. Let us just describe the expansion of $\mathbb{U}_\sigma$ when $\sigma = [2, \ldots, n, 1]$.

Define a function φ on permutations as follows, starting from $\varphi([1]) = 1$. For $\sigma \in \mathfrak{S}_n$, if $\sigma_n \neq n$, then $\varphi(\sigma) = \varphi(\sigma \setminus n))$, else

$$\varphi(\sigma) = \varphi(\sigma \setminus n) \frac{[1]\,[2n{-}\sigma_{n-1}{-}1]}{[n{-}1]\,[n{-}\sigma_{n-1}]} .$$

For example, $\varphi([1,3,4,2,5]) = \varphi([1,3,4,2]) \frac{[1][10-2-1]}{[4][5-2]} = \varphi([1,3,2]) \frac{[1][7]}{[4][3]} = \varphi([1,2]) \frac{[1][7]}{[4][3]} = \frac{[2]}{[1]} \frac{[1][7]}{[4][3]}$.

Proposition 1.9.2. *For any integer n one has*

$$\mathbb{U}_{2\ldots n1} = \sum_{\zeta \le [2\ldots n1]} \varphi(\zeta)\, \nabla_\zeta .$$

Proof. Supposing known the expansion $\mathbb{U}_{[2,\ldots,n-1,1,n]} = \sum c_\nu \nabla_\nu$, one obtains

$$\begin{aligned} \mathbb{U}_{[2,\ldots,n,1]} = \mathbb{U}_{[2,\ldots,n-1,1,n]}\, T_{n-1}(n{-}1) &= \sum c_\nu \nabla_\nu \left(T_{n-1}(\nu_{n-1}{-}n) + \frac{[1][2n{-}1{-}\nu_{n-1}]}{[n{-}1][n{-}\nu_{n-1}]} \right) \\ &= \sum c_\nu \left(\nabla_{\nu s_{n-1}} + \frac{[1][2n{-}1{-}\nu_{n-1}]}{[n{-}1][n{-}\nu_{n-1}]} \nabla_\nu \right) , \end{aligned}$$

which is the required property. QED

For example,

$$\mathbb{U}_{231} = \nabla_{231} + \frac{[2]}{[1]} \nabla_{132} + \frac{[1][4]}{[2]^2} \nabla_{213} + \frac{[3]}{[1]} \nabla_{123} ,$$

$$\begin{aligned} \mathbb{U}_{2341} = &\left(\nabla_{2341} + \frac{[2]}{[1]} \nabla_{1342} + \frac{[1][4]}{[2]^2} \nabla_{2143} + \frac{[3]}{[1]} \nabla_{1243} \right) \\ &+ \left(\frac{[1][6]}{[3]^2} \nabla_{2314} + \frac{[1][4]^2}{[3][2]^2} \nabla_{2134} + \frac{[5]}{[3]} \nabla_{1324} + \frac{[4]}{[1]} \nabla_{1234} \right) . \end{aligned}$$

The maximal elements $\mathbb{U}_\omega$, ∇_ω can be expressed in terms of the maximal divided difference ∂_ω, according to [33] :

Theorem 1.9.3. *Given n, let $\omega = [n, \ldots, 1]$, $\omega' = [n{-}1, \ldots, 1]$. Then the maximal elements $\mathbb{U}_\omega$ and ∇_ω have the following expressions*

$$\begin{aligned}
\mathbb{U}_\omega &= \mathbb{U}_{\omega'}\, T_{n-1}(n{-}1) \ldots T_2(2)\, T_1(1) && (1.9.6)\\
&= \mathbb{U}_{\omega'}\big(1 - t_2T_{n-1} + t_2^2T_{n-1}T_{n-2} - \cdots + (-t_2)^{n-1}T_{n-1}\ldots T_1\big) && (1.9.7)\\
&= \sum_{w\in\mathfrak{S}_n} (-t_2)^{\ell(w\omega)}T_w && (1.9.8)\\
&= \prod_{1\le i<j\le n} (t_1x_i + t_2x_j)\, \partial_\omega && (1.9.9)
\end{aligned}$$

$$\begin{aligned}
\nabla_\omega &= \nabla_{\omega'}\, T_{n-1}(1{-}n) \ldots T_2(-2)\, T_1(-1) && (1.9.10)\\
&= \nabla_{\omega'}\big(1 - t_1T_{n-1} + t_1^2T_{n-1}T_{n-2} - \cdots + (-t_1)^{n-1}T_{n-1}\ldots T_1\big) && (1.9.11)\\
&= \sum_{w\in\mathfrak{S}_n} (-t_1)^{\ell(w\omega)}T_w && (1.9.12)\\
&= \partial_\omega \prod_{1\le i<j\le n} (t_2x_i + t_1x_j) && (1.9.13)
\end{aligned}$$

Proof. The first expression for $\mathbb{U}_\omega$ and ∇_ω result from the definition of a Yang-Baxter element, choosing the factorization $\omega = \omega'\, s_{n-1} \ldots s_1$.

By recursion on n, one sees the equivalence of (1.9.11), (1.9.12), products being reduced.

All the operators occurring in the above formulas commute with multiplication with symmetric functions in $\mathfrak{Sym}(n)$, one can characterize them by their action on the Schubert basis $\{X_\sigma(\mathbf{x}, \mathbf{0}),\ \sigma \in \mathfrak{S}_n\}$ (see [108]).

Since ∇_i, $i = 1, \ldots, n{-}1$, can be factorized on the left from the RHS of (1.9.12), (1.9.13), these two RHS annihilate all Schubert polynomials, except $X_\omega = x_1^{n-1} \ldots x_n^0$. Therefore ∂_ω is a left factor of them.

Every element of $\mathcal{H}_n$ can be written uniquely as a sum $\sum_{w\in\mathfrak{S}_n} \partial_w P_w$ with coefficients P_w which are polynomials in $x_1, \ldots, x_n$. The RHS of (1.9.11) and of $\nabla_{\omega'}(-t_1)^{n-1}\, T_{n-1}(-1)\frac{1}{-t_1} \ldots T_1(-1)\frac{1}{-t_1}$ have same coefficient in ∂_ω. This coefficient is obtained by mere commutation : $f\nabla_i = f\partial_i(t_2x_i + t_1x_{i+1}) \sim \partial_i f^{s_i}(t_2x_i + t_1x_{i+1})$, the extra term $(f\partial_i)(t_2x_i + t_1x_{i+1})$ imposed by Leibniz formula cannot contribute to a reduced decomposition of ∂_ω. Therefore, formula (1.9.11) is true if it is true for $n{-}1$. The same reasoning applies to the factorization $\nabla_\omega = \nabla_{\omega'}\, T_{n-1}(1{-}n) \ldots T_1(-1)$ which has the same coefficient in ∂_ω than $\nabla_{\omega'}\nabla_{n-1} \ldots \nabla_1$. By symmetry, the properties of ∇_ω imply similar properties of $\mathbb{U}_\omega$. QED

Let $\lambda \in \mathbb{N}^\ell$ be a composition. Put $v = [0, \lambda_1, \lambda_1{+}\lambda_2, \ldots, \lambda_1 + \cdots + \lambda_\ell]$,

$$\Delta_\lambda^{t_1t_2} = \prod_{k=1}^{\ell} \prod_{v_{k-1}+1\le i<j\le v_k} (t_1x_i + t_2x_j) \qquad (1.9.14)$$

$$\Delta_\lambda^{t_2t_1} = \prod_{k=1}^{\ell} \prod_{v_{k-1}+1\le i<j\le v_k} (t_2x_i + t_1x_j)\,. \qquad (1.9.15)$$

Let ω_λ be the maximal element of the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$. Then, by direct product, one gets from the preceding theorem

$$\mathbb{U}_{\omega_\lambda} = \Delta_\lambda^{t_1 t_2}\, \partial_{\omega_\lambda} \tag{1.9.16}$$

$$\nabla_{\omega_\lambda} = \partial_{\omega_\lambda}\, \Delta_\lambda^{t_2 t_1}\,. \tag{1.9.17}$$

For example, for $\lambda = [3, 2]$, and $\mu \in \mathbb{N}^5$, the image of x^μ under

$$\nabla_{32154} = \sum_{\sigma \in \mathfrak{S}_{32}} (-t_1)^{\ell(\sigma)} T_\sigma = \partial_{32154} \Delta_{32}^{t_2 t_1}$$

is equal to the Schur function $s_{\mu - 43210}(\mathbf{x}_5)$ times $\Delta_{32}^{t_2 t_1}$.

1.10 B, C, D action on polynomials

As for type A, one transfers operations on vectors to operations on polynomials by acting on the exponents of monomials.

Thus, $s_i^B = s_i^C$ acts on x_i only by $x_i \to x_i^{-1}$, and s_i^D acts on x_{i-1}, x_i by $x_i \to x_{i-1}^{-1}$, $x_{i-1} \to x_i^{-1}$.

We also have divided differences, this time with a difference between types B and C :

$$\partial_i^B := (1 - s_i^B)\frac{1}{x_i^{1/2} - x_i^{-1/2}},\ \pi_i^B = x_i^{1/2}\,\partial_i^B,\ \widehat{\pi}_i^B = \partial_i^C x_i^{-1/2},\ i = 1 \dots n\,.$$

$$\partial_i^C := (1 - s_i^C)\frac{1}{x_i - x_i^{-1}},\ \pi_i^C = x_i\,\partial_i^C,\ \widehat{\pi}_i^C = \partial_i^C x_i^{-1},\ i = 1 \dots n\,.$$

$$\partial_i^D := (1 - s_i^D)\frac{1}{x_{i-1}^{-1} - x_i},\ \pi_i^D = \left(1 - s_i^D \frac{1}{x_{i-1}x_i}\right)\frac{1}{1 - \frac{1}{x_{i-1}x_i}},$$
$$\widehat{\pi}_i^D = (1 - s_i^D)\frac{1}{x_{i-1}x_i - 1},\ i = 2 \dots n\,.$$

As in type A, the above operators can be characterized in a simple manner, taking into account symmetries. For example, in type C, the divided differences $\partial_i^C, \pi_i^C, \widehat{\pi}_i^C$ commute with multiplication with functions symmetrical in $x_i, 1/x_i$ (which are functions of the variable $x_i^\bullet = x_i + x_i^{-1}$). It suffices to give their action on the basis $\{1, x_i\}$ of $\mathfrak{Pol}(x_i^\pm)$ as a free $\mathfrak{Pol}(x_i^\bullet)$ module :

$$\begin{array}{c|ccc} & \partial_i^C & \pi_i^C & \widehat{\pi}_i^C \\ 1 & 0 & 1 & 0 \\ x_i & 1 & x_i + x_i^{-1} & x_i^{-1} \end{array}\,.$$

For type D, say for $i = 2$, the space $\mathfrak{Pol}(x_1^\pm, x_2^\pm)$ is a free module of rank 4 over the D-invariants. One can take as a basis $1, x_1, x_2, x_2x_1^{-1}$, on which the divided differences act as follows :

$$\begin{array}{l|ccc} & \partial_2^D & \pi_2^D & \widehat{\pi}_2^D \\ 1 & 0 & 1 & 0 \\ x_1 & x_1x_2^{-1} & x_1 + x_2^{-1} & x_2^{-1} \\ x_2 & 1 & x_2 + x_1^{-1} & x_1^{-1} \\ x_2x_1^{-1} & 0 & x_2x_1^{-1} & 0 \end{array}\,.$$

For type $\heartsuit = B, C$, the divided differences for two consecutive indices, say

$1, 2$, satisfy braid relations[12] :

$$\begin{aligned} \pi_1\, \pi_2^\heartsuit\, \pi_1\, \pi_2^\heartsuit &= \pi_2^\heartsuit\, \pi_1\, \pi_2^\heartsuit\, \pi_1 \\ \widehat{\pi}_1\, \widehat{\pi}_2^\heartsuit\, \widehat{\pi}_1\, \widehat{\pi}_2^\heartsuit &= \widehat{\pi}_2^\heartsuit\, \widehat{\pi}_1\, \widehat{\pi}_2^\heartsuit\, \widehat{\pi}_1\,, \end{aligned}$$

but

$$\partial_2^C \partial_1 \partial_2^C \partial_1 \neq \partial_1 \partial_2^C \partial_1 \partial_2^C\,.$$

In type D, for $i \neq n-2$, then π_n^D commutes with π_i, and $\widehat{\pi}_n^D$ commutes with $\widehat{\pi}_i$, and

$$\pi_n^D\, \pi_{n-2}\, \pi_n^D = \pi_{n-2}\, \pi_n^D\, \pi_{n-2} \qquad \& \qquad \widehat{\pi}_n^D\, \widehat{\pi}_{n-2}\, \widehat{\pi}_n^D = \widehat{\pi}_{n-2}\, \widehat{\pi}_n^D\, \widehat{\pi}_{n-2}\,.$$

Notice that the squares satisfy the same relations than in type A :

$$\partial_i^\heartsuit \partial_i^\heartsuit = 0 \quad \& \quad \pi_i^\heartsuit \pi_i^\heartsuit = \pi_i^\heartsuit \quad \& \quad \widehat{\pi}_i^\heartsuit \widehat{\pi}_i^\heartsuit = -\widehat{\pi}_i^\heartsuit\,,\ \heartsuit = B, C, D\,.$$

Choosing as generators $s_1, \ldots, s_{n-1}, s_n^\heartsuit$, $\heartsuit = B, C, D$, one obtains by reduced products operators $\pi_w^\heartsuit$ and $\widehat{\pi}_w^\heartsuit$ indexed by the elements of the group. Of special importance are those corresponding to $w_0^\heartsuit$.

Proposition 1.10.1. *Let n be an integer, $\rho = [n{-}1, \ldots, 0]$, $x_i^\bullet = x_i + x_i^{-1}$, $i = 1, \ldots, n$. Write $\partial_i^\bullet$ for the divided differences relative to the alphabet $\mathbf{x}^\bullet = \{x_1^\bullet, \ldots, x_n^\bullet\}$. Then*

$$\begin{aligned} \pi_{w_0}^B &= x^\rho\, \pi_1^B \cdots \pi_n^B\, \partial_\omega^\bullet = x^\rho\, \partial_\omega^\bullet\, \pi_1^B \cdots \pi_n^B && (1.10.1) \\ \widehat{\pi}_{w_0}^B &= \widehat{\pi}_1^B \cdots \widehat{\pi}_n^B\, \partial_\omega^\bullet\, x^{-\rho} = \partial_\omega^\bullet\, \widehat{\pi}_1^B \cdots \widehat{\pi}_n^B\, x^{-\rho} && (1.10.2) \\ \pi_{w_0}^C &= x^\rho\, \pi_1^C \cdots \pi_n^C\, \partial_\omega^\bullet = x^\rho\, \partial_\omega^\bullet\, \pi_1^C \cdots \pi_n^C && (1.10.3) \\ &= x^{\rho+1^n} \Big(\sum (\text{-}1)^{\ell(w)} w\Big) \prod_{1\le i<j\le n} (x_i^\bullet - x_j^\bullet)^{-1} \prod_{1\le i\le n} (x_i - x_i^{-1})^{-1} && (1.10.4) \\ \widehat{\pi}_{w_0}^C &= \widehat{\pi}_1^C \cdots \widehat{\pi}_n^C\, \partial_\omega^\bullet\, x^{-\rho} = \partial_\omega^\bullet\, \widehat{\pi}_1^C \cdots \widehat{\pi}_n^C\, x^{-\rho} && (1.10.5) \end{aligned}$$

Notice that $\partial_\omega^\bullet = \partial_\omega \prod_{i<j\le n} (1 - x_i^{-1} x_j^{-1})^{-1}$ commutes with $\pi_1^B \cdots \pi_n^B$ and $\pi_1^C \cdots \pi_n^C$ because $x_i^\bullet$ commutes with π_i^B and π_i^C.

Consequently, images of $\pi_{w_0^B}$ and $\pi_{w_0^C}$ can be written as symmetric functions of $\mathbf{x}_n^\bullet$. For example, for $n = 3$, the image of x^{310} under $\pi_{w_0}^C$ is equal to

$$\begin{aligned} &\big(x_1^5 \pi_1^C\big)\big(x_2^2 \pi_2^C\big)\big(x_3^0 \pi_3^C\big) \\ &\qquad = \big((x_1^\bullet)^5 - 4(x_1^\bullet)^3 + 3x_1^\bullet\big)\big((x_2^\bullet)^2 - 1\big)\ \partial_{321}^\bullet \\ &\qquad\qquad = s_{310}(\mathbf{x}_3^\bullet) - 4s_{110}(\mathbf{x}_3^\bullet) - 3s_{000}(\mathbf{x}_3^\bullet)\,, \end{aligned}$$

[12] One has extra relations, like

$$\begin{aligned} \partial_1^C \pi_1 \partial_1^C \pi_1 &= \pi_1 \partial_1^C \pi_1 \partial_1^C \\ \partial_1^C \widehat{\pi}_1 \partial_1^C \widehat{\pi}_1 &= \widehat{\pi}_1 \partial_1^C \widehat{\pi}_1 \partial_1^C \end{aligned}$$

.

since $x^{310}x^{\rho} = x^{520}$, and $x_1^5\pi_1^C = (x_1^\bullet)^5 - 4(x_1^\bullet)^3 + 3x_1^\bullet$, $x_2^2\pi_2^C = (x_2^\bullet)^2 - 1$.

Let
$$\Theta_n^D = \frac{1}{2}(1+s_1^B)\cdots(1+s_n^B) + \frac{1}{2}(1-s_1^B)\cdots(1-s_n^B)\,.$$

Proposition 1.10.2. *The maximal divided differences for type D_n satisfy*

$$\pi_{w_0}^D = x^\rho\,\Theta_n^D\,\partial_\omega^\bullet \tag{1.10.6}$$
$$= \left(x^\rho \sum_w (-1)^{\ell(w)} w\right) \prod_{1\le i<j\le n} (x_i^\bullet - x_j^\bullet)^{-1} \tag{1.10.7}$$
$$\widehat{\pi}_{w_0}^D = \Theta_n^D\,\partial_\omega^\bullet\, x^{-\rho} = x^{-\rho}\,\pi_{w_0}^D\, x^{-\rho} \tag{1.10.8}$$

In type B or C, an alternating sum $\sum_{w\in W}(-1)^{\ell(w)}(x^v)^w$ may be represented as the determinant
$$\det\left(x_i^{v_j} - x_i^{-v_j}\right)_{i,j=1\ldots n}\,.$$
In type D, this alternating sum is equal to half of the sum of two determinants :
$$\det\left(x_i^{v_j} - x_i^{-v_j}\right)_{i,j=1\ldots n} + \det\left(x_i^{v_j} + x_i^{-v_j}\right)_{i,j=1\ldots n}\,,$$
the first determinant being null when some v_i is equal to 0. In particular
$$\sum_w (-1)^{\ell(w)}(x^\rho)^w = 2^{-1}\det\left(x_i^{n-j} + x_i^{j-n}\right)_{i,j=1\ldots n} = \prod_{1\le i<j\le n}(x_i^\bullet - x_j^\bullet)\,. \tag{1.10.9}$$

The groups of type B_n or D_n can be embedded into $\mathfrak{S}_{2n}$. However, relations between type B, C, D divided differences and divided differences relative to $\mathfrak{S}_{2n}$ are not straightforward. The next proposition describe $\pi_{w_0}^C$ in terms of $\mathfrak{S}_{2n}$, using the specialization $x_{2i-1} \to x_i$, $x_{2i} \to x_i^{-1}$, $1 \le i \le n$.

Proposition 1.10.3. *Given n, let $\zeta = (s_1\cdots s_{2n-1})(s_1\cdots s_{2n-3})\cdots(s_1s_2s_3)(s_1)$. Then*
$$\pi_{w_0}^C = \pi_\zeta\Big|_{\mathbf{x}\to\{x_1,x_1^{-1},x_2,x_2^{-1},\ldots\}}\,,$$
as operators on $\mathfrak{Pol}(\mathbf{x}_n)$.

Proof. The ring $\mathfrak{Pol}(\mathbf{x}_{2n})$ is a free-module over $\mathfrak{Sym}(\mathbf{x}_{2n})$, with basis $\{x^v : [0,\ldots,0] \le v \le [2n\text{–}1,\ldots,0]\}$. The submodule $\mathfrak{Pol}(\mathbf{x}_n)$ has basis $\{x^v : [0,\ldots,0] \le v \le [2n\text{–}1,\ldots,n,0,\ldots,0]\}$. One can as well take $\{x^v : [0^n] \le v \le [2n\text{–}1,\ldots,n]\}$, or, our present choice,
$$\{\, x^v : [1\text{–}2n,\ldots,-n] \le v \le [0^n]\,\}\,.$$
Specializing symmetric functions of $\mathbf{x}_{2n}$ into symmetric functions of $x_1, x_1^{-1},\ldots,x_n,x_n^{-1}$, one sees that the same set of monomials[13] span $\mathfrak{Pol}(\mathbf{x}_n)$ as a $\mathfrak{Sym}(\mathbf{x}_n^\bullet)$-module. Therefore it is sufficient to test the proposition on these monomials.

[13] but they are no more independent. For example, for $n = 2$, $x^{0,-2} = x^{-3,-1} - ax^{-2,-1} + bx^{-1,-1} - x^{0,0}$, with $a = x_1 + x_2 + x_1^{-1} + x_2^{-1}$, $b = x_1x_2 + x_1x_2^{-1} + x_2x_1^{-1} + 1 + x_1^{-1}x_2^{-1}$.

Since both $\pi^C_{w_0}$ and π_ζ admit the symmetrizer π_ω, $\omega = [n, \ldots, 1]$ as a left factor, the test can be restricted to all Schur functions of $\mathbf{x}^\vee := \{x_1^{-1}, \ldots, x_n^{-1}\}$ indexed by partitions contained in n^n.

Instead of enumerating partitions, one can introduce $\mathbf{y} = \{y_1, \ldots, y_n\}$ and test the single function

$$R(\mathbf{x}^\vee, \mathbf{y}) = \prod_{i,j=1}^{n} (x_i^{-1} - y_j)\,.$$

Let us first consider $R(\mathbf{x}^\vee, \mathbf{y})\,\pi^C_{w_0}$. The monomials x^u in the expansion of $R(\mathbf{x}^\vee, \mathbf{y})$ which give a non-zero contribution are those such that $u + \rho$, with $\rho = [n, \ldots, 1]$, has all its component different in absolute value. Since $[0, \ldots, 1{-}n] \le u + \rho \le \rho$, the vector $u + \rho$ must be a signed permutation of ρ, in which case $x^u \pi^C_{w_0} = \pm 1$. Therefore, the sum $\sum_w \pm\Big(x^\rho R(\mathbf{x}^\vee, \mathbf{y})\Big)^w (\Delta^C)^{-1}$, which expresses $R(\mathbf{x}^\vee, \mathbf{y})\,\pi^C_{w_0}$, is independent of $\mathbf{x}$. Specializing $\mathbf{x} = \mathbf{y}$, only the subsum

$$\sum_{w \in \mathfrak{S}_n} \pm\Big(x^\rho R(\mathbf{y}^\vee, \mathbf{y})\Big)^w (\Delta^C(\mathbf{y}))^{-1} = \sum_{w \in \mathfrak{S}_n} \pm\Big(x^\rho\Big)^w \cdot R(\mathbf{y}^\vee, \mathbf{y})(\Delta^C(\mathbf{y}))^{-1}$$

survives. After simplification, this subsum appears to be equal to

$$y_1 \cdots y_n \prod_{i<j\le n} (1 - y_i y_j)\,.$$

Let us now treat $\pi_\zeta = \pi_\omega(\pi_n \cdots \pi_{2n-1})\pi_\eta$, with $\pi_\eta = (\pi_{n-1} \cdots \pi_{2n-3}) \cdots (\pi_2\pi_3)(\pi_1)$. The symmetrizer π_ω preserves $R(\mathbf{y}^\vee, \mathbf{y})$, the operator $(\pi_n \cdots \pi_{2n-1})$ acts only on the factor $R(x_n^{-1}, \mathbf{y})$ and sends it to $(-1)^n y_1 \cdots y_n$. One is left with the computation of

$$R(\mathbf{x}', \mathbf{y})\,\pi_\eta\Big|_{x_{2i} = x_{2i-1}^{-1}}\,,$$

with $\mathbf{x}' = \{x_1^{-1}, \ldots, x_{n-1}^{-1}\}$. Assuming by induction the validity of the proposition for $n-1$, this last function is equal to $R(\mathbf{x}', \mathbf{y})\pi_{w_0'}$, with w_0' relative to C_{n-1}.

The monomials x^u appearing in the expansion of $R(\mathbf{x}', \mathbf{y})$ being such that $[-1, \ldots, -n{+}1] \le u + \rho' \le \rho'$, with $\rho' = [n{-}1, \ldots, 1]$, then for the same reason as above, the sum

$$\sum_w \pm\Big(x^{\rho'} R(\mathbf{x}', \mathbf{y})\Big)^w (\Delta^C(x_1, \ldots, x_{n-1}))^{-1}$$

does not depend on $\mathbf{x}$. Specializing $x_1 = y_1, \ldots, x_{n-1} = y_{n-1}$, the sum reduces to

$$\sum_{w \in \mathfrak{S}_{n-1}} \Big(y^{\rho'}\Big)^w R(\mathbf{y}', \mathbf{y}) \frac{1}{\Delta^C(y_1, \ldots, y_{n-1})} = y_1 \cdots y_{n-1} \prod_{i<j\le n-1} (1 - y_i y_j) R(\mathbf{y}', y_n)\,,$$

with $\mathbf{y}' = \{y_1^{-1}, \ldots, y_{n-1}^{-1}\}$. In final, the two operators send the test function $R(\mathbf{x}', \mathbf{y})$ to the same element, and therefore are equal. QED

For example, for $n = 2$, one has

$$x^{1100}(\pi_1\pi_2\pi_3)(\pi_1) = (x_1{+}x_2)(x_3{+}x_4) + x_1x_2$$

and this polynomial is transformed, by $\mathbf{x} \to [x_1, x_1^{-1}, x_2, x_2^{-1}]$, into

$$x^{11}\pi_1\pi_2^C\pi_1\pi_2^C = (x_1{+}x_1^{-1})(x_2{+}x_2^{-1}) + 1\,,$$

which is equal, as we shall see later, to $K^C_{-1,-1}$.

The two families of divided differences $\pi_i^\heartsuit$, $\widehat{\pi}_i^\heartsuit$ are related by the equations

$$\pi_i = 1 + \widehat{\pi}_i\,,\ i = 1, \ldots, n{-}1 \quad \& \quad \pi_n^\heartsuit = 1 + \widehat{\pi}_n^\heartsuit\,,\ \heartsuit = B, C, D\,.$$

For any element w of the Weyl group of type $\heartsuit$, by taking any reduced decomposition of it and the corresponding products of π_i's or $\widehat{\pi}_i$'s, one obtains an expansion of π_w in terms of $\widehat{\pi}_v$, and conversely, of $\widehat{\pi}_w$ in terms of π_v. From a simple property that followers of Bourbaki call the *exchange lemma*, which describes the growth of intervals for the Bruhat order with respect to $w \to ws_i$, one obtains the following relations between the two families of divided differences (given for type A in [99]).

Lemma 1.10.4. *For any element w of a Weyl group of type $\heartsuit = A, B, C, D$, one has the following sums over the Bruhat order :*

$$\pi_w = \sum_{v \le w} \widehat{\pi}_v \tag{1.10.10}$$

$$\widehat{\pi}_w = \sum_{v \le w} (-1)^{\ell(w)-\ell(v)} \pi_v\,. \tag{1.10.11}$$

For example, for type C, and $w = [2, -3, -1]$, then $w = s_3^C s_1 s_2 s_3^C$ and

$$\pi_w = (1 + \widehat{\pi}_3^C)(1 + \widehat{\pi}_1)(1 + \widehat{\pi}_2)(1 + \widehat{\pi}_3^C) = \widehat{\pi}_{123} + \big(\widehat{\pi}_{213} + \widehat{\pi}_{132} + \widehat{\pi}_{12\bar{3}}\big) \\ + \big(\widehat{\pi}_{231} + \widehat{\pi}_{21\bar{3}} + \widehat{\pi}_{13\bar{2}} + \widehat{\pi}_{1\bar{3}2}\big) + \big(\widehat{\pi}_{23\bar{1}} + \widehat{\pi}_{2\bar{3}1} + \widehat{\pi}_{1\bar{3}\bar{2}}\big) + \widehat{\pi}_{2\bar{3}\bar{1}}\,.$$

On the other hand, for type D, $w = [2, -3, -1] = s_1 s_3^D$, and

$$\pi_w = (1 + \widehat{\pi}_1)(1 + \widehat{\pi}_3^D) = \widehat{\pi}_{123} + \widehat{\pi}_{213} + \widehat{\pi}_{1\bar{3}\bar{2}} + \widehat{\pi}_{2\bar{3}\bar{1}}\,.$$

As a matter of fact, Stembridge [187] shows that the 0-Hecke algebra furnishes the easiest way to compute the Möbius function relative to the Bruhat order of Coxeter groups[14].

[14] the operators π_i and $\widehat{\pi}_i$ give two realizations of the 0-Hecke algebra, since $(\pi_i{-}0)(\pi_i{-}1) = 0$ and $(\widehat{\pi} - 0)_i(\widehat{\pi}_i + 1) = 0$.

1.11 Some operators on symmetric functions

Divided divided differences commute with multiplication with symmetric functions. They can nevertheless be used to build operators on symmetric functions, after breaking the initial symmetry, say for example, by sending x_1 to x_1^{-1}, or to qx_1, or using derivatives, then symmetrizing.

As a first example, let us use isobaric derivatives $\delta_i : f \to x_i \frac{d}{dx_i}(f)$, and more conveniently, symmetric functions in the alphabet $\daleth = \{\daleth_1 = \delta_1 - \frac{1}{2},\, \daleth_2 = \delta_2 - \frac{3}{2},\, \ldots,\, \daleth_n = \delta_n + \frac{1}{2} - n\}$.

The following lemma shows that symmetric functions in $\daleth$, followed by π_ω, act diagonally on Schur functions.

Lemma 1.11.1. *Let $g \in \mathfrak{Sym}(\mathbf{x}_n)$, $\lambda \in \mathbb{N}^n$ be a partition, $\mathcal{A}_\lambda$ be the alphabet $\{\lambda_1 - \frac{1}{2},\, \lambda_2 - \frac{3}{2},\, \ldots,\, \lambda_n + \frac{1}{2} - n\}$. Then $s_\lambda(\mathbf{x}_n) g(\daleth)\, \pi_\omega = g(\mathcal{A}_\lambda) s_\lambda(\mathbf{x}_n)$.*

Proof. Writing $\pi_\omega = x^\rho \partial_\omega$, one can commute x^ρ with $g(\daleth)$, at the cost of changing $\daleth$ into $\daleth' = \{\lambda_1 + \frac{1}{2} - n,\, \lambda_2 + \frac{1}{2} - n,\, \ldots,\, \lambda_n + \frac{1}{2} - n\}$, due to the fact that $(\delta_i - a)x_i = x_i\delta_i - a - 1$. Factorizing $\partial_\omega = \left(\sum_{\sigma \in \mathfrak{S}_n} \pm\sigma\right) \Delta(\mathbf{x}_n)^{-1}$, one can commute $\sum \pm\sigma$ with the symmetric function in $\daleth'$, thus obtaining

$$s_\lambda(\mathbf{x}_n) g(\daleth)\, \pi_\omega = s_\lambda(\mathbf{x}_n) x^\rho \sum \pm\sigma g(\daleth') \Delta(\mathbf{x}_n)^{-1} = s_\lambda(\mathbf{x}_n) \Delta(\mathbf{x}_n) g(\daleth') \Delta(\mathbf{x}_n)^{-1} .$$

The action of $g(\daleth')$ on $s_\lambda(\mathbf{x}_n)\Delta(\mathbf{x}_n)$, written as a determinant of powers of $x_1, \ldots, x_n$ is immediate, furnishing the result. QED

Since $p_1(\daleth)$ acts by multiplication by $d - n^2/2$ on homogeneous symmetric functions of degree d, the first interesting operators occur in degree 2. Indeed the operator $p_2(\daleth)\pi_\omega - \frac{1}{4}\binom{2n+1}{3}$ may be found in different places, as a Hamiltonian. It can be written, in terms of derivatives with respect to power sums, as the operator

$$\mathfrak{Sym} \ni f \to \sum_{i>0} \sum_{j>0} ij p_{i+j} \frac{d}{dp_i} \frac{d}{dp_j}(f) + (i+j) p_i p_j \frac{d}{dp_{i+j}}(f) .$$

As a second example, let us introduce two parameters α, β and consider the *Sekiguchi operator*

$$\Omega = (\alpha\delta + \beta) \ldots (\alpha\delta_n + \beta - n + 1)\, \pi_\omega ,$$

on symmetric functions of $\mathbf{x} = \mathbf{x}_n$. To explicit the action of Ω, we shall take as a linear basis of $\mathfrak{Sym}(\mathbf{x})$ the Schur functions in the alphabet $\mathbf{x}^\alpha = \frac{1}{\alpha}\mathbf{x}$. Equivalently, we introduce a second alphabet $\mathbf{y}$ of cardinality n, and compute

$$\sigma(\mathbf{x}^\alpha \mathbf{y})\Omega = \prod_i \prod_j (1 - x_i y_j)^{-1/\alpha}\, \Omega .$$

Since $(1 - x_i y)^{-1/\alpha}(\alpha\delta_i + \gamma) = x_i y (1 - x_i y)^{-1/\alpha - 1} + \gamma(1 - x_i y)^{-1/\alpha}$, one sees that there exists a function $F(\mathbf{x}, \mathbf{y})$ independent of α such that $\sigma(\mathbf{x}^\alpha \mathbf{y})\Omega =$

$F(\mathbf{x},\mathbf{y})\sigma\left((1+\frac{1}{\alpha})\mathbf{xy}\right)$. This function may be determined by putting $\alpha = 1$, and is thus equal to $\sigma(\mathbf{xy})\Omega\big|_{\alpha=1}\sigma(-2\mathbf{xy})$. We have seen just above that $\Omega\big|_{\alpha=1}$ may be written $\Delta(\mathbf{x})(\delta_1+\gamma)\dots(\delta_n+\gamma)$, with $\gamma = \beta+1-n$. Thanks to Cauchy, $\sigma(\mathbf{xy})\Delta(\mathbf{x}) = \frac{1}{\Delta(\mathbf{y})}\det\left(\frac{1}{1-x_iy_j}\right)$, and therefore

$$\sigma(\mathbf{xy})\Omega\big|_{\alpha=1} = \frac{1}{\Delta(\mathbf{y})}\det\left(\frac{\gamma+(1-\gamma)x_iy_j}{(1-x_iy_j)^2}\right)\frac{1}{\Delta(\mathbf{x})}\,,$$

and $F(X,Y)$ is the numerator of this last function.

As in the case of Gaudin determinant $\det\left((1-x_iy_j)^{-1}(1-x_iy_j+\gamma)^{-1}\right)$, or Izergin-Korepin determinant $\det\left((1-x_iy_j)^{-1}(1-qx_iy_j)^{-1}\right)$, one can write the quotient of the numerator of $\sigma(\mathbf{xy})\Omega\big|_{\alpha=1}$ by the two Vandermonde as a product of two rectangular matrices [101, 109]. Explicitly, let $M^e(\mathbf{x}_n)$ be the matrix

$$M^e(\mathbf{x}_n) = \Big[(\text{-}1)^{j-i}e_{j-i}(\mathbf{x})(\beta-n+2i-j)\Big]_{\substack{i=1\dots n\\ j=1..2n}}\,. \tag{1.11.1}$$

Then $F(\mathbf{x},\mathbf{y})$ is the determinant of the product of this matrix with $\Big[h_{i-j}(\mathbf{y})\Big]_{\substack{i=1..2n\\ j=1..n}}$.

For example, for $n=2$, $F(\mathbf{x},\mathbf{y})$ is the determinant of the product

$$\begin{bmatrix} e_0(\beta-1) & \text{-}e_1(\beta-2) & e_2(\beta-3) & 0\\ 0 & e_0\beta & \text{-}e_1(\beta-1) & e_2(\beta-2)\end{bmatrix}\begin{bmatrix} h_0 & 0\\ h_1 & h_0\\ h_2 & h_1\\ h_3 & h_2\end{bmatrix}\,,$$

where, by symmetry between $\mathbf{x}$ and $\mathbf{y}$, the h_i are the complete functions of one alphabet, and the e_i, of the other alphabet. In terms of products of Schur functions of $\mathbf{x}_2$ and $\mathbf{y}_2$, one has

$$\begin{aligned}F(\mathbf{x}_2,\mathbf{y}_2) = \beta(\beta\text{-}1) - (\beta\text{-}1)^2 s_1s_1 + 2s_{11}s_{11} + (\beta\text{-}1)(\beta\text{-}2)(s_2s_{11}+s_{11}s_2)\\ - (\beta\text{-}2)^2 s_{21}s_{21} + (\beta\text{-}2)(\beta\text{-}3)s_{22}s_{22}\,.\end{aligned}$$

The function $\sigma(\mathbf{x}^\alpha\mathbf{y})$ expand as $\sum S_v(\mathbf{x}^\alpha)S_v(\mathbf{y})$, sum over all (increasing) partitions in $\mathbb{N}^n$. Therefore, the image of $S_v(\mathbf{x}^\alpha)$ under Ω is equal to the coefficient of $S_v(\mathbf{y})$ in $F(\mathbf{x},\mathbf{y})\sigma((1+\alpha^{-1})\mathbf{xy})$, that is equal to

$$\sum_{u\uparrow} M^e_u S_{v/u}\left((1{+}\frac{1}{\alpha})\mathbf{x}\right) = \det\left(M^e\cdot\Big[S_{v_j+j-i}((1{+}\frac{1}{\alpha})\mathbf{x})\Big]_{\substack{i=1\dots 2n\\ j=1\dots n}}\right)\,, \tag{1.11.2}$$

denoting by M^e_u the minor of M on columns $u_1{+}1,\dots u_n{+}n$. The matrix M^e is in fact the sum of the two matrices

$$\Big[(\text{-}1)^{j-i}(b-n+i)e_{j-i}(\mathbf{x})\Big] \quad\text{and}\quad \Big[(\text{-}1)^{j-i}(i-j)e_{j-i}(\mathbf{x})\Big]\,.$$

Let $M^p(\mathbf{x}_n)$ be the $n\times\infty$ matrix of power sums

$$M^p(\mathbf{x}_n)=\begin{bmatrix} \beta{+}1{-}n & p_1(\mathbf{x}) & p_2(\mathbf{x}) & p_3(\mathbf{x}) & \cdots \\ 0 & \beta{+}2{-}n & p_1(\mathbf{x}) & p_2(\mathbf{x}) & \cdots \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & \beta & p_1(\mathbf{x}) & \cdots \end{bmatrix}.$$

Since $\sum(\text{-}1)^i e_i(\mathbf{x})\sigma((1+\alpha^{-1})\mathbf{x})=\sigma(\mathbf{x}^\alpha)$, and $\sum(\text{-}1)^i i e_i(\mathbf{x})\sigma((1+\alpha^{-1})\mathbf{x})=(p_1(\mathbf{x})+p_2(\mathbf{x})+\dots)\sigma(\mathbf{x}^\alpha)$, the product (1.11.2) can be transformed into the product

$$M^p(\mathbf{x}_n)\cdot\Big[S_{v_j+j-i}(\mathbf{x}^\alpha)\Big]_{\substack{i=1\dots\infty\\ j=1\dots n}}. \tag{1.11.3}$$

Using Newton's relations $\sum_{i=1}^\infty p_i(\mathbf{x})\sigma(\mathbf{x})=\sum_{i=0}^\infty iS_i(\mathbf{x})$, one obtains that $S_v(\mathbf{x}^\alpha)\Omega$ is equal to the determinant of

$$\Big[\big(\alpha(v_j+j-i)+\beta-n+i\big)S_{v_j+j-i}(\mathbf{x}^\alpha)\Big]_{i,j=1\dots n}. \tag{1.11.4}$$

For example,

$$S_{136}(\mathbf{x}^\alpha)\Omega=\begin{vmatrix} (1\alpha+\beta-2)S_1(\mathbf{x}^\alpha) & (4\alpha+\beta-2)S_4(\mathbf{x}^\alpha) & (8\alpha+\beta-2)S_8(\mathbf{x}^\alpha) \\ (0\alpha+\beta-1)S_1(\mathbf{x}^\alpha) & (3\alpha+\beta-1)S_3(\mathbf{x}^\alpha) & (7\alpha+\beta-1)S_7(\mathbf{x}^\alpha) \\ 0 & (2\alpha+\beta)S_2(\mathbf{x}^\alpha) & (6\alpha+\beta)S_6(\mathbf{x}^\alpha) \end{vmatrix}.$$

The shifts $\beta{-}n{+}i$ in (1.11.4) are constant by rows. The expansion by rows of the determinant expressing $S_v(\mathbf{x}^\alpha)\Omega$, starting from the bottom, may be written

$$\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\varphi\big((\lambda+\rho)\sigma-\rho\big)S^{(\lambda+\rho)\sigma-\rho}(\mathbf{x}^\alpha)\,,$$

with $\lambda=v\!\downarrow$, where, for $u\in\mathbb{N}^n$, $S^u(\mathbf{x}^\alpha)$ denotes the product of complete functions $S_{u_1}(\mathbf{x}^\alpha)\dots S_{u_n}(\mathbf{x}^\alpha)$, and $\varphi(u)=(\alpha u_1+\beta)\dots(\alpha u_n+\beta{+}1{-}n)$.

Introduce another alphabet $\mathbf{z}$, and denote $S2z$ the linear morphim

$$\mathfrak{Sym}(x)\ni s_\lambda(x^\alpha)\to\sum\pm z^{(\lambda+\rho)\sigma-\rho}\in\mathfrak{Pol}(\mathbf{z})\,,$$

by $z2S$ the linear morphism sending z^u onto the product $S^u(\mathbf{x}^\alpha)$.

The preceding computation may be interpreted as the following factorization of the Sekiguchi operator:

$$\mathfrak{Sym}\xrightarrow{S2z}\mathfrak{Pol}(\mathbf{z})\xrightarrow{z^v\to\varphi(v)z^v}\mathfrak{Pol}(\mathbf{z})\xrightarrow{z2S}\mathfrak{Sym}\,.$$

Let $\daleth=\{\daleth_1=\alpha\delta_1-\frac12,\ \daleth_2=\alpha\delta_2-\frac32,\ \dots,\ \daleth_n=\alpha\delta_n+\frac12{-}n\}$. The Sekiguchi operator may be written $\sum(\beta+\frac12)^{n-i}e_i(\daleth)\pi_\omega$, and therefore determines the action of each elementary function $e_i(\daleth)\pi_\omega$. Since $e_1(\daleth)$ acts as a scalar on homogeneous polynomials, one more generally knows the action of any linear combination of

$e_1(\daleth)^k e_i(\daleth)$, for example $e_1(\daleth)^2 - e_2(\daleth) = p_2(\daleth)$, $e_1(\daleth)^3 - 3e_1(\daleth)e_2(\daleth) + e_3(\daleth) = p_3(\daleth)$, $e_1(\daleth)e_2(\daleth) - e_3(\daleth) = s_{21}(\daleth)$.

Explicitly, for any polynomial f in $\daleth$, any $v \in \mathbb{N}^n$, let $\varphi_f(v) = f(\alpha v_1 - \frac{1}{2}, \ldots, \alpha v_n + \frac{1}{2} - n)$. Then the description of the action of the Sekiguchi operator entails

Lemma 1.11.2. *Let $f = p_2, p_3$ or s_{21}. Then the action of $f(\daleth)\pi_\omega$ on $\mathfrak{Sym}$ factorizes as*

$$\mathfrak{Sym} \xrightarrow{S2z} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{x^v \to \varphi_f(v) x^v} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{z2S} \mathfrak{Sym}\,.$$

The Sekiguchi operator preserves degrees. Expression (1.11.4) shows that is triangular in the basis $\{s_\lambda(\mathbf{x}^\alpha),\ \ell(\lambda) \le n\}$. Since φ takes distinct values on $\mathbb{N}^n$, the eigenspaces of Ω are 1-dimensional, their generators being the *Jack symmetric polynomials*. Since these polynomials are specializations of Macdonald polynomials, we postpone at this point any further comments about them. The operator $\left(p_2(\daleth) - \frac{1}{4}\binom{2n+1}{3}\right)\pi_\omega$ is also diagonal in the basis of Jack polynomials, with eigenvalues $\sum(\alpha\lambda_i + 1/2 - i)^2 - \frac{1}{4}\binom{2n+1}{3} = \alpha^2 \sum \lambda_i^2 + \alpha \sum (1-2i)\lambda_i$. It is in fact a rewriting of the Calogero-Sutherland Hamiltonian, and has been considered by physicists [67], see also [16]. To my knowledge, the operators corresponding to $p_3(\daleth)$ and $s_{21}(\daleth)$ have not been used, though they also diagonalize in the basis of Jack polynomials. Beware that the operator $p_4(\daleth)\pi_\omega$ does not act diagonally on Jack polynomials[15].

It is easy to transform isobaric factorized operators into degree-raising operators, by introducing inside the factorization of the operator the multiplication by a fixed polynomial. For example, let us see how to transform the first operator that we saw in this section into an operator deforming the product of Schur functions.

Let λ be a partition in $\mathbb{N}^n$. Then the operator $\Omega_\lambda = x^\lambda(\delta_1 + \beta)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega$ acting on $\mathfrak{Sym}(\mathbf{x}_n)$ may be rewritten

$$x^\lambda x^\rho (\sum \pm\sigma)(\delta_1 + \beta{+}1{-}n)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega \\ = s_\lambda(\mathbf{x}_n)\Delta(\mathbf{x}_n)(\delta_1 + \beta{+}1{-}n)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega\,,$$

and therefore the image of a Schur function $s_\mu(\mathbf{x}_n)$ under Ω_λ is equal to

$$\sum_\nu (s_\lambda s_\mu, s_\nu)(\nu_1 + \beta)\ldots(\nu_n + \beta{+}1{-}n)s_\nu(\mathbf{x}_n)\,,$$

where the coefficients $(s_\lambda s_\mu, s_\nu)$ are the structure constants appearing in $s_\lambda(\mathbf{x}_n)s_\mu(\mathbf{x}_n) = \sum_\nu (s_\lambda s_\mu, s_\nu)s_\nu(\mathbf{x}_n)$. We shall meet similar operators in the case of Macdonald polynomials.

One can also use the divided differences associated to types B, C, D to define operators on $\mathfrak{Sym}$.

[15] This is compatible with the fact that $p_4 = e_1^4 - 4e_1^2 e_2 + 4e_1e_3 + 2e_{2^2} - 4e_4$, the term e_2^2 preventing to apply the preceding considerations.

Let us first consider the action of $-x_1^{-k}\pi_1^B x_1^k$, $k \in \mathbb{Z}$, on functions of x_1.

Since $S_r(x_1{+}1)x_1^{-r/2}$ is invariant under s_1^B, one has

$$-S_r(x_1{+}1)x_1^{-k}\pi_1^B x_1^k = -[r{+}1]x_1^{r/2-k}\pi_1^B x_1^k = \begin{cases} x_1[r{+}1][2k{-}r{-}1] \text{ if } k > r/2 \\ -x_1^{2k-r}[r{+}1][r{+}1{-}2k] \text{ if } k \le r/2 \end{cases} ,$$

with $[j] = 1{+}x_1{+}\dots{+}x_1^j$.

One notices that the same functions can be obtained by combining ∂_1 with the specialization $x_2 = 1$. More precisely, one checks that for all $r \ge 0$, all $k \in \mathbb{Z}$, one has

$$-S_r(x_1{+}1)\, x_1^{-k}\pi_1^B x_1^{k-1} = S_{r(}(x_1^{-1} + x_2)\, x_1^{2k-1}\partial_1\Big|_{x_2=1} .$$

The next proposition shows how to extend this observation to any n, and will constitute our last example for this section.

Proposition 1.11.3. *Let $\lambda \in \mathbb{N}^n$ be a partition. Then one has, for any $k \in\in \mathbb{Z}$,*

$$(-1)^n s_\lambda(\mathbf{x}_n + 1)\, x_n^{-k}\pi_n^B x_n^k \pi_{n-1}\dots\pi_1(x_1\dots x_n)^{-1} = s_\lambda\, s_n^B x_1^{2k-1}\partial_1\dots\partial_n\Big|_{x_{n+1}=1} . \tag{1.11.5}$$

Proof. By recurrence on n, one sees that, for any symmetric function $f(x_1,\dots,x_n)$, one has

$$f(x_1,\dots,x_n)\, x_n^{-k}\pi_n^B x_n^k \pi_{n-1}\dots\pi_1 \\ = \sum_{i=1}^n f(\dots,\frac{1}{x_i},\dots)\frac{x_i^{2k-1}}{R(x_i,\mathbf{x}_n\setminus x_i)(1{-}x_i)} + f(x_1,\dots,x_n)\frac{1}{R(\mathbf{x}_n,1)} .$$

This is a Lagrange-type sum ([108, Th. 7.8.2]) which can be written

$$f(x_1,\dots,x_n)s_1^B x_i^{2k-1}(1{-}x_1)^{-1}\partial_1\dots\partial_{n-1} + f(x_1,\dots,x_n)\frac{1}{R(\mathbf{x}_n,1)} ,$$

but one can make this expression more symmetrical by considering the alphabet $x_1,\dots,x_{n+1}$, and by supposing[16] that f is the specialization $x_{n+1} = 1$ of a symmetric function of $x_1,\dots,x_{n+1}$, thus obtaining the stated identity. QED

For example, for $n = 3$, $\lambda = [1,0,0]$, $k = 3$, one has

$$-s_1(\mathbf{x}_3{+}1)\, x_3^{-3}\pi_3^B x_3^3 = (1{+}x_1{+}x_2)(x_3 + + \dots + x_3^5) + (x_3{+}x_3^2{+}x_3^3) ,$$

whose image under $\pi_2\pi_1$ is $\big(s_1(\mathbf{x}_3{+}1) + s_{21}(\mathbf{x}_3{+}1)\big)x_1x_2x_3$.

On the other hand,

$$(x_1^{-1} + x_2 + x_3 + x_4)x_1^5\,\partial_1\partial_2\partial_3 = s_1(\mathbf{x}_4) + s_{21}(\mathbf{x}_4) ,$$

and this agrees with the proposition.

[16] This is no restriction: $s_\lambda(\mathbf{x}_n) = s_\lambda(\mathbf{x}_{n+1} - 1)\big|_{x_{n+1}=1}$.

1.12 Weyl character formula

Irreducible characters for type $\heartsuit = A, B, C, D$ have been described by Weyl. For $\lambda \in \mathbb{N}^n$ dominant[17], Weyl's character formula reads

$$\chi_\lambda^\heartsuit = \frac{\sum_w (-1)^{\ell(w)} \left(x^{\lambda+\rho}\right)^w}{\sum_w (-1)^{\ell(w)} \left(x^{\rho}\right)^w}, \tag{1.12.1}$$

where $\rho = [n{-}1, \dots, 0]$ in type A, D, $\rho = [n, \dots, 1]$ in type C and $\rho = [n-\frac{1}{2}, \dots, \frac{1}{2}]$ in type B.

Using the factorization of the alternating sum of the elements of each group, one recognizes that the characters $\chi_\lambda^\heartsuit$ are equal to the image of x^λ under $\pi_{w_0}^\heartsuit$:

$$x^\lambda \, \pi_{w_0}^\heartsuit = \chi_\lambda^\heartsuit \,. \tag{1.12.2}$$

Each $\pi_{w_0}^\heartsuit$ has ∂_ω as a right factor. Since, for any functions $f_1(x), \dots, f_n(x)$, one has

$$f_1(x_1) \cdots f_n(x_n) \, \partial_\omega = \det(f_i(x_j)) / \det(x_i^{n-j})$$

one may write the numerators of Weyl character formula as the following determinants (still with $\lambda_n = 0$ for type D) :

$$\det(x_i^{\lambda_j+n-j}) \quad \text{type} \quad A \tag{1.12.3}$$

$$\det(x_i^{\lambda_j+n-j+1/2} - x_i^{-\lambda_j-n+j-1/2}) \quad \text{type} \quad B \tag{1.12.4}$$

$$\det(x_i^{\lambda_j+n-j+1} - x_i^{-\lambda_j-n+j-1}) \quad \text{type} \quad C \tag{1.12.5}$$

$$\frac{1}{2} \det(x_i^{\lambda_j+n-j} + x_i^{-\lambda_j-n+j}) \quad \text{type} \quad D \tag{1.12.6}$$

Let $\Delta(\mathbf{x}) = \prod_{1 \le i < j \le n} (x_i - x_j)$. Then the denominators $\Delta^A, \Delta^B, \Delta^C, \Delta^D$ of Weyl character formula are respectively equal to

$$\Delta^A = \Delta(\mathbf{x}),\ \Delta^B = \prod_i (\sqrt{x_i} - \frac{1}{\sqrt{x_i}}) \Delta(\mathbf{x}^\bullet),\ \Delta^C = \prod_i (x_i - \frac{1}{x_i}) \Delta(\mathbf{x}^\bullet),\ \Delta^D = \Delta(\mathbf{x}^\bullet),$$

still using the notation $\mathbf{x}^\bullet = \{x_1^\bullet, \dots, x_n^\bullet\}$, with $x_i^\bullet = x_i + x_i^{-1}$.

The numerators of Weyl's formula may also be written as determinants, so that the right hand side of Weyl's formula for type A, B, C, D, say in the case $\lambda = [3, 1, 0]$, would look like

$$\frac{\begin{vmatrix} x_1^5 & x_1^2 & 1 \\ x_2^5 & x_2^2 & 1 \\ x_3^5 & x_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}, \quad \frac{\begin{vmatrix} x_1^{11/2} - x_1^{-11/2} & x_1^{5/2} - x_1^{-5/2} & x_1^{1/2} - x_1^{-1/2} \\ x_2^{11/2} - x_2^{-11/2} & x_2^{5/2} - x_2^{-5/2} & x_2^{1/2} - x_2^{-1/2} \\ x_3^{11/2} - x_3^{-11/2} & x_3^{5/2} - x_3^{-5/2} & x_3^{1/2} - x_3^{-1/2} \end{vmatrix}}{\begin{vmatrix} x_1^{5/2} - x_1^{-5/2} & x_1^{3/2} - x_1^{-3/2} & x_1^{1/2} - x_1^{-1/2} \\ x_2^{5/2} - x_2^{-5/2} & x_2^{3/2} - x_2^{-3/2} & x_2^{1/2} - x_2^{-1/2} \\ x_3^{5/2} - x_3^{-5/2} & x_3^{3/2} - x_3^{-3/2} & x_3^{1/2} - x_3^{-1/2} \end{vmatrix}},$$

[17] For simplicity, we impose $\lambda_n = 0$ in type D, but we shall lift this restriction later.

$$\frac{\begin{vmatrix} {x_1}^6 - {x_1}^{-6} & {x_1}^3 - {x_1}^{-3} & x_1 - {x_1}^{-1} \\ {x_2}^6 - {x_2}^{-6} & {x_2}^3 - {x_2}^{-3} & x_2 - {x_2}^{-1} \\ {x_3}^6 - {x_3}^{-6} & {x_3}^3 - {x_3}^{-3} & x_3 - {x_3}^{-1} \end{vmatrix}}{\begin{vmatrix} {x_1}^3 - {x_1}^{-3} & {x_1}^2 - {x_1}^{-2} & x_1 - {x_1}^{-1} \\ {x_2}^3 - {x_2}^{-3} & {x_2}^2 - {x_2}^{-2} & x_2 - {x_2}^{-1} \\ {x_3}^3 - {x_3}^{-3} & {x_3}^2 - {x_3}^{-2} & x_3 - {x_3}^{-1} \end{vmatrix}}\,, \quad \frac{\begin{vmatrix} {x_1}^5 + {x_1}^{-5} & {x_1}^2 + {x_1}^{-2} & 1 \\ {x_2}^5 + {x_2}^{-5} & {x_2}^2 + {x_2}^{-2} & 1 \\ {x_3}^5 + {x_3}^{-5} & {x_3}^2 + {x_3}^{-2} & 1 \end{vmatrix}}{\begin{vmatrix} {x_1}^2 + {x_1}^{-2} & x_1 + {x_1}^{-1} & 1 \\ {x_2}^2 + {x_2}^{-2} & x_2 + {x_2}^{-1} & 1 \\ {x_3}^2 + {x_3}^{-2} & x_3 + {x_3}^{-1} & 1 \end{vmatrix}}\,.$$

When λ is an integral multiple of ρ, the numerator in Weyl's character formula is the image of the denominator under raising the variables to some power. Writing $k(\rho)$ for $[k\rho_1, k\rho_2, \ldots, k\rho_n]$, and $h_k(a{+}b)$ for the complete function of degree k in the variables a, b, one has

$$\begin{aligned}
\chi^A_{k(\rho)} &= \frac{\prod_{1\le i<j\le n}(x_i^{k+1} - x_j^{k+1})}{\prod_{1\le i<j\le n}(x_i - x_j)} = \prod_{1\le i<j\le n} h_k(x_i{+}x_j)\,,\\
\chi^D_{k(\rho)} &= \chi^A_{k(\rho)} \frac{\prod_{1\le i<j\le n}\left(1 - x_i^{-k-1}x_j^{-k-1}\right)}{\prod_{1\le i<j\le n} 1 - x_i^{-1}x_j^{-1}}\\
&= \prod_{1\le i<j\le n} h_k(x_i{+}x_j) h_k(1{+}x_i^{-1}x_j^{-1})\,,\\
\chi^B_{k(\rho)} &= \chi^D_{k(\rho)} \prod_{i=1}^n \frac{x_i^{(k+1)/2} - x_i^{-(k+1)/2}}{x_i^{1/2} - x_i^{-1/2}}\\
&= \prod_{i=1}^n h_k\left(\sqrt{x_i} + \frac{1}{\sqrt{x_i}}\right) \prod_{1\le i<j\le n} h_k(x_i{+}x_j) h_k(1{+}x_i^{-1}x_j^{-1})\,,\\
\chi^C_{k(\rho)} &= \chi^D_{k(\rho)} \prod_{i=1}^n \frac{x_i^{k+1} - x_i^{-k-1}}{x_i - x_i^{-1}}\\
&= \prod_{i=1}^n h_k\left(x_i + \frac{1}{x_i}\right) \prod_{1\le i<j\le n} h_k(x_i{+}x_j) h_k(1{+}x_i^{-1}x_j^{-1})\,.
\end{aligned}$$

For example, for $n = 2$, $k = 2$, one has

$$\chi^C_{42} = \left(x_1^2 + 1 + \frac{1}{x_1^2}\right)\left(x_2^2 + 1 + \frac{1}{x_2^2}\right)(x_1^2 + x_1x_2 + x_2^2)\left(1 + \frac{1}{x_1x_2} + \frac{1}{x_1^2x_2^2}\right)\,.$$

1.13 Macdonald Poincaré polynomial

The *length* of a reduced decomposition of an element w of a Weyl group is equal, using standard notions from the theory of *root systems*, to the number of roots in the intersection of $\mathcal{R}^+$ and $-w\mathcal{R}^+$.

Instead of enumerating inversions, let us define an *inversion weight* as follows. Embed the Weyl group of type B_n, C_n, D_n into $\mathfrak{S}_{2n}$. Given $w \in W$ and the corresponding $\sigma \in \mathfrak{S}_{2n}$, to a pair $(i,j) : 1 \le i < j \le n$, such that $\sigma_i > \sigma_j$ associate a factor h_{ji}. To a pair such that $\sigma_i > \sigma_{2n+1-j}$ associate a factor h_{ij}. Moreover, to all $i : 1 \le i \le n$ such that $w_i < 0$ associate a factor h_i in type B, and a factor h_{ii} in type C. The *inversion weight* $\mathcal{I}(w)$ of $w \in W$ is the product of these factors.

One can also define $\mathcal{I}(w)$ recursively by left multiplication by simple transpositions. Given w, s_k such that $\ell(s_k w) > \ell(w)$, then w and $s_k w$ either differ in two positions i,j or $(s_k w)_i = -w_i$. In that last case (which do not occur for type A or D), one has $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_i$ in type B and $= h_{ii}$ in type C. In the first case, if $w_i w_j > 0$ and $[\dots w_i \dots w_j \dots] \to [\dots w_j \dots w_i \dots]$, then $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_{ji}$. Otherwise, if $w_i w_j < 0$, then $[\dots w_i \dots w_j \dots] \to [\dots - w_j \dots - w_i \dots]$ and $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_{ij}$.

For example, for type C_4, one has the following chain of inversion factors :

$$[2,\overline{4},\overline{1},3] \xleftarrow{s_4^C\, h_{22}} [2,4,\overline{1},3] \xleftarrow{s_3\, h_{42}} [2,3,\overline{1},4] \xleftarrow{s_1\, h_{13}} [1,3,\overline{2},4]$$
$$\xleftarrow{s_2\, h_{23}} [1,2,\overline{3},4] \xleftarrow{s_3\, h_{34}} [1,2,\overline{4},3] \xleftarrow{s_4^C\, h_{33}} [1,2,4,3] \xleftarrow{s_3\, h_{43}} [1,2,3,4]$$

The inversions are more straightforward to read when writing the inverse elements :

$$[\overline{3},1,4,\overline{2}]^{-1} \xleftarrow{s_4^C\, h_{22}} [\overline{3},1,4,2]^{-1} \xleftarrow{s_3\, h_{42}} [\overline{3},1,2,4]^{-1} \xleftarrow{s_1\, h_{13}} [1,\overline{3},2,4]^{-1}$$
$$\xleftarrow{s_2\, h_{23}} [1,2,\overline{3},4]^{-1} \xleftarrow{s_3\, h_{34}} [1,2,4,\overline{3}]^{-1} \xleftarrow{s_4^C\, h_{33}} [1,2,4,3]^{-1} \xleftarrow{s_3\, h_{43}} [1,2,3,4]^{-1}$$

For each Weyl group of type $\heartsuit = A_{n-1}, B_n, C_n, D_n$, Macdonald defined the following kernel[18] $\mathcal{M}^\heartsuit$, introducing formal parameters h_{ji} :

$$\begin{aligned}
\mathcal{M}^A &= \prod_{1 \le i < j \le n} (1 - h_{ji} x_j x_i^{-1}) \\
\mathcal{M}^D &= \mathcal{M}^A \prod_{1 \le i < j \le n} (1 - h_{ij} x_i^{-1} x_j^{-1}) \\
\mathcal{M}^B &= \mathcal{M}^D \prod_{1 \le i \le n} (1 - h_i x_i^{-1}) \\
\mathcal{M}^C &= \mathcal{M}^D \prod_{1 \le i \le n} (1 - h_{ii} x_i^{-2})
\end{aligned}$$

[18] Of course, Macdonald does not mix types, but taking a pure combinatorial point of view leaves us this freedom.

For example, for A_2 and D_3, one has

$$\mathcal{M}^A = \left(1 - \frac{h_{21}\,x_2}{x_1}\right)\left(1 - \frac{h_{31}\,x_3}{x_1}\right)\left(1 - \frac{h_{32}\,x_3}{x_2}\right),$$

$$\mathcal{M}^D = \mathcal{M}^A \left(1 - \frac{h_{12}}{x_1\,x_2}\right)\left(1 - \frac{h_{13}}{x_1\,x_3}\right)\left(1 - \frac{h_{23}}{x_2\,x_3}\right).$$

The following theorem, due to Macdonald [147, Th.2.8], generalizes the enumeration of elements of a Weyl group according to their length.

Theorem 1.13.1. *For a Weyl group of type $\heartsuit = A, B, C, D$, with maximal element w_0, one has*

$$\mathcal{M}\,\pi^{\heartsuit}_{w_0} = \sum_{w\in W} \mathcal{I}(w)\,.$$

Proof. Each kernel, multiplied by $x^{\rho^\heartsuit}$ is a sum of monomials x^v, where the exponents respectively satisfy the conditions (componentwise comparison) : for type A, $[0,\ldots,0] \le v \le [n{-}1,\ldots,n{-}1]$,
for type B, $[1{-}n,\ldots,1{-}n] \le v + [\frac12,\ldots,\frac12] \le [n,\ldots,n]$,
for type C, $[-n,\ldots,-n] \le v \le [n,\ldots,n]$,
for type D, $[1{-}n,\ldots,1{-}n] \le v \le [n{-}1,\ldots,n{-}1]$.

Under the operator $\sum_w (-1)^{\ell(w)} \frac{1}{\Delta^\heartsuit}$, such monomials are sent to 0, or to ± 1 if they appear in the expansion of $\Delta^\heartsuit$. One checks that in that last case, the coefficient is indeed the inversion weight $\mathcal{I}(w)$. QED

For example, for type C_2, the contributing terms are

$$x^{2,1} - x^{2,-1}h_{22} + x^{1,-2}h_{12}h_{22} - x^{-1,-2}h_{11}\,h_{12}\,h_{22} - x^{1,2}h_{21} \\ + x^{-1,2}h_{21}\,h_{11} - x^{-2,1}h_{21}\,h_{11}\,h_{12} + x^{-2,-1}h_{21}\,h_{11}\,h_{12}\,h_{22}\,.$$

One could have decided[19] to denote the elements of the group by the element of the orbit of $\rho^\heartsuit$. In type A, one would have permutations of $[n{-}1,\ldots,0]$, in type B, signed permutations of $[n{-}\frac12,\ldots,\frac12]$, in type C, signed permutations of $[n,\ldots,1]$, and finally, in type D, signed permutations of $[n{-}1,\ldots,0]$.

The usual Poincaré polynomial is obtained by specializing all h_i, h_{ij} to q and thus is obtained by symmetrizing the "q-Vandermonde".

One could have taken an arbitrary subsum of the expansion of $\mathcal{M}^\heartsuit$. Macdonald's theorem states that the only terms surviving after symmetrization are those having for coefficient the inversion weight of an element of the group. The following theorem shows how to apply this property to generate intervals for the weak order.

For $v, w \in W$, write $w \ge_L v$ if the product $(wv^{-1})\,v$ is reduced, i.e. if $\ell(w) = \ell(wv^{-1}) + \ell(v)$. In that case $\mathcal{I}(v)$ is a factor of $\mathcal{I}(w)$. In the following statement,

[19] In type A, Cauchy considered the Vandermonde determinant, that he in fact introduced, as the generating function of permutations together with their signs, and consequently, the Vandermonde determinant as the "generic" determinant.

we shall use the same notation $\mathcal{I}(w)$ for the set of inversions and the inversion weight of $w \in W$.

Let $h^x_{ji} = h_{ji} x_j x_i^{-1}$, $j > i$, and $h^x_{ij} = h_{ij} x_i^{-1} x_j^{-1}$, $i \le j$, $h^x_i = h_i x_i^{-1}$.

Theorem 1.13.2. *Given a pair w, v such that $w \ge_L v$, then*

$$\prod_{\alpha \in \mathcal{I}(w) \setminus \mathcal{I}(v)} (1 - h^x_\alpha) \prod_{\alpha \in \mathcal{I}(v)} (-h^x_\alpha)\, \pi^\heartsuit_{w_0} = \sum_{u:\, w \ge_L u \ge_L v} \mathcal{I}(v)\,. \tag{1.13.1}$$

Proof. We already remarked that we have only to extract the products of h_{ji} which are inversion weights of elements of W. But $u \in W$ is such that $w \ge_L u$ if and only if $\mathcal{I}(u)$ divides $\mathcal{I}(w)$, thus u in the RHS if and only if it belongs to the left-order interval $[w, v]$. QED

It is interesting to notice that the interval $[1, w]$ for the Bruhat order can be obtained, thanks to Lemma 1.10.4, by taking any reduced decomposition $w = s_i \cdots s_j$ and evaluating the product $(1 + \widehat{\pi}_i) \cdots (1 + \widehat{\pi}_j)$. On the other hand, the preceding theorem gives the interval $[1, w]_L$ for the weak order by symmetrizing a factor of degree $\ell(w)$.

For example, for $w = [3, 4, 1, 2] \in \mathfrak{S}_4$, the initial interval for the Bruhat order is given by

$$\begin{aligned} \pi_{3412} = \pi_2\pi_3\pi_1\pi_2 &= (1+\widehat{\pi}_2)(1+\widehat{\pi}_3)(1+\widehat{\pi}_1)(1+\widehat{\pi}_3) \\ &= \widehat{\pi}_{3412} + \widehat{\pi}_{3214} + \widehat{\pi}_{3142} + \widehat{\pi}_{3124} + \widehat{\pi}_{2413} + \widehat{\pi}_{2314} + \widehat{\pi}_{2143} \\ &\quad + \widehat{\pi}_{2134} + \widehat{\pi}_{1432} + \widehat{\pi}_{1423} + \widehat{\pi}_{1342} + \widehat{\pi}_{1324} + \widehat{\pi}_{1243} + \widehat{\pi}_{1234}\,, \end{aligned}$$

while the initial interval for the left order is obtained by computing

$$\begin{aligned} &\left(1 - h_{31}\frac{x_3}{x_1}\right)\left(1 - h_{32}\frac{x_3}{x_2}\right)\left(1 - h_{41}\frac{x_4}{x_1}\right)\left(1 - h_{42}\frac{x_4}{x_2}\right)\pi_{4321} \\ &\qquad = 1 + h_{32} + h_{31}h_{32} + h_{32}h_{42} + h_{31}h_{32}h_{42} + h_{31}h_{41}h_{32}h_{42}\,, \end{aligned}$$

which translates, passing from the inversion weights to the permutations, into

$$[1,2,3,4], [1,3,2,4], [1,4,2,3], [2,3,1,4], [2,4,1,3], [3,4,1,2]\,.$$

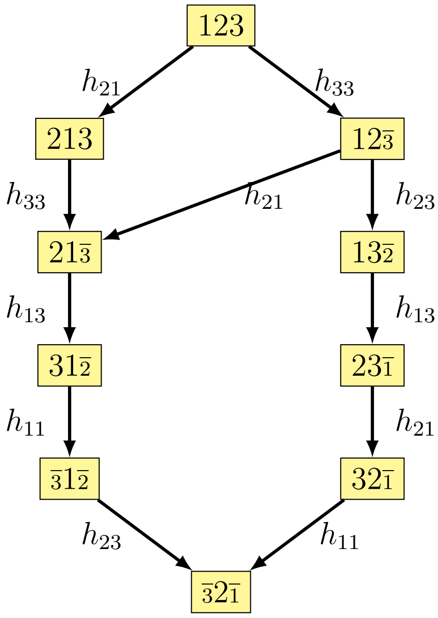
The Poincaré polynomial is obtained by specializing all h_α to q. For example, let $w = [5, 2, 4, 6, 1, 3]$, $v = [3, 1, 2, 5, 4, 6]$ in $\mathfrak{S}_6$. Then

$$\mathcal{I}([5,2,4,6,1,3])/\mathcal{I}([3,1,2,5,4,6] = h_{51}h_{52}h_{53}h_{61}h_{63}h_{64}\,,\ \mathcal{I}([3,1,2,5,4,6]) = h_{21}h_{31}h_{54}$$

and the polynomial of the interval is equal to

$$\begin{aligned} &\left(1 - h_{51}\frac{x_5}{x_1}\right)\left(1 - h_{52}\frac{x_5}{x_2}\right)\left(1 - h_{53}\frac{x_5}{x_3}\right)\left(1 - h_{61}\frac{x_6}{x_1}\right)\left(1 - h_{63}\frac{x_6}{x_3}\right)\left(1 - h_{64}\frac{x_6}{x_4}\right) \\ &\times \left(-\frac{x_2}{x_1}\right)\left(-\frac{x_3}{x_1}\right)\left(-\frac{x_5}{x_4}\right)\pi_{654321}\Big|_{h_{ji}=q} = q^6 + 2q^5 + 2q^4 + 3q^3 + 2q^2 + 2q + 1\,. \end{aligned}$$

We end by giving an example in type C, for $n = 3$, writing the interval and the inversions in the order they are created.



Thus, the Poincaré polynomial for this interval is equal to $1 + h_{21} + h_{33} + h_{23}h_{33} + h_{21}h_{33} + h_{13}h_{23}h_{33} + h_{21}h_{13}h_{33} + h_{21}h_{13}h_{11}h_{33} + h_{21}h_{13}h_{23}h_{33} + h_{21}h_{13}h_{23}h_{11}h_{33}$.

1.14 Poincaré with descents

For a Coxeter group W, the usual Poincaré polynomial is $\sum_{w\in W} q^{\ell(w)}$. We have already mentioned, for the classical Weyl groups of type A, B, C, D, the following factorizations of the Poincaré polynomial, denoting by $[i]$ the q-integer $(q^i-1)/(q-1)$:

- type A $[1]\,[2]\cdots[n]$,
- type BC $[2]\,[4]\cdots[2n]$,
- type D $[2]\,[4]\cdots[2n-2]\,[n]$.

The Poincaré polynomial is obtained from Macdonald's generating function $\mathcal{M}^{\heartsuit}\pi^{\heartsuit}_{w_0}$, in type $\heartsuit = A, B, C, D$, by specializing all parameters to q. But one can use more elaborate specializations. In particular, descents correspond to parameters $h_{i+1,i}$ in type A, together with h_n in type B, h_{nn} in type C, $h_{n-1,n}$ in type D, and can be treated differently than the other parameters.

Reiner [173] gives a generating function for the q-*Euler distribution* $\sum_{w\in W} t^{des(w)}q^{\ell(w)}$, for an infinite family of affine Coxeter groups.

In this section, we examine the question of introducing a function $\psi(w)$ depending on the set of descents $\mathcal{D}(w)$ of w, such that $\sum_{w\in W} q^{\ell(w)}\psi(w)$ still factorizes into simple factors.

Iwahori and Matsumoto [68] give a solution to this problem, choosing proper specializations of the parameters d_i into powers of q in the function

$$\sum_{w\in W} q^{-\ell(w)} \prod_{i\in\mathcal{D}(w)} d_i \,.$$

Stembridge and Waugh [186] use the corresponding affine groups to give a proof of Iwahori-Matsumoto formula which does not rely on the classification of root systems. For types A, B, C, D, the formula given by these authors[20] reads as follows.

Theorem 1.14.1. *For $\heartsuit = A, B, C, D$, the sum $\sum_{w\in W} q^{-\ell(w)} \prod_{i\in\mathcal{D}(w)} d_i$ is equal to*

$$\prod_{i=1}^{n-1}(1-q^{i(n-i)})(1-q^i)^{-1} \text{ in type } A\ ,\ \text{for } d_i = q^{i(n-i)} \tag{1.14.1}$$

$$2\prod_{i=1}^{n}(1-q^{i(2n-i)})(1-q^{2i-1})^{-1} \text{ in type } B\ ,\ \text{for } d_i = q^{i(2n-i)} \tag{1.14.2}$$

$$2\prod_{i=1}^{n-1}(1-q^{i(2n+1-i)})(1-q^{2i-1})^{-1}(1-q^{n(n+1)/2})(1-q^{2n-1})^{-1}$$
$$\text{in type } C\ ,\ \text{for } d_i = q^{i(2n+1-i)} \text{ except } d_n = q^{n(n+1)/2} \tag{1.14.3}$$

[20] Stembridge and Waugh write a formula which is valid for all finite Weyl groups in terms of the coefficients b_i appearing in the decomposition $2\rho = b_1\alpha_1 + \cdots + b_n\alpha_n$.

$$4\prod_{i=1}^{n-2}(1-q^{i(2n+1-i)})(1-q^{2i-1})^{-1}(1-q^{n(n-1)/2})^2(1-q^{2n-3})^{-1}(1-q^{n-1})^{-1}$$

$$\textit{in type } D\ ,\ \textit{for } d_i = q^{i(2n-1-i)}\ \textit{except } d_{n-1}=d_n=q^{n(n-1)/2}\,. \quad (1.14.4)$$

We give a more general formula for types A, B, C, D, specializing appropriately the Macdonald kernels defined in the preceding section.

For type A, we introduce parameters $y_0, y_1, \ldots, y_{n-1}$, and take

$$\mathcal{N}_n^A = \prod_{i+1<j\le n}\left(1-\frac{y_{i-1}}{y_i}\frac{x_j}{x_i}\right)\prod_{1\le i<n}\left(1-y_{i-1}y_i^{n-i-1}\frac{x_j}{x_i}\right) \quad (1.14.5)$$

For example, for $n=4$, one has

$$\mathcal{N}_4^A = \left(1-\frac{y_0x_3}{y_1x_1}\right)\left(1-\frac{y_0x_4}{y_1x_1}\right)\left(1-\frac{y_1x_4}{y_2x_2}\right)$$
$$\left(1-\frac{{y_1}^2y_0x_2}{x_1}\right)\left(1-\frac{y_2y_1x_3}{x_2}\right)\left(1-\frac{y_2x_4}{x_3}\right)$$

Introducing parameters a, z, d_{n-1}, d_n, we define

$$\mathcal{N}_n^B = \prod_{i=1}^{n-1}\left(1-a^iq^{i(n-i)}\frac{x_{i+1}}{qx_i}\right)\prod_{i<j-1}\left(1-\frac{x_j}{qx_i}\right)$$
$$\prod_{i=1}^{n-1}\left(1-\frac{z}{qx_i}\right)\left(1-\frac{za^n}{qx_n}\right)\prod_{1\le i<j\le n}\left(1-\frac{1}{qx_ix_j}\right) \quad (1.14.6)$$

$$\mathcal{N}_n^C = \prod_{i=1}^{n-1}\left(1-a^iq^{i(n-i)}\frac{x_{i+1}}{qx_i}\right)\prod_{i<j-1}\left(1-\frac{x_j}{qx_i}\right)$$
$$\prod_{i=1}^{n-1}\left(1-\frac{z}{qx_i^2}\right)\left(1-\frac{zd_n}{qx_n^2}\right)\prod_{1\le i<j\le n}\left(1-\frac{1}{qx_ix_j}\right) \quad (1.14.7)$$

$$\mathcal{N}_n^D = \prod_{i=1}^{n-2}\left(1-a^iq^{i(n-i)}\frac{x_{i+1}}{qx_i}\right)\prod_{i+1<j\le n}\left(1-\frac{x_j}{qx_i}\right)\prod_{1\le i<j\le n-1}\left(1-\frac{1}{qx_ix_j}\right)$$
$$\prod_{1\le i\le n-2}\left(1-\frac{1}{qx_ix_n}\right)\left(1-d_{n-1}\frac{x_{n-1}}{qx_n}\right)\left(1-d_n\frac{1}{qx_{n-1}x_n}\right) \quad (1.14.8)$$

Theorem 1.14.2. *The kernels $\mathcal{N}_n^A$, $\mathcal{N}_n^B$, $\mathcal{N}_n^C$ give the following generating functions:*

$$\mathcal{N}_n^A\pi_\omega = \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n}\left(\frac{y_{i-1}}{y_i}\right)^{\mathfrak{c}_i(\sigma)}\prod_{i\in\mathcal{D}(\sigma}y_i^{n-i}$$
$$= (1+y_0+\cdots+y_0^{n-1})(1+y_1+\cdots+y_1^{n-2})\cdots(1+y_{n-2})\,, \quad (1.14.9)$$

$\mathfrak{c}(\sigma)$ *being the code of* σ*.*

$$\mathcal{N}_n^B\,\pi_{w_0}^B = \sum_w q^{-\ell(w)} z^{m(w)} \prod_{i\in\mathcal{D}(w)} a^i q^{i(n-i)}$$
$$= (1{+}a{+}\cdots{+}a^{n-1})(1{+}aq{+}\cdots{+}(aq)^{n-2})$$
$$(1{+}aq^2{+}\cdots{+}(aq^2)^{n-3})\cdots(1+aq^{n-2})\prod_{i=1}^{n}(1+zaq^{-i})\,, \quad (1.14.10)$$

$m(w)$ *being the multiplicity of* s_n^B *in any reduced decomposition of* w*.*

$$\mathcal{N}_n^C\,\pi_{w_0}^C = \sum_w q^{-\ell(w)} z^{m(w)}\, d_n^{\epsilon(w)} \prod_{i\in\mathcal{D}(w)\setminus\{n\}} a^i q^{i(n-i)}$$
$$= (1{+}aq{+}\cdots{+}(aq)^{n-2})(1{+}aq^2{+}\cdots{+}(aq^2)^{n-3})\cdots(1+aq^{n-2})$$
$$\Big(a^{n-1}S_{n-1}(1{+}\frac{1}{a} - \bar{z}(q^{-1}{+}\cdots{+}q^{-n+1})$$
$$+ d_n z^n q^{-n(n+1)/2} S_{n-1}(1{+}a - \frac{1}{\bar{z}}(q{+}\cdots{+}q^{n-1})\Big)\,, \quad (1.14.11)$$

with $\epsilon(w)=1$ *if* $n\in\mathcal{D}(w)$*, and* $=0$ *otherwise,* $m(w)$ *being the multiplicity of* s_n^C *in any reduced decomposition of* w*.* [21]

Specializing y_i to q^i, which corresponds to taking $d_i = q^{i(n-i)}$ for a descent i, one obtains Iwahori-Matsumoto generating function for type A:

$$\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)} \prod_{i\in\mathcal{D}(\sigma} q^{i(n-i)} = n\,(1+q+\cdots+q^{n-2})$$
$$(1+q^2+\cdots+q^{2(n-3)})\cdots(1+q^{n-2}) \quad (1.14.12)$$

In type B, the specialization $a=q^n$, $z=1$ gives Iwahori-Matsumoto function. For example,

$$\mathcal{N}_3^B\pi_{w_0}^B = (1+a+a^2)(1+aq)(1+azq^{-1})(1+azq^{-2})(1+azq^{-3})$$

specializes to

$$2(1+q+\cdots+q^7)(1+q^3+q^6)$$

and, for $n=4$,

$$(1{+}a{+}a^2{+}a^3)(1{+}aq{+}a^2q^2)(1{+}aq^2)(1{+}azq^{-1})(1{+}azq^{-2})(1{+}azq^{-3})(1{+}azq^{-4})$$

[21] we use λ-rings notations. For any two alphabets $A=\sum a$, $B=\sum b$, one has $S_k(A-\bar{z}B) = \sum_{i=0}^k (-z)^i S_{k-i}(A)S_i(-B)$, where the $S_i(A)$ are the complete functions of A and $(-1)^i S_i(-B)$ are the elementary symmetric functions of B.

specializes to

$$2\,(1-q^{12})(1-q^{15})(1-q^{16})\,\big((1\text{-}q)(1\text{-}q^3)(1\text{-}q^5)\big)^{-1}\,.$$

In type C, the specialization $a=q^{n+1}$, $d_n=q^{n(n+1)/2}$, $z=1$ gives Iwahori-Matsumoto function. For example, for $n=3$, the generating function $\mathcal{N}_3^C\pi_{w_0}$, which is equal to

$$(1+aq)\left(\left(\frac{a^2}{q^4}+\frac{a^2}{q^5}+\frac{a^2}{q^3}\right)z^2+\left(\frac{a^2}{q}+\frac{a^2}{q^2}+\frac{a^2}{q^3}+\frac{a}{q^2}+\frac{a}{q^3}+\frac{a}{q}\right)z+a^2+a+1\right.$$
$$\left.+\,d_3\left(\left(\frac{a^2}{q^6}+\frac{a}{q^6}+q^{-6}\right)z^3+\left(q^{-4}+\frac{a}{q^3}+\frac{a}{q^4}+\frac{a}{q^5}+q^{-5}+q^{-3}\right)z^2+\left(q^{-3}+q^{-1}+q^{-2}\right)z\right)\right)$$

specializes to

$$2\,(1-q^6)^2(1-q^{10})\big((1\text{-}q)(1\text{-}q^3)(1\text{-}q^5)\big)^{-1}\,.$$

The theorem is proved by factorizing appropriately the operator $\pi^{\heartsuit}_{w_0}$, details may be found in the note [].

Chapter 2

Linear Bases for type A

2.1 Schubert, Grothendieck and Demazure

To interpolate a function $f(x_1)$ at points $y_1, y_2, \ldots,$, Newton [155] chose the basic polynomials $Y_0 = 1$, $Y_1 = (x_1 - y_1)$, $Y_2 = (x_1 - y_1)(x_1 - y_2), \ldots$ and found that the coefficients of $f(x_1)$ in this basis could be obtained by divided differences.

One can add the remark to Newton's computations that the Newton basis $Y_0, Y_1, Y_2, \ldots$ is invariant under the divided differences $\partial_i^{\mathbf{y}}$. Indeed, $Y_k \partial_k^{\mathbf{y}} = -Y_{k-1}$, and $Y_k \partial_i^{\mathbf{y}} = 0$ for $i \neq k$. It is therefore natural to generate bases of polynomials using the different operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i$ that are at our disposal. However, we also need starting points, i.e. polynomials such that them together with their descent will constitute a basis. In the case of non symmetric Macdonald polynomials, because one also has "raising operators" which increase degree, we need only one starting point, which is 1. For the other families of polynomials, the starting points will be associated to the diagrams of partitions, to the cost of having to check compatibility conditions between the different starting points.

Given $\lambda \in \mathbb{N}^n$ a partition (i.e. $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$), then

$$Y_\lambda := \prod_{i=1..n, j=1..\lambda_i} (x_i - y_j) \quad \& \quad G_\lambda := \prod_{i=1..n, j=1..\lambda_i} (1 - y_j x_i^{-1})$$

are the *dominant Schubert polynomials* and the *dominant Grothendieck polynomial* respectively, of index λ, and

$$K_\lambda = x^\lambda = \widehat{K}_\lambda$$

are the *dominant Demazure characters for type A*. We shall rather say *key polynomials* instead of Demazure characters [27] in reference to their combinatorial interpretation in terms of keys.

$$\underset{Y_{322}}{\begin{array}{|c|c|c}\hline x_3-y_1 & x_3-y_2 & \\ \hline x_2-y_1 & x_2-y_2 & \\ \hline x_1-y_1 & x_1-y_2 & \multicolumn{1}{c|}{x_1-y_3} \\ \hline\end{array}} \qquad \underset{K_{322}}{\begin{array}{|c|c|c}\hline x_3 & x_3 & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & \multicolumn{1}{c|}{x_1} \\ \hline\end{array}} \qquad \underset{G_{322}}{\begin{array}{|c|c|c}\hline 1-\frac{y_1}{x_3} & 1-\frac{y_2}{x_3} & \\ \hline 1-\frac{y_1}{x_2} & 1-\frac{y_2}{x_2} & \\ \hline 1-\frac{y_1}{x_1} & 1-\frac{y_2}{x_1} & \multicolumn{1}{c|}{1-\frac{y_3}{x_1}} \\ \hline\end{array}}$$

We define Schubert polynomials to be all[1] the non-zero images of the dominant Schubert polynomials under products of ∂_i's and Grothendieck polynomials[2] to be all the images of the dominant Grothendieck polynomials under products of π_i's. Similarly, the two types of key polynomials are defined by taking all images under products of π_i's or of $\widehat{\pi}_i$'s respectively.

Since the operators satisfy relations, we cannot index the polynomials by the choice of the starting point and the sequence of operators used. In fact, all these polynomials can be indexed by weights in $\mathbb{N}^n$, the recursive definition being

$$Y_{\dots,v_{i+1},v_i-1,\dots} = Y_v\,\partial_i \quad \& \quad G_{\dots,v_{i+1},v_i-1,\dots} = G_v\,\pi_i \text{ when } v_i > v_{i+1} \tag{2.1.1}$$

$$K_v\,\pi_i = K_{v\,s_i} \;\&\; \widehat{K}_v\,\widehat{\pi}_i = \widehat{K}_{v\,s_i},\text{ when } v_i > v_{i+1}\,. \tag{2.1.2}$$

Thus, the operators act on the indices just by sorting increasingly in the case of key polynomials, and by sorting and decreasing the biggest of the two components exchanged, in the case of Schubert and Grothendieck polynomials[3].

It is clear that these four families constitute linear bases of $\mathfrak{Pol}(n)$, because Y_v, K_v, $\widehat{K}_v$ have leading term[4] x^v, and G_v has leading term x^{-v}. However, it is unsatisfactory to have mere bases, one must be able to express a general polynomial

[1] There are dominant polynomials in the images of a dominant polynomial, in the Schubert and Grothendieck cases; therefore, one has to check consistency, as we already mentioned, but this easy.

[2] As a natural continuation of my work about syzygies of determinantal varieties, I had determined the classes, as polynomials, of the structure sheaves of the Schubert subvarieties of a flag manifold. It was a time where Grothendieck had some complaints about the world of mathematicians. I proposed to M.P. Schützenberger to call these classes *Grothendieck polynomials*, to which suggestion he readily agreed. They appear under the label G-polynomials in the paper[123] introducing them, the referee having disagreed with the terminology. The said referee fortunately forgot to extend his ban to future work. Moreover, Alexandre Grothendieck did not protest against this appellation.

[3] Choosing permutations as indexing sets, then the action is simply sorting. We did not give the case $v_i \le v_{i+1}$ because it is determined by the relations $\partial_i^2 = 0$, $\pi_i^2 = \pi_i$, $\widehat{\pi}_i^2 = -\widehat{\pi}_i$. Thus in that case,

$$Y_v\partial_i = 0,\; G_v\pi_i = G_v,\; K_v\pi = K_v,\; \widehat{K}_v\widehat{\pi}_i = -\widehat{K}_v\,.$$

[4] Notice that $x^{ji}\partial_1 = x^{j-1,i} + x^{j-2,i+1} + \cdots + x^{i,j-1}$ and that $x^{ji}\pi_1 = x^{j,i} + x^{j-1,i+1} + \cdots + x^{i,j}$. From this, it is easy to prove by induction that the monomials x^u appearing in Y_v, K_v are such that $u_n \le v_n$, $u_n + u_{n-1} \le v_n + v_{n-1}, \dots$. In particular, $u \le v$ for the *right lexicographic order*,

in term of these bases. We shall see how to do it in the next section, by defining a scalar product.

As examples of Schubert and Grothendieck polynomials, one obtains the following polynomials starting from the dominant ones Y_{210} and G_{210}.

$$Y_{210} = \begin{array}{ll} (x_2 - y_1) & \\ (x_1 - y_1) & (x_1 - y_2) \end{array}$$

$$Y_{110} = \begin{array}{ll} (x_2 - y_1) & \\ (x_1 - y_1) & \bullet \end{array} \qquad Y_{200} = \begin{array}{ll} \bullet & \\ (x_1{-}y_1) & (x_1{-}y_2) \end{array}$$

$$Y_{100} = \begin{array}{ll} \bullet & \\ x_1 - y_1 & \bullet \end{array} \qquad Y_{010} = x_1{+}x_2{-}y_1{-}y_2$$

$$Y_{000} = 1$$

$$G_{210} = \begin{array}{ll} (1 - \frac{y_1}{x_2}) & \\ (1 - \frac{y_1}{x_1}) & (1 - \frac{y_2}{x_1}) \end{array}$$

$$G_{110} = \begin{array}{ll} (1 - \frac{y_1}{x_2}) & \\ (1 - \frac{y_1}{x_1}) & \bullet \end{array} \qquad G_{200} = \begin{array}{ll} \bullet & \\ (1{-}\frac{y_1}{x_1}) & (1{-}\frac{y_2}{x_1}) \end{array}$$

$$G_{100} = \begin{array}{ll} \bullet & \\ 1 - \frac{y_1}{x_1} & \bullet \end{array} \qquad G_{010} = 1 - \frac{y_1 y_2}{x_1 x_2}$$

$$G_{000} = 1$$

For these two families, only the polynomial indexed by 010 is not dominant. However, in general Schubert and Grothendieck polynomials do not factorize, though they still have the same type of vanishing properties than the dominant ones.

Our starting Schubert polynomials are products of linear factors $x_i - y_j$. We shall be able to express general Schubert or Grothendieck polynomials as sums of

---

i.e. the order such that if $u < v$ then there exist k such that $u_i = v_i$ for $i = k{+}1, \ldots, n$ and $u_k < v_k$. Similarly, all monomials x^u appearing in the expansion of G_v are such that $-u_n \le -v_n$, $-u_n{-}u_{n-1} \le -v_n{-}v_{n-1}, \ldots$.

products of linear factors[5]. For example, using Leibnitz' formula, one obtains the sequence of polynomials

$$Y_{320} = \begin{array}{|c|c|c}\cline{1-2} x_2-y_1 & x_2-y_2 & \\ \hline x_1-y_1 & x_1-y_2 & \multicolumn{1}{c|}{x_1-y_3} \\ \hline \end{array}\,, \quad Y_{301} = \begin{array}{|c|c|c}\cline{1-1} x_3-y_2 & & \\ \hline x_1-y_1 & x_1-y_2 & \multicolumn{1}{c|}{x_1-y_3} \\ \hline \end{array} + \begin{array}{|c|c|c}\cline{1-1} x_2-y_1 & & \\ \hline x_1-y_1 & x_1-y_2 & \multicolumn{1}{c|}{x_1-y_3} \\ \hline \end{array}\,,$$

$$Y_{021} = \begin{array}{|c|c}\cline{1-1} x_3-y_2 & \\ \hline x_2-y_2 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline \end{array} + \begin{array}{|c|c}\cline{1-1} x_3-y_2 & \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline \end{array} + \begin{array}{|c|c}\cline{1-1} x_2-y_1 & \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline \end{array} + \begin{array}{|c|c}\cline{1-1} x_3-y_2 & \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_1-y_2} \\ \hline \end{array} + \begin{array}{|c|c}\cline{1-1} x_2-y_1 & \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_1-y_2} \\ \hline \end{array}\,,$$

and the last polynomial, Y_{021}, does not factorize anymore.

2.2 Using the y-variables

Some properties of Schubert and Grothendieck polynomials are easier to follow using permutations for the indexing. Given a permutation σ of code v, then one uses both notations $Y_v(\mathbf{x},\mathbf{y})$ and $X_\sigma(\mathbf{x},\mathbf{y})$ for the same Schubert polynomial, as well as $G_v(\mathbf{x},\mathbf{y})$ and $G_{(\sigma)}(\mathbf{x},\mathbf{y})$ for the same Grothendieck polynomial.

Both families satisfy a fundamental symmetry in $\mathbf{x},\mathbf{y}$. Indeed, given $i \le n-1$, denoting as usual $\omega = [n,\ldots,1]$, then it is immediate, because the statement reduces to compute the image of $(x_i - y_{n-i})$ or $(1 - y_{n-i}x_i^{-1})$, that

$$X_\omega(\mathbf{x},\mathbf{y})\,\partial_i^{\mathbf{x}} \;=\; -X_\omega(\mathbf{x},\mathbf{y})\,\partial_{n-i}^{\mathbf{y}} \tag{2.2.1}$$

$$G_{(\omega)}(\mathbf{x},\mathbf{y})\,\pi_i^{\mathbf{x}} \;=\; G_{(\omega)}(\mathbf{x},\mathbf{y})\,\pi_{n-i}^{\mathbf{1/y}}\,, \tag{2.2.2}$$

where $\pi_{n-i}^{\mathbf{1/y}}$ denotes the isobaric divided differences relative to $\mathbf{y}^\vee = \{y_1^{-1}, y_2^{-1},\ldots\}$.

By iteration, noticing that the symmetry is valid for $X_\omega(\mathbf{x},\mathbf{y})$ and $G_{(\omega)}(\mathbf{x},\mathbf{y})$, one obtains the following proposition.

Proposition 2.2.1. *The Schubert and Grothendieck polynomials satisfy the recursion*

$$X_{s_i\sigma}(\mathbf{x},\mathbf{y}) = -X_\sigma(\mathbf{x},\mathbf{y})\,\partial_i^{\mathbf{y}} \quad \& \quad G_{(s_i\sigma)}(\mathbf{x},\mathbf{y}) = G_{(\sigma)}(\mathbf{x},\mathbf{y})\,\pi_i^{\mathbf{1/y}}\,, \tag{2.2.3}$$

for i such that $\ell(s_i\sigma) \le \ell(\sigma)$, as well as the symmetry

$$X_\sigma(\mathbf{x},\mathbf{y}) = (-1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x}) \quad \& \quad G_{(\sigma)}(\mathbf{x},\mathbf{y}) = G_{(\sigma^{-1})}(\mathbf{y}^\vee,\mathbf{x}^\vee)\,. \tag{2.2.4}$$

Symmetry in consecutive variables can be seen on the indexing. Indeed, if i and v are such that $v_i \le v_{i+1}$, then Y_v and G_v are symmetrical in x_i, x_{i+1}, because they are equal to $Y_u\partial_i$ and $G_u\pi_i$ respectively, with $u = [\ldots, v_{i+1}+1, v_i, \ldots]$. Consequently, one has the following lemma.

[5] these expressions are not unique.

Lemma 2.2.2. *Let i, j, v be such that $v_i \le v_{i+1} \le \cdots \le v_j$. Then Y_v, G_v, K_v are symmetric in $x_i, \ldots, x_j$.*

In the case where $v \in \mathbb{N}^n$ is antidominant (i.e. $v = v\!\uparrow$), then Y_v, G_v, K_v are therefore symmetric in $x_1, \ldots, x_n$. In fact, let $\lambda = v\!\downarrow$ be the decreasing reordering of v. Then $K_v = x^\lambda \pi_\omega = x^{\lambda+\rho}\partial_\omega$ is equal to the Schur function $s_\lambda(\mathbf{x}_n)$, and $Y_v = Y_{\lambda+\rho}\partial_\omega$ specializes to $s_\lambda(\mathbf{x}_n)$ for $\mathbf{y} = \mathbf{0}$, because $Y_{\lambda+\rho}$ specializes to $x^{\lambda+\rho}$. The polynomial G_v, v antidominant, can also be considered as a deformation of a Schur function. It still possesses a determinantal expression. Geometrically, it is interpreted as the class of the structure sheaf of a Schubert variety in the Grothendieck ring of a Graßmannian and I described it in [94] by pure manipulation of determinants without using divided differences.

Let us call *Graßmannian* Schubert (resp. Grothendieck) polynomials. the polynomials indexed by antidominant v.

2.3 Flag complete and elementary functions

Both Schubert, Demazure and Grothendieck polynomials are non symmetric generalizations of the fundamental basis of symmetric functions that are Schur functions. In fact, the present notes will illustrate that many properties of the Schur basis can be extended to properties of the Y_v, K_v, G_v bases. But there are other bases of $\mathfrak{Sym}(\mathbf{x})$, particularly the products of elementary functions $e_i(\mathbf{x})$ and the products of complete functions $h_i(\mathbf{x})$. Let us generalize these into *flag elementary functions* and *flag complete functions*.

Definition 2.3.1. *For any r, any $v \in \mathbb{N}^r$, $v \le [r-1, \ldots, 0]$, let*

$$P_v = e_{v_1}(\mathbf{x}_{r-1}) \cdots e_{v_r}(\mathbf{x}_0)$$

and, for any n, any $v \in \mathbb{N}^n$, let

$$H_v = h_{v_1}(\mathbf{x}_1) \cdots h_{v_n}(\mathbf{x}_n)\,.$$

It is clear that $\{H_v : v \in \mathbb{N}^n\}$ is a linear basis of $\mathfrak{Pol}(\mathbf{x}_n)$, which is triangular in the basis of monomials. Identifying v and $0v$, one checks that $\cup_r\{P_v : v \in \mathbb{N}^r\}$ is also a linear basis of the space of polynomials in $x_1, x_2, \ldots$. Notice that the restriction on v eliminates the elementary functions which are null because of degree strictly higher than the cardinality of the alphabet. Beware that P_{v0} is different from P_v, because of the order we write the flag of alphabets. This change of convention for the indexing of the basis of flag elementary functions will be justified by the non-commutative extension of P_v.

It is not straightforward to express monomials in these two bases. For example,

$$\begin{aligned} x_2^2 &= P_{1,1,0,0} - P_{2,0,0,0} - P_{1,1,0} \\ &= (x_1+x_2+x_3)(x_1+x_2) - (x_1x_3+x_1x_2+x_2x_3) - (x_1+x_2)x_1 \end{aligned}$$

$$x_2^2 = H_{0,2} - H_{1,1} = (x_1^2 + x_1x_2 + x_2^2) - x_1(x_1+x_2)\,.$$

We shall obtain such expansions by using a scalar product on polynomials.

More generally, monomials can be written as flag Schur functions. Let $v \in \mathbb{N}^n$, $u = [v_n, \dots, v_1]$. Then [108, 1.4.10]

$$x^v = S_u(\mathbf{x}_n, \dots, x_1) = \Big| h_{u_j+j-i}(\mathbf{x}_{n+1-j}) \Big|\,. \tag{2.3.1}$$

For example,

$$x^{0,3,1,2} = S_{2,1,3,0}(\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) = \begin{vmatrix} h_2(\mathbf{x}_4) & h_2(\mathbf{x}_3) & h_5(\mathbf{x}_2) & h_3(\mathbf{x}_1) \\ h_1(\mathbf{x}_4) & h_1(\mathbf{x}_3) & h_4(\mathbf{x}_2) & h_2(\mathbf{x}_1) \\ h_0(\mathbf{x}_4) & h_0(\mathbf{x}_3) & h_3(\mathbf{x}_2) & h_1(\mathbf{x}_1) \\ 0 & 0 & h_2(\mathbf{x}_2) & h_0(\mathbf{x}_1) \end{vmatrix}\,.$$

Expanding by columns (but from the right!), one finds the expression of the monomial in the H-basis :

$$\begin{aligned} x^{0,3,1,2} &= H_{0,3,1,2} - H_{1,2,1,2} - H_{0,4,0,2} + H_{2,2,0,2} - H_{0,3,2,1} \\ &\quad + H_{1,2,2,1} + H_{0,5,0,1} - H_{3,2,0,1} + H_{0,4,2,0} - H_{2,2,2,0} - H_{0,5,1,0} + H_{3,2,1,0}\,. \end{aligned}$$

The following proposition illustrates that Schur functions in $\mathbf{x}_n$ can also be easily expressed in these two bases, using flags of alphabets[6] in the Jacobi-Trudi determinants.

Proposition 2.3.2. *Let v be the increasing reordering of a partition λ, $u \in \mathbb{N}^r$ be the reordering of the conjugate $\lambda^\sim$. Then the Schur function $s_\lambda(\mathbf{x}_n)$, also denoted $S_v(\mathbf{x}_n)$, is equal to both determinants*

$$S_v(\mathbf{x}_1/\mathbf{x}_2/\dots/\mathbf{x}_n) = \Big| h_{v_j+j-i}(\mathbf{x}_i) \Big|$$
$$\text{and} \quad \Lambda_u(\mathbf{x}_{n+r-1}/\mathbf{x}_{n+r-2}/\dots/\mathbf{x}_n) = \Big| e_{u_j+j-i}(\mathbf{x}_{n+r-i}) \Big|\,. \tag{2.3.2}$$

The expansions of these determinants furnishes the required expressions of $s_\lambda(\mathbf{x}_n)$. For example, for $n = 3$, $\lambda = [4,2]$, one has $\lambda^\sim = [2,2,1,1]$ and

$$s_{42}(\mathbf{x}_3) = S_{024}(\mathbf{x}_1/\mathbf{x}_2/\mathbf{x}_3) \quad = \quad \begin{vmatrix} h_0(\mathbf{x}_1) & h_3(\mathbf{x}_1) & h_6(\mathbf{x}_1) \\ 0 & h_2(\mathbf{x}_2) & h_5(\mathbf{x}_2) \\ 0 & h_1(\mathbf{x}_3) & h_4(\mathbf{x}_3) \end{vmatrix}$$

$$\begin{vmatrix} e_1(\mathbf{x}_6) & e_2(\mathbf{x}_6) & e_4(\mathbf{x}_6) & e_5(\mathbf{x}_6) \\ e_0(\mathbf{x}_5) & e_1(\mathbf{x}_5) & e_3(\mathbf{x}_5) & e_4(\mathbf{x}_5) \\ 0 & e_0(\mathbf{x}_4) & e_2(\mathbf{x}_4) & e_3(\mathbf{x}_4) \\ 0 & 0 & e_1(\mathbf{x}_3) & e_2(\mathbf{x}_3) \end{vmatrix} \quad = \quad \Lambda_{1122}(\mathbf{x}_6/\mathbf{x}_5/\mathbf{x}_4/\mathbf{x}_3)\,,$$

[6] but this time, flags are constant by rows.

which entails

$$s_{4,2}(\mathbf{x}_3) = H_{0,2,4} - H_{0,5,1} = P_{1,1,2,2,0,0,0} - P_{1,1,3,1,0,0,0} - P_{2,0,2,2,0,0,0} \\ + P_{4,0,0,2,0,0,0} + P_{2,0,3,1,0,0,0} + P_{1,4,0,1,0,0,0} - P_{1,3,0,2,0,0,0} - P_{5,0,0,1,0,0,0} \, .$$

Given i, there is at most one component of the function P_v and of the function H_v which is not symmetrical in x_i, x_{i+1}. Since

$$e_k(\mathbf{x}_i)\partial_i = \big(e_k(\mathbf{x}_{i-1}) + x_i e_{k-1}(\mathbf{x}_{i-1})\big)\,\partial_i = e_{k-1}(\mathbf{x}_{i-1})$$

and

$$h_k(\mathbf{x}_i)\,\pi_i = h_k(\mathbf{x}_{i+1})\,,$$

the image of $P_v = \cdots e_k(\mathbf{x}_i)e_\ell(\mathbf{x}_{i-1})\cdots$ under ∂_i is a flag $\cdots e_{k-1}(\mathbf{x}_{i-1})e_\ell(\mathbf{x}_{i-1})\cdots$ which is not permitted if $(k-1)\ell \neq 0$. Similarly, the image of $H_v = \cdots h_k(\mathbf{x}_i)h_\ell(\mathbf{x}_{i+1})\cdots$ under π_i,which is $\cdots h_k(\mathbf{x}_{i+1})h_\ell(\mathbf{x}_{i+1})\cdots$, is also illegal if $k\ell \neq 0$.

But, from the case of order 2 of (2.3.2), one has, with $\alpha = \min(k\text{-}1,\ell)$ and $\beta = \max(k\text{-}1,\ell)$,

$$e_{k-1}(\mathbf{x}_{i-1})e_\ell(\mathbf{x}_{i-1}) = \Big(e_\alpha(\mathbf{x}_i)e_\beta(\mathbf{x}_{i-1}) + e_{\alpha-1}(\mathbf{x}_i)e_{\beta+1}(\mathbf{x}_{i-1}) + \cdots \\ + e_0(\mathbf{x}_i)e_{\beta+\alpha}(\mathbf{x}_{i-1})\Big) - \Big(e_{\beta+1}(\mathbf{x}_i)e_{\alpha-1}(\mathbf{x}_{i-1}) + \cdots + e_{\beta+\alpha}(\mathbf{x}_i)e_0(\mathbf{x}_{i-1})\Big)\,,$$

and, with $\alpha = \min(k,\ell)$, $\beta = \max(k,\ell)$,

$$h_k(\mathbf{x}_{i+1})h_\ell(\mathbf{x}_{i+1}) = \Big(h_\alpha(\mathbf{x}_i)h_\beta(\mathbf{x}_{i+1}) + \cdots + h_0(\mathbf{x}_i)h_{\beta+\alpha}(\mathbf{x}_{i+1})\Big) \\ - \Big(h_{\beta+1}(\mathbf{x}_i)h_{\alpha-1}(\mathbf{x}_{i+1}) + \cdots + h_{\beta+\alpha}(\mathbf{x}_i)h_0(\mathbf{x}_{i+1})\Big)\,.$$

This entails the following actions of ∂_i and π_i.

Lemma 2.3.3. *Let n, i be two positive integers, $0 < i < n$, $v \in \mathbb{N}^n$ being such that $v \leq [n\text{-}1,\ldots,0]$, $\alpha = \min(v_{n-i}-1, v_{n-i+1})$, $\beta = \max(v_{n-i}-1, v_{n-i+1})$. Then*

$$P_{\bullet\bullet v_{n-i},v_{n-i+1}\bullet\bullet}\,\partial_i = \sum_{j=0}^{\alpha} P_{\bullet\bullet\alpha-j,\beta+j\bullet\bullet} - \sum_{j=1}^{\alpha} P_{\bullet\bullet\beta+j,\alpha-j\bullet\bullet}\,. \tag{2.3.3}$$

For $v \in \mathbb{N}^n$, $\alpha = \min(v_i, v_{i+1})$, $\beta = \max(v_i, v_{i+1})$, one has

$$H_{\bullet\bullet v_i,v_{i+1}\bullet\bullet}\,\pi_i = \sum_{j=0}^{\alpha} H_{\bullet\bullet\alpha-j,\beta+j\bullet\bullet} - \sum_{j=1}^{\alpha} H_{\bullet\bullet\beta+j,\alpha-j\bullet\bullet}\,. \tag{2.3.4}$$

For example,

$$\begin{aligned} P_{52\,03210}\,\partial_6 &= P_{24\,03210} + P_{15\,03210} + (P_{06\,03210}) - P_{51\,03210} - P_{60\,03210}\,, \\ H_{9\,26\,99}\,\pi_2 &= H_{9\,26\,99} + H_{9\,17\,99} + H_{9\,08\,99} - H_{9\,71\,99} - H_{9\,80\,99}\,, \end{aligned}$$

the term $P_{0603210}$ being null because $e_6(\mathbf{x}_5) = 0$.

2.4 Three scalar products

Let us first look for a scalar product on $\mathfrak{Pol}(n)$ compatible with the product structure and with degree.

When $n = 1$,

$$(f(x_1), g(x_1)) = CT\left(f(x_1), g(\frac{1}{x_1})\right),$$

where CT means "constant term", is a good candidate. Generalizing to $(f, g) = CT\left(f(x_1, \ldots, x_n), g(\frac{1}{x_1}), \ldots, \frac{1}{x_n})\right)$ means considering the ring of polynomials as a tensor product of rings of polynomials in 1 variable, a rather poor structure. Reversing the order of variables in the function g is not enough, one needs a kernel to link the variables.

We define

$$(f, g) = CT\Big(f(x_1, \ldots, x_n) g(x_n^{-1}, \ldots, x_1^{-1}) \prod_{1 \le i < j \le n} (1 - x_i x_j^{-1})\Big), \tag{2.4.1}$$

and write $\Omega_n = \prod_{1 \le i < j \le n} (1 - x_i x_j^{-1})$ for the kernel.

Explicitely, for two monomials, $(x^u, x^v) = (x^{u_1 - v_n, \ldots, u_n - v_1}, 1)$ and $(x^v, 1) \neq 0$ only when x^{-v} appears in the expansion of Ω_n. In that case $(x^v, 1) = \pm 1$ according to the sign x^{-v} has in Ω_n.

Similar definitions and properties hold for the root systems of type B, C, D (see later sections) with appropriate kernels $\Omega_n^B, \Omega_n^C, \Omega_n^D$.

For $n = 3$, one has

$$\Omega_3 = x^{000} - x^{1,-1,0} - x^{0,1,-1} + x^{2,-1,-1} + x^{1,1,-2} - x^{2,0,-2}$$

and therefore

$$(x^{000}, 1) = 1 = (x^{-2,1,1}, 1) = (x^{-1,-1,2}, 1) \;\&\; (x^{-1,1,0}, 1) = -1 = (x^{0,-1,1}, 1) = (x^{-2,0,2}, 1),$$

the other monomials being orthogonal to 1 (one has enumerated the positive and negative roots for type A_2).

Notice that, for symmetric functions, Weyl has defined the scalar product

$$(f, g)^{Weyl} = \frac{1}{n!} CT\Big(f(x_1, \ldots, x_n) g(x_1^{-1}, \ldots, x_n^{-1})\, \Omega_n^2\Big).$$

We shall see that in the case of Schur functions

$$(s_\lambda, s_\mu) = (s_\lambda, s_\mu)^{Weyl} = \delta_{\lambda,\mu},$$

so that the restriction of all these scalar products to symmetric functions coincides with the usual scalar product with respect to which Schur functions constitute an orthonormal basis.

However, we have also to use the structure of $\mathfrak{Pol}(n)$ as a free $\mathfrak{Sym}(n)$-module. We define for $f, g \in \mathfrak{Pol}(n)$,

$$(f,g)^{\partial} := fg\,\partial_\omega \quad \& \quad (f,g)^{\pi} := fg\,\pi_\omega\,.$$

These quadratic forms take values in $\mathfrak{Sym}(n)$ and are $\mathfrak{Sym}(n)$-linear.

The main properties of all these quadratic forms is the compatibility with the operators used to define the different bases.

Proposition 2.4.1. *For $i:\ 1 \le i \le n{-}1$,*

- *π_i is adjoint to π_{n-i} with respect to $(\ ,\ )$,*
- *∂_i is self-adjoint with respect to $(\ ,\ )^{\partial}$,*
- *π_i is self-adjoint with respect to $(\ ,\ )^{\pi}$.*

Proof. Let us check that all these statements reduce to the case $n = 2$.

$$(f\partial_i, g)^{\partial} = \big((f\partial_i)g\big)\,\partial_\omega = (f\partial_i\, g)\partial_i\partial_{s_i\omega} = \big((f\partial_i)(g\partial_i)\big)\partial_{s_i\omega}\,.$$

The last expression being symmetrical in f, g, one has, indeed, $(f\partial_i, g)^{\partial} = (f, g\partial_i)^{\partial}$. The same computation applies to the case $(\ ,\ )^{\pi}$.

The kernel Ω_n can be written $\Omega'\,(1 - x_i x_{i+1}^{-1})$, with Ω' symmetrical in x_i, x_{i+1}, and one can first compute the constant term in x_i, x_{i+1}. Let us write $f = f_1 + x_{i+1}f_2$, $g(x_n^{-1}, \ldots, x_1^{-1}) = h(x_1, \ldots, x_n) = g_1 + x_{i+1}g_2$, with f_1, f_2, g_1, g_2 invariant under s_i. The difference $f\pi_i h - h\pi_i f = f\widehat{\pi}_i h - h\widehat{\pi}_i f$ is equal to $(f_1g_2 - g_1f_2)x_{i+1}$. Therefore the constant term

$$\begin{aligned} CT_{x_i,x_{i+1}}&\Big((f\pi_i h - h\pi_i f)\,(1 - x_i/x_{i+1})\Omega'\Big) \\ &= CT_{x_i,x_{i+1}}\Big((f\widehat{\pi}_i h - h\widehat{\pi}_i f)\,(1 - x_i/x_{i+1})\Omega'\Big) \\ &= CT_{x_i,x_{i+1}}\Big((x_{i+1} - x_i)\,(f_1g_2 - g_1f_2)\Omega'\Big) \end{aligned}$$

is null, because the function inside parentheses is antisymmetrical in x_i, x_{i+1}. Taking into account the transformation $x_i \to x_{n+1-i}^{-1}$, this nullity proves that π_i is adjoint to π_{n-i}. QED

Thanks to Proposition 2.4.1, the scalar products $(f, s_\lambda(\mathbf{x}_n))$ can be rewritten as scalar products with dominant monomials. Indeed $s_\lambda(\mathbf{x}_n) = x^\lambda\pi_\omega$, and therefore

$$\big(f, s_\lambda(\mathbf{x}_n)\big) = \big(f, x^\lambda\pi_\omega\big) = \big(f\pi_\omega, x^\lambda\pi_\omega\big) = \big(f\pi_\omega, x^\lambda\big)\,.$$

On the other hand,

$$\big(f, s_\lambda(\mathbf{x}_n)\big)^{\partial} = (f,1)^{\partial}s_\lambda(\mathbf{x}_n) \quad \& \quad \big(f, s_\lambda(\mathbf{x}_n)\big)^{\pi} = (f,1)^{\pi}s_\lambda(\mathbf{x}_n)\,,$$

since these last two scalar products are $\mathfrak{Sym}(\mathbf{x}_n)$-linear.

2.5 Kernels

With a scalar product and a basis defined by self-adjoint operators, it is easy to find the adjoint basis. Once more, it is sufficient to understand the case $n = 2$.

Lemma 2.5.1. *Let $i \in \{1, \ldots, n{-}1\}$, $D_i = \pi_i$, $\widehat{D}_i = \widehat{\pi}_i$ (resp. $D_i = \partial_i = \widehat{D}_i$). Let $f, g \in \mathfrak{Pol}(n)$, $f' = fD_i$, $g' = g\widehat{D}_i$. Then the two equalities $(f, g)^D = 0$, $(f', g)^D = 1$ imply that $(f', g')^D = 0$ and that $(f, g')^D = 1$.*

Proof. Consider first the case $D_i = \pi_i$ and write $f = f_1 + x_{i+1}f_2$, $g = g_1 + x_{i+1}g_2$. Then $f' = f_1$, $g' = g(\pi_i - 1) = g_1 - g$. Consequently,

$$(f, g')^\pi = (f_1, g_1)^\pi - (f, g)^\pi = (f', g)^\pi = 1 \ \& \ (f', g')^\pi = (f_1, g_1)^\pi - (f_1, g)^\pi = 0\,.$$

The computation is similar for $D_i = \partial_i$. QED

This lemmma will allow propagating orthogonality relations. But to produce a hen, we need an egg, or conversely.

Let

$$\Theta_n^Y := \prod_{1 \le i < j \le n} (y_i - x_j) \qquad \& \qquad \Theta_n^G := \prod_{1 \le i < j \le n} (1 - x_j y_i^{-1})\,.$$

Lemma 2.5.2. *Let $v : \mathbf{0} \le \rho = [n{-}1, \ldots, 0]$. Then*

$$(Y_v, \Theta_n^Y)^\partial = 0 = (G_v, \Theta_n^G)^\pi\,,$$

except for $v = \mathbf{0}$, in which case

$$(Y_{\mathbf{0}}, \Theta_n^Y)^\partial = 1 = (G_{\mathbf{0}}, \Theta_n^G)^\pi\,.$$

Proof. By definition, $(f(\mathbf{x}), \Theta_n^Y)^\partial = f(\mathbf{x})\Theta_n^Y\, \partial_\omega$ for any polynomial $f(\mathbf{x})$. If this polynomial belong to the span of $x^v : v \le \rho$, then $f(\mathbf{x})\Theta_n^Y$ belong to the span of $x^v : v \le [n{-}1, \ldots, n{-}1]$ and its image under ∂_ω is a symmetric polynomial of degree 0 (only the monomials which are a permutation of x^ρ have a non-zero image). On the other hand, the scalar product can also be written as a sum :

$$(Y_v, \Theta_n^Y)^\partial = \sum_\sigma (\text{-}1)^{\ell(\sigma)} \big(Y_v \Theta_n^Y\big)^\sigma \frac{1}{\Delta(\mathbf{x})}\,.$$

Since this is a function of degree 0 in $\mathbf{x}$, one can specialize $\mathbf{x} = \mathbf{y}$ without changing its value. However, all $(\Theta_n^Y)^\sigma$ then vanish, except for the identity, in which case Θ_n^Y specializes to Δ. Therefore,[7] $(Y_v, \Theta_n^Y)^\partial = Y_v(\mathbf{y}, \mathbf{y}) = \delta_{v,\mathbf{0}}$.

The proof is similar for Grothendieck polynomials. QED

[7] The vanishing of $Y_v(\mathbf{y}, \mathbf{y})$, which is evident for dominant v, is proved following an induction which in fact furnishes more specializations. Thus we do not prove it at this point, but refer to Corollary 3.1.3 below.

2.6 Adjoint Schubert and Grothendieck polynomials

The ring $\mathfrak{Pol}(n)$ is a free $\mathfrak{Sym}(n)$-module with bases $\{x^v : v \le \rho\}$ and $\{x^{-v} : v \le \rho\}$ (one takes Laurent polynomials in the second case). Therefore $\{Y_v : v \le \rho\}$ and $\{G_v : v \le \rho\}$ are two linear bases. Starting with $\widehat{Y}_\rho := \Theta_n^Y$ and $\widehat{G}_\rho := \Theta_n^G$, instead of Y_ρ and G_ρ, one generates recursively two other bases

$$\widehat{Y}_{\dots,v_{i+1},v_i-1,\dots} = \widehat{Y}_v\, \partial_i \quad \& \quad \widehat{G}_{\dots,v_{i+1},v_i-1,\dots} = \widehat{G}_v\, \widehat{\pi}_i \text{ when } v_i > v_{i+1}\,. \tag{2.6.1}$$

Here are these bases for $n = 3$.

$$\widehat{Y}_{210} = \begin{array}{ll}(y_1 - x_3) & (y_2 - x_3)\\ (y_1 - x_2) & \end{array}$$

$$\widehat{Y}_{110} = (y_1 - x_3)(y_2 - x_3) \qquad \widehat{Y}_{200} = \begin{array}{l}(y_1 - x_3)\\ (y_1 - x_2)\end{array}$$

$$\widehat{Y}_{100} = y_1 + y_2 - x_2 - x_3 \qquad \widehat{Y}_{010} = y_1 - x_3$$

$$\widehat{Y}_{000} = 1$$

$$\widehat{G}_{210} = \begin{array}{ll}(1 - \frac{x_3}{y_1}) & (1 - \frac{x_3}{y_2})\\ (1 - \frac{x_2}{y_1}) & \end{array}$$

$$\widehat{G}_{110} = \frac{x_2}{y_2}(1 - \frac{x_3}{y_1})(1 - \frac{x_3}{y_2}) \qquad \widehat{G}_{200} = \frac{x_3}{y_2}\begin{array}{l}(1 - \frac{x_3}{y_1})\\ (1 - \frac{x_2}{y_1})\end{array}$$

$$\widehat{G}_{100} = \frac{x_3}{y_1}\left(1 - \frac{x_2 x_3}{y_1 y_2}\right) \qquad \widehat{G}_{010} = \frac{x_2 x_3}{y_1 y_2}\left(1 - \frac{x_3}{y_1}\right)$$

$$\widehat{G}_{000} = \frac{x_2 x_3^2}{y_1^2 y_2}$$

Lemmas 2.5.1, 2.5.2 give the following pairs of adjoint bases.

Theorem 2.6.1. *The bases* $\{Y_v : v \le \rho\}$ *and* $\{\widehat{Y}_v : v \le \rho\}$ *are adjoint with respect to* $(\ ,\ )^\partial$*. The bases* $\{G_v : v \le \rho\}$ *and* $\{\widehat{G}_v : v \le \rho\}$ *are adjoint with respect to* $(\ ,\ )^\pi$*.*

More precisely, the pairing is

$$(Y_v, \widehat{Y}_u)^\partial = \delta_{v,\rho-u} = (G_v, \widehat{G}_u)^\pi\,. \tag{2.6.2}$$

The two bases $\{\widehat{Y}_v\}$ and $\{\widehat{G}_v\}$ can in fact be easily obtained as images of $\{Y_v\}$ and $\{G_v\}$ respectively. Indeed, Ω is obtained from Y_ρ by reversing the alphabet $\mathbf{x}$, but divided differences satisfy

$$\omega\,\partial_i\,\omega = -\partial_{n-i}\,. \tag{2.6.3}$$

Similarly, let $\clubsuit$ be the involution $x_i \to x_{n+1-i}^{-1}$. Then

$$\clubsuit\,\pi_i\,\clubsuit = \pi_{n-i} \quad \& \quad \omega x^{-\rho}\,\pi_i\,x^\rho\omega = -\widehat{\pi}_{n-i}\,. \tag{2.6.4}$$

Extend the involution to codes of permutations : $u\clubsuit = v$ if and only iff the corresponding permutations σ, ζ, are such that $\omega\sigma\omega = \zeta$. Then, the relations (2.6.3, 2.6.4) induce

Lemma 2.6.2. *The adjoint polynomials $\widehat{Y}_v$ and $\widehat{G}_v$ are related to the original ones by*

$$\left(\widehat{Y}_v\right)^\omega = (-1)^{|v|}\,Y_{v\clubsuit} \quad \& \quad \left(\widehat{G}_v\right)^\omega = (-1)^{|v|}\,G_{v\clubsuit}\frac{x^\rho}{y^\rho}\,. \tag{2.6.5}$$

As a consequence, for any $\sigma, \zeta \in \mathfrak{S}_n$, one has

$$\Big(X_\sigma(\mathbf{x},\mathbf{y})\,,\; X_\zeta(\mathbf{x}^\omega,\mathbf{y})\Big)^\partial \;=\; (-1)^{\ell(\zeta)}\delta_{\sigma,\zeta\omega} \tag{2.6.6}$$

$$\left(G_{(\sigma)}(\mathbf{x},\mathbf{y})\,,\; \left(\frac{x^\rho}{y^\rho}G_{(\zeta)}(\mathbf{x},\mathbf{y})\right)^\omega\right)^\pi \;=\; (-1)^{\ell(\zeta)}\delta_{\sigma,\zeta\omega} \tag{2.6.7}$$

The decomposition of any polynomial in the Schubert or Grothendieck basis can easily be computed using the scalar products with their adjoint bases. Here is the matrix of change of basis between monomials $x^v : 0 \geq v \geq [-2,-1,0]$ and Grothendieck polynomials :

	000	100	010	200	110	210
$1/x^{000}$	1	0	0	0	0	0
$1/x^{100}$	$\frac{1}{y_1}$	$-\frac{1}{y_1}$	0	0	0	0
$1/x^{010}$	$\frac{1}{y_2}$	$\frac{1}{y_1}$	$-\frac{1}{y_2}$	0	$-\frac{1}{y_1}$	0
$1/x^{200}$	$\frac{1}{y_1^2}$	$-\frac{y_2+y_1}{{y_1}^2y_2}$	0	$\frac{1}{y_1y_2}$	0	0
$1/x^{110}$	$\frac{1}{y_1y_2}$	0	$-\frac{1}{y_1y_2}$	0	0	0
$1/x^{210}$	$\frac{1}{{y_1}^2y_2}$	$-\frac{1}{{y_1}^2y_2}$	$-\frac{1}{{y_1}^2y_2}$	$\frac{1}{{y_1}^2y_2}$	$\frac{1}{{y_1}^2y_2}$	$-\frac{1}{{y_1}^2y_2}$

2.7 Bases adjoint to elementary and complete functions

Expanding the kernels Θ_n^Y and Θ_n^G, one finds the bases adjoint to monomials, for the two scalar products $(\,,\,)^\partial$ and $(\,,\,)^\pi$.

Proposition 2.7.1. *Given n, let $\mathbf{x}^\vee = \{x_1^{-1}, \dots, x_n^{-1}\}$. Then for any $u, v : u \le \rho$, $v \le \rho$ one has*

$$\left(P_{\rho-v}(\mathbf{x})\,,\, x^{u\omega}\right)^{\partial} = (\text{-}1)^{|v|}\delta_{v,u} = \left(P_v(\mathbf{x}^\vee)\,,\, x^{u\omega}\right)^{\pi}. \tag{2.7.1}$$

The basis adjoint to $\{H_v : v \le \rho\}$ requires a little more work, because the monomials appearing in the expansion of H_v do not respect the condition that their exponent be majorized by ρ. We first some technical properties of divided differences.

Lemma 2.7.2. *Let $a, b, k, n \in \mathbb{N}$ be such that $1 \le k < n$, $0 \le a, b \le n\text{–}k$. Then*

$$S_{1^b}(\mathbf{x}_n - x_k) S_a(\mathbf{x}_k) \partial_k \dots \partial_1 = \begin{cases} (\text{-}1)^b \ \textit{if} \quad a\text{+}b = n\text{–}k \\ 0 \quad \textit{otherwise} \end{cases} \tag{2.7.2}$$

Proof. One expands $S_{1^b}(\mathbf{x}_n - x_k) = S_{1^b}(\mathbf{x}_n) - x_k S_{1^b}(\mathbf{x}_n) + \cdots + (\text{-}x_k)^b$. On the other hand, $x_k^i S_a(\mathbf{x}_k) = S_{a+i}(\mathbf{x}_k) - \sum x^u$, sum over monomials x^u, $u \in \mathbb{N}^k$ such that $u_k \le i\text{–}1$. The initial function is therefore equal to

$$\left(S_{1^b}(\mathbf{x}_n) S_a(\mathbf{x}_k) - S_{1^{b-1}}(\mathbf{x}_n) S_{a+1}(\mathbf{x}_k) - \cdots + (\text{-}1)^b S_0(\mathbf{x}_n) S_{a+b}(\mathbf{x}_k)\right) - \sum c_u x^u\,,$$

with $c_u \in \mathfrak{Sym}(\mathbf{x}_n)$ and $u\mathbb{N}^k$ such that $u_k \le b\text{–}1 < n\text{–}k$. The extra monomials x^u are sent to 0 by $\partial_k \dots \partial_{n-1}$ for degree reason. The sum inside parentheses is sent to

$$\begin{aligned} &S_{1^b}(\mathbf{x}_n) S_{a-n+k}(\mathbf{x}_k) - S_{1^{b-1}}(\mathbf{x}_n) S_{a+1-n+k}(\mathbf{x}_k) - \dots \\ &\qquad + (\text{-}1)^b S_0(\mathbf{x}_n) S_{a+b-n+k}(\mathbf{x}_k) = (\text{-}1)^b S_{a+b-n+k}(\mathbf{x}_n - \mathbf{x}_n)\,. \end{aligned}$$

This last function is different from 0 only in the case $S_0(\mathbf{x}_n\text{–}\mathbf{x}_n) = 1$, that is only for $a\text{+}b = n\text{–}k$. QED

Proposition 2.7.3. *Given n, for any $v \le \rho$, let $\widehat{H}_v = S_{1^{v_1}}(\mathbf{x}_n - x_1) S_{1^{v_2}}(\mathbf{x}_n - x_2) \dots S_{1^{v_{n-1}}}(\mathbf{x}_n - x_{n-1})$. Then*

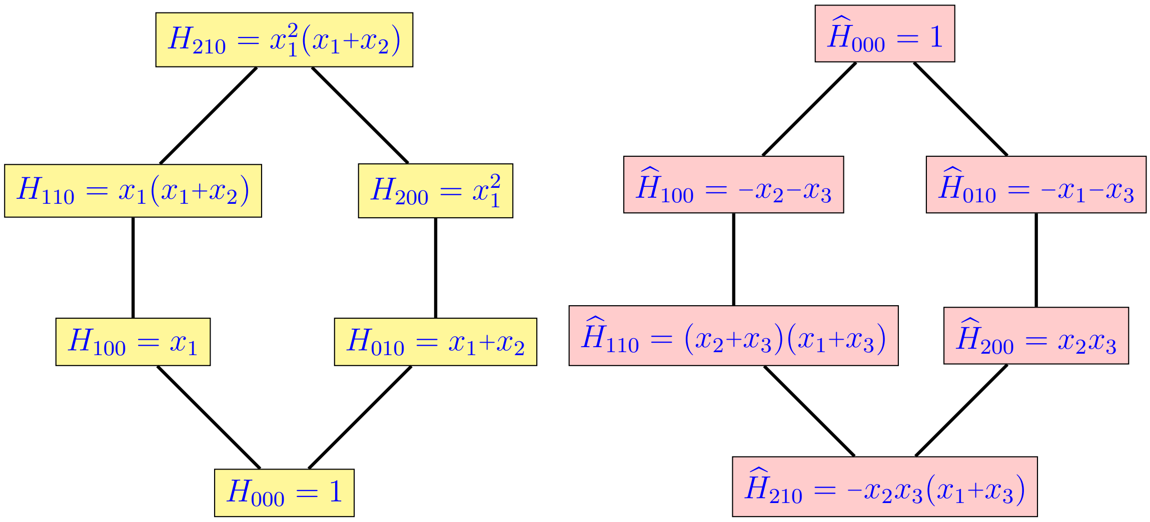
$$\left(\widehat{H}_v\,,\, H_u\right)^{\partial} = (\text{-}1)^{|v|}\delta_{v,\rho-u}\,,\ u, v \le \rho\,. \tag{2.7.3}$$

Proof. Factorize $\partial_\omega = (\partial_{n-1})(\partial_{n-2}\partial_{n-1}) \dots (\partial_1 \dots \partial_{n-1})$. By decreasing induction on k, one has to compute

$$\begin{aligned} &\Big(S_{1^{v_1}}(\mathbf{x}_n - x_1) \dots S_{1^{v_k}}(\mathbf{x}_n - x_k)\Big)\Big(S_{v_1}(\mathbf{x}_1) \dots S_{v_k}(\mathbf{x}_k)\Big)\partial_k \dots \partial_{n-1} \\ &\qquad = f\,(S_{1^{v_k}}(\mathbf{x}_n - x_k) S_{v_k}(\mathbf{x}_k))\,\partial_k \dots \partial_{n-1}\,, \end{aligned}$$

with f symmetrical in $x_k, \dots, x_n$, and therefore commuting with $\partial_k \dots \partial_{n-1}$. Eq. 2.7.2 forces the equality $v_k\text{+}u_k = n\text{–}k$, to have non nullity, and we can proceed with $k\text{–}1$. QED

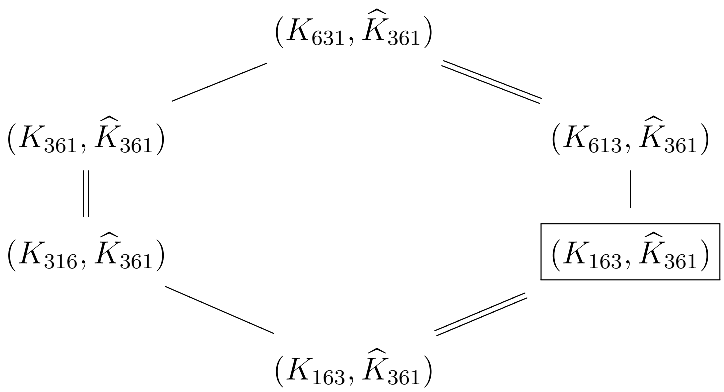
For example, for $n = 3$, one has the following pair of adjoint bases.



2.8 Adjoint key polynomials

The two families $\{Y_v : v \in \mathbb{N}^n\}$, $\{G_v : v \in \mathbb{N}^n\}$ are bases of $\mathfrak{Pol}(n)$ (as a vector space). We have also given two other bases, $\{K_v : v \in \mathbb{N}^n\}$ and $\{\widehat{K}_v : v \in \mathbb{N}^n\}$, that are in fact adjoint with respect to (,), as states the next theorem.

First, one checks that for any partition λ, then $(K_v, x^\lambda) = 0$, except when $v = \lambda\omega = [\lambda_n, \ldots, \lambda_1]$, in which case $(K_{\lambda\omega}, x^\lambda) = 1$ (cf. [44, Cor 12]). Using that π_i is adjoint to π_{n-i}, this allows to compute any $(K_v, \widehat{K}_u)$. For example, writing in a box the non-zero scalar products, the knowledge of all $(K_v, \widehat{K}_{361})$



determines all $(K_v, \widehat{K}_{316})$

$$\begin{tikzcd}
 & (K_{631}, \widehat{K}_{316}) \arrow[dl, dash] \arrow[dr, equal] & \\
(K_{316}, \widehat{K}_{316}) \arrow[d, equal] & & \boxed{(K_{613}, \widehat{K}_{316})} \arrow[d, dash] \\
(K_{316}, \widehat{K}_{316}) \arrow[dr, dash] & & (K_{163}, \widehat{K}_{316}) \arrow[dl, equal] \\
 & (K_{163}, \widehat{K}_{316}) &
\end{tikzcd}$$

In conclusion, one has the following property (cf. [44, Th 15]) :

Theorem 2.8.1. *Given* $u, v \in \mathbb{N}^n$*, then* $(K_v, \widehat{K}_u) = 0$*, except* $(K_v, \widehat{K}_{v\omega}) = 1$.

In particular, if λ is dominant, then $(K_v, x^\lambda) = 0$, except if $v = \lambda\omega$, in which case K_v is a Schur function.

Notice that the pairing, for Schubert and Grothendieck polynomials, is also the reversing $\sigma \to \sigma\omega$, when indexing these polynomials by permutations, but not when using codes.

2.9 Reproducing kernels for Schubert and Grothendieck polynomials

In the theory of orthogonal polynomials in one variable one finds it convenient to make use of *reproducing kernels* $K_n(x,y) = P_0(x)P_0(y) + \cdots + P_n(x)P_n(y)$, associated to a family of polynomials $P_0(x), P_1(x), \ldots$ of degree $0, 1, \ldots$, which are orthonormal with respect to a linear functional $f \to \int f$. The name "reproducing" comes from the property that

$$\int f(x) K_n(x,y) = f(y)$$

whenever f is a polynomial of degree $\le n$.

The *Cauchy kernel* $\prod_{x\in\mathbf{x}, y\in\mathbf{y}}(1-xy)^{-1}$ plays a similar role in the theory of symmetric polynomials. It does not require much effort nor imagination to deduce from the preceding section kernels corresponding to the bases $\{Y_v\}$, $\{G_v\}$ or $\{K_v\}$. Write $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$ for the identification of any symmetric function of x_n with the same symmetric function of $\mathbf{y}_n$.

Theorem 2.9.1. *For any $v : \mathbf{0} \le v \le \rho$, one has*

$$\big(\Theta_n^Y\,,\, x^v\big)^{\partial} = y^v \quad \& \quad \big(\Theta_n^G\,,\, x^{-v}\big)^{\pi} = y^{-v}\,. \tag{2.9.1}$$

For any Laurent polynomial f in x_n, one has, modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$,

$$\big(\Theta_n^Y\,,\, f(\mathbf{x})\big)^{\partial} \equiv f(\mathbf{y}) \quad \& \quad \big(\Theta_n^G\,,\, f(\mathbf{x})\big)^{\pi} \equiv f(\mathbf{y})\,. \tag{2.9.2}$$

The two kernels expand as follows

$$\Theta_n^Y(\mathbf{x},\mathbf{z}) = \prod_{1\le i<j\le n} (z_i - x_j) \;=\; \sum_{v\le\rho} Y_v(\mathbf{z},\mathbf{y})\, \widehat{Y}_{\rho-v}(\mathbf{x},\mathbf{y}) \tag{2.9.3}$$

$$\Theta_n^G(\mathbf{x},\mathbf{z}) = \prod_{1\le i<j\le n} (1 - x_j z_i^{-1}) \;=\; \sum_{v\le\rho} G_v(\mathbf{z},\mathbf{y})\, \widehat{G}_{\rho-v}(\mathbf{x},\mathbf{y}) \tag{2.9.4}$$

There is no real need of a proof. The reproducing property has been obtained in the course of proving Lemma 2.5.2. Taking coefficients in $\mathfrak{Sym}(\mathbf{x}_n)$, one obtains (2.9.2) from (2.9.1). The function $\Theta_n^Y(\mathbf{x},\mathbf{z})$ belongs to the span of $\{z^u x^{v\omega} : u, v \le \rho\}$, and therefore can be written

$$\Theta_n^Y(\mathbf{x},\mathbf{z}) = \sum_{u,v} c_{u,v}(\mathbf{y})\, Y_u(\mathbf{z},\mathbf{y}) \widehat{Y}_{\rho-v}(\mathbf{x},\mathbf{y})\,.$$

Therefore, for any $v \le \rho$, one has $\big(\Theta_n^Y(\mathbf{x},\mathbf{z}), Y_v(\mathbf{x},\mathbf{y})\big)^{\partial} = \sum_u c_{u,v}(\mathbf{y})\, Y_u(\mathbf{z},\mathbf{y})$. However, the reproducing property shows that this is also equal to $Y_v(\mathbf{z},\mathbf{y})$ and this proves (2.9.3), the case of Grothendieck polynomials being similar. QED

For example, for $n=2$, one has

$$\begin{aligned}\Theta_2^G(\mathbf{x},\mathbf{z}) = 1 - x_2/z_1 &= G_{00}(\mathbf{z},\mathbf{y})\widehat{G}_{10}(\mathbf{x},\mathbf{y}) + G_{10}(\mathbf{z},\mathbf{y})\widehat{G}_{00}(\mathbf{x},\mathbf{y}) \\ &= 1\cdot\left(1-\frac{x_2}{y_1}\right) \qquad + \qquad \left(1-\frac{y_1}{z_1}\right)\cdot\frac{x_2}{y_1}\,.\end{aligned}$$

For $n=3$, Maple computes

$$\begin{aligned}\Theta_3^Y(\mathbf{x},\mathbf{z}) &= (z_1-x_2)\,(z_1-x_3)\,(z_2-x_3) = -\,(-y_1+x_2)\,(-y_1+x_3)\,(-y_2+x_3)\\ &+ (z_2-y_2+z_1-y_1)\,(-y_1+x_3)\,(-y_1+x_2) - (-z_1+y_2)\,(z_1-y_1)\,(y_1-x_3)\\ &+ (z_1-y_1)\,(-y_2+x_3)\,(-y_1+x_3) + (-y_1+z_2)\,(z_1-y_1)\,(-x_2+y_2+y_1-x_3)\\ &- (-z_1+y_2)\,(z_1-y_1)\,(-y_1+z_2)\;.\end{aligned}$$

The essential property of $\Theta_n^Y(\mathbf{x},\mathbf{y})$ and $\Theta_n^G(\mathbf{x},\mathbf{y})$ is that $\Theta_n^Y(\mathbf{y}^\sigma,\mathbf{y})$ and $\Theta_n^G(\mathbf{y}^\sigma,\mathbf{y})$ both vanish when σ is different from the identity. Along the same lines as for Θ_n^Y and Θ_n^G, one sees that the kernels $Y_\rho(\mathbf{x},\mathbf{y})$ and $G_\rho(\mathbf{x},\mathbf{y})$ satisfy a twisted reproduction property :

$$\big(Y_\rho(\mathbf{x},\mathbf{y})\,,\,f(\mathbf{x})\big)^\partial \equiv f(\mathbf{y}^\omega) \quad \& \quad \big(G_\rho(\mathbf{x},\mathbf{y})\;\;,\,f(\mathbf{x})\big)^\pi \equiv f(\mathbf{y}^\omega)\,, \tag{2.9.5}$$

modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$, the equivalence being replaced by an equality when f belongs to the span of $\{x^v : [0,\dots,0] \le v \le [0,\dots,n{-}1]\}$. For example,

$$(G_{210}(\mathbf{x},\mathbf{y}), x_3^2)^\pi = \left(1-\frac{y_1}{x_1}\right)\left(1-\frac{y_2}{x_1}\right)\left(1-\frac{y_1}{x_2}\right) x_3^2\,\pi_{321} = y_1^2\,.$$

Notice that, using (2.2.4) and (2.6.5), exchanging the role of $\mathbf{y}$ and $\mathbf{x}$, one can rewrite (2.9.4) into

$$\sum_{v\le\rho} (\text{-}1)^{|v|} G_v(\mathbf{x},\mathbf{z})\, G_{\rho-v}(\mathbf{x},\mathbf{y}) = Y_\rho(\mathbf{z},\mathbf{y})\, x^{-\rho}\,. \tag{2.9.6}$$

By taking the image of (2.9.3) under products of ∂_i's and the image of (2.9.4) under products of $\widehat{\pi}_i$'s, one obtains decompositions of general $\widehat{Y}_v$ or general $\widehat{G}_v$, and by involution, of general Y_v and G_v. Let us detail these decompositions in the next sections.

2.10 Cauchy formula for Schubert

Given $u,v,w\in\mathbb{N}^n$, majorized by ρ, write $w = u\odot v$ iff and only the permutations $\sigma(w),\sigma(u),\sigma(v)$ of which they are the codes, are such that $\sigma(w)=\sigma(u)\sigma(v)$ and the product is reduced[8]. With this notation one has the following Cauchy formula for Schubert polynomials (given in [97] for $\mathbf{y}=\mathbf{0}$).

[8] i.e. such that lengths add: $\ell(\sigma(w)) = \ell(\sigma(u)) + \ell(\sigma(v))$. Notice that the product of two permutations η,ν is *reduced* if and only if $\partial_\eta\partial_\nu = \partial_{\eta\nu}$.

Theorem 2.10.1. *Let σ be a permutation in $\mathfrak{S}_n$, $w \in \mathbb{N}^n$ be its code. Then*

$$Y_w(\mathbf{x},\mathbf{z}) = \sum_{u,v:\, u\odot v=w} Y_u(\mathbf{y},\mathbf{z})Y_v(\mathbf{x},\mathbf{y}) \tag{2.10.1}$$

$$X_\sigma(\mathbf{x},\mathbf{z}) = \sum_{\eta,\nu:\, \partial_\eta\partial_\nu=\partial_\sigma} X_\eta(\mathbf{y},\mathbf{z})\, X_\nu(\mathbf{x},\mathbf{y})\,. \tag{2.10.2}$$

Proof. One starts from the formula in the case $\sigma=\omega$, which is a rewriting of (2.9.3) using (2.6.5). Supposing (2.10.2) to be true for σ, let i be such that $\ell(\sigma s_i)<\ell(\sigma)$. The terms in the RHS are of two types: either $\ell(\nu s_i)<\ell(\nu)$, or not. These last terms are such that $X_\nu(\mathbf{x},\mathbf{y})\partial_i=0$. Therefore the image of (2.10.2) under ∂_i is

$$X_{\sigma s_i}(\mathbf{x},\mathbf{z}) = \sum_{\eta,\zeta:\, \partial_\eta\partial_\zeta=\partial_{\sigma s_i}} X_\eta(\mathbf{y},\mathbf{z})\, X_\zeta(\mathbf{x},\mathbf{y})\,,$$

with $\zeta=\nu s_i$. QED

For example, for $w=[0,3,1]$, one has the following expansion of $Y_{031}(\mathbf{x},\mathbf{z})$, writing $Y_u\,Y_v$ for $Y_u(\mathbf{y},\mathbf{z})Y_v(\mathbf{x},\mathbf{y})$

$$Y_{031}(\mathbf{x},\mathbf{z}) = \begin{tikzcd}[column sep=small]
 & & Y_{0000}Y_{031} \arrow[dl, no head] \arrow[dr, dashed, no head] & & \\
 & Y_{0100}Y_{030} \arrow[dr, dashed, no head] & & Y_{0001}Y_{021} \arrow[dl, no head] \arrow[dr, equal] & \\
 & & Y_{0101}Y_{020} \arrow[d, equal] & & Y_{002}Y_{011} \arrow[d, no head] \\
 & & Y_{012}Y_{010} \arrow[dr, no head] & & Y_{03}Y_{001} \arrow[dl, equal] \\
 & & & Y_{031}Y_{000} &
\end{tikzcd} ,$$

or, indexing by permutations,

$$X_{15324}(\mathbf{x},\mathbf{z}) = \begin{tikzcd}[column sep=small]
 & & X_{12345}X_{15324} \arrow[dl, no head] \arrow[dr, dashed, no head] & & \\
 & X_{13245}X_{15234} \arrow[dr, dashed, no head] & & X_{12354}X_{14325} \arrow[dl, no head] \arrow[dr, equal] & \\
 & & X_{13254}X_{14235} \arrow[d, equal] & & X_{12534}X_{13425} \arrow[d, no head] \\
 & & X_{13524}X_{13245} \arrow[dr, no head] & & X_{15234}X_{12435} \arrow[dl, equal] \\
 & & & X_{15324}X_{12345} &
\end{tikzcd} .$$

In these last conventions, the edges are simple transpositions: $X_\eta X_{s_i\zeta} \to X_{\eta s_i}X_\zeta$.

Notice that the above decomposition of $Y_\rho(\mathbf{x},\mathbf{z}) = \prod_{i+j\le n}(x_i - z_j)$, becomes similar, when specializing $\mathbf{y} = \mathbf{0}$, to the Cauchy expansion of the resultant $\prod_{i,j\le n}(x_i - z_j)$ in terms of Schur functions in $\mathbf{x}$ and in $\mathbf{z}$. In fact, let m, r be two integers such that $r + m < n$. Then the special case of (2.10.1) for $w = r^m$, $\mathbf{y} = \mathbf{0}$ is

$$Y_{r^m}(\mathbf{x},\mathbf{z}) = \sum_{u,v:\, u\odot v = w} Y_u(\mathbf{0},\mathbf{z})Y_v(\mathbf{x},\mathbf{0}) = \sum_{\lambda\le r^m} (\text{-}1)^{|\mu|} s_\mu(\mathbf{z}_r) s_\lambda(\mathbf{x}_m)\,, \qquad (2.10.3)$$

sum over all pairs of partitions λ, μ such that the conjugate of μ is $[r-\lambda_m, \ldots, r-\lambda_1]$.

2.11 Cauchy formula for Grothendieck

The analogous formula for Grothendieck polynomials is not more complicated. Instead of taking reduced products, i.e. products $\partial_\eta\partial_\nu \neq 0$, one has to use products in the 0-Hecke algebra, of the type $\pi_\eta\pi_\nu$.

Theorem 2.11.1. *Let σ be a permutation in $\mathfrak{S}_n$, $\omega = [n, \ldots, 1]$.*

$$\widehat{G}_{(\sigma)}(\mathbf{x},\mathbf{z}) = \sum_{\zeta\in\mathfrak{S}_n} G_{(\zeta)}(\mathbf{z},\mathbf{y})\widehat{G}_{(\omega\zeta)}(\mathbf{x},\mathbf{y})\,\widehat{\pi}_{\omega\sigma} \qquad (2.11.1)$$

$$\frac{y^\rho}{z^\rho} G_{(\sigma)}(\mathbf{x},\mathbf{z}) = \sum_{\zeta} (\text{-}1)^{\ell(\zeta)} G_{(\zeta\omega)}(\mathbf{z},\mathbf{y}) \left(G_{(\zeta)}(\mathbf{x},\mathbf{y})\pi_{(\omega\sigma)}\right) \qquad (2.11.2)$$

Proof. The first formula is the image of (2.9.4) under $\widehat{\pi}_{\omega\sigma}$, the second is the image of the case $\sigma = \omega$, which is a rewriting of (2.9.6), under $\pi_{\omega\sigma}$. QED

For example, for $n = 3$, writing G_v for $G_v(\mathbf{z},\mathbf{y})$ and $\widehat{G}_v$ for $\widehat{G}_v(\mathbf{x},\mathbf{y})$, the image of $\widehat{G}_{210}(\mathbf{x},\mathbf{z}) = \sum_v G_v \widehat{G}_{210-v}$ under $\widehat{\pi}_1$ is

$$\widehat{G}_{110}(\mathbf{x},\mathbf{z}) = (G_{110} - G_{210})\,\widehat{G}_{000} + (G_{010} - G_{200})\,\widehat{G}_{010} + (G_{000} - G_{100})\,\widehat{G}_{110}\,,$$

then under $\widehat{\pi}_2$,

$$\widehat{G}_{100}(\mathbf{x},\mathbf{z}) = (G_{010} - G_{200} - G_{110} + G_{210})\,\widehat{G}_{000} + (G_{000} - G_{100})\,\widehat{G}_{100}\,.$$

2.12 Divided differences as scalar products

Since the ∂_i's are self-adjoint with respect to $(\,,\,)^\partial$, and the π_i's are self-adjoint with respect to $(\,,\,)^\pi$, one can use (2.9.1) to express any $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$.

Proposition 2.12.1. *Let $f \in \mathfrak{Pol}(\mathbf{x}_n, \mathbf{y}_n)$, $\sigma \in \mathfrak{S}_n$, and $\mathbf{z} = \mathbf{z}_n$ be an extra alphabet. Then*

$$f\,\partial_\sigma = \Big(f\,,\, X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega)\Big)^\partial \Big|_{\mathbf{z}=\mathbf{x}} \tag{2.12.1}$$

$$f\,\pi_\sigma = \Big(f\,,\, G_{(\omega\sigma^{-1})}(\mathbf{x}, \mathbf{z})\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}^\omega} \tag{2.12.2}$$

$$f\,\widehat{\pi}_\sigma = \Big(f\,,\, \widehat{G}_{(\omega\sigma^{-1})}(\mathbf{x}, \mathbf{z})\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}} \tag{2.12.3}$$

Proof. The proofs of the three assertions are similar, let us consider only the first one.

$$\begin{aligned}(-1)^{\ell(\omega\sigma)} X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega) = X_{\sigma^{-1}\omega}(\mathbf{x}, \mathbf{z})\,\omega = X_\omega(\mathbf{x}, \mathbf{z})\partial_{\omega\sigma^{-1}\omega}\,\omega \\ = X_\omega(\mathbf{x}, \mathbf{z})\omega\big(\omega\partial_{\omega\sigma^{-1}\omega}\omega\big) = X_\omega(\mathbf{x}^\omega, \mathbf{z})\partial_{\sigma^{-1}}(-1)^{\ell(\sigma)}\end{aligned}$$

and therefore one has

$$\Big(f\,,\, X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega)\Big)^\partial = \Big(f\,,\, (-1)^{\ell(\omega)} X_\omega(\mathbf{x}^\omega, \mathbf{z})\partial_{\sigma^{-1}}\Big)^\partial = \Big(f\partial_\sigma,\,,\, X_\omega(\mathbf{z}, \mathbf{x}^\omega)\Big)^\partial .$$

Specializing $\mathbf{z} = \mathbf{x}$ and using the reproducing property (2.9.1), one gets (2.12.1). QED

For example, for $n = 3$, $\sigma = [2,3,1]$, one has $\omega\sigma = [2,1,3]$, $\omega\sigma^{-1} = [1,3,2]$, and

$$\begin{aligned}
f\partial_{231} = f\partial_1\partial_2 &= \Big(f\,,\, X_{213}(\mathbf{z}, \mathbf{x}^\omega)\Big)^\partial \Big|_{\mathbf{z}=\mathbf{x}} = \Big(f\,,\, z_1 - x_3\Big)^\partial \Big|_{\mathbf{z}=\mathbf{x}} \\
f\pi_{231} = f\pi_1\pi_2 &= \Big(f\,,\, G_{(132)}(\mathbf{x}, \mathbf{z})\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}^\omega} = \Big(f\,,\, 1 - \frac{z_1 z_2}{x_1 x_2}\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}^\omega} \\
f\widehat{\pi}_{231} = f\widehat{\pi}_1\widehat{\pi}_2 &= \Big(f\,,\, \widehat{G}_{(132)}(\mathbf{x}, \mathbf{z})\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}} = \Big(f\,,\, \frac{x_2 x_3}{z_1 z_2}\Big(1 - \frac{x_3}{y_1}\Big)\Big)^\pi \Big|_{\mathbf{z}=\mathbf{x}} .
\end{aligned}$$

2.13 Divided differences in terms of permutations

Let $D = \sum_{\zeta\in\mathfrak{S}_n} \zeta c_\zeta(\mathbf{x}_n)$ be a sum of permutations with coefficients which are rational functions in $\mathbf{x}_n$. Any function $f(\mathbf{x}_n, \mathbf{y}_n)$ which vanish in all specializations $\mathbf{x}_n^\sigma = \mathbf{y}_n$, except in $\mathbf{x}_n = \mathbf{y}_n$, can be used to determine the coefficients $c_\zeta(\mathbf{x}_n)$. Indeed, putting $g(\mathbf{x}_n, \mathbf{y}_n) = f(\mathbf{x}_n, \mathbf{y}_n)\, D$, one has $g(\mathbf{x}_n, \mathbf{y}_n) = \sum_\zeta f(\mathbf{x}_n^\zeta, \mathbf{y}_n)\, c_\zeta(\mathbf{x}_n)$, and therefore

$$g(\mathbf{x}_n, \mathbf{x}_n^\zeta) = f(\mathbf{x}_n^\zeta, \mathbf{x}_n^\zeta)\, c_\zeta(\mathbf{x}_n)\,. \tag{2.13.1}$$

The kernels Θ_n^Y, Θ_n^G have the required vanishing properties. In consequence the operators $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$ can be expressed in terms of specializations of Schubert or Grothendieck polynomials, and one obtains the following expansions (the expression of the coefficients are not unique, due to the many symmetries of Schubert and Grothendieck polynomials).

Proposition 2.13.1. *Given $\sigma \in \mathfrak{S}_n$, the divided differences $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$ are equal to the following sums of permutations :*

$$\begin{aligned}
\partial_\sigma \prod_{i<j\le n} (x_i - x_j) &= \sum_{\zeta\le\sigma} (-1)^{\ell(\zeta)} \zeta\, X_{\omega\sigma}(\mathbf{x}_n, \mathbf{x}_n^{\zeta^{-1}\omega}) && (2.13.2)\\
\pi_\sigma &= \sum_{\zeta\le\sigma} \zeta\, f_\sigma(\mathbf{x}_n^{\zeta^{-1}}, \mathbf{x}_n^{\omega}) && (2.13.3)\\
\widehat{\pi}_\sigma \prod_{i<j\le n} \left(1 - \frac{x_i}{x_j}\right) &= \sum_{\zeta\le\sigma} \zeta\, G_{(\sigma\omega)}(\mathbf{x}_n^\omega, \mathbf{x}_n^{\zeta^{-1}})\,, && (2.13.4)
\end{aligned}$$

with $f_\sigma(\mathbf{x}_n, \mathbf{y}_n) = G_{(\omega\sigma^{-1})}(\mathbf{x}_n, \mathbf{y}_n) \prod_{i<j\le n} (1 - x_j x_i^{-1})^{-1}$.

For example

$$\begin{aligned}
\partial_1\partial_2 &= \big(s_1 s_2\, (x_1{-}x_2) - s_2\, (x_1{-}x_2) - (x_1{-}x_3)\, s_1 + (x_1{-}x_3)\big) \frac{1}{\Delta(\mathbf{x}_3)}\\
\pi_1\pi_2 &= s_1 s_2 \frac{x_3^2}{(x_1{-}x_3)(x_2{-}x_3)} - s_2 \frac{x_1 x_3}{(x_1{-}x_3)(x_2{-}x_3)} - s_1 \frac{x_2^2}{(x_1{-}x_2)(x_2{-}x_3)}\\
&\qquad + \frac{x_1 x_2}{(x_1{-}x_2)(x_2{-}x_3)}\\
\widehat{\pi}_1\widehat{\pi}_2 &= (s_1 s_2 - s_2) \frac{x_3^2}{(x_1{-}x_3)(x_2{-}x_3)} + (1 - s_1) \frac{x_2 x_3}{(x_1{-}x_2)(x_2{-}x_3)}\,.
\end{aligned}$$

One can compare these expressions to those given in the preceding section. In fact, they can be obtained by mere expansion of

$$\begin{aligned}
\partial_1\partial_2 &= (1-s_1)\frac{1}{x_1-x_2}\,(1-s_2)\frac{1}{x_2-x_3} \\
\pi_1\pi_2 &= \left(s_1\frac{x_2}{x_2-x_1}+\frac{x_1}{x_1-x_2}\right)\left(s_2\frac{x_3}{x_3-x_2}+\frac{x_2}{x_2-x_3}\right) \\
\widehat{\pi}_1\widehat{\pi}_2 &= (s_1-1)\frac{1}{1-x_1x_2^{-1}}\,(s_2-1)\frac{1}{1-x_2x_3^{-1}}\,.
\end{aligned}$$

This is essentially the method followed by Kostant and Kumar [83, 84], but with this method properties of the resulting coefficients are more difficult to extract than when specializing polynomials in two sets of variables. For example we shall see later that the inverse transition matrices, from permutations to the different types of divided differences, involve the same coefficients as the transition matrices, and this fact can easily be obtained from properties of Schubert and Grothendieck polynomials.

The leading term of π_σ and $\widehat{\pi}_\sigma$, i.e. the coefficient of σ, is obtained by mere commutation. Taking a reduced decomposition $\sigma = s_i s_j s_h \cdots s_k$, then this leading term is

$$\begin{aligned}
&s_i\frac{1}{1-x_ix_{i+1}^{-1}}\,s_j\frac{1}{1-x_jx_{j+1}^{-1}}\,s_h\cdots s_k\frac{1}{1-x_kx_{k+1}^{-1}} \\
&\qquad = s_i\cdots s_k\,\left(\frac{1}{1-x_ix_{i+1}^{-1}}\right)^{s_js_h\cdots s_k}\left(\frac{1}{1-x_jx_{j+1}^{-1}}\right)^{s_h\cdots s_k}\cdots\frac{1}{1-x_kx_{k+1}^{-1}}\,.
\end{aligned}$$

In the language of root systems, this property reads as follows.

Lemma 2.13.2. *Let Φ^+, Φ^- be the positive (resp. negative) roots of the root system of type A_{n-1}. Then, in the basis of permutations, π_σ and $\widehat{\pi}_\sigma$ have leading term*

$$F(\sigma) := \prod_{\alpha\in\Phi^+\cap\sigma\Phi^-}\frac{1}{1-e^\alpha}\,.$$

This leading term intervenes in geometry, for what concerns the postulation of Schubert varieties.

Let $\lambda\in\mathbb{N}^n$ be dominant weight , v be a permutation of λ, $\sigma\in\mathfrak{S}_n$ be of minimum length such that $v=\lambda\sigma$. One defines the limit $m\to\infty$ of $K_{mv}x^{-mv}$ to be

$$(1-zx^\lambda)^{-1}\pi_\sigma(1-zx^v)\big|_{z=x^{-v}}\,.$$

Expanding π_σ in terms of permutations, one has

$$(1-zx^\lambda)^{-1}\pi_\sigma(1-zx^v) = F(\sigma)+\sum_{\zeta<\sigma}\frac{1-zx^v}{1-zx^{\lambda\zeta}}\,c_\sigma^\zeta\,,$$

with coefficients c_σ^ζ obtained in (2.13.3). The hypothesis on the pair λ, σ insures that all terms, but the first one, vanish under the specialization $z = x^{-v}$. One thus recovers in the special case of type A a property due to Peterson and Kumar in the more general context of Kac-Moody algebras.

Corollary 2.13.3. *Let $\lambda \in \mathbb{N}^n$ be dominant, $\sigma \in \mathfrak{S}_n$ be of minimum length modulo the stabilizer of λ. Then the common limit $m \to \infty$ of $x^{m\lambda}\pi_\sigma x^{-m\lambda\sigma}$ and $x^{m\lambda}\widehat{\pi}_\sigma x^{-m\lambda\sigma}$ is equal to*

$$\prod_{\alpha\in\Phi^+\cap\sigma\Phi^-} \frac{1}{1-e^\alpha}\,.$$

For example, for $\lambda = [2,1,0]$, $v = [1,0,2]$, one has $\sigma = s_1 s_2$ and the limit of $K_{m,0,2m}x^{-m,0,-2m}$ and $\widehat{K}_{m,0,2m}x^{-m,0,-2m}$ is equal to $\big((1-x_1x_3^{-1})(1-x_2x_3^{-1})\big)^{-1}$. The limit of $K_{0,0,m}x^{0,0,-m} = S_m(x_1+x_2+x_3)x_3^{-m} = S_m(x_1x_3^{-1}+x_2x_3^{-1}+1)$ is also $\big((1-x_1x_3^{-1})(1-x_2x_3^{-1})\big)^{-1}$, in accordance with the fact that σ is still equal to s_1s_2.

2.14 Schubert, Grothendieck and Demazure as commutation factors

One could obtain the expression of permutations in terms of divided differences by iterating Leibnitz formula, starting with expressions like

$$s_2s_1s_2 = \big(1+\partial_2(x_3-x_2)\big)\big(1+\partial_1(x_2-x_1)\big)\big(1+\partial_2(x_3-x_2)\big)\,.$$

Let us specially examine the commutation with ∂_ω or π_ω. For example,

$$\begin{aligned}
&\partial_1 x_2 = x_1\partial_1 - 1\\
&\qquad\partial_2\partial_1\partial_2 x_2x_3^2 = \partial_2x_3\partial_1x_2\partial_2x_3 = (x_2\partial_2-1)(x_1\partial_1-1)(x_2\partial_2-1) = \dots\\
&\qquad\qquad = x^{210}\partial_2\partial_1\partial_2 - x^{200}\partial_1\partial_2 - x^{110}\partial_2\partial_1 + x^{100}\partial_1 + (x^{100}+x^{010})\partial_2 - 1\,.
\end{aligned}$$

This case shows a disymmetry which can be cured by using Schubert polynomials instead of monomials :

$$\begin{aligned}
\partial_2\partial_1\partial_2 x_2x_3^2 = {}& Y_{210}(\mathbf{x},\mathbf{0})\partial_1\partial_2 - Y_{200}(\mathbf{x},\mathbf{0})\partial_2\partial_1\partial_2 - Y_{110}(\mathbf{x},\mathbf{0})\partial_2\partial_1\\
&+ Y_{100}(\mathbf{x},\mathbf{0})\partial_1 + Y_{010}(\mathbf{x},\mathbf{0})\partial_2 - Y_{000}(\mathbf{x},\mathbf{0})\,.
\end{aligned}$$

The following theorem states that Schubert and Grothendieck polynomials do occur in the commutation of some element with ∂_ω or π_ω. Notice that this gives a generation which does not require division.

Theorem 2.14.1. *Fixing n, with $\rho=[n{-}1,\ldots,0]$, one has*

$$\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)}X_\sigma(\mathbf{x},\mathbf{y})\,\partial_{\sigma^{-1}} \;=\; \partial_\omega X_\omega(\mathbf{y},\mathbf{x}^\omega) \tag{2.14.1}$$

$$\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma\omega)}\partial_\sigma\, X_\sigma(\mathbf{x},\mathbf{y}) \;=\; X_\omega(\mathbf{y},\mathbf{x}^\omega)\partial_\omega \tag{2.14.2}$$

$$\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)}x^\rho G_{(\sigma)}(\mathbf{x},\mathbf{y})\,\pi_{\sigma^{-1}} \;=\; \pi_\omega\, X_\omega(\mathbf{y},\mathbf{x}^\omega) \tag{2.14.3}$$

$$\sum_{\sigma\in\mathfrak{S}_n}(-1)^{\ell(\sigma)}\pi_\sigma\, G_{(\sigma)}(\mathbf{x},\mathbf{y})x^\rho \;=\; X_\omega(\mathbf{y},\mathbf{x}^\omega)\,\pi_\omega\,. \tag{2.14.4}$$

Proof. (2.14.1) and (2.14.2) are equivalent, by left-right symmetry of the Leibnitz relations. Let us prove (2.14.2). The factor $X_\omega(\mathbf{y},\mathbf{x}^\omega)$ is the reproducing kernel Θ_n^Y, and therefore (2.14.2) can be proved by checking that, for any $f(\mathbf{x})$ in the linear span of $\langle x:\ 0\le v\le\rho\rangle$, one has

$$\sum(-1)^{\ell(\sigma)}f(x)\partial_\sigma X_\sigma(\mathbf{x},\mathbf{y})=f(y)\,.$$

Introducing an extra alphabet $\mathbf{z}$, one needs a single check,

$$X_\omega(\mathbf{y},\mathbf{z})=\sum_\sigma(-1)^{\ell(\sigma)}X_\omega(\mathbf{x},\mathbf{z})\partial_\sigma X_\sigma(\mathbf{x},\mathbf{y})=\sum_\sigma(-1)^{\ell(\sigma)}X_{\omega\sigma}(\mathbf{x},\mathbf{z})X_\sigma(\mathbf{x},\mathbf{y})\,.$$

But this is the Cauchy formula

$$X_\omega(\mathbf{y},\mathbf{z})=\sum_\sigma X_{\omega\sigma}(\mathbf{x},\mathbf{z})X_{\sigma^{-1}}(\mathbf{y},\mathbf{x})\,.$$

Similarly, (2.14.4) is proved by checking the action on $G_{(\omega)}(\mathbf{x},\mathbf{z})$. Thanks to (2.9.6), one has

$$\begin{aligned}G_{(\omega)}(\mathbf{x},\mathbf{z})\sum(-1)^{\ell(\sigma)}\pi_\sigma G_{(\sigma)}(\mathbf{x},\mathbf{y})\,x^\rho &= \sum(-1)^{\ell(\sigma)}G_{(\omega\sigma)}(\mathbf{x},\mathbf{z})G_{(\sigma)}(\mathbf{x},\mathbf{y})\,x^\rho\\ &= X_\omega(\mathbf{y},\mathbf{z})\,.\end{aligned}$$

On the other hand, $X_\omega(\mathbf{y},\mathbf{x}^\omega)y^{-\rho}=\Theta_n^G$ is a reproducing kernel with respect to π_ω, and therefore, one has

$$G_{(\omega)}(\mathbf{x},\mathbf{z})X_\omega(\mathbf{y},\mathbf{x}^\omega)\,\pi_\omega=G_{(\omega)}(\mathbf{y},\mathbf{z})y^\rho\,.$$

In final, the images of $G_{(\omega)}(\mathbf{x},\mathbf{z})$ under the two sides of (2.14.4) are equal. QED

By specialisation of $\mathbf{y}$, one obtains the following commutations :

$$\sum(-1)^{\ell(\sigma)}\partial_\sigma\, X_\sigma(\mathbf{x},\mathbf{0}) \;=\; x^{01\ldots n-1}\,\partial_\omega \tag{2.14.5}$$

$$\sum(-1)^{\ell(\sigma)}x^\rho G_{(\sigma)}(\mathbf{x},\mathbf{1})\,\pi_{\sigma^{-1}} \;=\; \pi_\omega\,(1{-}x_2)\ldots(1{-}x_n)^{n-1}\,. \tag{2.14.6}$$

For example, for $n=3$, one has

$$\pi_1\pi_2\pi_1\,(1\text{-}x_2)(1\text{-}x_3)^2 = \begin{cases} \text{-}(x_1\text{-}1)^2(x_2\text{-}1)\pi_1\pi_2\pi_1 \\ -x^{210}G_{210}\pi_1\pi_2\pi_1 \end{cases}$$

$$\begin{cases} x_1(x_1\text{-}1)(x_2\text{-}1)\pi_2\pi_1 \\ x^{210}G_{110}\,\pi_2\pi_1 \end{cases} \qquad\qquad \begin{cases} x_2(x_1\text{-}1)^2\pi_1\pi_2 \\ x^{210}G_{200}\,\pi_1\pi_2 \end{cases}$$

$$\begin{cases} \text{-}x_1x_2(x_1\text{-}1)\pi_1 \\ \text{-}x^{210}G_{100}\,\pi_1 \end{cases} \qquad\qquad \begin{cases} \text{-}x_1(x_1x_2\text{-}1)\pi_2 \\ \text{-}x^{210}G_{010}\,\pi_2 \end{cases}$$

$$x_1^2x_2$$

Given n, using on products of divided differences and rational functions in $\mathbf{x}$ the *double reversal*

$$\partial_i P \partial_j \ldots \partial_k Q \to Q^\omega \partial_{n-k} \ldots P^\omega \partial_{n-i}\,,$$

one transforms (2.14.3) into

$$X_\omega(\mathbf{x},\mathbf{y})\,\partial_\omega = \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma G_{(\omega\sigma\omega)}(\mathbf{x}^\omega,\mathbf{y})\,. \tag{2.14.7}$$

For example,

$$\begin{aligned} X_{321}(\mathbf{x},\mathbf{y})\partial_{321} &= \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_3(1\text{-}y_1x_3^{-1})(1\text{-}y_1x_2^{-1})(1\text{-}y_2x_3^{-1}) + \widehat{\pi}_1\widehat{\pi}_2(1\text{-}y_1x_3^{-1})(1\text{-}y_2x_3^{-1}) \\ &+ \widehat{\pi}_2\widehat{\pi}_1(1\text{-}y_1x_3^{-1})(1\text{-}y_1x_2^{-1}) + \widehat{\pi}_1(1\text{-}y_1y_2x_3^{-1}x_2^{-1}) + \widehat{\pi}_2(1\text{-}y_1x_3^{-1}) + 1 \\ &= \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 G_{(321)}(\mathbf{x}^\omega,\mathbf{y}) + \widehat{\pi}_1\widehat{\pi}_2 G_{(312)}(\mathbf{x}^\omega,\mathbf{y}) \\ &+ \widehat{\pi}_2\widehat{\pi}_1 G_{(231)}(\mathbf{x}^\omega,\mathbf{y}) + \widehat{\pi}_1 G_{(132)}(\mathbf{x}^\omega,\mathbf{y}) + \widehat{\pi}_2 G_{(213)}(\mathbf{x}^\omega,\mathbf{y}) + 1\,. \end{aligned}$$

Notice that pushing the coefficients on the right in $X_\zeta(\mathbf{0},\mathbf{x}^\omega)\,\partial_\omega$, for any $\zeta\in\mathfrak{S}_n$, can be obtained by expanding $X_\omega(\mathbf{y},\mathbf{x}^\omega)$ in (2.14.2).

In fact, $X_\omega(\mathbf{y},\mathbf{x}^\omega)$ may be thought as the generating function of a linear basis of $\mathfrak{Pol}(\mathbf{x}_n)$ as a $\mathfrak{Sym}(\mathbf{x}_n)$-free module. Hence Formula 2.14.2 implies that for any function $g(\mathbf{x}_n)$, one has

$$g(\mathbf{x}_n^\omega)\,\partial_\omega = \sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\partial_\sigma\left(g(\mathbf{x}_n)\,\partial_{\omega\sigma}\right). \tag{2.14.8}$$

When restricting the action of $g(\mathbf{x}_n^\omega)\,\partial_\omega$ to functions having partial symmetries, one reduces summation (2.14.8), as in the next case.

Corollary 2.14.2. *Let $m \le n$, $r = n{-}m$, $k \ge 0$. For any partition $\lambda \le r^m$, denote*
$$\partial^\lambda = \left(\partial_m \dots \partial_{m+\lambda_1-1}\right) \dots \left(\partial_1 \dots \partial_{\lambda_m-1}\right).$$
Then the restriction of the action of $Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m}$ to $\mathfrak{Sym}(m,r)$ is equal to
$$Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m} = \sum_{\lambda\le r^m} (\text{-}1)^{|\mu|}\partial^\lambda\, Y_{0^{\mu_r},k-\mu_r,0^{\mu_{r-1}-\mu_r},k-\mu_{r-1},\dots,0^{\mu_1-\mu_2},k-\mu_1}(\mathbf{x},\mathbf{y})\,, \tag{2.14.9}$$
denoting by μ the partition which is conjugate to $[r{-}\lambda_m,\dots,r{-}\lambda_1]$.

Proof. The operators $X_\omega(\mathbf{x}^\omega,\mathbf{y})\partial_\omega$ and $Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m}$ have the same action on $\mathfrak{Sym}(m,r)$, up to sign. Moreover, the permutations σ which are not minimal in their coset $(\mathfrak{S}_m\times\mathfrak{S}_r)\,\sigma$ annihilate elements of $\mathfrak{Sym}(m,r)$, and therefore disappear from summation (2.14.8). QED

For example, for $n=5, m=2$, writing $\begin{array}{|c|c|c|}\hline 2&3&\cdot\cdot\\\hline 1&\cdot\cdot\\\cline{1-2}\end{array}$ for $(\partial_2\partial_3\dots)(\partial_1\dots)$, one has

$$\begin{aligned}
Y_{666}(\mathbf{x}^\omega,\mathbf{y})\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2&3\\\hline\end{array} &= \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2&3\\\hline\end{array}Y_{666} - \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2\\\cline{1-2}\end{array}Y_{6605} + \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}Y_{66004}\\
&+\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1\\\cline{1-1}\end{array}Y_{6055} - \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}Y_{60504} - \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}Y_{0555} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}Y_{60044}\\
&+\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}Y_{05504} - \begin{array}{|c|}\hline 2\\\hline\end{array}Y_{05044} + \boxed{}\,Y_{00444}\,.
\end{aligned}$$

Formula 2.14.4 :
$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\pi_\sigma\, G_{(\sigma)}(\mathbf{x},\mathbf{y}) = X_\omega(\mathbf{y},\mathbf{x}^\omega)\,\pi_\omega x^{-\rho}$$
can be rewritten
$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\pi_\sigma \left(G_{(\omega)}(\mathbf{x},\mathbf{y})\pi_{\omega\sigma}\right) = (\text{-}1)^{\ell(\omega)}x^{\rho\omega}G_{(\omega)}(\mathbf{x}^\omega,\mathbf{y})\pi_\omega x^{-\rho}\,,$$
and implies that, for any function $g(\mathbf{x}_n)$, one has
$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\omega\sigma)}\pi_\sigma \left(g(\mathbf{x}_n)\pi_{\omega\sigma}\right) = x^{\rho\omega}g(\mathbf{x}_n^\omega)\,\pi_\omega x^{-\rho} = g(\mathbf{x}_n^\omega)\,\widehat{\pi}_\omega\,. \tag{2.14.10}$$

Using, thanks to (2.6.4), that $\pi_\sigma = (\text{-}1)^{\ell(\sigma)}x^\rho\omega\,\widehat{\pi}_{\omega\sigma\omega}\,\omega x^{-\rho}$, putting $\zeta=\omega\sigma\omega$, $h=(x^\rho g)^\omega$, this last equation can be transformed into
$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\omega\sigma)}\widehat{\pi}_\sigma \left(h(\mathbf{x}_n)\widehat{\pi}_{\omega\sigma}\right) = h(\mathbf{x}_n^\omega)\,\widehat{\pi}_\omega\,. \tag{2.14.11}$$

Taking $g(\mathbf{x}_n) = x^\lambda = h(\mathbf{x}_n)$, with λ dominant, one obtains key polynomials by commutation :

Theorem 2.14.3. *Given an integer n and a partition $\lambda \in \mathbb{N}^n$, then one has*

$$\sum\nolimits_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\pi_{\omega\sigma}\left(K_\lambda\pi_\sigma\right) = \left(x^{\rho+\lambda}\right)^\omega \pi_\omega x^{-\rho} \tag{2.14.12}$$

$$\sum_{\sigma\in\mathfrak{S}_n,\,\sigma\, min} (\text{-}1)^{\ell(\sigma)}\widehat{\pi}_{\omega\sigma}\,\widehat{K}_{\lambda\sigma} = x^{\lambda\omega}\,\widehat{\pi}_\omega\,, \tag{2.14.13}$$

the sum being limited, in the second expression, to the permutations minimum in their coset modulo the stabilizer of λ.

For example, for $\lambda = [3,1,0]$, one has

$$\pi_2\pi_1\pi_2 K_{310} - \pi_1\pi_2 K_{130} - \pi_2\pi_1 K_{301} + \pi_1 K_{103} + \pi_2 K_{031} - K_{013} = x^{025}\,\pi_{321}/x^{210}\,,$$

and for $\lambda = [1,0,0]$, one has

$$\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\,\widehat{K}_{100} - \widehat{\pi}_1\widehat{\pi}_2\,\widehat{K}_{010} + \widehat{\pi}_1\widehat{K}_{001} = x^{001}\,\widehat{\pi}_{321}\,.$$

Using (1.4.8), one rewrites (2.14.13) into the following commutation of π_ω with a dominant monomial :

$$\pi_\omega\, x^\lambda = \sum_{\sigma\in\mathfrak{S}_n,\,\sigma\, min} \widehat{K}_{\lambda\sigma}(\mathbf{x}^\omega)\pi_{\omega\sigma}\,, \tag{2.14.14}$$

sum over all permutations σ which are of minimum length in their coset modulo the stabilizer of λ.

For example,

$$\begin{aligned}\pi_1\pi_2\pi_1\, x_1^2 &= x^{002}\pi_1\pi_2\pi_1 + \left(x^{020}+x^{011}\right)\pi_1\pi_2 + \left(x^{200}+x^{110}+x^{101}\right)\pi_2\\ &= \widehat{K}_2(\mathbf{x}^\omega)\pi_1\pi_2\pi_1 + \widehat{K}_{02}(\mathbf{x}^\omega)\pi_1\pi_2 + \widehat{K}_{002}(\mathbf{x}^\omega)\pi_2\,.\end{aligned}$$

Taking in (2.14.10) $g(\mathbf{x}_n) = G_\lambda(\mathbf{x},\mathbf{y})$, with λ dominant, one obtains again Grothendieck polynomials by commutation :

$$G_\lambda(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_\omega = \sum\nolimits_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\pi_{\sigma^{-1}\omega}\left(G_\lambda(\mathbf{x},\mathbf{y})\pi_\sigma\right). \tag{2.14.15}$$

For example, for $\lambda = [1,1,0]$, one has

$$\begin{aligned}\left(1-y_1x_2^{-1}\right)\left(1-y_1x_3^{-1}\right)\widehat{\pi}_{321} &= (\pi_2\pi_1\pi_2 - \pi_1\pi_2)\left(1-y_1x_1^{-1}\right)\left(1-y_1x_2^{-1}\right)\\ &\quad + (-\pi_2\pi_1+\pi_1)\left(1-y_1x_1^{-1}\right) + (\pi_2-1)\\ &= (\pi_2\pi_1\pi_2-\pi_1\pi_2)G_{110} + (-\pi_2\pi_1+\pi_1)G_{100} + (\pi_2-1)G_{000}\,.\end{aligned}$$

Thanks to the symmetry (1.4.8), one deduces from the preceding formula the expression of the product of π_ω with a dominant Grothendieck polynomial in terms of $\widehat{\pi}_\sigma$:

$$\pi_\omega\, G_{(\lambda}(\mathbf{x},\mathbf{y}) = \sum\nolimits_{\sigma\in\mathfrak{S}_n} \left(G_\lambda(\mathbf{x},\mathbf{y})\pi_\sigma\right)^\omega \widehat{\pi}_{\omega\sigma^{-1}}\,. \tag{2.14.16}$$

For example, for $n = 3$, one has

$$\pi_{321} G_{210}(\mathbf{x}, \mathbf{y}) = G_{210}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 \widehat{\pi}_2 \widehat{\pi}_1 + G_{200}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_2 \widehat{\pi}_1 + G_{110}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 \widehat{\pi}_2 \\ + G_{010}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_2 + G_{000}(\mathbf{x}^\omega, \mathbf{y})\,.$$

The expression of $\pi_\omega G_\lambda(\mathbf{x}, y)$ can be reduced when λ has repeated parts, i.e. when there exists i such that $G_\lambda(\mathbf{x}, y)\pi_i = G_\lambda(\mathbf{x}, y)$. Thus

$$\pi_{321} G_{110}(\mathbf{x}, \mathbf{y}) = G_{110}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 \widehat{\pi}_2 \widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_2 \widehat{\pi}_1 + G_{110}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 \widehat{\pi}_2 \\ + G_{000}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_2 + G_{000}(\mathbf{x}^\omega, \mathbf{y})$$

can be written, by right multiplication with π_1, as

$$\pi_{321} G_{110}(\mathbf{x}, \mathbf{y}) = G_{110}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_1 \widehat{\pi}_2 \pi_1 + G_{100}(\mathbf{x}^\omega, \mathbf{y})\, \widehat{\pi}_2 \pi_1 + G_{000}(\mathbf{x}^\omega, \mathbf{y})\, \pi_1\,.$$

2.15 Cauchy formula for key polynomials

The usual Cauchy formula is the expansion of $\prod_{i,j\le n}(1-x_iy_j)^{-1}$ in terms of Schur functions. We are going to see that "half" the Cauchy kernel $\prod_{i+j\le n+1}(1-x_iy_j)^{-1}$ expands in terms of key polynomials.

Notice first that

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\cdots(1-x_1\cdots x_ny_1\cdots y_n)} = \sum_\lambda x^\lambda y^\lambda$$

is the generating function of dominant monomials $x^\lambda y^\lambda$ in $\mathbf{x}$ and $\mathbf{y}$. Its image under the product of the two symmetrizers $\pi_\omega^{\mathbf{x}}\,\pi_\omega^{\mathbf{y}}$ transforms this equality into

$$\prod_{i,j\le n}(1-x_iy_j)^{-1} = \sum_\lambda S_\lambda(\mathbf{x})S_\lambda(\mathbf{y})\,.$$

We can use the same starting point, but symmetrize partially in $\mathbf{x}$ and $\mathbf{y}$. Let $\Xi_n := \sum_{\sigma\in\mathfrak{S}_n}\widehat{\pi}_\sigma^x\,\pi_{\sigma\omega}^y$. Filtering the set of permutations according to the position of n, one gets the following factorization (we refer to [44] for more details).

Lemma 2.15.1. *We have*

$$\Xi_n = \Xi_{n-1}\left(\sum_{i=0}^{n-1}\widehat{\pi}_{[n-1:i]}^x\,\pi_{[n-1:n-1-i]}^y\right), \tag{2.15.1}$$

where $\pi_{[n-1:i]} := \pi_{n-1}\,\pi_{n-2}\cdots\pi_{n-i}$.

For example, the element Ξ_4 factorizes as

$$\Xi_4 = \Xi_3\left(\pi_3^y\pi_2^y\pi_1^y + \widehat{\pi}_3^x\pi_3^y\pi_2^y + \widehat{\pi}_3^x\widehat{\pi}_2^x\pi_3^y + \widehat{\pi}_3^x\widehat{\pi}_2^x\widehat{\pi}_1^x\right).$$

From the definition of key polynomials, the image under Ξ_n of $\sum_\lambda x^\lambda y^\lambda$ is equal to a sum of products of $K_v(\mathbf{y})$, $\widehat{K}_u(\mathbf{x})$. More precisely

$$\sum_\lambda x^\lambda y^\lambda\,\Xi_n = \sum_v K_v(\mathbf{y})\widehat{K}_{v\omega}(\mathbf{x})\,.$$

Using no more, but repeatedly, that

$$f(1-x_ig)^{-1}\pi_i^{\mathbf{x}} = f(1-x_ig)^{-1}(1-x_{i+1}g)^{-1}$$

when f,g belong to $\mathfrak{Sym}(x_i,x_{i+1})$, one checks that the image of $(1-x_1y_1)^{-1}(1-x_1x_2y_1y_2)^{-1}\cdots$ under Ξ_n is equal to $\prod_{i+j\le n+1}(1-x_iy_j)^{-1}$ [44, Prop 3]. Hence the following kernel.

Theorem 2.15.2. *For every n one has*

$$\prod_{i+j\le n+1}(1-x_iy_j)^{-1} = \sum_{v\in\mathbb{N}^n} K_v(\mathbf{y})\widehat{K}_{v\omega}(\mathbf{x})\,.$$

For example, for $n=2$, one has

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)}(\pi_1^y+\widehat{\pi}_1^x) = \frac{1}{(1-x_1y_1)(1-x_1y_2)}+\frac{y_1x_2}{(1-x_1y_1)(1-x_2y_1)}$$
$$= \frac{1}{(1-x_1y_1)(1-x_1y_2)(1-x_2y_1)} = 1+\sum_{i\le j} K_{ij}(\mathbf{y})x^{ji} + \sum_{j>i} y^{ji}\widehat{K}_{ij}(\mathbf{x})\,,$$

the key polynomials $K_{ij}(\mathbf{y})$ being Schur functions in y_1, y_2, while $\widehat{K}_{ij}(\mathbf{x}) = K_{ij}(\mathbf{x}) - x^{ji}$, when $i \le j$.

2.16 π and $\widehat{\pi}$-reproducing kernels

We have shown in (2.9.2) a reproducing property of the operator $f \to (f\,,\,\Theta_n^G)^\pi$. Let us rewrite it without using the scalar product $(\;,\;)^\pi$. Let

$$ {}^{\widehat{\pi}}\Theta_n^G \;=\; \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_{\sigma^{-1}}\, G_{(\sigma)}(\mathbf{z},\mathbf{x}) \tag{2.16.1}$$

$$ {}^{\pi}\Theta_n^G \;=\; \sum_{\sigma\in\mathfrak{S}_n} \pi_{\sigma^{-1}}\, \widehat{G}_{(\sigma)}(\mathbf{z},\mathbf{x}^\omega) \tag{2.16.2}$$

For example, for $n=3$, one has

$$ {}^{\widehat{\pi}}\Theta_3^G = 1 + \widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right) + \widehat{\pi}_2\left(1-\frac{x_1x_2}{z_1z_2}\right) + \widehat{\pi}_1\widehat{\pi}_2\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_2}{z_1}\right) + \widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_1}{z_2}\right) + \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_2}{z_1}\right)\left(1-\frac{x_1}{z_2}\right). $$

With the alphabets $\mathbf{z},\mathbf{x}^\omega,\mathbf{y}$ instead of $\mathbf{x},\mathbf{y},\mathbf{z}$, Formula 2.9.4 reads

$$ \Theta_n^G = \sum_{v\le\rho} G_v(\mathbf{y},\mathbf{x}^\omega)\widehat{G}_{\rho-v}(\mathbf{z},\mathbf{x}^\omega)\,. $$

Indexing by permutations, using the symmetry $G_{(\sigma)}(\mathbf{x},y^\vee)\clubsuit = G_{(\sigma^{-1})}(\mathbf{y},\mathbf{x}^\omega)$ given in (2.2.4), and the conjugation $\clubsuit\pi_i\clubsuit = \pi_{n-i}$, one rewrites this last formula as

$$ \Theta_n^G(\mathbf{z},\mathbf{y}) \;=\; \sum_{v\le\rho} \Theta_n^G(\mathbf{x},\mathbf{y})\pi_{\sigma^{-1}}\, \widehat{G}_{(\sigma)}(\mathbf{z},\mathbf{x}^\omega) \tag{2.16.3}$$

$$ \;=\; \Theta_n^G(\mathbf{x},\mathbf{y})\,{}^{\pi}\Theta_n^G\,, \tag{2.16.4}$$

In other words, for any $v\,:\,[0,\ldots,0] \le v \le [0,\ldots,n{-}1] = \rho^\omega$, one has the reproducing property $x^v\,{}^{\pi}\Theta_n^G = z^v$. Equivalently, (2.16.4) rewrites as

$$ \widehat{G}_\rho(\mathbf{x},\mathbf{y})\,{}^{\pi}\Theta_n^G = \widehat{G}_\rho(\mathbf{z},\mathbf{y})\,. \tag{2.16.5}$$

A similar computation shows that for $0\le v\le\rho$, one has $x^{-v}\,{}^{\widehat{\pi}}\Theta_n^G = z^{-v}$, or, equivalently,

$$ G_\rho(\mathbf{x},\mathbf{y})\,{}^{\widehat{\pi}}\Theta_n^G = G_\rho(\mathbf{z},\mathbf{y})\,. \tag{2.16.6}$$

These two sets of monomials are bases of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free $\mathfrak{Sym}(\mathbf{x}_n)$-module, and therefore the reproducing property extends to the full space, after identifying $\mathfrak{Sym}(\mathbf{x}_n)$ and $\mathfrak{Sym}(\mathbf{z}_n)$. In final, one has

Proposition 2.16.1. *For any $f\in\mathfrak{Pol}(\mathbf{x}_n)$ one has*

$$ f(\mathbf{x}_n)^{\pi}\Theta_n^G \equiv f(\mathbf{z}_n) \equiv f(\mathbf{x}_n)^{\widehat{\pi}}\Theta_n^G\,, \tag{2.16.7}$$

modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{z}_n)$.

Notice that the two operators ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ are not equal. Thus

$$\begin{aligned} x_2{}^{\widehat{\pi}}\Theta_2^G &= x_2\left(1+\widehat{\pi}_1(1-\frac{x_1}{z_1})\right) = x_1x_2z_1^{-1}\,, \\ x_2{}^{\pi}\Theta_2^G &= x_2\left(\frac{z_2}{x_2}+\pi_1(1-\frac{z_2}{x_2})\right) = z_2\,, \end{aligned}$$

evaluating modulo $\mathfrak{Sym}(\mathbf{x}_2)=\mathfrak{Sym}(\mathbf{z}_2)$ being necessary to insure equality.

Notice also that the two formulas $x^v\,{}^{\pi}\Theta_n^G = z^v$ for $0\le v\le \rho^\omega$ and $x^{-v}\,{}^{\widehat{\pi}}\Theta_n^G = z^{-v}$ for $0\le v\le\rho$ show that both operators ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ take values in $\mathfrak{Sym}(\mathbf{x}_n)\otimes \mathfrak{Pol}(\mathbf{z}_n)$.

In the case $n=2$, one can rewrite ${}^{\widehat{\pi}}\Theta_2^G = \pi_1-\partial_1\frac{x_1x_2}{z_1}$, ${}^{\pi}\Theta_2^G=\pi_1-\partial_1 z_2$. This prompts us to define, for any i,

$$\widehat{\theta}_i = \pi_i - \partial_i\frac{x_ix_{i+1}}{z_i} \quad \& \quad \theta_i=\pi_i-\partial_i z_{i+1}\,.$$

These operators do not satisfy the braid relations if the parameters z_i are not all equal. Let us show however, that one can use them to factorize ${}^{\widehat{\pi}}\Theta_n^G$ and ${}^{\pi}\Theta_n^G$.

The action of $\theta_2\theta_1\theta_2$ on $\widehat{G}_{210}(\mathbf{x},\mathbf{y})$ is such that each step is of the type $(1-x_{i+1}y_j^{-1})f\theta_i = (1-z_{i+1}y_j^{-1})f$, with f symmetrical in x_i,x_{i+1}. Therefore one has $\widehat{G}_{210}(\mathbf{x},\mathbf{y})\theta_2\theta_1\theta_2 = \widehat{G}_{210}(\mathbf{z},\mathbf{y})$, and, more generally,

$$\widehat{G}_\rho(\mathbf{x},\mathbf{y})\,(\theta_{n-1})(\theta_{n-2}\theta_{n-1})\dots(\theta_1\dots\theta_{n-1}) = \widehat{G}_\rho(\mathbf{z},\mathbf{y})\,.$$

One checks similarly that

$$G_\rho(\mathbf{x},\mathbf{y})\,(\widehat{\theta}_1)(\widehat{\theta}_2\widehat{\theta}_1)\dots(\widehat{\theta}_{n-1}\dots\widehat{\theta}_1) = G_\rho(\mathbf{z},\mathbf{y})\,.$$

Hence, these two products of operators have the same action on $\mathfrak{Pol}(\mathbf{x}_n)$ than ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ respectively, and one has the following proposition.

Proposition 2.16.2. *Given n, one has the factorizations*

$$\begin{aligned} {}^{\pi}\Theta_n^G &= (\theta_{n-1})(\theta_{n-2}\theta_{n-1})\dots(\theta_1\dots\theta_{n-1}) && (2.16.8) \\ {}^{\widehat{\pi}}\Theta_n^G &= (\widehat{\theta}_1)(\widehat{\theta}_2\widehat{\theta}_1)\dots(\widehat{\theta}_{n-1}\dots\widehat{\theta}_1)\,. && (2.16.9) \end{aligned}$$

2.17 Decompositions in the affine Hecke algebra

The elementary constituents of all the operators that we have used so far in type A are divided differences, together with "multiplication by elements of $\mathfrak{Rat}(\mathbf{x})$", the ring of rational functions in $\mathbf{x}$. One could as well take permutations and elements of $\mathfrak{Rat}(\mathbf{x})$. Indeed, the algebras generated by $\{\partial_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{s_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{\pi_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{T_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$ all coincide. With M.P. Schützenberger, we call it *algebra of divided differences*, Bourbaki prefers *produit croisé de l'algèbre du groupe symétrique et de* $\mathfrak{Rat}(\mathbf{x})$, Kostant and Kumar use the expression *smash product*, and finally, the terminology *affine Hecke algebra for type* A puts the emphasis on the elements T_i.

Every element of this algebra is uniquely written as a sum $\sum_{\sigma\in\mathfrak{S}_n} \partial_\sigma R^\partial_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \sigma R^s_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma R^\pi_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma R^{\widehat{\pi}}_\sigma$, or $\sum_{\sigma\in\mathfrak{S}_n} T_\sigma R^T_\sigma$ respectively, choosing to put the coefficients on the right. Symmetry properties like (1.4.8) allow to pass from the right module structure to the left one.

We show in (3.3.1), as a consequence of the multivariate Newton interpolation formula, how to pass from divided differences to permutations using Schubert polynomials, or conversely in (3.3.3). In fact, this type of expansions uses only the obvious fact that the kernel $\Theta^Y(\mathbf{x},\mathbf{y})$ vanish for all specializations $\mathbf{y} = \mathbf{x}^\zeta$, except when ζ is the identity. Instead of $\Theta^Y(\mathbf{x},\mathbf{y})$, one could as well use as a kernel $Y_\rho(\mathbf{x},\mathbf{y})$, $G_\rho(\mathbf{x},\mathbf{y})$, or $\widehat{G}_\rho(\mathbf{x},\mathbf{y})$, the non vanishing being obtained for the identity or for the maximal permutation according to the choice of the kernel.

More generally, given any $f(\mathbf{x}_n) \in \mathfrak{Pol}(\mathbf{x}_n)$, let $\Theta^f(\mathbf{x},\mathbf{y}) = f(\mathbf{x}_n)\Theta^Y_n$. Then for any element $\nabla = \sum_\sigma \sigma R^s_\sigma$, one has $\Theta^f(\mathbf{x},\mathbf{y})\nabla = \sum_\sigma \Theta^f(\mathbf{x}^\sigma,\mathbf{y}) R^s_\sigma$, and therefore the coefficients are such that

$$R^s_\sigma = \Theta^f(\mathbf{x},\mathbf{y})\sigma^{-1}\nabla\Big|_{\mathbf{y}=\mathbf{x}} \frac{1}{f(\mathbf{x}_n)\Delta(\mathbf{x}_n)} \,.$$

Similar expressions hold for the other coefficients $R^\partial_\sigma, R^\pi_\sigma, R^{\widehat{\pi}}_\sigma$.

As a matter of fact, some of the formulas in preceding sections may be interpreted as identities in the affine Hecke algebra. For example, taking $\mathbf{z} = \mathbf{x}^\zeta$ in (2.16.7), one obtains the expansion of any permutation in the basis $\{\pi_\sigma\}$ or $\{\widehat{\pi}_\sigma\}$.

Let us summarize the main expansions, that will be needed later, of any element ∇ of the affine Hecke algebra.

$$\nabla = \sum_{\sigma\in\mathfrak{S}_n} \partial_\sigma \left(X_{\sigma^{-1}}(\mathbf{x},\mathbf{y})\nabla\Big|_{\mathbf{y}=\mathbf{x}} \right) \tag{2.17.1}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \sigma \left(\prod_{1\le i<\le j\le n} (x_i - y_j)^{\sigma^{-1}} \nabla\Big|_{\mathbf{y}=\mathbf{x}} \right) \tag{2.17.2}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma \left(\widehat{G}_{(\sigma^{-1})}(\mathbf{x},\mathbf{y}^\omega)\nabla\Big|_{\mathbf{y}=\mathbf{x}} \right) \tag{2.17.3}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma \left(G_{(\sigma^{-1})}(\mathbf{x},\mathbf{y})\nabla\Big|_{\mathbf{y}=\mathbf{x}} \right) . \tag{2.17.4}$$

For example,

$$\begin{aligned} s_1 s_2 = \sum_{\sigma\in\mathfrak{S}_3} \widehat{\pi}_{\sigma^{-1}}\, G_\sigma(\mathbf{x}^{s_1 s_2},\mathbf{y})\Big|_{\mathbf{y}=\mathbf{x}} &= 1 + \widehat{\pi}_2 \frac{(x_3 x_1 - y_1 y_2)}{x_3 x_1} + \widehat{\pi}_1 \left(1 - \frac{y_1}{x_3}\right) \\ &+ \widehat{\pi}_2\widehat{\pi}_1 \left(1 - \frac{y_1}{x_1}\right)\left(1 - \frac{y_1}{x_3}\right) + \widehat{\pi}_1\widehat{\pi}_2 \left(1 - \frac{y_1}{x_3}\right)\left(1 - \frac{y_2}{x_3}\right) \\ &+ \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 \left(1 - \frac{y_1}{x_3}\right)\left(1 - \frac{y_2}{x_3}\right)\left(1 - \frac{y_1}{x_1}\right)\Big|_{\mathbf{y}=\mathbf{x}} \\ &= 1 + \widehat{\pi}_2 \left(1 - \frac{x_2}{x_3}\right) + \widehat{\pi}_1 \left(1 - \frac{x_1}{x_3}\right) + \widehat{\pi}_1\widehat{\pi}_2 \left(1 - \frac{x_1}{x_3}\right)\left(1 - \frac{x_2}{x_3}\right) . \end{aligned}$$

Specific cases of the above expansions appear in the literature. Kostant and Kumar [83] consider the transition matrices $\{\sigma\} \leftrightarrow \{\partial_\sigma\}$. Berline and Vergne [7], Arabia [1], Kostant and Kumar [84] consider the transition matrices $\{\sigma\} \leftrightarrow \{\pi_\sigma\}$. Kumar shows in [89] how to relate the entries of these last matrices (which are specializations of Grothendieck polynomials) to the singularities of Schubert varieties.

Notice that the above expansions are obtained by specializing polynomials in $\mathbf{x}, \mathbf{y}$. These polynomials are not unique. For example, instead of (2.17.3), one could use as well

$$\nabla = \sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma \left(G_{(\omega)}(\mathbf{x},\mathbf{y})\widehat{\pi}_{\omega\sigma^{-1}}\nabla\Big|_{\mathbf{y}=\mathbf{x}} \right) .$$

Let us mention in final the interest of expressing the basis of the usual Hecke algebra (with normalization $(T_i - t_1)(T_i - t_2) = 0$) in terms of the basis $\{\widehat{\pi}_\sigma\}$. For example, for $n = 3$, one has

$$T_1 = \widehat{\pi}_1 \frac{(x_2 t_1 + x_1 t_2)}{x_2} + t_1 \quad \& \quad T_2 = \widehat{\pi}_2 \frac{(x_3 t_1 + x_2 t_2)}{x_3} + t_1$$

$$T_1T_2 = \widehat{\pi}_1\widehat{\pi}_2\tfrac{(x_3t_1+x_2t_2)(x_1t_2+x_3t_1)}{x_3{}^2} + \widehat{\pi}_2\tfrac{(x_3t_1+x_2t_2)t_1}{x_3} + \widehat{\pi}_1\tfrac{\left(-x_1t_2{}^2+x_3t_1{}^2\right)}{x_3} + t_1{}^2$$
$$T_2T_1 = \widehat{\pi}_2\widehat{\pi}_1\tfrac{(x_2t_1+x_1t_2)(x_1t_2+x_3t_1)}{x_3x_2} + \widehat{\pi}_2\tfrac{\left(-x_1t_2{}^2+x_3t_1{}^2\right)}{x_3} + \widehat{\pi}_1\tfrac{t_1(x_2t_1+x_1t_2)}{x_2} + t_1{}^2$$

$$\begin{aligned}T_1T_2T_1 = \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 & \frac{(x_3t_1+x_2t_2)\,(x_1t_2+x_3t_1)\,(x_2t_1+x_1t_2)}{x_2x_3{}^2} \\ & + \widehat{\pi}_1\widehat{\pi}_2\frac{(x_3t_1+x_2t_2)\,t_1\,(x_1t_2+x_3t_1)}{x_3{}^2} + \widehat{\pi}_2\widehat{\pi}_1\frac{(x_2t_1+x_1t_2)\,(x_1t_2+x_3t_1)\,t_1}{x_3x_2} \\ & + \widehat{\pi}_2\frac{(-x_1t_2{}^2+x_3t_1{}^2)\,t_1}{x_3} + \widehat{\pi}_1\frac{(-x_1t_2{}^2+x_3t_1{}^2)\,t_1}{x_3} + t_1^3\,.\end{aligned}$$

and these expansions specialize to the expression of permutations in the basis $\{\widehat{\pi}_\sigma\}$ for $t_1 = 1$, $t_2 = -1$, the coefficients being then specializations of Grothendieck polynomials.

Chapter 3

Properties of Schubert polynomials

3.1 Schubert by vanishing properties

To have linear bases, we could have considered only the special case where $\mathbf{y} = \mathbf{0}$ in the case of Schubert polynomials, and $\mathbf{y} = \mathbf{1}$ in the case of Grothendieck polynomials. But doing so, we would lose many interesting specialization properties that these polynomials possess, and that can be used to characterize them easily, as we are going to see in this section for Schubert polynomials.

Given a permutation σ (considered as an element of $\mathfrak{S}_\infty$, whose code is v), let $\langle v\rangle = \mathbf{y}^\sigma = [y_{\sigma_1}, \ldots, y_{\sigma_n}]$.

We call $\langle v\rangle$ a *spectral vector*[1] and write $f\big(\langle v\rangle\big)$ for the specialisation of $f \in \mathfrak{Pol}(\mathbf{x}_n, \mathbf{y})$ in $x_1 = y_{\sigma_1}, \ldots, x_n = y_{\sigma_n}$.

Theorem 3.1.1. *Given $v \in \mathbb{N}^n$, and σ such that $v = \text{\c{}}(\sigma)$, then the Schubert polynomial $Y_v(\mathbf{x}, \mathbf{y})$ is the only polynomial in the space of degree $\leq |v|$ in $\mathbf{x}_n$ such that*

$$Y_v(\langle u\rangle, \mathbf{y}) = 0\,,\ u \neq v,\ |u| \leq |v| \qquad (3.1.1)$$

$$Y_v\big(\langle v\rangle, \mathbf{y}\big) = \Cap(v) := \prod_{i<j,\,\sigma_i>\sigma_j} (y_{\sigma_i} - y_{\sigma_j}) \qquad (3.1.2)$$

The specialization $\Cap(v)$ is called the *inversion polynomial* of σ. We shall also denote it $\Cap(\sigma)$ when no ambiguity is to be feared.

Proof. First, it is straightforward that the dominant Schubert polynomials, which are products of linear factors, satisfy both (3.1.1, 3.1.2).

[1] We use the same term as for the Yang-Baxter equation, because these two uses are related in several ways. Notice that $\mathbf{x}^{s_1} = [x_2, x_1, x_3, \ldots]$, $\mathbf{x}^{s_1 s_2} = [x_2, x_3, x_1, \ldots] = [x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}]$, with $\sigma = s_1 s_2 = [2, 3, 1]$. We are acting on the components of the vector $[x_1, x_2, \ldots]$. On the other hand, the action on the right on exponents of monomials: $x_1^\sigma = x^{[100]s_1 s_2} = x^{001} = x_3$, $x_2^\sigma = x^{[010]s_1 s_2} = x^{100} = x_1$, $x_3^\sigma = x^{[001]s_1 s_2} = x^{010} = x_2$ involves the inverse permutation $[3, 1, 2]$.

Therefore, we have just to check the behaviour of these conditions with respect to divided differences.

Lemma 3.1.2. *Let $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, i be such that $v_i > v_{i+1}$. Suppose that Y_v satisfies (3.1.1, 3.1.2). Then $Y_v\,\partial_i$ also satisfies (3.1.1, 3.1.2) for the index $v' = [v_1, \ldots, v_{i-1}, v_{i+1}, v_i{-}1, v_{i+2}, \ldots, v_n]$, which is the code of σs_i.*

Proof. Write $Y_v = f(x_i, x_{i+1}) - x_{i+1}g(x_i, x_{i+1})$, with $f, g \in \mathfrak{Sym}(x_i, x_{i+1})$. Let us check that g is the polynomial defined by (3.1.1, 3.1.2) for the index index v'.

If Y_v vanishes in $[x_i, x_{i+1}] = [a, b]$ and $[x_i, x_{i+1}] = [b, a]$, with $a \neq b$, then g inherits these vanishings: $g(a, b) = g(b, a) = 0$. On the other hand, in the points $\langle v\rangle$ and $\langle v'\rangle$, one has

$$\begin{aligned} Y_v\big(\langle v\rangle, \mathbf{y}\big) = ⋒(v) &= f(y_{\sigma_i}, y_{\sigma_{i+1}}) - y_{\sigma_{i+1}} g(y_{\sigma_i}, y_{\sigma_{i+1}}) \\ Y_v\big(\langle v'\rangle, \mathbf{y}\big) = 0 &= f(y_{\sigma_i}, y_{\sigma_{i+1}}) - y_{\sigma_i} g(y_{\sigma_i}, y_{\sigma_{i+1}})\,. \end{aligned}$$

Therefore $g(y_{\sigma_{i+1}}, y_{\sigma_i}) = ⋒(v)\big(y_{\sigma_i} - y_{\sigma_{i+1}}\big)^{-1}$ is the inversion polynomial of σs_i, and g satisfies the conditions (3.1.1, 3.1.2). This proves the lemma. But $Y_v\partial_i = -x_{i+1}g\partial_i = g$, and therefore g is the Schubert polynomial of index v'. This proves the theorem. QED

For example,

$$Y_{2010}(\mathbf{x}, \mathbf{y}) = (x_1 - y_1)(x_1 - y_2)(x_2 + x_3 - y_1 - y_2)$$

is characterized, among all polynomials in x_1, x_2, x_3, x_4 of degree no more than 3, by the vanishing in all $\mathbf{x}_4 = y^{\zeta}$, $\zeta \in \mathfrak{S}_4$, $\ell(\zeta) \le 3$, $\zeta \neq \sigma = [3, 1, 4, 2]$, and by the normalization

$$Y_{2010}(\mathbf{y}^{\sigma}, \mathbf{y}) = (y_3 - y_1)(y_3 - y_2)(y_1 + y_4 - y_1 - y_2) = ⋒([2, 0, 1, 0])\,.$$

A consequence of the theorem is the following vanishing property (which evident only for dominant polynomials), corresponding to $\langle \mathbf{0}\rangle = [y_1, y_2, \ldots, y_n]$.

Corollary 3.1.3. *For any $v \neq [0, \ldots, 0]$, one has $Y_v(\mathbf{y}, \mathbf{y}) = 0$.*

3.2 Multivariate interpolation

We have already used several times the vanishing in $\mathbf{x} = \mathbf{y} = \langle \mathbf{0}\rangle$, this property is better understood as a special case of (3.1.1).

Notice that the polynomials $Y_k = (x_1 - y_1)\cdots(x_1 - y_k)$ are the interpolation polynomials that Newton used in his famous interpolation formula. The next theorem states that the Schubert polynomials are precisely the universal coefficients in the generalization of Newton's formula to several variables (this theorem could be deduced from the Cauchy formula that we gave in Th. 2.10.2.

Given $v \in \mathbb{N}^n$, let ∂^v be any product of divided differences[2] such that $Y_v \partial^v = Y_{0\ldots0}$. It is easy to see that for any $u \neq v$, then $Y_u \partial^v$ is either 0 or a Schubert polynomial of index $\neq [0,\ldots,0]$.

Theorem 3.2.1 (MultivariateNewton)**.** *For any $f \in \mathfrak{Pol}(\mathbf{x},\mathbf{y})$, one has the expansion*

$$f(\mathbf{x}) = \sum_{v\in\mathbb{N}^n} f(\mathbf{x})\partial^v\big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x},\mathbf{y})\,. \tag{3.2.1}$$

Proof. Test the statement on the Schubert basis. In that case, $f(\mathbf{x})\partial^v$ is either 0 or a Schubert polynomial, whose specialization in $\mathbf{x}=\mathbf{y}$ (i.e. in the point $\langle 0\ldots 0\rangle$) is $\neq 0$ (and equal to 1) iff $f(\mathbf{x}) = Y_v$. QED

The preceding theorem gives the expansion of any polynomial in the Schubert basis, the coefficients being all the non-zero images under divided differences. In particular, one can take the key polynomials, or the Grothendieck polynomials[3]. For example, the polynomial K_{021} has only 6 non-zero images under divided differences, the images under $1, \partial_2, \partial_3, \partial_2\partial_3, \partial_3\partial_2, \partial_3\partial_2\partial_2$. Writing the coefficients in y as key polynomials, one has

$$\begin{aligned} K_{021}(\mathbf{x}) = K_0(\mathbf{y})Y_{0,2,1} + K_{0,1}(\mathbf{y})Y_{0,2} + K_{0,1,1}(\mathbf{y})Y_{0,1} \\ + K_{0,0,1}(\mathbf{y})Y_{0,1,1} + K_{0,2}(\mathbf{y})Y_{0,0,1} + K_{0,2,1}(\mathbf{y})Y_0\,. \end{aligned}$$

In the case where f is a polynomial in x_1 (and $\mathbf{y}$) only, the only non-zero divided differences are $f\partial_1$, $f\partial_1\partial_2$, $f\partial_1\partial_2\partial_3, \ldots$, and the theorem is the original theorem of Newton, apart from notations :

$$\begin{aligned} f(x_1) &= f(y_1) + f\partial_1\, Y_1 + f\partial_1\partial_2\, Y_2 + f\partial_1\partial_2\partial_3\, Y_3 + \cdots \\ &= f(y_1) + f\partial_1(x_1-y_1) + f\partial_1\partial_2(x_1-y_1)(x_1-y_2) + \cdots \end{aligned} \tag{3.2.2}$$

The interpolation of functions $f(x_1,x_2)$ of two variables reads

$$\begin{aligned} f(x_1,x_2) = f(y_1,y_2)Y_{00} + f\partial_2 Y_{01} + f\partial_1 Y_{10} + f\partial_2\partial_3 Y_{02} + f\partial_2\partial_1 Y_{11} \\ + f\partial_1\partial_2 Y_{20} + f\partial_2\partial_3\partial_4 Y_{03} + f\partial_2\partial_3\partial_1 Y_{12} + f\partial_2\partial_1\partial_2 Y_{21} + f\partial_1\partial_2\partial_3 Y_{30} + \ldots \end{aligned}$$

In the case that $f(x_1,x_2)$ is symmetrical, then $f\partial_1 = 0$, and only the terms $Y_{i,j}$, $i \le j$, which are those symmetrical in x_1, x_2, survive in the preceding formula:

$$\begin{aligned} f(x_1,x_2) = f(y_1,y_2)Y_{00} + f\partial_2 Y_{01} + f\partial_2\partial_3 Y_{02} + f\partial_2\partial_1 Y_{11} + f\partial_2\partial_3\partial_4 Y_{03} \\ + f\partial_2\partial_3\partial_1 Y_{12} + f\partial_2\partial_3\partial_4\partial_5 Y_{04} + f\partial_2\partial_3\partial_4\partial_1 Y_{13} + f\partial_2\partial_3\partial_1\partial_2 Y_{22} + \ldots \end{aligned}$$

[2] Take any reduced decomposition $s_i s_j \cdots s_k$ of σ, with σ of code v. Then $\partial_k \cdots \partial_j \partial_i$ is such product.

[3] after some change of variables, like $x_i \to 1/x_i$ or $x_i \to 1/(1-x_i)$, to transform Grothendieck polynomials into polynomials in $\mathbf{x}$, and not in $x_1^{-1}, x_2^{-1}, \ldots$.

Interpolation methods can also be used in the theory of symmetric polynomials. If $f(\mathbf{x}_n)$ belongs to $\mathfrak{Sym}(\mathbf{x}_n)$, then only anti-dominant indices occur in the RHS of (3.2.1). In other words, Newton's interpolation give an expansion of symmetric polynomials in terms of Graßmannian Schubert polynomials.

For example, the Schur function $s_{32}(\mathbf{x}_3)$, which is equal to $Y_{023}(\mathbf{x}, \mathbf{0})$, has the following expansion in terms of Graßmannian Schubert polynomials (writing $Y_u Y_v$ for $Y_u(\mathbf{y}, \mathbf{0}) Y_v(\mathbf{x}, \mathbf{y})$) :

$$\begin{aligned} s_{32}(\mathbf{x}_3) = Y_{023}(\mathbf{x}, \mathbf{0}) &= Y_{000}Y_{023} - Y_{00001}Y_{022} - Y_{001}Y_{013} + Y_{00101}Y_{012} \\ &+ Y_{011}Y_{003} - Y_{01101}Y_{002} - Y_{00201}Y_{011} + Y_{01201}Y_{001} - Y_{02201}Y_{000} \,. \end{aligned}$$

Such expansions have been considered by Chen and Louck [20] and by Olshanski and Okounkov [162], in the case where $\mathbf{y} = \{0, 1, 2, \ldots\}$ or $\mathbf{y} = \{q^0, q^1, q^2, \ldots\}$ (in which case the polynomials are called *factorial Schur functions*).

Newton interpolation is compatible with symmetry by blocks. Indeed, let $f(\mathbf{x}) \in \mathfrak{Sym}(m, n, p, \ldots)$, i.e. $f(\mathbf{x})$ is a function which is symmetrical in $x_1, \ldots, x_m$, symmetrical in $x_{m+1}, \ldots, x_{m+n}$, &c. Then $f(\mathbf{x}) = \sum c_v Y_v(\mathbf{x}, \mathbf{y})$, the set of indices v being restricted to those such that $v_1 \le \cdots \le v_m$, $v_{m+1} \le \cdots \le v_{m+n}$, &c., i.e. to those v for which $Y_v(\mathbf{x}, \mathbf{y})$ belongs to $\mathfrak{Sym}(m, n, p, \ldots)$. Otherwise, there would exist a divided difference ∂_i annihilating $f(\mathbf{x})$ and not $\sum c_v Y_v$. For example, if $f \in \mathfrak{Sym}(3, 4, 2)$, then the interpolation

$$f(\mathbf{x}) = \sum f(\mathbf{x}) \partial^v \big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x}, \mathbf{y})$$

involves only the $v \in \mathbb{N}^9$ such that $v_1 \le v_2 \le v_3$, $v_4 \le v_5 \le v_6 \le v_7$, $v_8 \le v_9$.

3.3 Permutations versus divided differences

Fashion has changed since Newton, and it may seem of little interest to interpolate functions by polynomials. In fact, classical interpolation theory may be thought as a way of producing algebraic identities involving polynomials or rational functions in several variables. In this interpretation, it still begs the right to exist, even to expand. Moreover, one can disguise interpolation under a more sophisticated terminology.

For example, consider the problem of expressing a permutation $\sigma \in \mathfrak{S}_n$, considered as an operator on $\mathfrak{Pol}(\mathbf{x}_n)$, in terms of divided differences. The image of (3.2.1) under σ is

$$f(\mathbf{x}^\sigma) = \sum_{v \in \mathbb{N}^n} f(\mathbf{x}) \partial^v \big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x}^\sigma, \mathbf{y}) \,.$$

Putting $\mathbf{y} = \mathbf{x}$ gives the following property obtained by Kostant and Kumar [83] in the more general context of Kac-Moody groups (they call the algebra of divided differences the *nil Hecke ring*).

Proposition 3.3.1. *Any permutation $\sigma \in \mathfrak{S}_n$ expands, in terms of divided differences, as*

$$\sigma = \sum_{v \leq \rho} \partial^v \, Y_v(\mathbf{x}^\sigma, \mathbf{x})\,. \tag{3.3.1}$$

For example,

$$s_2 s_1 = 1 + \partial_2(x_3 - x_1) + \partial_1(x_2 - x_1) + \partial_2\partial_1(x_3 - x_1)(x_2 - x_1)\,,$$

$$s_2 s_1 s_3 = 1 + \partial_1(x_2 - x_1) + \partial_2(x_4 - x_1) + \partial_3(x_4 - x_3) + \partial_2\partial_3(x_4 - x_3)(x_4 - x_1)$$
$$+\partial_1\partial_3(x_2-x_1)(x_4-x_3)+\partial_2\partial_1(x_2-x_1)(x_4-x_1)+\partial_2\partial_1\partial_3(x_2-x_1)(x_4-x_1)(x_4-x_3)\,.$$

Conversely, one may express divided differences in terms of permutations, and more generally, any linear combination with rational coefficients in $\mathbf{x}$.

Lemma 3.3.2. *Let n be an integer, $\Theta^Y(\mathbf{x},\mathbf{y}) := \prod_{1\leq i<j\leq n}(y_i - x_j)$ as before, and $\hbar = \sum_{\sigma\in\mathfrak{S}_n} \sigma\, h_\sigma$ be a sum with rational coefficients h_σ in $\mathbf{x}$. Then*

$$\Theta^Y(\mathbf{x},\mathbf{y})\,\hbar\big|_{\mathbf{y}=\mathbf{x}^\sigma} = (-1)^{\ell(\sigma)}\, h_\sigma \prod_{1\leq i<j\leq n} (x_i - x_j)\ . \tag{3.3.2}$$

Proof. We have already used that $\Theta^Y(\mathbf{x},\mathbf{x}^\zeta)$ vanishes for all permutations ζ different from the identity. Therefore $\Theta^Y(\mathbf{y}^\sigma,\mathbf{y}^\zeta)$ vanishes except for $\zeta=\sigma$, and the sum $\Theta^Y(\mathbf{x},\mathbf{y})\,\hbar = \sum \Theta^Y(\mathbf{x}^\sigma,\mathbf{y})h_\sigma$ reduces to a single term when specializing $\mathbf{y}$ to a permutation of $\mathbf{x}$. QED

We can take now $\hbar = \partial_\tau$. Then

$$\Theta^Y(\mathbf{x},\mathbf{y})\partial_\tau = X_\omega(\mathbf{x}^\omega,\mathbf{y})\partial_\tau = X_\omega(\mathbf{x},\mathbf{y})\,\omega\partial_\tau\omega\,\omega$$
$$= (-1)^{\ell(\tau)} X_\omega(\mathbf{x},\mathbf{y})\,\partial_{\omega\tau^{-1}\omega}\omega = (-1)^{\ell(\tau)} X_{\tau^{-1}\omega}(\mathbf{x}^\omega,\mathbf{y})\,.$$

In final, one has the following expression of ∂_τ [108, Prop. 10.2.5] :

Proposition 3.3.3. *Let $\tau\in\mathfrak{S}_n$. Let $\partial_\tau = \sum \zeta\, c^\tau_\zeta$ be the expression of ∂_τ in terms of permutations. Then*

$$(-1)^{\ell(\zeta)}\, c^\tau_\zeta = (-1)^{\ell(\omega\tau)}\, X_{\tau^{-1}\omega}(\mathbf{x}^{\omega\zeta},\mathbf{x})\frac{1}{\Delta(\mathbf{x})} = X_{\omega\tau}(\mathbf{x},\mathbf{x}^{\omega\zeta})\frac{1}{\Delta(\mathbf{x})}\ . \tag{3.3.3}$$

Notice that, apart from signs and the factor $\Delta(\mathbf{x})$, the entries of the transition matrix from permutations to divided differences, and its inverse, are the same.

Here are the two transition matrices for $n=3$, to be read by rows, coding $x_1-x_2=12$, $x_1-x_3=13$, $x_2-x_3=23$:

	1	∂_2	∂_1	$\partial_1\partial_2$	$\partial_2\partial_1$	$\partial_1\partial_2\partial_1$
1	1	0	0	0	0	0
s_2	1	23	0	0	0	0
s_1	1	0	12	0	0	0
s_2s_1	1	23	13	0	$23\cdot 13$	0
s_1s_2	1	13	12	$13\cdot 12$	0	0
$s_1s_2s_1$	1	13	13	$12\cdot 13$	$13\cdot 23$	$12\cdot 13\cdot 23$

	1	s_2	s_1	s_1s_2	s_2s_1	$s_1s_2s_1$
Δ	$12\cdot 13\cdot 23$	0	0	0	0	0
$\partial_2\Delta$	$-12\cdot 13$	$12\cdot 13$	0	0	0	0
$\partial_1\Delta$	$-13\cdot 23$	0	$13\cdot 23$	0	0	0
$\partial_2\partial_1\Delta$	13	-13	-23	0	23	0
$\partial_1\partial_2\Delta$	13	-12	-13	12	0	0
$\partial_1\partial_2\partial_3\Delta$	-1	1	1	-1	-1	1

Pairs of permutations τ, σ such that the specialisation $X_\tau(\mathbf{x}^\sigma, \mathbf{x})$ is not a divisor of the Vandermonde correspond singularities of Schubert varieties. There are only two singularities when $n = 4$. One of them occurs in the expansion of $\partial_2\partial_3\partial_1\partial_2$, which involves the specializations of $X_{2143} = (x_1 - x_2)(x_1 + x_2 + y_3 - y_1 - y_2 - y_3)$, among which one finds $(x_1 - x_4)^2$.

The full expansion of $\partial_2\partial_3\partial_1\partial_2$ is

$$(1-s_2)\Big(\frac{x_1-x_4}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(-x_3+x_1)(x_1-x_2)}$$
$$-s_1\frac{1}{(x_3-x_4)(x_2-x_3)(-x_3+x_1)(x_1-x_2)}-s_3\frac{1}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(x_1-x_2)}$$
$$+s_3s_2\frac{1}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(x_1-x_3)}+s_1s_2\frac{1}{(x_2-x_4)(x_2-x_3)(x_1-x_3)(x_1-x_2)}$$
$$+s_1s_3\frac{1}{(x_3-x_4)(x_2-x_3)(x_1-x_4)(x_1-x_2)}-s_1s_3s_2\frac{1}{(x_2-x_4)(x_2-x_3)(x_1-x_4)(-x_3+x_1)}\Big)$$

The other singularity, when $n = 4$, occurs for $\partial_3\partial_2\partial_1\partial_2\partial_3$, which requires specializing $X_{1324} = x_1 + x_2 - y_1 - y_2$:

$$\partial_3\partial_2\partial_1\partial_2\partial_3\,\Delta = (1-s_1)(1-s_3)\Big((x_1+x_2-x_3-x_4) - s_2(x_1-x_4) + s_2s_3(x_1-x_3)$$
$$+ s_2s_1(x_2-x_4) - s_2s_1s_3(x_2-x_3)\Big)\,.$$

On could obtain the expansion of a reduced product $\partial_i\cdots\partial_j$ by writing it as $(1-s_i)(x_i-x_{i+1})^{-1}\cdots(1-s_j)(x_j-x_{j+1})^{-1}$ and enumerating all subwords of $s_i\cdots s_j$. This is the method followed by Kostant and Kumar [83]. We prefer relating the coefficients to Schubert polynomials, in particular because the number of subwords of a reduced decomposition of a permutation σ is far greater than the number of permutations in the interval $[1, \sigma]$.

Since the coefficients c_ζ^τ in (3.3.3) must vanish when ζ does not belong to the interval $[1, \tau]$, one obtains the following characterization of the Ehresmann-Bruhat by vanishing properties of Schubert polynomials, which generalizes (3.1.1).

Proposition 3.3.4. *Given n and two permutations $\sigma, \zeta \in \mathfrak{S}_n$, then $X_\sigma(\mathbf{x}^\zeta, \mathbf{x}) \neq 0$ if and only if $\sigma \le \zeta$ with respect to the Ehresmann-Bruhat order.*

Graßmannian Schubert polynomials $Y_v : v \in \mathbb{N}^n$, $v = v\!\uparrow$ are symmetrical in $x_1, \ldots, x_n$. One does not need to specialize them in all permutations of $y_1, y_2, \ldots,$

but, by symmetry, only in $\langle u\rangle = [y_{\sigma_1}, \dots, y_{\sigma_n}]$ with σ of code $u0\dots0$ such that $u = u\!\uparrow$. In that case, the last proposition becomes :

Corollary 3.3.5. *Let $u, v \in \mathbb{N}^n$ be anti-dominant. Then $Y_v(\langle u\rangle, \mathbf{y}) \neq 0$ if and only if $v \leq u$ (componentwise).*

This property is given by Okounkov [158] in the case where $\mathbf{y} = \{0, 1, 2, \dots\}$.

3.4 Wronskian of symmetric functions

Given a positive integer r, and r functions f_i of a single variable, the determinant $|f_i(x_j|$ is divisible by the Vandermonde in $x_1, x_2, \dots,$ and the quotient may be thought as a discrete analogue of the Wronskian [108, Prop. 9.3.1].

Writing $f_i(x_j) = f_i(x_1)s_1 \dots s_{j-1}$, and using (3.3.1), one sees that

$$\left|f_i(x_j)\right|_{i,j=1,\dots,r} \prod_{r\geq j>i\geq 1} (x_j - x_i)^{-1} = \left|f_i x_1\, \partial_1 \dots \partial_{j-1}\right|_{i,j=1,\dots,r} .$$

The same formula (3.3.1) may be applied to symmetric functions, replacing the integer r by a partition. Let $\lambda \in \mathbb{N}^n$ be a partition. To a family of symmetric functions $f_1(\mathbf{x}_n), f_2(\mathbf{x}_n), \dots$ of cardinality the number of partitions contained in λ, we shall associate a Wronskian $W_\lambda(f_i)$.

For each $\mu \subseteq \lambda$, let σ^μ be the Graßmannian permutation of code $\mu\!\uparrow$. Thanks to (3.3.1), every symmetric function $f(\mathbf{x}_n)$ satisfies

$$f\left(x_{\sigma_1^\mu}, \dots, x_{\sigma_n^\mu}\right) = f(\mathbf{x}_n) + \dots + f\partial^{\mu\uparrow} \Cap (\sigma^\mu) .$$

Therefore, a determinant $\left|f_i\left(\mathbf{x}_n^{\sigma^\mu}\right)\right|$ may be transformed, by multiplication by a unitriangular matrix, into the determinant $\left|f_i(\mathbf{x}_{n)}\partial^{\mu\uparrow} \Cap (\sigma^\mu)\right|$.

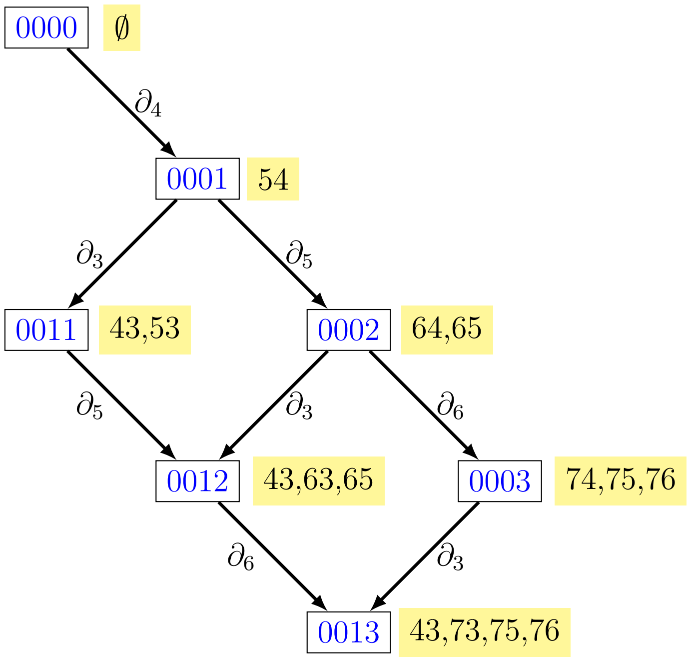
Definition 3.4.1. *Given a partition $\lambda \in \mathbb{N}^n$, and a family of symmetric functions $f_i(\mathbf{x}_n)$ of cardinality the number N of partitions contained in λ, then the* Wronskian *is*

$$W_\lambda(f_i(\mathbf{x}_n)) = \left|f_i \partial^{\mu\uparrow}\right|_{\substack{i=1\dots N\\ \mu\subseteq\lambda}} .$$

The preceding analysis has shown that the Wronskian is equal to

$$\left|f_i\left(\mathbf{x}_n^{\sigma^\mu}\right)\right| \frac{1}{\prod_{\mu\subseteq\lambda} \Cap(\sigma^\mu)} .$$

For example, let $n = 4$, $\lambda = [3, 1, 0, 0]$. Then the family $\{\mu\!\uparrow\}$, as well as the inversion polynomials $\Cap(\sigma^\mu)$, are displayed on the next figure (writing ji instead of $x_j - x_i$). The family $\{\partial^{\mu\uparrow}\}$ is the set of paths from the origin.



In the case where the family $\{f_{i(\mathbf{x}_n)}\}$ is the set of Schur functions $\{s_\mu(\mathbf{x}_n) : \mu \subseteq \lambda\}$, the Wronskian is unitriangular, and thus its determinant is equal to 1.

In the case of a rectangular partition $\lambda \subseteq r^n$, the sets $\{\sigma^\mu(\mathbf{x}_n)\}$ are all the subsets of cardinality n of $\{x_1, \ldots, x_{n+r}\}$. Given any $f \in \mathfrak{Sym}(\mathbf{x}_n)$, and $i : 1 \leq i \leq n{+}r{-}1$, then the set $\{f^{\mu\uparrow}\}$ is such that, either $f^{\mu\uparrow}$ and $f^{\mu\uparrow}\partial_i$ occur simultaneously, or $f^{\mu\uparrow}\partial_i = 0$. Thanks to the Leibnitz formula, this forces the Wronskian $W_{r^n}(f_1, f_2, \ldots)$ to be annihilated by all ∂_i, $i = 1, \ldots, n{+}r{-}1$. In other words, the Wronskian is a symmetric function when λ is a rectangular partition. Moreover, any inversion (j, i), $n{+}r \geq j > i \geq 1$, occurs $\binom{n+r-2}{n-1}$ times in the set of Graßmannian permutations $\{\sigma^\mu\}$.

In summary, one has the following lemma.

Lemma 3.4.2. *Let n, r be two positive integers, let $f_1, \ldots, f_N$, with $N = \binom{n+r}{n}$, belong to $\mathfrak{Sym}(\mathbf{x}_{n+r})$. Then*

$$\frac{1}{\prod_{n+r\geq j>i\geq 1}(x_j - x_i)^{\binom{n+r-2}{n-1}}} \Big| f_i(X) \Big|_{\substack{i=1\ldots N \\ X\subset\{x_1,\ldots,x_{n+r}\}}} = W_{r^n}(f_1, \ldots, f_N)$$

is a symmetric function of $x_1, \ldots, x_{n+r}$.

For example, for $n = r = 2$, the Wronskian

$$W_{22}\big(Y_0(\mathbf{x},\mathbf{0}), Y_{01}(\mathbf{x},\mathbf{0}), Y_{11}(\mathbf{x},\mathbf{0}), Y_{03}(\mathbf{x},\mathbf{0}), Y_{23}(\mathbf{x},\mathbf{0}), Y_{34}(\mathbf{x},\mathbf{0})\big)$$

$$= \begin{array}{c|cccccc}
 & 1 & \partial_2 & \partial_2\partial_1 & \partial_2\partial_3 & \partial_2\partial_3\partial_1 & \partial_2\partial_3\partial_1\partial_2 \\
\hline
Y_0 & Y_0 & 0 & 0 & 0 & 0 & 0 \\
Y_{01} & Y_{01} & Y_0 & 0 & 0 & 0 & 0 \\
Y_{11} & Y_{11} & Y_1 & Y_0 & 0 & 0 & 0 \\
Y_{03} & Y_{03} & Y_{002} & 0 & Y_{0001} & 0 & 0 \\
Y_{23} & Y_{23} & Y_{202} & Y_{012} & Y_{2001} & Y_{0101} & Y_{0001} \\
Y_{35} & Y_{35} & Y_{304} & Y_{024} & Y_{3003} & Y_{0203} & Y_{0013}
\end{array}$$

is equal to

$$Y_{0001}\big(Y_{0101}Y_{0013} - Y_{0203}Y_{0001}\big) = Y_{0001}^2 Y_{0113}\,.$$

3.5 Yang-Baxter and Schubert

One can degenerate Yang-Baxter bases of Hecke algebras into bases of the algebra of divided differences. However, instead of taking products of factors of the type ∂_i+1/c, let us take factors $1+c\partial_i$. Accordingly, given a spectral vector $[y_1,\ldots,y_n]$, one defines recursively a Yang-Baxter basis $\mho^\partial_\sigma$, starting from 1 for the identity permutation, by

$$\mho^\partial_{\sigma s_i} = \mho^\partial_\sigma \left(1+\partial_i \left(y_{\sigma_{i+1}} - y_{\sigma_i}\right)\right) \text{ for } \sigma_i < \sigma_{i+1}\,. \tag{3.5.1}$$

For example,

$$\begin{aligned}\mho^\partial_{321} &= (1+\partial_1(y_2-y_1))(1+\partial_2(y_3-y_1))(1+\partial_1(y_3-y_2))\\ &= 1+\partial_1(y_3-y_1)+\partial_2(y_3-y_1)+\partial_1\partial_2(y_2-y_1)(y_3-y_1)\\ &\qquad + \partial_2\partial_1(y_3-y_2)(y_3-y_1)+\partial_1\partial_2\partial_2(y_2-y_1)(y_3-y_1)(y_3-y_2)\end{aligned}$$

One remarks that the coefficients are the same as in the expression of $\sigma = [3,2,1]$ in terms of divided differences.

The following proposition shows that this property is true in general, and that the coefficients are still specialisations of Schubert polynomials.

Theorem 3.5.1. *The matrix of change of basis between* $\{\mho^\partial_\sigma\}$ *and* $\{\partial_\sigma\Delta(\mathbf{y})\}$, *and its inverse, have entries which are specializations of Schubert polynomials :*

$$\mho^\partial_\sigma = \sum_{\nu\le\sigma} \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y})\,, \tag{3.5.2}$$

$$\partial_\nu\,\Delta(\mathbf{y}) = \sum \mho^\partial_\sigma\, X_{\omega\nu}(\mathbf{y},\mathbf{y}^{\omega\sigma})\ . \tag{3.5.3}$$

Proof. Let σ and i be such that $\ell(\sigma)<\ell(\sigma s_i)$. Suppose known the expansion

$$\mho^\partial_\sigma = \sum_\nu \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y}) + \partial_{\nu s_i}\, X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y})\ ,$$

with $\nu:\ \ell(\nu)<\ell(\nu s_i)$. Then its product by $1+(y_{\sigma_{i+1}}-y_{\sigma_i})\partial_i$ is

$$\sum_\nu \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y}) + \partial_{\nu s_i}\left(X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y}) + X_\nu(\mathbf{y}^\sigma,\mathbf{y})(y_{\sigma_{i+1}}-y_{\sigma_i})\right)\ ,$$

and the identities

$$X_\nu(\mathbf{y}^{\sigma s_i},\mathbf{y}) \qquad \& \qquad X_{\nu s_i}(\mathbf{y}^{\sigma s_i},\mathbf{y}) = X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y}) + X_\nu(\mathbf{y}^\sigma,\mathbf{y})(y_{\sigma_{i+1}}-y_{\sigma_i})$$

give a similar expansion for $\mathcal{Y}_{\sigma s_i}$. QED

Notice that to expand products of factors $1+\partial_i(x_{i+1}-x_i)$, one has used the Leibnitz relations while in the present case the coefficients (in $\mathbf{y}$) commute with the operators acting on $\mathbf{x}$.

The analogy between Yang-Baxter elements and permutations can be materialised by acting on a proper element, as shows the following proposition.

Proposition 3.5.2. *For any $\sigma \in \mathfrak{S}_n$, one has*

$$X_\omega(\mathbf{x}, \mathbf{y}^\omega)\, \mho^\partial_\sigma = X_\omega(\mathbf{x}, \mathbf{y}^{\sigma\omega}) \tag{3.5.4}$$

Proof. In the step by step action of the factorised element $\mho^\partial_\sigma$, each step is of the type, $f(x_i - y_k)(1 + \partial_i(y_k - y_j) = f(x_i - y_j)$, $f \in \mathfrak{Sym}(x_i, x_{i+1})$. QED

For example, for $\sigma = [3, 4, 1, 2]$, writing the non-symmetric factor in a box, one has $\mho^\partial_{3412} = \big(1 + \partial_2(y_3 - y_2)\big)\big(1 + \partial_1(y_3 - y_1)\big)\big(1 + \partial_3(y_4 - y_2)\big)\big(1 + \partial_2(y_4 - y_1)\big)$ and

$$\begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_3 & \boxed{x_2{-}y_3} & \\ x_1{-}y_4 & x_2{-}y_4 & x_3{-}y_4 \end{array} \xrightarrow{1+\partial_2(y_3-y_2)} \begin{array}{lll} x_1{-}y_2 & & \\ \boxed{x_1{-}y_3} & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_4 & x_3{-}y_4 \end{array} \xrightarrow{1+\partial_1(y_3-y_1)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_4 & \boxed{x_3{-}y_4} \end{array}$$

$$\xrightarrow{1+\partial_3(y_4-y_2)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & \boxed{x_2{-}y_4} & x_3{-}y_2 \end{array} \xrightarrow{1+\partial_2(y_4-y_1)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_1 & x_3{-}y_2 \end{array}$$

$$= X_{4321}(\mathbf{x}, \mathbf{x}^{2143})\,.$$

The general properties of Yang-Baxter bases induce properties of specialisations of Schubert polynomials.

The symmetry (1.8.4) entails

$$(\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) = X_{\omega\nu\omega}(y^{\omega\sigma\omega}, \mathbf{y}^\omega)\,. \tag{3.5.5}$$

Each of the equations (1.8.9) and (1.8.10) gives in turn

$$\sum_\nu (\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) = \Delta(\mathbf{y})\, \delta_{\sigma,\zeta\omega}\,, \tag{3.5.6}$$

but this is a special case of Cauchy formula

$$\sum_\nu (\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) \;=\; \sum_\nu X_{\nu^{-1}}(\mathbf{y}, \mathbf{y}^\sigma) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) \;=\; X_\omega(\mathbf{y}^\zeta, \mathbf{y}^\sigma)\,.$$

The quadratic form $(\ ,\ )^{\mathcal{H}}$ defined in (1.8.5) degenerates into the form

$$(f\ ,\ g)^{\mathcal{H}00} = f\, g^\vee \cap \partial_\omega\,, \tag{3.5.7}$$

still denoting $f \to f^\vee$ be the anti-automorphism of the algebra of divided differences induced by $(\partial_\sigma)^\vee = \partial_{\sigma^{-1}}$.

Property (1.9.5) becomes

Proposition 3.5.3. *The Yang-Baxter bases associated to the spectral vectors $[y_1, \dots, y_n]$ and $[y_n, \dots, y_1]$ satisfy the relations*

$$\left(\mho^{\partial,\mathbf{y}}_\sigma\ ,\ \mho^{\partial,\mathbf{y}\omega}_\zeta\right)^{\mathcal{H}00} = \delta_{\sigma,\omega\zeta}\, \Delta(\mathbf{y}^\sigma)\,. \tag{3.5.8}$$

For example, for $\sigma = \zeta = [2,3,1]$, one has to take the product of

$$\mho_{231}^{\partial,\mathbf{y}} = 1 + \partial_1(y_2 - y_1) + \partial_2(y_3 - y_1) + \partial_1\partial_2(y_2 - y_1)(y_3 - y_1)$$

and

$$\left(\mho_{231}^{\partial,\mathbf{y}\omega}\right)^{\vee} = 1 + \partial_1(y_2 - y_3) + \partial_2(y_1 - y_3) + \partial_2\partial_1(y_2 - y_3)(y_1 - y_3)\,.$$

The coefficient of ∂_{321} in this product is equal to $(y_2-y_1)(y_3-y_1)(y_2-y_3)+(y_2-y_1)(y_2-y_3)(y_1-y_3) = 0$, and this proves that $\left(\mho_{231}^{\partial,\mathbf{y}}\,,\,\mho_{231}^{\partial,\mathbf{y}\omega}\right)^{\mathcal{H}00} = 0$.

3.6 Distance 1 and multiplication

The ring $\mathfrak{Sym}(\mathbf{x})$ has a linear basis consisting of Schur functions. Its multiplicative structure is determined by the *Pieri formulas*, i.e. by the products of Schur functions by the elementary (or complete) symmetric functions. In the non-symmetric case, the requirement to recover the ring structure is easier. Polynomials being sums of monomials, and monomials being products of variables, we need only describe the images of the different bases under multiplication by $x_1, x_2, \ldots$.

Our bases being obtained by the use of ∂_i's or π_i's, we could use the commutation properties of these operators with multiplication by a single variable.

In the case of Schubert polynomials, let us rather use interpolation methods. This time, it will be more convenient to index polynomials by permutations, passing from the notation Y_v to the notation X_σ, where v is the code $\mathfrak{c}(\sigma)$ of σ.

Definition 3.6.1. *$v \in \mathbb{N}^n$ is a* successor *of u if $|v| = |u| + 1$ & $Y_u(\langle v \rangle, \mathbf{y}) \neq 0$. Given two permutations ζ, σ, then ζ is a* successor *of σ iff this is so for their codes.*

Theorem 3.6.2. *A permutation ζ, of code v, is a successor of σ iff $\zeta\sigma^{-1}$ is a transposition (a, b), and $\ell(\zeta) = \ell(\sigma) + 1$. In that case,*

$$X_\sigma(\langle v \rangle, \mathbf{y}) = \Cap(v)\,(y_{\zeta_b} - y_{\zeta_a})^{-1} .$$

Proof. If $u = \mathfrak{c}(\sigma)$ is dominant, then it is immediate to write the specializations of Y_u and check the proposition in that case. Let us therefore suppose that there exists i such that $u_i < u_{i+1}$, and let η be such that $\mathfrak{c}(\eta) = [u_1, \ldots, u_{i-1}, u_{i+1}{+}1, u_i, u_{i+2}, \ldots, u_n]$. Since for any permutation ζ of code v, one has

$$\big(X_\eta(\langle v \rangle, \mathbf{y}) - (X_\eta(\langle v \rangle), \mathbf{y})^{s_i}\big)\,(y_{\zeta_i} - y_{\zeta_{i+1}})^{-1} = X_\sigma(\langle v \rangle, \mathbf{y})\,,$$

ζ can be a successor of σ only if $\zeta = \eta$, or if ζs_i is a successor of η. In the first case,

$$X_\sigma(\langle v \rangle, \mathbf{y}) = X_\eta(\langle v \rangle, \mathbf{y})(y_{\eta_i} - y_{\eta_{i+1}})^{-1} = \Cap(v))\,,$$

while in the second,

$$\frac{-X_\eta(\langle v \rangle^{s_i}, \mathbf{y})}{y_{\zeta_i} - y_{\zeta_{i+1}}} = \frac{\Cap(\mathfrak{c}(\zeta s_i))}{(y_{\zeta_{i+1}} - y_{\zeta_i})(y_{\zeta_b} - y_{\zeta_a})} = \frac{\Cap(\mathfrak{c}(\zeta))}{y_{\zeta_b} - y_{\zeta_a}}\,,$$

and this proves the proposition. QED

Corollary 3.6.3 (Monk formula [81])**.** *Given $v \in \mathbb{N}^n$, $\sigma = \langle v \rangle$, $k \in \{1, \ldots, n\}$, then*

$$(x_k - y_{\sigma_k})X_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{j>k} X_{\sigma\tau_{k,j}}(\mathbf{x}, \mathbf{y}) - \sum_{j<k} X_{\sigma\tau_{k,j}}(\mathbf{x}, \mathbf{y})\,, \tag{3.6.1}$$

summed over all transpositions $\tau_{k,j}$ such that $\ell(\sigma\tau_{k,j}) = \ell(\sigma) + 1$.

Proof. The polynomial $(x_k - y_{\sigma_k})X_\sigma(\mathbf{x},\mathbf{y})$ belongs to the linear span of $Y_w : |w| = |v|+1$, because it is of degree $|v|+1$ and vanishes in all $\mathbf{y}^{\langle w\rangle} : |w| \le |v|$. Writing it $\sum c_\zeta X_\zeta(\mathbf{x},\mathbf{y})$, and testing all the specializations $\mathbf{y}^\zeta$, one finds that the permutations appearing in the sum are exactly the successors of σ such that $y_{\zeta_k} \neq y_{\sigma_k}$. QED

Instead of multiplying by x_k, on can equivalently multiply by $x_1 + \cdots + x_k$ at once, obtaining the following Pieri formula generalizing the product of a Schur function by the elementary symmetric function of degree 1.

Corollary 3.6.4 (Degree 1 Pieri formula)**.** *Given $n, k : k \le n$, $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, $i \in \{1,\dots,n\}$, then*

$$(x_1 + \cdots + x_k - y_{\sigma_1} - \cdots - y_{\sigma_k})\, X_\sigma(\mathbf{x},\mathbf{y}) = \sum_{1\le i\le k<j} X_{\sigma\tau_{i,j}}(\mathbf{x},\mathbf{y})\,, \tag{3.6.2}$$

summed over transpositions $\tau_{i,j}$ such that $\ell(\sigma\tau_{i,j}) = \ell(\sigma)+1$.

One can iterate Monk formula. Let us call *k-path of length r* a sequence of permutations $\sigma^0, \sigma^1, \dots, \sigma^r$ such that $\ell(\sigma^{i+1}) = \ell(\sigma^i) + 1$ and $(\sigma^{i+1})^{-1}\sigma^i)$ is a transposition (k,j).

A k-path can be denoted by the sequence $[a_r, \dots, a_0]$ of values permuted, with

$$a_0 = (\sigma^0)_k,\ a_1 = (\sigma^1)_k,\ \dots, a_r = (\sigma^r)_k\,.$$

For $i = 1, \dots, r$, each permutation $\sigma^i(\sigma^0)^{-1}$ is a cycle $(a_i \dots a_1 a_0)$. The following proposition shows that the multiplication by a power of x_k can be described in terms of k-paths, the coefficients being complete functions $S_j(\,)$ of the variables y_i indexed by the values permuted.

Proposition 3.6.5. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$, $m \in \mathbb{N}$. Then, modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$, one has*

$$x_k^m\, X_\sigma(\mathbf{x},\mathbf{y}) = \sum \epsilon\, S_{m-1-r}(y_{a_0}, \dots, y_{a_r})\, X_{\tau_{a_r,a_{r-1}}\cdots\tau_{a_1a_0}\sigma}(\mathbf{x},\mathbf{y})\,, \tag{3.6.3}$$

sum over the k-paths of length $\le m$, the sign being given by the number of times $\tau_{a_i,a_{i-1}}$ transposes a value at position smaller than k.

Proof. Multiplying by x_k^m, using (3.6.1), involves enumerating paths with possible loops $\sigma^i = \sigma^{i+1}$ having weight y_j, with $j = (\sigma^i)_k$. The proposition results from grouping all the paths differing only by their loops, this explaining that the coefficient be a complete function. Each application of Monk formula possibly involves increasing the size of the symmetric group. One avoids that by using the ideal generated by the identification of symmetric functions in $\mathbf{x}_n$ with the same symmetric functions in $\mathbf{y}_n$. QED

The following tree describes the product $x_2^3\, X_{31425}(\mathbf{x},\mathbf{y})$, writing each permutation ζ above the coefficient of $X_\zeta(\mathbf{x},\mathbf{y})$.

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c}3\boxed{1}425\\ y_1^3\end{array} \arrow[dl, dash] \arrow[dr, dash] & & \\
\begin{array}{c}3\boxed{2}415\\ S_2(y_1,y_2)\end{array} \arrow[d, dash] & & \begin{array}{c}3\boxed{4}125\\ S_2(y_1,y_4)\end{array} \arrow[d, dash] \arrow[dr, dash] & \\
\begin{array}{c}3\boxed{4}215\\ y_1+y_2+y_3\end{array} \arrow[d, dash] \arrow[dr, dash] & & \begin{array}{c}4\boxed{3}125\\ -y_1-y_3-y_4\end{array} \arrow[d, dash] & \begin{array}{c}3\boxed{5}124\\ y_1+y_4+y_5\end{array} \arrow[d, dash] \\
\begin{array}{c}4\boxed{3}215\\ -1\end{array} & \begin{array}{c}3\boxed{5}214\\ 1\end{array} & \begin{array}{c}4\boxed{5}123\\ -1\end{array} & \begin{array}{c}5\boxed{3}124\\ -1\end{array}
\end{tikzcd}$$

or, for the readers who prefer one-dimensional formulas,

$$x_2^3 X_{31425} = y_1^3 X_{3142} + (y_1^2 + y_4^2 + y_1 y_4) X_{3412} + (y_1^2 + y_1 y_2 + y_2^2) X_{3241} + (y_1 + y_5 + y_4) X_{35124}$$
$$-(y_3 + y_1 + y_4) X_{4312} + (y_4 + y_1 + y_2) X_{3421} - X_{45123} + X_{361245} - X_{53124} + X_{35214} - X_{4321}\,.$$

3.7 Pieri formula for Schubert polynomials

The Italian geometer Pieri described the intersection of a Schubert cycle by a "special" one in the cohomology ring of the Grassmannian. In modern terms, he described the product of a Schur function by an elementary or complete function, the remarkable property being that there is no multiplicity in Pieri formula.

Let us generalize Pieri's result to Schubert polynomials, the presence of extra variables $\mathbf{y}$ allowing to interpret the intersection numbers 1 as complete functions of degree 0.

Our starting point will be the following case.

Lemma 3.7.1. *Let* $n, k, r \in \mathbb{N}$, $\rho = [n{-}1, \ldots, 0]$, $m = \max(n{-}k, 0)$ *and* $\mathbf{y}^{\heartsuit} = \{y_{m+1}, y_{m+2}, y_{m+3}, \ldots\}$. *Then*

$$Y_\rho(\mathbf{x}, \mathbf{y}) Y_{0^{k-1}r}(\mathbf{x}, \mathbf{z}) = Y_\rho(\mathbf{x}, \mathbf{y}) Y_{0^{k-1}r}(\mathbf{y}^{\heartsuit}, \mathbf{z}) + \sum_{i=1}^{k} \sum_{j=1}^{r} Y_{\rho+[0^{i-1}j0^{n-k}]}(\mathbf{x}, \mathbf{y}) Y_{0^{k-1+j}r-j}(\mathbf{y}^{\heartsuit}, \mathbf{z}) . \quad (3.7.1)$$

Proof. One uses Newton's interpolation (3.2.1) on the product fg, with $f = Y_\rho(\mathbf{x}, \mathbf{y})$, $g = Y_{0^{k-1}r}(\mathbf{x}, \mathbf{z})$, using Leibnitz' formula (1.4.2). The images of f under products of divided differences are 0 or Schubert polynomials that one has to specialize in $\mathbf{x} = \mathbf{y}$. Only $Y_{0\ldots 0}$ subsists. Let us first suppose that $n \le k$. In a sum $\sum_{\epsilon_i, \ldots \epsilon_h \in \{0,1\}} \big(f \partial_i^{\epsilon_i} \partial_j^{\epsilon_j} \cdots \partial_h^{\epsilon_h}\big) \big(g s_i^{\epsilon_i} \partial_i^{1-\epsilon_i} s_j^{\epsilon_j} \partial_j^{1-\epsilon_j} \cdots s_h^{\epsilon_h} \partial_h^{1-\epsilon_h}\big)$ there remains only divided differences ∂_i, $i < n$ acting on f, s_i preserving g, and products $\partial_k \partial_{k+1} \cdots \partial_{k+j-1}$ acting on g and sending it to $Y_{0^{k-1+j}r-j}(\mathbf{x}, \mathbf{z})$.

In final, for $n = 3 = k$ for example, the only non-zero contributions in Newton's formula are for $\partial_2 \partial_1 \partial_2 (\partial_3 \partial_4 \cdots)$, $\partial_2 (\partial_3 \partial_4 \cdots) \partial_1 \partial_2$ and $(\partial_3 \partial_4 \cdots) \partial_2 \partial_1 \partial_2$, and this corresponds indeed to the RHS of (3.7.1).

In the case where $n > k$, writing $\mathbf{y}^{\heartsuit} = \{y_{n-k}, y_{k+1}, \ldots\}$, one factors $Y_\rho(x, y) = Y_{(n-k)^k, n-k-1, \ldots, 0}(\mathbf{x}, \mathbf{y})\, Y_{k-1, \ldots, 0}(\mathbf{x}, \mathbf{y}^{\heartsuit})$, and write the interpolation for the product $Y_{k-1, \ldots, 0}(\mathbf{x}, \mathbf{y}^{\heartsuit}) Y_{0^{k-1}r}(\mathbf{x}, \mathbf{z})$. QED

For example, for $n = 5$, $k = 3$, $r = 2$, one has $\mathbf{y}^{\heartsuit} = \{y_3, y_4, \ldots\}$ and

$$Y_{43210}(\mathbf{x}, \mathbf{y}) Y_{002}(\mathbf{x}, \mathbf{z}) = Y_{43210}(\mathbf{x}, \mathbf{y}) Y_{002}(\mathbf{y}^{\heartsuit}, \mathbf{z}) + \big(Y_{53210}(\mathbf{x}, \mathbf{y}) + Y_{44210}(\mathbf{x}, \mathbf{y}) + Y_{43310}(\mathbf{x}, \mathbf{y})\big) \times Y_{0001}(\mathbf{y}^{\heartsuit}, \mathbf{z}) + \big(Y_{63210}(\mathbf{x}, \mathbf{y}) + Y_{45210}(\mathbf{x}, \mathbf{y}) + Y_{43410}(\mathbf{x}, \mathbf{y})\big) .$$

To describe the general Pieri formula, it is convenient to index Schubert polynomials by permutations, and generalize consecutivity in the Bruhat order.

Given an integer k, a pair of permutations $\sigma, \eta : \sigma \le \eta$ is called a k-*soulèvement* of degree indexsoulèvement $\ell(\eta) {-} \ell(\sigma)$ if each cycle ζ_i in the cycle-decomposition $\eta \sigma^{-1} = \zeta_1 \cdots \zeta_m$ is of the type $\zeta_i = (\alpha, \delta, \gamma, \ldots, \beta)$ with $\delta > \gamma > \cdots > \beta > \alpha$, $\{\delta, \ldots, \alpha\} \cap \{\sigma_1, \ldots, \sigma_k\} = \{\alpha\}$ and $\ell(\eta) = \ell(\sigma) + (\#\zeta_1 - 1) + \cdots + (\#\zeta_m - 1)$. Denote furthermore $\mathbf{y}^{\sigma,\eta} = \{y_{\sigma_1}, \ldots, y_{\sigma_k}\} \cup \{y_i : i \in \{\zeta_1\} \cup \cdots \cup \{\zeta_m\}\}$.

For example the pair $\sigma = [5,2,7,4,1,6,8,3,9]$, $\eta = [6,2,9,4,3,5,7,1,8]$) is a 5-soulèvement of degree $1+1+2 = \ell(\eta)-\ell(\sigma)$, because $\eta\sigma^{-1} = (1,3)(5,6)(7,9,8)$, and $\mathbf{y}^{\sigma,\eta} = \{y_5,y_2,y_7,y_4,y_1\} \cup \{y_1,y_3\} \cup \{y_5,y_6\} \cup \{y_7,y_9,y_8\}$
$= \{y_5,y_2,y_7,y_4,y_1,y_6,y_8,y_9\}$.

Theorem 3.7.2. *Let $n,k,r \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$. Then*

$$X_\sigma(\mathbf{x},\mathbf{y})Y_{0^{k-1}r}(\mathbf{x},\mathbf{z}) = \sum_\eta X_\eta(\mathbf{x},\mathbf{y})Y_{0^{k-1+j}r-j}(\mathbf{y}^{\sigma,\eta},\mathbf{z})\,, \tag{3.7.2}$$

sum over all k-soulèvements (σ,η) of degree $j=0,\ldots,r$.

Proof. The divided differences in $\mathbf{y}$ send $X_{n\ldots1}(\mathbf{x},\mathbf{y})$ onto any $X_\sigma(\mathbf{x},\mathbf{y})$, up to sign. Thus, the theorem can be proved by decreasing induction on $\ell(\sigma)$, checking the evolution of the RHS of (3.7.6) under a simple divided difference in $\mathbf{y}$, starting from (3.7.1). QED

For example of the recursion, the term $X_{3471256}(\mathbf{x},\mathbf{y})Y_{0^52}(y_3,y_1,y_5,y_4,y_7,y_6)$ occurs in the expansion of $X_{31542}(\mathbf{x},\mathbf{y})Y_{005}(\mathbf{x},\mathbf{z})$, and the permutation $[3,4,7,1,2,5,6][3,1,5,4,2]^{-1}$ is equal to the product of cycles $(1,4)(5,7,6)$. Under $-\partial_2^y$, this term gives, in the expansion of $X_{21543}(\mathbf{x},\mathbf{y})Y_{005}(\mathbf{x},\mathbf{z})$ the two terms $X_{3471256}(\mathbf{x},\mathbf{y})Y_{0^61}(y_2,y_1,y_5,y_4,y_3,y_7,y_6)$ and $X_{2471356}(\mathbf{x},\mathbf{y})Y_{0^52}(y_2,y_1,y_5,y_4,y_7,y_6)$, in accordance with

$$[3,4,7,1,2,5,6][2,1,5,4,3]^{-1} = (1,4)(2,3)(5,7,6)\,,$$
$$[2,4,7,1,3,5,6][3,1,5,4,2]^{-1} = (1,4)(5,7,6)\,.$$

The product of a Schubert polynomial by the elementary symmetric functions of $x_1,\ldots,x_k$ can be described similarly. In fact, instead of starting by the product of $Y_\rho(\mathbf{x},\mathbf{y})$ by $\sum_0^k(\mathbf{x},\mathbf{0})(-z)^{k-i} = Y_{1^k}(\mathbf{x},z) = \prod_1^k(x_i-z)$, one can multiply $Y_v(\mathbf{x},\mathbf{y})$ by $\prod_1^k(x_i-z_i)$ under some hypothesis on v. The elementary step is the following, which transforms multiplication by x_i into an action of a divided difference in $\mathbf{y}$.

Let v be dominant, $i=1$ or i be such that $v_{i-1} > v_i+1$, $v' = v+[0^{i-1}1]$, $j = v_i+1$. Then

$$Y_v(\mathbf{x},\mathbf{y})(x_i-z) = Y_{v'}(\mathbf{x},\mathbf{y}) + (y_j-z)Y_v(\mathbf{x},\mathbf{y}) = Y_{v'}(\mathbf{x},\mathbf{y})\,(z-y_{j+1})\partial_j^{\mathbf{y}}\,.$$

For example, for $v=[4,2,1]$, $i=2$, ignoring the factors which are invariant under $s_2^{\mathbf{y}}$, one has

$$Y_{421}(\mathbf{x},\mathbf{y})(x_2-y_3)(z-y_4) = \begin{array}{llll} \boxed{} & & & \\ \boxed{} & \boxed{} & \boxed{x_2-y_3} & \boxed{z-y_4} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{array} \xrightarrow{\partial_3^{\mathbf{y}}} Y_{421}(\mathbf{x},\mathbf{y})(x_2-z)\,.$$

By iteration, for an integer k and v dominant such that $v_1 > \cdots > v_k > v_{k+1}$, one has

$$\begin{aligned}
&Y_v(\mathbf{x},\mathbf{y})(x_1{-}z_1)\cdots(x_k{-}z_k)\\
&\quad = Y_{v+1^{k-1}}(\mathbf{x},\mathbf{y})\big((z_{k-1}{-}y_{v_{k-1}+2})\partial^{\mathbf{y}}_{v_{k-1}+1}\big)\cdots\big((z_1{-}y_{v_1+2})\partial^{\mathbf{y}}_{v_1+1}\big)(x_k{-}z_k)\\
&\qquad = Y_{v+1^{k-1}}(\mathbf{x},\mathbf{y})(x_k{-}z_k)\,\big((z_{k-1}{-}y_{v_{k-1}+2})\partial^{\mathbf{y}}_{v_{k-1}+1}\big)\cdots\\
&\qquad\quad = Y_{v+1^k}(\mathbf{x},\mathbf{y})\big((z_k{-}y_{v_k+2})\partial^{\mathbf{y}}_{v_k+1}\big)\cdots\big((z_1{-}y_{v_1+2})\partial^{\mathbf{y}}_{v_1+1}\big)\,,
\end{aligned}$$

and one obtains the following lemma.

Lemma 3.7.3. *Let v be dominant, k be such that $v_1 > \cdots > v_k > v_{k+1}$, $u = v + [1^k]$. Then one has*

$$\begin{aligned}
Y_v(\mathbf{x},\mathbf{y})(x_1{-}z_1)\cdots(x_k{-}z_k) &= Y_u(\mathbf{x},\mathbf{y})(z_k{-}y_{u_k+1})\partial^{\mathbf{y}}_{u_k}\cdots(z_1{-}y_{u_1+1})\partial^{\mathbf{y}}_{u_1}\\
&= Y_u(\mathbf{x},\mathbf{y})(z_k{-}y_{u_k+1})\cdots(z_1{-}y_{u_1+1})\partial^{\mathbf{y}}_{u_k}\cdots\partial^{\mathbf{y}}_{u_1}\,. \qquad (3.7.3)
\end{aligned}$$

As a corollary, one has, for $\rho = [n{-}1,\ldots,0]$ and $k \le n$,

$$Y_\rho(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{y},\mathbf{z}) = \sum_{0^k\le u\le 1^k} Y_{\rho+1^k-u}(\mathbf{x},\mathbf{y})(y_n{-}z_1)^{u_1}\cdots(y_{n-k+1}{-}z_1)^{u_k}\,. \qquad (3.7.4)$$

Using the divided differences in $\mathbf{z}$, this formula implies the following analog of (3.7.1).

Lemma 3.7.4. *Let n, r, k be three integers, $0 \le r \le k \le n$, $\rho = [n{-}1,\ldots,0]$. For $u \in [0,1]^k$, denote $\mathbf{y}^{\langle u\rangle} = \{y_{n+1-i} : i$ such that $u_i = 1, 1 \le i \le k\}$. Then*

$$\begin{aligned}
&Y_\rho(\mathbf{x},\mathbf{y})Y_{0^r1^{k-r}}(\mathbf{y},\mathbf{z})\\
&\qquad = \sum_{0^k\le u\le 1^k} (\text{-}1)^{|u|-r}Y_{\rho+1^k-u}(\mathbf{x},\mathbf{y})S_{|u]-r}(z_1{+}\cdots{+}z_{r+1} - \mathbf{y}^{\langle u\rangle})\,. \qquad (3.7.5)
\end{aligned}$$

For example,

$$\begin{aligned}
Y_{4321}(\mathbf{x},\mathbf{y})Y_{011}(\mathbf{x},\mathbf{z}) &= Y_{4431}(\mathbf{x},\mathbf{y}) + Y_{5331}(\mathbf{x},\mathbf{y}) + Y_{5421}(\mathbf{x},\mathbf{y})\\
&\quad - Y_{5321}(\mathbf{x},\mathbf{y})(z_1{+}z_2 - y_4{-}y_3) - Y_{4421}(\mathbf{x},\mathbf{y})(z_1{+}z_2 - y_5{-}y_3)\\
&\quad - Y_{4331}(\mathbf{x},\mathbf{y})(z_1{+}z_2 - y_5{-}y_4) + Y_{4321}(\mathbf{x},\mathbf{y})S_2(z_1{+}z_2 - y_5{-}y_4{-}y_3)\,.
\end{aligned}$$

The general product $Y_v(\mathbf{x},\mathbf{y})Y_{0^r1^{k-r}}(\mathbf{x},\mathbf{z})$ requires mirroring the notion of souléve-ment. Given an integer k, a pair of permutations $\sigma, \eta : \sigma \le \eta$ is called a k-*soulèvement gauche* of degree $\ell(\eta){-}\ell(\sigma)$ if each cycle ζ_i in the cycle-decomposition $\eta\sigma^{-1} = \zeta_1\cdots\zeta_m$ is of the type $\zeta_i = ((\alpha,\beta,\gamma,\ldots,\delta)$ with $\alpha < \beta < \gamma\cdots < \delta$, $\{\alpha,\ldots,\delta\}\cap\{\sigma_1,\ldots,\sigma_k\} = \{\delta\}$, and $\ell(\eta) = \ell(\sigma)+(\#\zeta_1-1)+\cdots+(\#\zeta_m-1)$. For a pair (σ,η) of permutations, denote $\mathbf{y}^{[\sigma,\eta]} = \{y_{\sigma_1},\ldots,y_{\sigma_k}\}\cap\{y_{\eta_1},\ldots,y_{\eta_k}\}$. Then one has the following Pieri formula for multiplication by elementary symmetric functions.

Theorem 3.7.5. *Let $n, k, r \in \mathbb{N}$, $0 \le r \le k \le n$, $\sigma \in \mathfrak{S}_n$. Then*

$$X_\sigma(\mathbf{x}, \mathbf{y}) Y_{0^r 1^{k-r}}(\mathbf{x}, \mathbf{z}) = \sum_\eta (-1)^{k-r+\ell(\sigma)-\ell(\eta)} X_\eta(\mathbf{x}, \mathbf{y}) S_{k-r+\ell(\sigma)-\ell(\eta)}(\mathbf{z}_{r+1} - \mathbf{y}^{[\sigma,\eta]}) , \tag{3.7.6}$$

sum over all k-soulèvements gauches (σ, η) of degree $j = 0, \ldots, r$.

For example, in the product $Y_{241596837}(\mathbf{x}, \mathbf{y}) Y_{0111}(\mathbf{x}, \mathbf{z})$, one has the term $-Y_{34195827}(\mathbf{x}, \mathbf{y})(z_1 + z_2 - \mathbf{y}_1 - y_4)$, the cycle decomposition of $\eta\sigma^{-1}$ being $(2,3)(5,9)$, and the intersection $\{y_{\sigma_1}, \ldots, y_{\sigma_4}\} \cap \{y_{\eta_1}, \ldots, y_{\eta_4}\}$ being $\{y_1, y_4\}$.

3.8 Products of Schubert polynomials by operators on y

We have described in the preceding sections Pieri multiplications by the combinatorics of *soulèvements*. Let us rather use now operators in $\mathbf{y}$. For example, the elementary products $Y_{201}(\mathbf{x},\mathbf{0})Y_{11}(\mathbf{x},\mathbf{0}) = Y_{32}(\mathbf{x},\mathbf{0})+Y_{311}(\mathbf{x},\mathbf{0})$, $Y_{101}(\mathbf{x},\mathbf{0})Y_{11}(\mathbf{x},\mathbf{0}) = Y_{31}(\mathbf{x},\mathbf{0}) + Y_{22}(\mathbf{x},\mathbf{0}) + Y_{211}(\mathbf{x},\mathbf{0})$ can be rewritten

$$\begin{aligned} Y_{201}(\mathbf{x},\mathbf{0})Y_{11}(\mathbf{x},\mathbf{0}) &= Y_{321}(\mathbf{x},\mathbf{0})(-\partial_1^{\mathbf{y}}-\partial_2^{\mathbf{y}}) \\ Y_{101}(\mathbf{x},\mathbf{0})Y_{11}(\mathbf{x},\mathbf{0}) &= Y_{321}(\mathbf{x},\mathbf{0})(-\partial_1^{\mathbf{y}}-\partial_2^{\mathbf{y}})(-\partial_2^{\mathbf{y}}-\partial_3^{\mathbf{y}}) \\ &= Y_{321}(\mathbf{x},\mathbf{0})\big(\partial_1^{\mathbf{y}}\partial_2^{\mathbf{y}} + \partial_1^{\mathbf{y}}\partial_3^{\mathbf{y}} + \partial_2^{\mathbf{y}}\partial_3^{\mathbf{y}}\big)\,, \end{aligned}$$

indicating that the elements $\delta_i = \partial_i + \partial_{i+1}$ play a role in the multiplication of Schubert polynomials.

For them, the braid relations of order 3 are still valid:

$$\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1} = (\partial_i\partial_{i+1}+\partial_i\partial_{i+2}+\partial_{i+1}\partial_{i+2})\partial_i + (\partial_i\partial_{i+2}+\partial_{i+1}\partial_{i+2})\partial_{i+1}\,.$$

In the case of order 2, one has $\delta_i\delta_j = \delta_j\delta_i$ if $|j-i| > 2$, but

$$\delta_i\delta_{i+2} - \delta_{i+2}\delta_i = \partial_{i+1}\partial_{i+2} - \partial_{i+2}\partial_{i+1}\,.$$

Note, however, that the elements δ_i do not satisfy a Hecke relation, but that

$$\delta_i^2 = \partial_i\partial_{i+1} + \partial_{i+1}\partial_i\ ,\ \ \delta_i^3 = 2\partial_i\partial_{i+1}\partial_i\ ,\ \ \delta_i^4 = 0\,.$$

Given two positive numbers r, p, let $\varphi([r],p) = \delta_p\cdots\delta_{p+r-1}$, and for a partition $\lambda \in \mathbb{N}^\ell$, let

$$\varphi(\lambda,p) = \varphi([\lambda_1],p)\varphi([\lambda_2],p-1)\cdots\varphi([\lambda_\ell],p-\ell+1)\,. \tag{3.8.1}$$

For example, the product $\varphi([3,3,1],3)$ is obtained by reading by the successive rows of the display $\begin{smallmatrix}\delta_3&\delta_4&\delta_5\\ \delta_2&\delta_3&\delta_4\\ \delta_1&&\end{smallmatrix}$. In fact, one easily checks that $\delta_1,\delta_2,\ldots$ satisfy the nilplactic relations (cf. 6.9.1)

$$\delta_i\delta_{i+1}\delta_{i-1} = \delta_i\delta_{i-1}\delta_{i+1}\,,$$

and this allows to pass from the row-reading of the array to its column-reading. Hence $\varphi([3,3,1],3)$ is also equal to $(\delta_3\delta_2\delta_1)(\delta_4\delta_3)(\delta_5\delta_4)$.

By induction on r, one checks that

$$\varphi([r],p) = \sum_{j=p}^{p+r}(\partial_p\cdots\partial_{j-1})(\partial_{j+1}\cdots\partial_{p+r})\,.$$

We shall not use the fact that, more generally, the non-zero terms in the expansion of $\varphi(\lambda,p)$ are in bijection with the partitions whose diagram is contained in the diagram of λ. For example, reading the diagrams by rows, i standing for ∂_i, one has 9 terms in the expansion of $\varphi([3,2],2)$. The partitions are obtained by reordering the lengths of the rows of the left part of diagram (as cut by the bullets).

$$\varphi([3,2],2) = \underset{\emptyset}{\begin{array}{|c|c|c|c|}\hline \bullet & 3 & 4 & 5 \\ \hline \bullet & 2 & 3 & \\ \cline{1-3}\end{array}} + \underset{1}{\begin{array}{|c|c|c|c|}\hline \bullet & 3 & 4 & 5 \\ \hline 1 & \bullet & 3 & \\ \cline{1-3}\end{array}} + \underset{11}{\begin{array}{|c|c|c|c|}\hline 2 & \bullet & 4 & 5 \\ \hline 1 & \bullet & 3 & \\ \cline{1-3}\end{array}} + \underset{2}{\begin{array}{|c|c|c|c|}\hline \bullet & 3 & 4 & 5 \\ \hline 1 & 2 & \bullet & \\ \cline{1-3}\end{array}}$$

$$+ \underset{21}{\begin{array}{|c|c|c|c|}\hline 2 & \bullet & 4 & 5 \\ \hline 1 & 2 & \bullet & \\ \cline{1-3}\end{array}} + \underset{3}{\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & \bullet \\ \hline \bullet & 2 & 3 & \\ \cline{1-3}\end{array}} + \underset{22}{\begin{array}{|c|c|c|c|}\hline 2 & 3 & \bullet & 5 \\ \hline 1 & 2 & \bullet & \\ \cline{1-3}\end{array}} + \underset{31}{\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & \bullet \\ \hline 1 & \bullet & 3 & \\ \cline{1-3}\end{array}} + \underset{32}{\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & \bullet \\ \hline 1 & 2 & \bullet & \\ \cline{1-3}\end{array}}$$

When v is dominant, and i is such that $v_{i-1} > v_i+1$ (or $i=1$), then

$$Y_v(\mathbf{x},\mathbf{y})(x_i - z) = Y_u(\mathbf{x},\mathbf{y}) + (y_{u_i} - z)Y_v(\mathbf{x},\mathbf{y}) = Y_u(\mathbf{x},\mathbf{y})\big(1 + \partial^{\mathbf{y}}_{u_i}(z - y_{u_i})\big)\,, \quad (3.8.2)$$

with $u = v + [0^{i-1}, 1]$. One has therefore to introduce the operators

$$D_i(z) = 1 + \partial_i^{\mathbf{y}}(z - y_i)$$

depending on extra indeterminates z.

Let us show that these operators allow to express the product of two dominant Schubert polynomials $Y_\lambda(\mathbf{x},\mathbf{y})Y_\mu(\mathbf{x},\mathbf{z})$. We have first to reinterpret the construction of the canonical reduced decomposition of a permutation from its code. Let $u \in \mathbb{N}^\ell$. Fill the diagram of u by consecutive numbers upwards in each column, starting with the column number at the bottom, as in section 1.1. Then $D^u(\mathbf{z})$ is the product obtained by reading the diagram by successive columns, from top to bottom, interpreting an entry i at level j as $D_i(z_j)$. For example, $u = [3,0,1,2]$ gives the diagram $\begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & & & \boxed{5} \\ \boxed{1} & & \boxed{3} & \boxed{4} \\ \hline \end{array}$ and the product

$$D^{3012}(\mathbf{z}) = \big(D_3(z_3)D_2(z_2)D_1(z_1)\big)\big(\,\big)\big(D_3(z_1)\big)\big(D_5(z_2)D_4(z_1)\big)\,.$$

Decomposing $Y_\mu(\mathbf{x},\mathbf{z})$ into products of factors of the type $(x_1 - z_j)\cdots(x_k - z_j)$, and applying repeatedly remark (3.8.2), one obtains the following description of the product of two dominant Schubert polynomials.

Proposition 3.8.1. *Let λ, μ be dominant, ζ be the permutation of code $\lambda+\mu$, σ be the permutation of code λ, and u be the code of $\zeta\sigma^{-1}$. Then*

$$Y_\lambda(\mathbf{x},\mathbf{y})Y_\mu(\mathbf{x},\mathbf{z}) = Y_{\lambda+\mu}(\mathbf{x},\mathbf{y})D^u(\mathbf{z})\,. \quad (3.8.3)$$

For example, for $\lambda = [4,3,3,2,1]$, $\mu = [2,2,1,1]$, one has $\lambda+\mu = [6,5,4,3,1]$, $\zeta = [7,6,5,4,2,1,3]$, $\sigma = [5,4,6,3,2,1]$, $\zeta\sigma^{-1} = [1,2,4,6,7,5,3]$, $u = [0,0,1,2,2,1]$, and $D^u(\mathbf{z}) = D_3(z_1)\cdot D_5(z_2)D_4(z_1)\cdot D_6(z_2)D_5(z_1)\cdot D_6(z_1)$. Hence

$$Y_{43321}(\mathbf{x},\mathbf{y})Y_{2211}(\mathbf{x},\mathbf{z}) = Y_{65431}(\mathbf{x},\mathbf{y})D_3(z_1)D_5(z_2)D_4(z_1)D_6(z_2)D_5(z_1)D_6(z_1)\,.$$

We are now in position to rewrite the product of a Schubert polynomial by an elementary symmetric function, $\varphi^{\mathbf{y}}(\lambda,p)$ standing for a product of operators on $\mathbf{y}$ instead of $\mathbf{x}$.

Proposition 3.8.2. *Let $k \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$ such*[4] *that $n \notin \{\sigma_1, \ldots, \sigma_k\}$, v be the code of σ, ζ be the permutation of maximal length in the coset $\sigma\mathfrak{S}_{k|n-k}$. Let λ be the partition $\#\{i : \zeta_i < \zeta_j, i = 1, \ldots, k, j = k+2, \ldots, n\}$. Let moreover $u = [\sigma_1, \ldots, \sigma_k] \uparrow$. Then*

$$X_\sigma(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},\mathbf{z}) = (-1)^{|\lambda|} X_{\omega\zeta^{-1}\sigma}(\mathbf{x},\mathbf{y})\, \varphi^{\mathbf{y}}(\lambda, n-k-1) D_{u_1}(z_1) \cdots D_{u_k}(z_1)\,. \quad (3.8.4)$$

Proof. . One first shows that

$$X_\zeta(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},\mathbf{z}) = (-1)^{|\lambda|} X_{\omega}(\mathbf{x},\mathbf{y})\, \varphi^{\mathbf{y}}(\lambda, n-k-1) D_{u_1}(z_1) \cdots D_{u_k}(z_1)\,,$$

by induction on the last part of λ. Since $\partial^{\mathbf{x}}_{\zeta^{-1}\sigma}$ commutes with $Y_{1^k}(\mathbf{x},\mathbf{z})$, one has

$$\begin{aligned} X_\sigma(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},\mathbf{z}) &= X_\zeta(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},\mathbf{z})\partial^{\mathbf{x}}_{\zeta^{-1}\sigma} \\ &= (-1)^{|\lambda|} X_{\omega}(\mathbf{x},\mathbf{y})\, \partial^{\mathbf{x}}_{\zeta^{-1}\sigma}\varphi^{\mathbf{y}}(\lambda, n-k-1) D_{u_1}(z_1) \cdots D_{u_k}(z_1) \\ &= (-1)^{|\lambda|} X_{\omega\zeta^{-1}\sigma}(\mathbf{x},\mathbf{y})\, \varphi^{\mathbf{y}}(\lambda, n-k-1) D_{u_1}(z_1) \cdots D_{u_k}(z_1)\,. \end{aligned}$$

QED

For example, for $\sigma = [1,4,2,5,3]$, $k = 3$, one has $\zeta = [4,2,1,5,3]$, $\omega\zeta^{-1}\sigma = [3,5,4,2,1]$, $\lambda = [2]$, and

$$\begin{aligned} X_{14253}(\mathbf{x},\mathbf{y})Y_{111}(\mathbf{x},\mathbf{z}) &= X_{35421}(\mathbf{x},\mathbf{y})(\partial^{\mathbf{y}}_1\partial^{\mathbf{y}}_2 + \partial^{\mathbf{y}}_1\partial^{\mathbf{y}}_3 + \partial^{\mathbf{y}}_2\partial^{\mathbf{y}}_3)D_1(z_1)D_2(z_1)D_4(z_1) \\ &= \big(X_{25413}(\mathbf{x},\mathbf{y}) + X_{25341}(\mathbf{x},\mathbf{y})\big)D_1(z_1)D_2(z_1)D_4(z_1) \\ = X_{25413} + X_{25341} &+ (y_4-z_1)\,X_{24513} + (y_1-z_1)\,X_{15342} + (y_4-z_1)\,X_{24351} + (y_1-z_1)\,X_{15423} \\ &+ (y_4-z_1)\,(y_1-z_1)\,X_{14352} + (y_2-z_1)\,(y_1-z_1)\,X_{15243} \\ &+ (y_4-z_1)\,(y_1-z_1)\,X_{14523} + (y_4-z_1)\,(y_2-z_1)\,(y_1-z_1)\,X_{14253}\,. \end{aligned}$$

One can rewrite the preceding formula using that

$$X_{\omega\zeta^{-1}\sigma}(\mathbf{x},\mathbf{y}) = (-1)^{\ell(\sigma)+\ell(\zeta)} X_\omega(\mathbf{x},\mathbf{y})\partial^{\mathbf{y}}_{\omega\sigma^{-1}\zeta\omega}\,,$$

so as to start from a dominant polynomial.

Combining with formula (3.8.3) expressing the product of two dominant Schubert polynomials, one obtains the product of a general Schubert polynomial by a dominant one in $\mathbf{x}, \mathbf{z}$, and by using divided differences in $\mathbf{z}$, the product of two Schubert polynomials.

Theorem 3.8.3. *With the notations of (3.8.4), let $\rho = [n-1, \ldots, 0]$, $\omega = [n, \ldots, 1]$. Let moreover μ be a partition of length k, $\eta = \mu - 1^k$, ξ be the permutation of code $\rho+\eta$, and finally w be the code of $\xi\omega$. Then*

$$\begin{aligned} X_\sigma(\mathbf{x},\mathbf{y})Y_\mu(\mathbf{x},\mathbf{z}) = (-1)^{|\lambda|+\ell(\sigma)+\ell(\zeta)} & Y_{\rho+\eta} D^w(z_2, \ldots, z_{\mu_1}) \\ & \partial^{\mathbf{y}}_{\omega\sigma^{-1}\zeta\omega}\varphi^{\mathbf{y}}(\lambda, n-k-1) D_{u_1}(z_1) \cdots D_{u_k}(z_1)\,. \quad (3.8.5) \end{aligned}$$

[4]if not, one adds a fixed point $n+1$ to σ.

Let v be such that there exists a permutation ν such that $Y_\mu(\mathbf{x},\mathbf{z})\partial^{\mathbf{z}}_\nu = (-1)^{\ell(\nu)} Y_v(\mathbf{x},\mathbf{z})$. Then

$$X_\sigma(\mathbf{x},\mathbf{y})Y_v(\mathbf{x},\mathbf{z}) = (-1)^{|\lambda|+\ell(\sigma)+\ell(\zeta)+\ell(\nu)} Y_{\rho+\eta} D^w(z_2,\dots,z_{\mu_1})$$
$$\partial^{\mathbf{y}}_{\omega\sigma^{-1}\zeta\omega}\varphi^{\mathbf{y}}(\lambda, n{-}k{-}1) D_{u_1}(z_1)\cdots D_{u_k}(z_1)\,\partial^{\mathbf{z}}_\nu\,. \quad (3.8.6)$$

Continuing the preceding example, one has

$$X_{14253}(\mathbf{x},\mathbf{y})Y_{331}(\mathbf{x},\mathbf{z}) = Y_{4321}(\mathbf{x},\mathbf{y})\partial^{\mathbf{y}}_4\partial^{\mathbf{y}}_3\varphi([2],1)D_1(z_1)D_2(z_1)D_4(z_1)Y_{22}(\mathbf{x},\{z_2,z_3\})$$
$$= Y_{6521}(\mathbf{x},\mathbf{y})D_5(z_3)D_6(z_3)D_4(z_2)D_5(z_2)\partial^{\mathbf{y}}_4\partial^{\mathbf{y}}_3\varphi([2],1)D_1(z_1)D_2(z_1)D_4(z_1)\,.$$

The image of this equation under $\partial^{\mathbf{z}}_3$ gives a non-dominant product:

$$-X_{14253}(\mathbf{x},\mathbf{y})Y_{231}(\mathbf{x},\mathbf{z}) = Y_{6521}(\mathbf{x},\mathbf{y})\big(\partial^{\mathbf{y}}_5+\partial^{\mathbf{y}}_6+\partial^{\mathbf{y}}_5\partial^{\mathbf{y}}_6(z_3+z_4-y_5-y_6)\big)$$
$$D_4(z_2)D_5(z_2)\partial^{\mathbf{y}}_4\partial^{\mathbf{y}}_3\varphi([2],1)D_1(z_1)D_2(z_1)D_4(z_1)\,.$$

Formula (3.8.6) for the product of two Schubert polynomials involves using divided differences in $\mathbf{z}$. Thanks to Leibnitz, one needs only to know that $D_j(z_i)\partial^{\mathbf{z}}_i = \partial^{\mathbf{y}}_j$ and $D_j(z_{i+1})\partial^{\mathbf{z}}_i = -\partial^{\mathbf{y}}_j$ to eliminate the divided differences in $\mathbf{z}$. In all, the product of two Schubert polynomials $X_\sigma(\mathbf{x},\mathbf{0})Y_v(\mathbf{x},\mathbf{0})$ in $\mathbf{x}$ only is computed by starting from a dominant ancestor $Y_\mu(\mathbf{x},\mathbf{z})$ of $Y_v(\mathbf{x},\mathbf{z})$, taking the image of (3.8.5) under an appropriate product of divided differences in $\mathbf{z}$, then specializing all z_i to 0, i.e. replacing all $D_j(z_i)$ by $D_j(0) = 1-\partial^{\mathbf{y}}_j y_j = -y_{j+1}\partial_j$. However, evaluating a mixture of $\partial^{\mathbf{y}}_i$ and $y_{i+1}\partial^{\mathbf{y}}_i$ is not straightforward.

Let us follow another strategy by first pushing all the operators $D_j(z_i)$ to the right, iterating the Pieri formula.

First notice that (3.8.4) implies that the expression

$$X_\sigma(\mathbf{x},\mathbf{0})x_1\cdots x_k = \sum X_\zeta(\mathbf{x},\mathbf{0})$$

extends to

$$X_\sigma(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},\mathbf{z}) = \sum X_\zeta(\mathbf{x},\mathbf{y})D_{u_1}(z_1)\cdots D_{u_k}(z_1)\,.$$

Moreover, since $u_1,\dots,u_k$ is increasing, coefficients can be pushed to the right and the product $D_{u_1}(z_1)\cdots D_{u_k}(z_1)$ is equal to

$$\sum_{\epsilon_i=0,1} (\partial^{\mathbf{y}}_{u_1})^{\epsilon_1}\cdots(\partial^{\mathbf{y}}_{u_k})^{\epsilon_k}(z_1-y_{u_1})^{\epsilon_1}\cdots(z_1-y_{u_k})^{\epsilon_k}\,.$$

For a given ζ, some $\partial^{\mathbf{y}}_i$ possibly annihilate $X_\zeta(\mathbf{x},\mathbf{y})$. Thus there exists a minimal subsequence $u^\zeta_\sigma = [v_1,\dots,v_r]$ of $[u_1,\dots,u_k]$ such that

$$X_\zeta(\mathbf{x},\mathbf{y})D_{u_1}(z_1)\cdots D_{u_k}(z_1) = X_\zeta(\mathbf{x},\mathbf{y})D^k_{\sigma,\zeta}(z_1)\,,$$

with $D^k_{\sigma,\zeta}(z_1) = D_{v_1}(z_1)\cdots D_{v_r}(z_1)$. For example,

$$\begin{aligned}X_{3165247}(\mathbf{x},\mathbf{y})Y_{111}(\mathbf{x},\mathbf{z}) &= X_{6573124}(\mathbf{x},\mathbf{y})\varphi([2,2,1],3)D_1(z_1)D_3(z_1)D_6(z_1)\\ &= X_{4275136}(\mathbf{x},\mathbf{y})D_1(z_1)D_3(z_1)D_6(z_1) + X_{4571236}(\mathbf{x},\mathbf{y})D_3(z_1)D_6(z_1)\\ &\qquad + X_{5273146}(\mathbf{x},\mathbf{y})D_1(z_1)D_6(z_1) + X_{5371246}(\mathbf{x},\mathbf{y})D_6(z_1)\,.\end{aligned}$$

Let us write $\sigma \xrightarrow{k} \zeta$ for such a *Pieri pair* of permutations (which was called *k-soulèvement gauche of degree k* in the preceding section). One can iterate the Pieri multiplication

$$\begin{aligned}X_\sigma(\mathbf{x},\mathbf{y})Y_{1^k}(\mathbf{x},z_1)Y_{1^r}(\mathbf{x},z_2) &= \sum_\zeta X_\zeta(\mathbf{x},\mathbf{y})D^k_{\sigma,\zeta}(z_1)Y_{1^r}(\mathbf{x},z_2)\\ &= \sum_\zeta X_\zeta(\mathbf{x},\mathbf{y})Y_{1^r}(\mathbf{x},z_2)D^k_{\sigma,\zeta}(z_1)\,,\end{aligned}$$

and therefore, assuming $r \le k$, one has

$$X_\sigma(\mathbf{x},\mathbf{y})Y_{2^r1^{k-r}}(\mathbf{x},\mathbf{z}) = \sum_\tau\sum_\zeta X_\tau(\mathbf{x},\mathbf{y})D^r_{\zeta,\tau}(z_2)\,D^k_{\sigma,\zeta}(z_1)$$

sum over all *Pieri paths* $\sigma \xrightarrow{k} \zeta \xrightarrow{r} \tau$.

By iteration, one obtains

Proposition 3.8.4. *Let σ be a permutation, λ be a partition, $\mu = \lambda^\sim$ be its conjugate, $m = \lambda_1$. Then*

$$X_\sigma(\mathbf{x},\mathbf{y})Y_\lambda(\mathbf{x},\mathbf{z}) = \sum_{\zeta_1,\ldots,\zeta_m} X_{\zeta_m}(\mathbf{x},\mathbf{y})\,D^{\mu_m}_{\zeta_m,\zeta_{m-1}}(z_m)\cdots D^{\mu_1}_{\zeta_1,\zeta_0}(z_1)\,, \tag{3.8.7}$$

sum over all Pieri chains

$$\sigma = \zeta_0 \xrightarrow{\mu_1} \zeta_1 \xrightarrow{\mu_2} \zeta_2 \cdots \xrightarrow{\mu_m} \zeta_m\ .$$

Continuing the preceding example, writing $X_\sigma \begin{array}{l}\boxed{a}\\ \boxed{b}\boxed{c}\end{array}$ for $X_\sigma(\mathbf{x},\mathbf{y})D_a(z_2)D_b(z_1)D_c(z_1)$, one has

$$\begin{aligned}Y_{2032}(\mathbf{x},\mathbf{y})Y_{221}(\mathbf{x},\mathbf{z}) &= X_{316524}(\mathbf{x},\mathbf{y})Y_{221}(\mathbf{x},\mathbf{z})\\
&= X_{6723145}\begin{array}{l}\boxed{5}\\ \boxed{1}\boxed{6}\end{array} + X_{5671234}\begin{array}{l}\boxed{4}\boxed{5}\\ \boxed{3}\boxed{6}\end{array} + X_{7425136}\begin{array}{l}\boxed{\bullet}\\ \boxed{1}\boxed{3}\boxed{6}\end{array} + X_{5741236}\begin{array}{l}\boxed{4}\\ \boxed{3}\boxed{6}\end{array}\\
&+ X_{7345126}\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\boxed{6}\end{array} + X_{7451236}\begin{array}{l}\boxed{3}\\ \boxed{6}\end{array} + X_{5374126}\begin{array}{l}\boxed{2}\boxed{4}\\ \boxed{1}\boxed{3}\boxed{6}\end{array} + X_{7352146}\begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{6}\end{array}\\
&+ X_{5724136}\begin{array}{l}\boxed{4}\\ \boxed{1}\boxed{3}\boxed{6}\end{array} + X_{7523146}\begin{array}{l}\boxed{\bullet}\\ \boxed{1}\boxed{6}\end{array} + X_{6731245}\begin{array}{l}\boxed{5}\\ \boxed{6}\end{array} + X_{5472136}\begin{array}{l}\boxed{4}\\ \boxed{1}\boxed{3}\boxed{6}\end{array}\\
&\qquad + X_{6372145}\begin{array}{l}\boxed{2}\boxed{5}\\ \boxed{1}\boxed{6}\end{array} + X_{6471235}\begin{array}{l}\boxed{3}\boxed{5}\\ \boxed{6}\end{array} + X_{7531246}\begin{array}{l}\boxed{\bullet}\\ \boxed{6}\end{array}\,.\end{aligned}$$

Formula (3.8.5) would on the other hand give

$$Y_{2032}(\mathbf{x},\mathbf{y})Y_{221}(\mathbf{x},\mathbf{z}) = Y_{6542}(\mathbf{x},\mathbf{y})D_5(z_2)D_6(z_2)\varphi([2,2,1],3)D_1(z_1)D_3(z_1)D_6(z_1)\,.$$

By image of (3.8.7) under divided differences in $\mathbf{z}$, and specializing $\mathbf{y}$ and $\mathbf{z}$ to 0, one obtains the product of two general Schubert polynomials in $\mathbf{x}$ only. However, cancellations occur, the different Pieri paths cannot be considered independently of each other, and more work is needed to produce a positive combinatorial rule.

Fomin and Kirillov [40] describe the product of two Schubert polynomials by introducing some quadratic algebras and evaluating Schubert polynomials in Dunkl-type operators.

3.9 Transition for Schubert polynomials

The right-hand side of Monk formula (3.6.1) involves two sets W_+, W_- of permutations:

$$(x_k - y_{\sigma_k}) X_\sigma(\mathbf{x},\mathbf{y}) = \sum_{\zeta\in W_+} X_\zeta(\mathbf{x},\mathbf{y}) - \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,,$$

Let us call *transition* the case where W_+ is a singleton, rewriting the equation

$$X_\zeta(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k}) X_\sigma(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,, \qquad (3.9.1)$$

the set W_- depending on the pair (k,ζ), or equivalently, the pair (k,σ) as described in (3.6.1).

For example,

$$\begin{aligned} X_{52186347}(\mathbf{x},\mathbf{y}) &= (x_2{-}y_1) X_{51286347}(\mathbf{x},\mathbf{y}) \\ &= (x_4{-}y_7) X_{5217634}(\mathbf{x},\mathbf{y}) + X_{5271634}(\mathbf{x},\mathbf{y}) \\ &\qquad + X_{5712634}(\mathbf{x},\mathbf{y}) + X_{7215634}(\mathbf{x},\mathbf{y}) \\ &= (x_5{-}y_4) X_{52184367}(\mathbf{x},\mathbf{y}) + X_{52481367}(\mathbf{x},\mathbf{y}) + X_{54182367}(\mathbf{x},\mathbf{y}) \end{aligned}$$

Transitions are compatible with Young subgroups. Indeed, let ζ belong to $\mathfrak{S}_{r|n-r}$. Then $\zeta = \zeta'\zeta''$, where ζ' fixes $r{+}1,\ldots,n$ and ζ'' fixes $1,\ldots,r$. Any transition for ζ' induces a transition for ζ. A transition

$$X_{\zeta'}(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k}) X_\sigma(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,,$$

all the permutations ν fix $r{+}1,\ldots,n$, and therefore one has the transition

$$X_\zeta(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k}) X_{\sigma\zeta''}(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_{\nu\zeta''}(\mathbf{x},\mathbf{y}) \,. \qquad (3.9.2)$$

By recurrence on the length of ζ', one obtains the following factorisation property of Schubert polynomials.

Corollary 3.9.1. *Let ζ belong to a Young subgroup, and $\zeta = \zeta'\zeta''$ its corresponding factorisation. Then*

$$X_\zeta(\mathbf{x},\mathbf{y}) = X_{\zeta'}(\mathbf{x},\mathbf{y})\, X_{\zeta''}(\mathbf{x},\mathbf{y}) \,. \qquad (3.9.3)$$

Transitions may be used recursively to decompose Schubert polynomials into sums of "shifted monomials" $\prod(x_i - y_j)$, stopping the process when arriving at dominant polynomials.

Among all transitions for a given ζ, let us choose the one for which k is maximum, and call it *maximal transition*. For this transition, let us rather index polynomials by codes instead of permutations. Let $v \in \mathbb{N}^n$ be the code of ζ, and k be such that that $v_k > 0$, $v_{k+1} = 0 = \cdots = v_n$. Let $v' = v - [0^{k-1}10^{n-k}]$

and $\sigma = \langle v' \rangle$. In other words, $x^v = x^{v'} x_k$, with k maximal. Then the maximal transition rewrites as

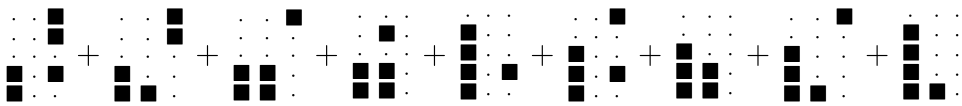
$$Y_v(\mathbf{x}, \mathbf{y}) = (x_k - y_{\sigma_k}) Y_{v'}(\mathbf{x}, \mathbf{y}) + \sum\nolimits_u Y_u(\mathbf{x}, \mathbf{y}) \;, \tag{3.9.4}$$

summed over all u such that $|u| = |v|$ and $\langle u \rangle \sigma^{-1}$ is a transposition τ_{ik} with $i < k$.

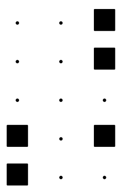
For example, starting with $v = [2, 0, 3]$, $\langle v' \rangle = \sigma = [3, 1, 5, 2, 4]$, one has the following sequence of transitions :

$$\begin{array}{rcl} Y_{203}(\mathbf{x}, \mathbf{y}) & = & (x_3 - y_5) Y_{202}(\mathbf{x}, \mathbf{y}) + Y_{230}(\mathbf{x}, \mathbf{y}) + Y_{401}(\mathbf{x}, \mathbf{y}) \,, \\ Y_{230}(\mathbf{x}, \mathbf{y}) & = & (x_2 - y_4) Y_{220}(\mathbf{x}, \mathbf{y}) + Y_{320}(\mathbf{x}, \mathbf{y}) \,, \\ Y_{401}(\mathbf{x}, \mathbf{y}) & = & (x_3 - y_2) Y_{400}(\mathbf{x}, \mathbf{y}) + Y_{410}(\mathbf{x}, \mathbf{y}) \,, \\ \cdots & & \cdots \end{array}$$

that one terminates when attaining dominant indices. Finally, writing each shifted monomial as a diagram of black squares in the Cartesian plane $\big($ a square in column i, row j corresponds to a factor $(x_i - y_j)$ $\big)$, the polynomial $Y_{203}(\mathbf{x}, \mathbf{y})$ reads



the first diagram, for example, coding the product



$$\Rightarrow \quad \begin{array}{ccc} \cdot & \cdot & (x_3 - y_5) \\ \cdot & \cdot & (x_3 - y_4) \\ \cdot & \cdot & \cdot \\ (x_1 - y_2) & \cdot & (x_3 - y_2) \\ (x_1 - y_1 \cdot & \cdot & \end{array} \;.$$

We shall give in the sequel a different combinatorial description of Schubert polynomials in terms of tableaux.

Fomin and Kirillov [39] give configurations from which one reads a different decomposition of Schubert polynomials into shifted monomials.

3.10 Branching rules

Let us ignore the term $(x_k - y_{\sigma_k}) Y_{v'}(\mathbf{x}, \mathbf{y})$ in the maximal transition formula (3.9.4) and write

$$Y_v \to \sum_u \qquad \text{or} \qquad X_\sigma \to \sum_\zeta X_\zeta \,, \tag{3.10.1}$$

where the u's or ζ's are described in (3.9.4).

However, if v is dominant, then $Y_v = (x_k - y_{\sigma_k}) Y_{v'}$ and it would not be very informative to write $Y_v \to 0$. Let us introduce the equivalence $v \sim [0, v]$, allowing the concatanation of $0's$ on the left, which corresponds to identify $\mathfrak{S}_n$ and its image $\mathfrak{S}_1 \times \mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$.

We can now iterate (3.10.1), producing an infinite graph.

Let us examine more closely the case where a permutation σ has only one successor. Write this permutation $\sigma = A\,2\,B\,4\,C\,3\,D$, with $2 < 3 < 4$, A, B, C, D being factors[5] such that $C\,3\,D$ is increasing, $D > 4$ and $B \cap [2, \ldots, 3] = \emptyset$. The successors of σ are all the permutations obtained by exchanging 3 in $A\,2\,B\,3\,C\,4\,D$ with a letter on its left such that length increases by 1 only. The permutation $\zeta = A\,3\,B\,2\,C\,4\,D$ fulfills this requirement, and if B does not contain any letter smaller than 2, then it is the unique successor of σ.

This indicates that permutations avoiding the pattern 2143 play a special role. Let us say that σ is *vexillary*[6] if there does not exist $i, j, k, l :\ \sigma_j < \sigma_i < \sigma_l < \sigma_k$. A *vexillary code* is the code of a vexillary permutation.

We have just seen that if σ is vexillary, then it has only one successor in a transition. In terms of codes, transition for vexillary codes reads as follows (eventually transforming v into $[0, v]$).

Lemma 3.10.1. *Let $v = [A\,b\,D\,c] \in \mathbb{N}^n$ be a vexillary code, with $c \neq 0$, the letter b being the rightmost occurence of the maximal value in $\{AbD\} \cap \{0, 1, \ldots, c{-}1\}$. Let $v' = [A\,b\,D\,c{-}1]$, $u = [A\,c\,D\,b]$, $\sigma = \langle v' \rangle$, $k = \sigma_n$. Then v' and u are vexillary codes, and*

$$Y_v(\mathbf{x}, \mathbf{y}) = (x_n - y_k) Y_{v'}(\mathbf{x}, \mathbf{y}) + Y_u(\mathbf{x}, \mathbf{y})\,. \tag{3.10.2}$$

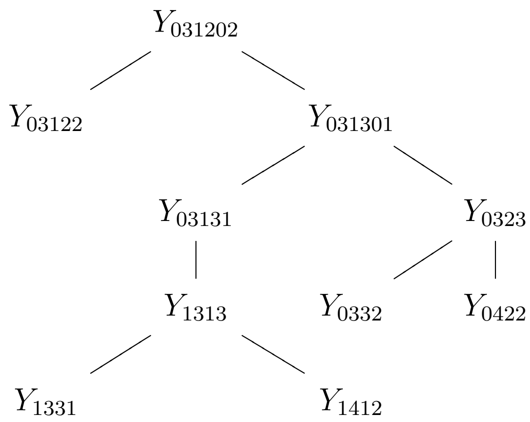
With this rule, here is the graph originating from the vexillary code $[0, 1, 2, 8, 2, 7, 6, 4]$:

$$\begin{aligned}
&[0,1,2,8,2,7,6,4] \to [0,1,2,8,4,7,6,2] \to [0,2,2,8,4,7,6,1] \\
&\qquad \to [1,2,2,8,4,7,6] \to [1,2,2,8,6,7,4] \to [1,2,4,8,6,7,2] \\
&\qquad\qquad \to [2,2,4,8,6,7,1] \sim [0,2,2,4,8,6,7,1] \\
&\qquad \to [1,2,2,4,8,6,7] \to [1,2,2,4,8,7,6] \to [1,2,2,6,8,7,4] \\
&\qquad\qquad\qquad \to [1,2,4,6,8,7,2] \to [2,2,4,6,8,7,1] \to \ldots
\end{aligned}$$

Since a vexillary code has only one successor, one can truncate any transition graph, stopping at each vexillary code. For example, for $v = [0, 3, 1, 2, 0, 2]$, the transition graph is :

[5] σ is considered as a word, and the letters $2, 3, 4$ are not necessarily consecutive in the alphabet. One requires only that $2 < 3 < 4$.

[6] There are a lot of flags in a flag variety, but M.P. Schützenberger and I needed still more, to describe the properties of certain permutations. This is why we introduced the latin root “vexillum”, which survived a first period of drought and flourished afterwards.



Garsia [50] studies in detail this transition tree.

3.11 Vexillary Schubert polynomials

To a permutation σ, with code $v \in \mathbb{N}^n$, one associates two partitions $\mu, \lambda \in \mathbb{N}^n$ as follows. Let $w \in \mathbb{N}^n$ be such that $w_i = \max(j : j \geq i, v_j \geq v_i)$. Then μ, is the decreasing reordering of w and λ be the minimum dominant weight such that Y_v is the image of Y_λ under a product of divided differences.

The next property shows that vexillary Schubert polynomials can be expressed as a multi-Schur function.

Proposition 3.11.1. *Let v be a vexillary code, μ and λ be the associated partitions defined just above. Then*

$$Y_v(\mathbf{x}, \mathbf{y}) = S_{v\uparrow}(\mathbf{x}_{\mu_1} - \mathbf{y}_{\lambda_n}, \ldots, \mathbf{x}_{\mu_n} - \mathbf{y}_{\lambda_1}) \,. \tag{3.11.1}$$

Proof. Normalize v by suppressing terminal 0's, so that one may suppose $r = v_n \neq 0$. Then the transition formula (3.10.2) states that

$$Y_v(\mathbf{x}, \mathbf{y}) = (x_n - y_k) Y_{v'}(\mathbf{x}, \mathbf{y}) + Y_u(\mathbf{x}, \mathbf{y})$$

Suppose the proposition to be true for v', by induction on weight, and u. The two Schur functions differ in only one column the sum being

$$(x_n - y_k) S_{\bullet, r-1, \bullet}(\bullet, \mathbf{x}_n - \mathbf{y}_{k-1}, \bullet) + S_{\bullet, r, \bullet}(\bullet, \mathbf{x}_{n-1} - \mathbf{y}_{k-1}, \bullet) \,.$$

Since for any j, any A (here, $A = \mathbf{x}_{n-1} - \mathbf{y}_{k-1}$), one has

$$(x_n - y_k) S_{j-1}(A + x_n) + S_j(A) = S_j(A + x_n - y_k)$$

this sum is equal to the expected multiSchur function $S_{\bullet, r, \bullet}(\bullet, \mathbf{x}_n - \mathbf{y}_k, \bullet)$. One initiates the proposition by the Grasmannian case, where the determinant is obtained as the image of $Y_\lambda(\mathbf{x}, \mathbf{y})$ under ∂_ω. QED

For example, for $v = [0, 2, 7, 2, 4, 5, 5, 4]$ one has $w[8, 8, 3, 8, 8, 7, 7, 8]$, $\mu = [8^5 7^2 3]$, $\lambda = [9^3 7^2 3^2 0]$,

$$\begin{aligned}
&Y_{02724554}(\mathbf{x}, \mathbf{y}) \\
&\quad = S_{02244557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_8{-}\mathbf{y}_7}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_3{-}\mathbf{y}_9) \\
&\quad = (x_8{-}y_7) Y_{027245530000}(\mathbf{x}, \mathbf{y}) + Y_{027445520000}(\mathbf{x}, \mathbf{y}) \\
&\quad = (x_8{-}y_7) S_{02234557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_8{-}\mathbf{y}_6}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_3{-}\mathbf{y}_9) \\
&\qquad + S_{02244557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_7{-}\mathbf{y}_6}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_3{-}\mathbf{y}_9) \,.
\end{aligned}$$

In the case of only one non-zero component, one has

$$Y_{0^{n-1}\,k}(\mathbf{x}, \mathbf{y}) = S_k(\mathbf{x}_n - \mathbf{y}_{k+n-1}) \,,$$

two of the indices appearing in the complete function determine the third one. The entries of the determinant (3.11.1) are not exactly of this type, but nevertheless, we are going to replace complete functions by Schubert polynomials.

Let us first modify (3.11.1), in the case of repeated parts of $v\!\uparrow$. Transform each block of columns

$$S_{\bullet\bullet k^r\bullet\bullet}(\bullet\bullet,\ \mathbf{x}_n\text{-}\mathbf{y}_m, \ldots, \mathbf{x}_n\text{-}\mathbf{y}_m,\ \bullet\bullet)$$

into

$$S_{\bullet\bullet k^r\bullet\bullet}(\bullet\bullet,\ \mathbf{x}_n\text{-}\mathbf{y}_m, \mathbf{x}_n\text{-}\mathbf{y}_{m+1}, \ldots, \mathbf{x}_n\text{-}\mathbf{y}_{m+r-1},\ \bullet\bullet)\,.$$

This amounts to adding to some columns a linear combination of the ones on its left, and does not change the value of the determinant.

The bottom row of the new determinant is

$$[Y_{0^{\mu_1-1}u_1}(\mathbf{x},\mathbf{y}),\ Y_{0^{\mu_2-1}u_2}(\mathbf{x},\mathbf{y}),\ \ldots, Y_{0^{\mu_n-1}u_n}(\mathbf{x},\mathbf{y})\,]\,,$$

with $u = v\!\uparrow -\rho$ (defining $Y_{0^j k}(\mathbf{x},\mathbf{y}) = 0$ when $k < 0$). If the bottom element of a column is $Y_{0^{r-1}k}(\mathbf{x},\mathbf{y}) = S_k(\mathbf{x}_r\text{-}\mathbf{y}_m)$, the last but one is

$$\begin{aligned} S_{k+1}(\mathbf{x}_r\text{-}\mathbf{y}_m) &= S_{k+1}(\mathbf{x}_r\text{-}\mathbf{y}_{m+1}) + y_{m+1}S_k(\mathbf{x}_r\text{-}\mathbf{y}_m) \\ &= Y_{0^{r-1}k+1}(\mathbf{x},\mathbf{y}) + y_{m+1}Y_{0^{r-1}k}(\mathbf{x},\mathbf{y})\,. \end{aligned}$$

However, since the index $m{+}1$ is different in each column, one cannot transform the last but one row into a row of Schubert polynomials by adding to it a multiple of the last row. This can be overcome by introducing a truncation map ϕ on polynomials in $\mathbf{x}, \mathbf{y}$. Given $f(\mathbf{x}, y_1, \ldots, y_m)$, of positive degree in y_m, define

$$\phi\big(f(\mathbf{x}, y_1, \ldots, y_m)\big) = f(\mathbf{x}, y_1, \ldots, y_{m-1}, 0)\,.$$

Thus the complete function $S_{k+1}(\mathbf{x}_r - \mathbf{y}_{k+r-1})$ can be written $\phi\big(Y_{0^{r-1}k+1}(\mathbf{x},\mathbf{y})\big)$, and more generally, the determinant expressing a vexillary Schubert polynomial can be expressed as a determinant with entries of the type $\phi^i\big(Y_{0^{r-1}k+i}(\mathbf{x},\mathbf{y})\big)$.

In summary, one has the following expression of a vexillary Schubert polynomial.

Proposition 3.11.2. *Let $v \in \mathbb{N}^n$ be a vexillary code, μ be the partition associated to it as in (3.11.1), $u = v\!\uparrow -\rho$. Then*

$$Y_v(\mathbf{x},\mathbf{y}) = \det\Big|\phi^{n-i}\big(Y_{0^{\mu_j-1}u_j+n-i}\big)\Big|\,. \tag{3.11.2}$$

Continuing the preceding example, for $v = [0,2,7,2,4,5,5,4]$ one has $n = 8$, $\mu = [8^5 7^2 3]$, $u = [-7,-4,-3,0,1,3,4,7]$ and $Y_v(\mathbf{x},\mathbf{y})$ is equal to the determinant

$$\Big|\phi^{8-i}\big(Y_{0^7,1-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^7,4-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^7,5-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^7,8-i}(\mathbf{x},\mathbf{y})\big),$$
$$\phi^{8-i}\big(Y_{0^7,9-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^6,11-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^6,12-i}(\mathbf{x},\mathbf{y})\big),\ \phi^{8-i}\big(Y_{0^2,15-i}(\mathbf{x},\mathbf{y})\big)\Big|\,.$$

3.12 Schubert and hooks

We have seen that vexillary Schubert polynomials can be expressed as determinants. It is therefore natural to have recourse to the theory of minors to obtain properties of the different families of polynomials we have seen so far. We have already used Binet-Cauchy formula for the minors of a product of two matrices, or Jacobi's formula for the minors of the adjoint of a matrix.

Another powerful relation is Bazin formula for determinants of minors. Let M be na $n \times \infty$ matrix. Given $v \in \mathbb{N}^n$, denote by $[v]$ the minor of order n of M taken on columns $v_1, \ldots, v_n$. Let $r \leq n$, $A, B \in \mathbb{N}^r$, $C \in \mathbb{N}^{n-r}$. Then Bazin formula [108, p.188] is

$$\det\Big|[A \setminus a, b, C]\Big|_{a\in A, b\in B} = [A, C]^{r-1}\,[B, C] \tag{3.12.1}$$

It is remarked in [131] that the expression of a Schur function $s_\lambda(\mathbf{x}_n)$ as a determinant of hook-Schur functions is a direct corollary of Bazin formula for the matrix $M = \Big[s_{j-i}(\mathbf{x}_n)\Big]_{i=1,\ldots,n;\, j=1,\ldots,\infty}$.

More general matrices produce analog formulas for different generalizations of Schur functions, as illustrated by Macdonald [149]. In fact, Macdonald 9th variation (see also [146, Ex.21, p.57]) is precisely a direct proof of Bazin formula in the special case where the minor $[A, C]$ is equal to 1. As a corollary of it, Olshanski, Regev and Vershik [164, Prop.3.1] give the expression of a Graßmannian Schubert polynomial in terms of hooks. We are going to show that Bazin formula applies to any vexillary Schubert polynomial, but we have first to precise what is a hook in Schubert calculus.

For Schubert, an “elementary condition“ meant a Schubert subvariety of a Graßmannian indexed by a hook partition. He expressed the class of a general Schubert variety in the cohomology ring as a determinant of hooks. Giambelli in his thesis [55] explicited the cohomology ring (in fact the Chow ring) of a Graßmannian as a ring of symmetric polynomials, with linear basis the classes of Schubert varieties indexed by a partition contained in a fixed rectangle, identifying them with Schur functions (defined as determinants of complete or elementary symmetric functions). Thus Giambelli's contribution, for what concerns Schur functions, is rather in the equality between the Jacobi-Trudi determinant and the determinant of hook-Schur functions which now bears his name, but is due to Schubert. We shall nevertheless keep the terminology *Giambelli determinant* for the determinant of hooks equal to a Schur function.

Changing in a Giambelli determinant every hook Schur function $Y_{0^c1^ba}(\mathbf{x}, \mathbf{0})$ into $Y_{0^c1^ba}(\mathbf{x}, \mathbf{y})$, one notices on a few examples that the new determinant becomes equal to a Graßmannian Schubert polynomial (this is not true for the Jacobi-Trudi determinant). So one can expect vexillary Schubert polynomials to be amenable to hooks, but one has to extract more information from the code that the Frobenius coding of the partition obtained by reordering it.

. Thus let us define a *hook Schubert polynomial* to be a polynomial $Y_v(\mathbf{x},\mathbf{y})$, where all components v_i belong to $\{0,1\}$ except at most one, and such that v be a vexillary code. For example, $[1,1,3],[1,3,1],[3,1,1],[3,0,1,1]$, $[3,0,0,1,1]$ are allowable, but not $[1,1,0,3]$, nor $[3,0,0,0,1,1]$.

For $k\geq 1$, $w=[0^\gamma,1^\beta]$, let $n=\beta+\gamma+1$. For $j\leq n-1$, let $v=[w_1,\ldots,w_{j-1},k,w_j,\ldots,w_{n-1}]$. If v is vexillary, then the determinant 3.11.1 can easily be transformed into

$$Y_v(\mathbf{x},\mathbf{y})=S_{1^\beta k}(\mathbf{x}_n-\mathbf{y}_{\gamma+1},\ldots,\mathbf{x}_n-\mathbf{y}_{\gamma+1},\mathbf{x}_j-\mathbf{y}_{k+j-1})\,,$$

determinant that one can denote $s_{k-1|\beta}(n|j)$, in accordance with the notation used in the case of Schur functions [146, p.47].

Going back to Bazin, let $v\in\mathbb{N}^n$ be a vexillary code, $u=v\uparrow-\rho$, $r=\#\{j: u_j>0\}$. Let us enlarge the matrix 3.11.2 into a matrix of order $n\times(n+r)$. One transforms the sequence $\boxed{\begin{smallmatrix}\mu_1\\ u_1\end{smallmatrix}},\ldots,\boxed{\begin{smallmatrix}\mu_n\\ u_n\end{smallmatrix}}$ into the sequence

$$\boxed{\begin{matrix}\mu_1\\ u_1\end{matrix}}\;\begin{bmatrix}\mu_1\\ u_1{+}1\end{bmatrix},\ldots,\boxed{\begin{matrix}\mu_2\\ u_2\end{matrix}},\begin{bmatrix}\mu_2\\ u_2{+}1\end{bmatrix},\ldots,\begin{bmatrix}\mu_{n-r}\\ 0\end{bmatrix},\boxed{\begin{matrix}\mu_{n-r+1}\\ u_{n-r+1}\end{matrix}},\ldots,\boxed{\begin{matrix}\mu_n\\ u_n\end{matrix}},$$

by inserting in the sequence $u_1,\ldots,u_n$ the values in $[u_1,\ldots,0]$ missing, and correspondingly, completing the exponents by keeping the first exponent μ_j on its left. Let ν be the new upper sequence, and η the lower sequence. Define $M_v(\mathbf{x},\mathbf{y})$ to be the matrix

$$M_v(\mathbf{x},\mathbf{y})\Big[\phi^{n-i}\big(Y_{0^{\nu_j-1}\eta_j+n-i}\big)\Big]_{i=1,\ldots,n;\,j=1,\ldots,n+r}\,. \tag{3.12.2}$$

Thus, for the same $v=[0,2,7,2,4,5,5,4]$ as above, the sequence

$$\boxed{\begin{matrix}8\\ \bar7\end{matrix}}\boxed{\begin{matrix}8\\ \bar4\end{matrix}}\boxed{\begin{matrix}8\\ \bar3\end{matrix}}\boxed{\begin{matrix}8\\ 0\end{matrix}}\boxed{\begin{matrix}8\\ 1\end{matrix}}\boxed{\begin{matrix}7\\ 3\end{matrix}}\boxed{\begin{matrix}7\\ 4\end{matrix}}\boxed{\begin{matrix}3\\ 7\end{matrix}}$$

is transformed into

$$\boxed{\begin{matrix}8\\ \bar7\end{matrix}}\;\begin{bmatrix}8\\ \bar6\end{bmatrix}\;\begin{bmatrix}8\\ \bar5\end{bmatrix}\;\boxed{\begin{matrix}8\\ \bar4\end{matrix}}\boxed{\begin{matrix}8\\ \bar3\end{matrix}}\;\begin{bmatrix}8\\ \bar2\end{bmatrix}\;\begin{bmatrix}8\\ \bar1\end{bmatrix}\;\boxed{\begin{matrix}8\\ 0\end{matrix}}\boxed{\begin{matrix}8\\ 1\end{matrix}}\boxed{\begin{matrix}7\\ 3\end{matrix}}\boxed{\begin{matrix}7\\ 4\end{matrix}}\boxed{\begin{matrix}3\\ 7\end{matrix}},$$

and therefore $\nu=[8^9,7^2,3]$ and $\eta=[-7,\ldots,0,1,3,4,7]$.

The Giambelli determinant for $s_{v\downarrow}(\mathbf{x}_n)$ used the Bazin formula for the matrix obtained from $M_v(\mathbf{x},\mathbf{y})$ by erasing ϕ and replacing each $Y_{0^a k}(\mathbf{x},\mathbf{y})$ by $s_k(\mathbf{x}_n)$. Taking the same minors as in the case of Schur functions [131] gives a determinantal expression of the vexillary Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$, where the entries $s_{\alpha|\beta}(\mathbf{x}_n)$ have been replaced by some hook Schubert polynomials. Making precise which hooks appear is the subject of the next proposition, after introducing two more vectors associated to a vexillary code.

Let v be vexillary, and r be the rank of the partition $v\downarrow$. Let $b_1,b_2,\ldots$ be the levels of the bottom boxes of each of the non-void columns of the Rothe diagram of v (taking matrix coordinates), written in decreasing order. The first vector that we need is $t^v = [t_1, \ldots, t_r] = [b_1{-}b_1, b_1{-}b_2, \ldots, b_1{-}b_r]$. The second vector starts from the code $c = [c_1, c_2, \ldots]$ of v, that is $c_i = \#\{j : j > i, v_i > v_j\}$, as when v is a permutation. Let $d^v = [d_1, \ldots, d_r]$ be the truncation of $c\!\downarrow$ to its first r components. Then Bazin formula gives the following determinantal expression of vexillary Schubert polynomial.

Proposition 3.12.1. *Let $v \in \mathbb{N}^n$ be a vexillary code, $\lambda = v\!\downarrow$, $(\alpha|\beta)$ be the Frobenius notation of λ, and t^v, d^v be the two vectors defined just above. For every $i, j \le r$, let*

$$w^j = [0^{n-1-\beta_j-t_j} 1^{\beta_j} 0^{t_j}] \quad \& \quad w^{ij} = [w^j_1, \ldots, w^j_{n-1-d_i},\ \alpha_i{+}1,\ w^j_{n-d_i}, \ldots, w^j_{n-1}]\,.$$

Then

$$Y_v(\mathbf{x},\mathbf{y}) = \det\Big|Y_{w^{ij}}(\mathbf{x},\mathbf{y})\Big|_{i,j=1,\ldots,r} \tag{3.12.3}$$

$$= \det\Big|s_{\alpha_i|\beta_j}(n{-}d_i|n{-}t_j)\Big|_{i,j=1,\ldots,r} \tag{3.12.4}$$

For example, for $v = [0, 2, 7, 2, 4, 5, 5, 4]$, one has $(\alpha|\beta) = (6320|6521)$, $r = 4$, $n = 8$, $c = [0, 0, 5, 0, 0, 1, 1, 0]$, $d^v = [5, 1, 1, 0]$, $t^v = [0, 0, 0, 0]$,

$$Y_v(\mathbf{x},\mathbf{y}) = \begin{vmatrix} Y_{0171^5}(\mathbf{x},\mathbf{y}) & Y_{0^271^5}(\mathbf{x},\mathbf{y}) & Y_{0^270^31^2}(\mathbf{x},\mathbf{y}) & Y_{0^270^41}(\mathbf{x},\mathbf{y}) \\ Y_{01^541}(\mathbf{x},\mathbf{y}) & Y_{0^21^441}(\mathbf{x},\mathbf{y}) & Y_{0^5141}(\mathbf{x},\mathbf{y}) & Y_{0^641}(\mathbf{x},\mathbf{y}) \\ Y_{01^531}(\mathbf{x},\mathbf{y}) & Y_{0^21^431}(\mathbf{x},\mathbf{y}) & Y_{0^5131}(\mathbf{x},\mathbf{y}) & Y_{0^631}(\mathbf{x},\mathbf{y}) \\ Y_{01^61}(\mathbf{x},\mathbf{y}) & Y_{0^21^51}(\mathbf{x},\mathbf{y}) & Y_{0^51^21}(\mathbf{x},\mathbf{y}) & Y_{0^611}(\mathbf{x},\mathbf{y}) \end{vmatrix}$$

$$= \begin{vmatrix} s_{6|6}(3|8) & s_{6|5}(3|8) & s_{6|2}(3|8) & s_{6|1}(3|8) \\ s_{3|6}(7|8) & s_{3|5}(7|8) & s_{3|2}(7|8) & s_{3|1}(7|8) \\ s_{2|6}(7|8) & s_{2|5}(7|8) & s_{2|2}(7|8) & s_{2|1}(7|8) \\ s_{0|6}(8|8) & s_{0|5}(8|8) & s_{0|2}(8|8) & s_{0|1}(8|8) \end{vmatrix}.$$

The Giambelli determinant is compatible with transitions. For example, one has

$$Y_{57604311}(\mathbf{x},\mathbf{y}) = \begin{vmatrix} s_{6|6}(2|8) & s_{6|3}(2|6) & s_{6|2}(2|6) & s_{6|0}(2|5) \\ s_{4|6}(3|8) & s_{4|3}(3|6) & s_{4|2}(3|6) & s_{4|0}(3|5) \\ s_{2|6}(3|8) & s_{2|3}(3|6) & s_{2|2}(3|6) & s_{2|0}(3|5) \\ s_{0|6}(5|8) & s_{0|3}(5|6) & s_{0|2}(5|6) & s_{0|0}(5|5) \end{vmatrix}$$

The transition $Y_{\boxed{5}7\boxed{6}04311}(\mathbf{x},\mathbf{y}) = (x_3 - y_7)Y_{\boxed{5}7\boxed{5}04311}(\mathbf{x},\mathbf{y}) + Y_{\boxed{6}7\boxed{5}04311}(\mathbf{x},\mathbf{y})$ amounts to decompose the preceding determinant as the sum

$$(x_3{-}y_7)\begin{vmatrix} s_{6|6}(2|8) & s_{6|3}(2|6) & s_{6|2}(2|6) & s_{6|0}(2|5) \\ s_{3|6}(3|8) & s_{3|3}(3|6) & s_{3|2}(3|6) & s_{3|0}(3|5) \\ s_{2|6}(3|8) & s_{2|3}(3|6) & s_{2|2}(3|6) & s_{2|0}(3|5) \\ s_{0|6}(5|8) & s_{0|3}(5|6) & s_{0|2}(5|6) & s_{0|0}(5|5) \end{vmatrix}$$

$$+ \begin{vmatrix} s_{6|6}(2|8) & s_{6|3}(2|6) & s_{6|2}(2|6) & s_{6|0}(2|5) \\ s_{4|6}(2|8) & s_{4|3}(2|6) & s_{4|2}(2|6) & s_{4|0}(2|5) \\ s_{2|6}(3|8) & s_{2|3}(3|6) & s_{2|2}(3|6) & s_{2|0}(3|5) \\ s_{0|6}(5|8) & s_{0|3}(5|6) & s_{0|2}(5|6) & s_{0|0}(5|5) \end{vmatrix},$$

each element of the second row decomposing as

$$s_{4|b}(3|N) = (x_3 - y_7)s_{3|b}(3|N) + s_{4|b}(2|N)\,.$$

The order of the Giambelli determinant can decrease by 1 in a transition, but this case also can be followed on the determinants. For example, the transition $Y_{576\boxed{04}311}(\mathbf{x},\mathbf{y}) = (x_5 - y_5)Y_{5760\boxed{03}311}(\mathbf{x},\mathbf{y}) + Y_{576\boxed{40}311}(\mathbf{x},\mathbf{y})$ gives the sum

$$(x_5{-}y_5)\begin{vmatrix} s_{6|6}(2|8) & s_{6|3}(2|6) & s_{6|2}(2|6) \\ s_{4|6}(3|8) & s_{4|3}(3|6) & s_{4|2}(3|6) \\ s_{2|6}(3|8) & s_{2|3}(3|6) & s_{2|2}(3|6) \end{vmatrix} + \begin{vmatrix} s_{6|6}(2|8) & s_{6|3}(2|6) & s_{6|2}(2|6) & s_{6|0}(2|4) \\ s_{4|6}(3|8) & s_{4|3}(3|6) & s_{4|2}(3|6) & s_{4|0}(3|4) \\ s_{2|6}(3|8) & s_{2|3}(3|6) & s_{2|2}(3|6) & s_{2|0}(3|4) \\ s_{0|6}(4|8) & s_{0|3}(4|6) & s_{0|2}(4|6) & s_{0|0}(4|4) \end{vmatrix}.$$

In fact, the determinants expressing $Y_{57604311}(\mathbf{x},\mathbf{y})$ and $Y_{57640311}(\mathbf{x},\mathbf{y})$ differ only by their South-East entry, respectively $s_{0|0}(5|5)$ and $s_{0|0}(4|4)$. Since $s_{0|0}(5|5) - s_{0|0}(4|4) = x_5{-}y_5$, the difference of the two determinants is equal to the minor of this entry times $x_5{-}y_5$.

3.13 Stable part of Schubert polynomials

In the theory of symmetric functions, one usually prefers to eliminate variables by taking the projective limit $\mathfrak{Sym}(\mathbf{x}_\infty)$ of the ring $\mathfrak{Sym}(x_1,\dots,x_n)$, which amounts to using infinite alphabets.

In terms of Schubert polynomials, the embedding $\mathfrak{Sym}(\mathbf{x}_n) \hookrightarrow \mathfrak{Sym}(\mathbf{x}_{n+1})$ translates into the transformation $Y_v(\mathbf{x},\mathbf{0}) \to Y_{0v}(\mathbf{x},\mathbf{0})$ for v antidominant. This leads to define the *stable part* $\mathcal{S}t(Y_v)$ of a Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$, as

$$\mathcal{S}t(Y_v) = Y_{0^N v}(\mathbf{x},\mathbf{y})\Big|_{x_j=0=y_j,\, j>N},$$

with N big enough, and consider it as an element of $\mathfrak{Sym}(\mathbf{x}_\infty)\otimes\mathfrak{Sym}(\mathbf{y}_\infty)$.

We first need to analyze the transformation $Y_v(\mathbf{x},\mathbf{y}) \to Y_{0v}(\mathbf{x},\mathbf{y})$ to compare $Y_{0^N v}(\mathbf{x},\mathbf{y})$ and $Y_{0^{N+1} v}(\mathbf{x},\mathbf{y})$ and precise what "N big enough" means.

Lemma 3.13.1. *Let $v\in\mathbb{N}^n$, $v\le[n,\dots,1]$. Then*

$$\begin{aligned} Y_v(\mathbf{x},\mathbf{0})\pi^x_n\dots\pi^x_1 &= Y_{0v}(\mathbf{x},\mathbf{0}) && (3.13.1)\\ Y_v(\mathbf{x},\mathbf{y})\pi^x_n\dots\pi^x_1\pi^y_n\dots\pi^y_1 &= Y_{0v}(\mathbf{x},\mathbf{y})\,. && (3.13.2)\end{aligned}$$

Proof. By trivial commutation, one writes $\pi^x_n\dots\pi^x_1 = x_n\dots x_1\partial^x_n\dots\partial^x_1$, and one uses that $Y_v(\mathbf{x},\mathbf{0})x_n\dots x_1 = Y_{v+1^n}(\mathbf{x},\mathbf{0})$ when $v\in\mathbb{N}^n$. This proves the first statement. Writing $Y_v(\mathbf{x},\mathbf{y})$ as a sum $\sum c_{u,u'}Y_u(\mathbf{x},\mathbf{0})Y_{u'}(\mathbf{y},\mathbf{0})$, one obtains that $Y_v(\mathbf{x},\mathbf{y})\pi^x_n\dots\pi^y_1$ is equal to $\sum c_{u,u'}Y_{0u}(\mathbf{x},\mathbf{0})Y_{0u'}(\mathbf{y},\mathbf{0})$, that is, to $Y_{0v}(\mathbf{x},\mathbf{y})$. QED

Lemma 3.13.2. *Let $f\in\mathfrak{Pol}(\mathbf{x}_n)\otimes\mathfrak{Pol}(\mathbf{y}_m)$, $\omega_n=[n,\dots,1]$, $\omega_m=[m,\dots,1]$, $\pi_{n\times n}=(\pi_n\dots\pi_{2n-1})\dots(\pi_1\dots\pi_n)$. Then*

$$f\,\pi^x_{\omega_n}\pi^y_{\omega_m} = f\,\pi^x_{n\times n}\pi^y_{m\times m}\Big|_{x_i=0,i>n,\,y_j=0,j>m}. \qquad (3.13.3)$$

Proof. Any monomial x^v, $v\in\mathbb{N}^n$, can be written $x^v = S_{v\omega}(\mathbf{x}_n,\mathbf{x}_{n-1},\dots,\mathbf{x}_1)$, and its image under $\pi_n\dots\pi_{2n-1}$ is equal to $S_{v\omega}(\mathbf{x}_{2n},\mathbf{x}_{n-1},\dots,\mathbf{x}_1)$, which is sent to $S_{v\omega}(\mathbf{x}_{2n},\mathbf{x}_{2n-1},\mathbf{x}_{n-2},\dots)$ under $\pi_{n-1}\dots\pi_{2n-2}$. In final, $x^v\pi_{n\times n}$ is equal to $S_{v\omega}(\mathbf{x}_{2n},\mathbf{x}_{2n-1},\dots,\mathbf{x}_{n+1})$, and this function restricts to $S_{v\omega}(\mathbf{x}_n) = x^v\pi_{\omega_n}$. QED

For $v\le[n,\dots,1]$, the stable part of $Y_v(\mathbf{x},\mathbf{y})$ is obtained by computing $Y_{0^n v}(\mathbf{x},\mathbf{y})$, which is the image of $Y_v(\mathbf{x},\mathbf{y})$ under $(\pi^x_n\dots\pi^x_1)\dots(\pi^x_{2n-1}\dots\pi^x_1)\,(\pi^y_n\dots\pi^y_1)\dots(\pi^y_{2n-1}\dots\pi^y_1)$ according to (3.13.2). But the product of divided differences can be rewritten $\pi^x_{1,\dots,n,2n,\dots,n+1}\pi^y_{1,\dots,n,2n,\dots,n+1}\pi^x_{n\times n}\pi^y_{n\times n}$. The first two factors preserve functions of $\mathbf{x}_n$ and $\mathbf{y}_n$. Therefore,

$$Y_{0^n v}(\mathbf{x},\mathbf{y}) = Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{n\times n}\pi^y_{n\times n}\,.$$

Using (3.13.3), one sees that

$$\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) = Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{n\times n}\pi^y_{n\times n}\,. \qquad (3.13.4)$$

A transition

$$Y_v(\mathbf{x},\mathbf{y}) = (x_k - y_j) Y_{v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_u(\mathbf{x},\mathbf{y})$$

entails a transition

$$Y_{0^n v}(\mathbf{x},\mathbf{y}) = (x_{k+n} - y_{j+n}) Y_{0^n v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_{0^n u}(\mathbf{x},\mathbf{y})\,.$$

Therefore transitions may be used to compute stable parts :

$$\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) = \mathcal{S}t(Y_{0^n v}(\mathbf{x},\mathbf{y})) = \sum_{u\in\mathcal{U}} \mathcal{S}t(Y_u(\mathbf{x},\mathbf{y}))\,. \tag{3.13.5}$$

The determinantal expression of a vexillary polynomial, for $v \le [n,\dots,1]$, shows that its stable part is equal to

$$\mathcal{S}t(Y_{0^n v}(\mathbf{x},\mathbf{y})) = S_{v\uparrow}(\mathbf{x}_n - \mathbf{y}_n)\,.$$

One can in fact relax the condition on v. If $Y_\lambda(\mathbf{x},\mathbf{y})$ is a dominant ancestor of $Y_v(\mathbf{x},\mathbf{y})$, with $v \in \mathbb{N}^n$ and $m=\lambda_1$, then $Y_v(\mathbf{x},\mathbf{y})$ is a polynomial in $x_1,\dots,x_n$ and $y_1,\dots y_m$. Using (3.13.2) and (3.13.3), one sees[7] that

$$Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{\omega_n}\pi^y_{\omega_m} = S_{v\uparrow}(\mathbf{x}_n - \mathbf{y}_m)\,. \tag{3.13.6}$$

In summary, one has the following three ways of determining the stable part of a Schubert polynomial.

Theorem 3.13.3. *Let $v\in\mathbb{N}^n$, Y_λ be a dominant ancestor of Y_v, $m=\lambda_1$. Let $Y_{0v}(\mathbf{x},\mathbf{y}) = (x_k - y_j)Y_{0v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_u(\mathbf{x},\mathbf{y})$ be a transition. Then*

$$\begin{aligned}
\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) &= Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{\omega_n}\pi^y_{\omega_m} && (3.13.7)\\
&= Y_{0^{n+m}v}(\mathbf{x},\mathbf{y})\Big|_{x_i=0, i>n,\, y_j=0, j>m} && (3.13.8)\\
&= \sum_{u\in\mathcal{U}} \mathcal{S}t(Y_u(\mathbf{x},\mathbf{y}))\,. && (3.13.9)
\end{aligned}$$

For example, the transition graph for $v=[0,3,1,2,0,2]$ given above has five terminal vertices: $Y_{03122}, Y_{1331}, Y_{1412}, Y_{0332}, Y_{0422}$, and this implies that

$$\begin{aligned}
\mathcal{S}t(Y_{031202}(\mathbf{x},\mathbf{y})) = s_{3221}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{3311}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{4211}(\mathbf{x}_\infty - \mathbf{y}_\infty)\\
+ s_{332}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{422}(\mathbf{x}_\infty - \mathbf{y}_\infty)\ .
\end{aligned}$$

[7] The action of $\pi^x_{\omega_n}$ on the determinant of complete functions of $\mathbf{x}_k - \mathbf{y}_j$ expressing $Y_v(\mathbf{x},\mathbf{y})$ consists in replacing all $\mathbf{x}_k$ by $\mathbf{x}_n$. The action of $\pi^y_{\omega_m}$ is much more delicate, one has to use that some determinants of complete functions in $\mathbf{x}_k - \mathbf{y}_j$ can be written as determinants of complete functions in $\mathbf{y}_j - \mathbf{x}_k$ (cf. [94]). For example, the equality $X_\sigma(\mathbf{x},\mathbf{y}) = (-1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x})$ gives such a transformation of determinants in the vexllary case. We have bypassed this transformation by using $Y_v(\mathbf{x},\mathbf{y}) \to Y_{0^N v}(\mathbf{x},\mathbf{y})$.

We shall see later that

$$Y_{031202}(\mathbf{x},\mathbf{0}) = K_{31202} + K_{31301} + K_{41201} + K_{323} + K_{422}\,.$$

Since evidently the image under π_ω of a key polynomial is a Schur function, the decomposition of a Schubert polynomial (specialized in $\mathbf{y}=\mathbf{0}$) into key polynomials is still another way of computing its stable part.

A special case of the determination of the stable part of a vexillary Schubert polynomial is the *Sergeev-Pragacz formula* showing that a Schur function of a difference of alphabets $\mathbf{x}_n$–$\mathbf{y}_m$ can be obtained by symmetrization of a product of differences x_i–y_j. Indeed, let $\lambda \in \mathbb{N}^n$ be dominant, $m \geq \lambda_1$. Then

$$Y_\lambda(\mathbf{x},\mathbf{y})\pi^x_{\omega_n}\pi^y_{\omega_m} = S_{\lambda\uparrow}(\mathbf{x}_n,\mathbf{y}_m)\,. \tag{3.13.10}$$

For example, writing the explicit expression of π_ω a a sum over the symmetric group, one has

$$\begin{aligned}S_{024}(\mathbf{x}_3-\mathbf{y}_4) &= Y_{420}\,\pi^x_{321}\pi^y_{4321}\\ &= \frac{1}{\Delta(x_1,x_2,x_3)\Delta(y_1,y_2,y_3,y_4)} \sum_{\sigma\in\mathfrak{S}^x_3,\zeta\in\mathfrak{S}^y_4} (-1)^{\ell(\sigma)+\ell(\zeta)}\big(x^{210}y^{3210}\,Y_\lambda\big)^{\sigma\zeta}\,.\end{aligned}$$

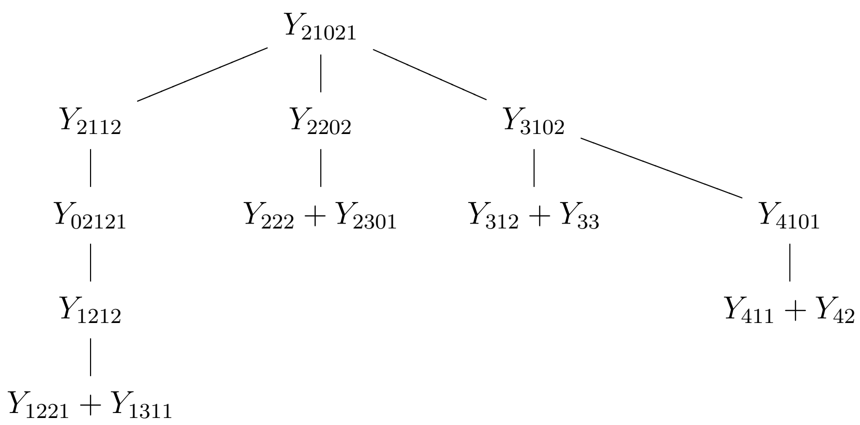
3.14 Schubert and the Littlewood-Richardson rule

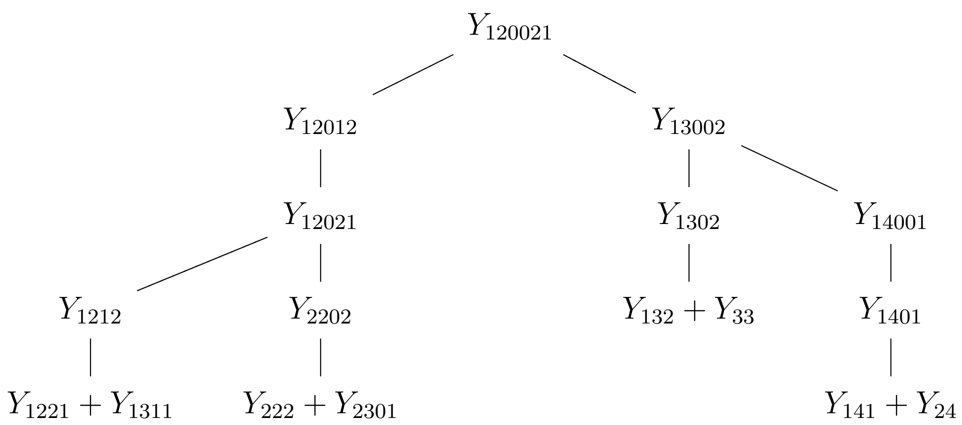
When a permutation $\sigma \in \mathfrak{S}_n$ belongs to a Young subgroup $\mathfrak{S}_{n'} \times \mathfrak{S}_{n''}$, the Schubert polynomial $X_\sigma(\mathbf{x}, \mathbf{y}) = Y_{v',v''}(\mathbf{x}, \mathbf{y})$ factorizes. This factorization is compatible with the restriction[8] of $Y_{0^N,v',v''}(\mathbf{x}, \mathbf{y})$ to $\mathbf{x}_N, \mathbf{y}_N$, and therefore in that case

$$\mathcal{S}t(Y_v(\mathbf{x}, \mathbf{y})) = \mathcal{S}t(Y_{v'}(\mathbf{x}, \mathbf{y}))\, \mathcal{S}t(Y_{v''}(\mathbf{x}, \mathbf{y}))\,.$$

In particular, when the Schubert polynomial factorizes into two vexillary Schubert polynomials, then its stable part is the product of two Schur functions. Since the stable part can be computed by transition, this observation furnishes many ways, different from the usual *Littlewood-Richardson rule*, of computing the product of Schur functions.

For example, to compute the square of s_{21}, one can start with any $v = v'v''$, with $v', v'' \in \{[2,1,0], [2,0,1,0], [1,2,0,0]\}$. Here are two possible transition graphs, starting with $[2,1,0,2,1,0]$ or $[2,1,0,1,2,0,0]$, which are the codes of the permutations $[3,2,1,6,5,4] \in \mathfrak{S}_3 \times \mathfrak{S}_3$ and $[3,2,1,5,7,4,6] \in \mathfrak{S}_3 \times \mathfrak{S}_4$, and stopping at vexillary codes.





[8] using symmetization is more delicate, since symmetrization does not commute vith product in general.

Both graphs imply that

$$s_{21}s_{21} = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211} \ .$$

Chapter 4

Products and transitions for Grothendieck and Keys

4.1 Monk formula for type A key polynomials

Instead of considering the multiplication by each x_i in the key basis, let us describe the multiplication by

$$\xi = \xi_n^A = y_1x_1 + \cdots + y_nx_n\,.$$

This element is invariant under the symmetric group acting on x_i and y_i simultaneously, and therefore, for any permutation σ, one has $(\xi)^{\sigma^x} = (\xi)^{(\sigma^y)^{-1}}$.

Since key polynomials are obtained by applying on dominant monomials the operators π_σ, $\sigma \in \mathfrak{S}_n$, we essentially need to describe the products $\pi_\sigma\xi$, that we shall write

$$\pi_\sigma\xi = x_1\varphi_\sigma^1 + \cdots + x_n\varphi_\sigma^n\,.$$

The commutation relations $\pi_ix_i = x_{i+1}\pi_i + x_i$, $\pi_ix_{i+1} = x_i\pi_i - x_i = x_i\widehat{\pi}_i$, $\pi_1\ldots\pi_ix_{i+1} = x_1\widehat{\pi}_1\ldots\widehat{\pi}_i$ imply

$$\begin{aligned}\pi_1\ldots\pi_{k-1}\xi &= \pi_1\ldots\pi_{k-2}\,(\xi)^{s_{k-1}^y}\pi_{k-1} + \pi_1\ldots\pi_{k-2}x_{k-1}(y_{k-1}-y_k)\\ &= \pi_1\ldots\pi_{k-3}\,(\xi)^{s_{k-2}^ys_{k-1}^y}\pi_{k-2}\pi_{k-1}\\ &\quad + \pi_1\ldots\pi_{k-3}x_{k-2}\pi_{k-1}(y_{k-2}-y_k) + x_1\widehat{\pi}_1\ldots\widehat{\pi}_{k-3}(y_{k-1}-y_k)\,.\end{aligned}$$

Iterating and grouping the coefficients of y_k, one obtains

$$\begin{aligned}\pi_1\ldots\pi_{k-1}\xi = (\xi)^{s_1^y\ldots s_{k-1}^y}\pi_1\ldots\pi_{k-1} + x_1\Big(&\widehat{\pi}_1\ldots\widehat{\pi}_{k-1}y_k + \widehat{\pi}_1\ldots\widehat{\pi}_{k-2}y_{k-1}\\ &+ \widehat{\pi}_1\ldots\widehat{\pi}_{k-3}y_{k-2}\pi_{k-1} + \widehat{\pi}_1\ldots\widehat{\pi}_{k-4}y_{k-3}\pi_{k-2}\pi_{k-1} + \cdots + y_1\pi_2\ldots\pi_k\Big)\,.\end{aligned} \tag{4.1.1}$$

Given a permutation $\sigma \in \mathfrak{S}_n$, let us write it $\sigma = \zeta s_1\ldots s_{k-1}$, with $\zeta \in \mathfrak{S}_{1\times n-1}$. Relation 4.1.1 entails

$$\varphi_\sigma^i = \left(\varphi_\zeta^i\right)^{s_1^y\ldots s_{k-1}^y}\pi_1\ldots\pi_{k-1}\ ,\ i \geq 2$$

$$\varphi^1_\sigma = \pi_\zeta\Big(y_1\pi_2\ldots\pi_k+\cdots+\widehat{\pi}_1\ldots\widehat{\pi}_{k-1}y_k\Big)\,,$$

while $\varphi^1_\zeta = \pi_\zeta y_1$.

These recursions furnish an induction on n for the products $K_v\xi$.

Proposition 4.1.1. *Let $v\in\mathbb{N}^n$, $\lambda = v\downarrow$, $\sigma\in\mathfrak{S}_n$, $\zeta\in\mathfrak{S}_{1\times n-1}$ be such that $K_v\pi_\sigma = x^\lambda\pi_\zeta\pi_1\ldots\pi_{k-1}$. Then*

$$K_v\xi = \Big(x^\lambda\pi_\zeta\xi\Big|_{y_1=0}\Big)^{s^y_1\ldots s^y_{k-1}} + \\ x^\lambda x_1\pi_\zeta\Big(y_1\pi_2\ldots\pi_k+\widehat{\pi}_1y_2\pi_3\ldots\pi_k+\cdots+\widehat{\pi}_1\ldots\widehat{\pi}_{k-1}y_k\Big)\,. \quad (4.1.2)$$

For example, when $v=[1,3,5,7]$, one has $\lambda=[7,5,3,1]$, $\sigma=[4,3,2,1]$, $\zeta=[1,4,3,2]$. Supposing known that

$$K_{7135}\,\xi - y_1K_{8135} = \big(y_4K_{7136}+(y_3-y_4)K_{7163}+(y_2-y_3)K_{7613}\big) \\ +\big(y_3K_{7145}+(y_2-y_3)K_{7415}\big)+y_2K_{7235}\,,$$

one obtains

$$x^{7531}\big(x_2\varphi^2_{4321}+x_3\varphi^3_{4321}+x_4\varphi^4_{4321}\big) = \big(y_3K_{1367}+(y_2-y_3)K_{1637} \\ +(y_1-y_2)K_{6137}\big)+\big(y_2K_{1457}+(y_1-y_2)K_{4157}\big)+y_1K_{2357}\,,$$

while

$$x^{7531}x_1\varphi^1_{4321} = K_{7135}x_1\big(y_1\pi_2\pi_3+\widehat{\pi}_1y_2\pi_3+\widehat{\pi}_1\widehat{\pi}_2y_3+\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_3y_4\big) \\ = y_4K_{1358}+(y_3-y_4)K_{1385}+(y_2-y_3)K_{1835}+(y_1-y_2)K_{8135}\,,$$

the sum of these two terms being equal to $K_{1357}\xi$.

A fully explicit Monk formula would require finding combinatorial objects compatible with the above recursion, as well as a justification of the fact that the coefficients seem to be of the type y_i or (y_i-y_j) only. For example,

$$K_{20424}\,\xi = y_5K_{20425}+(y_3-y_5)K_{20524}+(y_2-y_3)K_{25024}+(y_1-y_3)K_{50224} \\ +(y_4-y_2)K_{32404}+(y_3-y_2)K_{52024}+y_4K_{20434}+y_2K_{21424} \\ +(y_1-y_4)K_{30424}+(y_4-y_5)K_{20452}+(y_5-y_4)K_{20542}+(y_2-y_4)K_{23404}\,.$$

4.2 Product $G_v\,x_1\ldots x_k$

We first need to extend the Ehresmann-Bruhat order to weights. Let $u,v\in\mathbb{N}^n$ be permuted of each other. Then $u\geq v$ if and only if for $k=1,\ldots,n$ one has $[u_1,\ldots,u_k]\uparrow\geq[v_1,\ldots,v_k]\uparrow$ componentwise.

Given $v \in \mathbb{N}^n$, $k \le n$, let

$$\mathcal{C}(v,k) = \{u:\ u \ge v\ \&\ \big(\forall i \ne k, us_i \ge v \text{ implies } us_i \ge u\big)\}\,.$$

In other words, $\mathcal{C}(v,k)$ is the set of weights above v which are minimum in the intersection of their coset modulo $\mathfrak{S}_{k\times n-k}$ with the interval $[v,[n\dots 1]]$.

Using these sets, we define two operations $\circledast, \circledcirc$. Given $v \in \mathbb{N}^n$, $k \le n$, $z \in \mathbb{N}^k$, let $u \in \mathcal{C}(v,k)$ be such that $[u_1,\dots,u_k]\uparrow = [z_1,\dots,z_k]\uparrow$ if it exists. In that case, define

$$v \circledcirc z = u \quad \& \quad v \circledast z = u + [1^k 0^{n-k}]\,.$$

Otherwise put $v \circledcirc z = \emptyset = v \circledast z$.

For example, for $v = [3,5,1,6,2,4]$, $z = [6,3,2]$, one has

$$v \circledcirc z = [3,6,2,5,1,4] \quad \& \quad v \circledast z = [4,7,3,5,1,4]\,.$$

We have given in Lemma 1.4.2 the normal reordering of products of the type $\pi_\sigma x_1 \cdots x_k$. These reorderings provide the decomposition of $G_v x_1 \cdots x_k$ and $K_v x_1 \cdots x_k$ in the Grothendieck or key basis respectively, in terms of punched diagrams.

Let us index Grothendieck polynomials by permutations, putting $G_\emptyset = 0$, and let us introduce the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$ generated by $e_i(\mathbf{x}_n) - e_i(\mathbf{y}_n)$, $i = 1 \dots n$.

Theorem 4.2.1. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$. Then, modulo the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$, one has*

$$G_{(\sigma)} x_1 \cdots x_k \equiv \sum_{\tau \in \mathcal{C}(\sigma,k)} y_{\tau_1} \cdots y_{\tau_k}\, G_{(\tau)} = \sum_{z \in \mathbb{N}^k:\, n \ge z_1 > \cdots z_k} y_{z_1} \cdots y_{z_k}\, G_{(\sigma \circledcirc z)}\,. \tag{4.2.1}$$

Proof. Let ζ be the maximal permutation in the coset $\sigma\mathfrak{S}_{k\times(n-k)}$. Then

$$G_{(\sigma)} x_1 \cdots x_k = G_{(\omega)} \pi_{(\omega\zeta)} \pi_{(\zeta^{-1}\omega\sigma)}\, x_1 \cdots x_k = G_{(\omega)} \pi_{(\omega\zeta)}\, x_1 \cdots x_k\, \pi_{(\zeta^{-1}\omega\sigma)}\,.$$

Thanks to (1.4.7), the product $\pi_{(\omega\zeta)} x_1 \cdots x_k$ is equal to a sum $\sum x^{\mathcal{U}} \pi^{\mathcal{U}}$ over some punched diagrams. However, for any i, one has[1] $\prod_{j=1}^{i} \prod_{h=1}^{n-i} (x_i - y_j) \equiv 0$, hence $G_{(\omega)}(1 - y_{n+1-i} x_i^{-1}) \equiv 0$, that is, $G_{(\omega)} x_i \equiv G_{(\omega)} y_{n+1-i}$. Therefore $G_{(\sigma)} x_1 \cdots x_k$ is congruent to a sum $\sum_\tau c_\tau G_{(\tau)}$, with c_τ a monomial in $\mathbf{y}_n$ of degree k. It remains, but we shall not do it, to check the equivalence between enumerating punched diagrams and permutations in $\mathcal{C}(\sigma,k)$. QED

[1] For every $i \le n$, one has $\prod_{j=1}^{i} \prod_{h=1}^{n-i} (x_i - y_j) = S_{(n+1-i))^i}(\mathbf{x}_i - \mathbf{y}_{n-i}) = S_{(n+1-i))^i}\big((\mathbf{y}_n - \mathbf{y}_{n+1-i}) - (\mathbf{x}_n - \mathbf{x}_i) + (\mathbf{x}_n - \mathbf{y}_n)\big) \equiv S_{(n+1-i))^i}\big((\mathbf{y}_n - \mathbf{y}_{n+1-i}) - (\mathbf{x}_n - \mathbf{x}_i)\big)$. his last function is null because the cardinality of $\mathbf{y}_n - \mathbf{y}_{n+1-i}$ is $< i$ and the cardinality of $\mathbf{x}_n - \mathbf{x}_i$ is $< n+1-i$. For example, for $n=5$, $i=2$, $S_{44}(\mathbf{x}_2 - \mathbf{y}_4) \equiv S_{44}\big(y_5 - (x_3 + x_4 + x_5)\big) = 0$.

For example, for $\sigma = [4,2,1,5,3]$, and $k=3$, then $G_{(42153)} = G_{(54321)}\pi_1\pi_3\pi_2\pi_4\pi_3$ and one has to enumerate the punched 122-diagrams to describe the product $G_{(42153)}\, x_1x_2x_3 = G_{31010}\, x_1x_2x_3 =$

$$
\begin{aligned}
&\left(x_2x_4x_5 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \to y_4y_2y_1 G_{(42153)} \right) + \left(x_1x_4x_5 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array} \to y_5y_2y_1 G_{(52143)} \right) \\
&+ \left(x_2x_3x_5 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \to y_4y_3y_1 G_{(43142)} \right) + \left(x_1x_2x_5 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & \bullet & 3 \\ \hline \end{array} \to y_5y_4y_1 G_{(45123)} \right) \\
&+ \left(x_2x_3x_4 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline 1 & 2 & 3 \\ \hline \end{array} \to y_4y_3y_2 G_{(42351)} \right) + \left(x_1x_2x_4 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array} \to y_5y_4y_2 G_{(42513)} \right) \\
&+ \left(x_1x_3x_5 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array} \to y_5y_3y_1 G_{(53142)} \right) + \left(x_1x_3x_4 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline \bullet & 2 & 3 \\ \hline \end{array} \to y_5y_3y_2 G_{(52341)} \right) \\
&\qquad\qquad + \left(x_1x_2x_3 \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array} \to y_5y_4y_3 G_{(43512)} \right) .
\end{aligned}
$$

One obtains the products $G_{(\eta)}\, x_1x_2x_3$, for any η in the coset $\sigma\mathfrak{S}_{3\times 2}$, by taking the image of the preceding expansion under products of π_i's, $i \neq 3$. For example, $G_{(24153)}\, x_1x_2x_3 = G_{(42153)}\, x_1x_2x_3\pi_1$ results from sorting each permutation τ in the preceding sum into $[[\tau_1,\tau_2]\uparrow, \tau_3, \tau_4, \tau_5]$.

The number of terms in (4.2.1) is equal to the number of strict partitions $z \in \mathbb{N}^k$ between u and $[n, \ldots, n{+}1{-}k]$, where $u = [\sigma_1, \ldots, \sigma_k]\downarrow$, or, equivalently, the number of partitions containing $[u_1{-}n, \ldots, u_n{-}1]$ and contained in $[(n{-}k)^k]$.

The original Schubert calculus involved Graßmannians, and, in our terms, Schubert and Grothendieck polynomials indexed by Graßmannian permutations. For any Graßmannian permutation σ, corresponding to the partition $\mu = [\sigma_k{-}k, \ldots, \sigma_1{-}1]$, any r, the number of terms in the expansion of $G_{(\sigma)}\,(x_1\cdots x_k)^r$ is the dimension of some space of sections, and is called a *postulation number*. From what precedes, it is equal to the number of increasing chains of partitions $\mu^0 = \mu \le \mu^1 \le \cdots \le \mu^k \le \mu^{k+1} = [(n{-}k)^k]$. This number has a determinantal formula proved by Hodge, with some help from Littlewood.

For example, the product $G_{(145236)}(x_1x_2x_3)^2$ involves 46 chains of strict partitions $[541] \leq \mu^1 \leq \mu^2 \leq [654]$ (represented as two-columns Young tableaux) :

$$\left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&1\\\hline\end{array}\right) G_{(145236)} + \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&1\\\hline\end{array}\right) G_{(146235)} + \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&1\\\hline\end{array}\right) G_{(156234)}$$

$$+ \left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 2&2\\\hline\end{array}\right) G_{(245136)} + \left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 2&3\\\hline\end{array}\right) G_{(345126)}$$

$$+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 2&2\\\hline\end{array}\right) G_{(246135)}$$

$$+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&2\\\hline\end{array}\right) G_{(256134)}$$

$$+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 2&3\\\hline\end{array}\right) G_{(346125)}$$

$$+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&3\\\hline\end{array}\right) G_{(356124)}$$

$$+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 4&4\\\hline\end{array}\right) G_{(456123)}$$

4.3 Product $K_v\, x_1 \ldots x_k$

The computations of $K_v\, x_1 \cdots x_k$ and $G_v\, x_1 \cdots x_k$ are similar, and use the same equivalence, detailed in the appendix, between enumerating punched diagrams and describing sets $\mathcal{C}(v, k)$. It translates into the following theorem for what concerns key polynomials.

Theorem 4.3.1. *Let $v \in \mathbb{N}^n$, $k \leq n$. Then*

$$K_v\, x_1 \cdots x_k = \sum_{u \in \mathcal{C}(v,k)} K_{u+[1^k, 0^{n-k}]} = \sum_z K_{v \circledast z}\,, \tag{4.3.1}$$

sum over all $z \in \mathbb{N}^k$, $z = z\uparrow$, z subword of $v\uparrow$.

For example, for v=[2132], $k = 2$, we frame the elements of $\mathcal{C}([2132])$ inside the interval $[2132, 3221]$, and figure the intersection of this interval with cosets modulo $\mathfrak{S}_{2\times 2}$.

$$\begin{tikzcd}[column sep=small]
 & & {[3221]} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & {[2321]} \arrow[dr, dash] & & {[3212]} \arrow[dl, dash] & \\
\boxed{[2231]} & & \boxed{[2312]} & & \boxed{[3122]} \\
 & & \boxed{[2132]} & &
\end{tikzcd}$$

On the other side, the subwords of length 2 of $v\!\uparrow = [1223]$ are 12, 13, 22, 23 and one has $v \circledast 12 = [2132] + [1100]$, $v \circledast 22 = [2231] + [1100]$, $v \circledast 13 = [3122] + [1100]$, $v \circledast 23 = [2312] + [1100]$, so that

$$\begin{aligned} K_{2132}\, x_1x_2 &= K_{2132+1100} + K_{2231+1100} + K_{3122+1100} + K_{2312+1100} \\ &= K_{3232} + K_{3331} + K_{4222} + K_{3412}\,. \end{aligned}$$

Notice that

$$\begin{aligned} K_{2132}x_1x_2 = K_{3221}\pi_1\pi_3\pi_2\, x_1x_2 = x^{3221}x_2x_4 \begin{array}{cc} & \boxed{3} \\ \boxed{1} & \boxed{2} \end{array} + x^{3221}x_1x_3 \begin{array}{cc} & \boxed{3} \\ \boxed{1} & \boxed{\bullet} \end{array} \\ + x^{3221}x_1x_2 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{1} & \boxed{2} \end{array} + x^{3221}x_1x_4 \begin{array}{cc} & \boxed{3} \\ \boxed{\bullet} & \boxed{2} \end{array} + x^{3221}x_1x_3 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{\bullet} & \boxed{2} \end{array}\,, \end{aligned}$$

but that the term $x^{3221}x_1x_3 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{\bullet} & \boxed{2} \end{array} = x^{4231}\pi_2 = 0$ disappears.

Dominant monomials can be written as products of fundamental weights $x_1 \cdots x_k$. Iterating (4.2.1) and (4.3.1), one obtains the product of a Grothendieck or a key polynomial by any dominant monomial. The rule will however take (later) a more satisfactory formulation when stated in terms of the plactic monoid.

4.4 Relating the two products

Let us show how to relate the products $G_{(\sigma)}x^\lambda$ and $K_u x^\lambda$.

Proposition 4.4.1. *Let $\sigma \in \mathfrak{S}_n$, $\lambda \in \mathbb{N}^n$ be a partition, $r \geq \lambda_1$, and $u = [r\sigma_1, \ldots, r\sigma_n]$. Then $K_u x^\lambda = \sum_w K_w$ is a sum without multiplicities and $G_{(\sigma)}x^\lambda$ is a sum over the same weights :*

$$G_{(\sigma)}x^\lambda = \sum_w y^{\langle w\rangle} G_{\zeta(w)}\,,$$

with $\zeta(w) = \big[\lfloor w_1/r\rfloor, \ldots, \lfloor w_n/r\rfloor\big]$, $z = w\uparrow$, $\langle w\rangle = [z_1{-}r, \ldots, z_n{-}r]$.

Proof. The product by x^λ is a chain of λ_1 multiplications by monomials of the type $x_1 \cdots x_k$. From the preceding theorems, it can be written in terms of the operators $x^t \pi_\eta$, with $t \le [\lambda_1, \dots, \lambda_1]$. The hypothesis on u is such that each $u\!\downarrow +t$ is dominant, and therefore, gives the key polynomial indexed by $[u\!\downarrow +t]\,\eta$. On the other hand, the same operator $x^t \pi_\eta$ contributes to a Grothendieck polynomial multiplied by the monomial in y of exponent $[t_n \dots, t_1]$. QED

The following table describes the product $G_{3142}x^{2200}$ as the same time, taking $r = 3$, as the product $K_{9,3,12,6}x^{2200}$.

G_{4321}	y^{0112}	$K_{14,10,7,3}$
G_{3142}	y^{2020}	$K_{11,5,12,6}$
G_{3421}	y^{0121}	$K_{11,13,7,3}$
G_{4312}	y^{1012}	$K_{14,10,4,6}$
G_{3241}	$y^{1120} + y^{0220}$	$K_{11,8,12,3} + K_{11,7,12,4}$
G_{4132}	$y^{2011} + y^{2002}$	$K_{14,5,9,6} + K_{13,5,10,6}$
G_{3412}	$y^{1021} + y^{0022}$	$K_{11,14,3,6} + K_{11,13,4,6}$
G_{4231}	$y^{0211}+y^{0202}+y^{1102}+y^{1111}$	$K_{13,8,10,3} + K_{14,8,9,3} + K_{14,7,9,4} + K_{13,7,10,4}$

Of special importance is the case of multiplication by $x^{k\dots 1}$. Let us show in the next lemma a case where it is of interest to mix bases.

Lemma 4.4.2. *Let $k \le n$, $u \in \mathbb{N}^n$ be such that $u_1 \ge \cdots \ge u_k$, $u_{k+1} \ge \cdots \ge u_n$. Then*

$$\widehat{K}_u\, x_1^k \cdots x_{k-1}^2 x_k = Y_{u+[k,\dots,1,0^{n-k}]}(\mathbf{x}, \mathbf{0})\,.$$

Proof. The hypothesis on u implies that, with $\lambda = u\downarrow$, there exists a strictly increasing $v \in \mathbb{N}^k$ such that

$$\begin{aligned}\widehat{K}_u &= \widehat{K}_\lambda\,(\widehat{\pi}_{v_1} \cdots \widehat{\pi}_1)(\widehat{\pi}_{v_2} \cdots \widehat{\pi}_2) \cdots (\widehat{\pi}_{v_k} \cdots \widehat{\pi}_k)\\ &= \widehat{K}_\lambda\,(\partial_{v_1} \cdots \partial_1 x_2 \cdots x_{v_1+1})(\partial_{v_2} \cdots \partial_2 x_3 \cdots x_{v_1+1}) \cdots (\partial_{v_k} \cdots \partial_k x_{k+1} \cdots x_{v_k+1})\end{aligned}$$

Using repeatedly that $(\partial_j \cdots \partial_i x_{i+1} \cdots x_{j+1}) x_1 \cdots x_i = x_1 \cdots x_{j+1}\_j \cdots \partial_i$, one can transfer all monomials to the left and obtain

$$\widehat{K}_u\, x_1^k \cdots x_k = x^\lambda (x_1 \cdots x_{v_1+1}) \cdots (x_1 \cdots x_{v_k+1})\,(\partial_{v_1} \cdots \partial_1) \cdots (\partial_{v_k} \cdots \partial_k)\,.$$

This is the image of a dominant monomial under a product of divided differences, hence the lemma after identifying the index of the Schubert polynomial. QED

4.5 Product with $(x_1 \dots x_k)^{-1}$

The original formulas of Pieri involved intersection of Schubert varieties with special Schubert varieties corresponding to elementary symmetric functions. At the level of Grothendieck polynomials, one has to consider products of Grothendieck polynomials with some special ones, for example with $G_{0^{k-1},1} = 1 - y_1 \cdots y_k x_1^{-1} \cdots x_k^{-1}$. This is not what we have done in (4.2.1), having taken $x_1 \cdots x_k$ intead of its inverse. Let us repair this in the next theorem, which can be found in [99, Th 6.4].

Theorem 4.5.1. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$. Let $\zeta \in \mathfrak{S}_n$ be such that $[\zeta_1, \ldots, \zeta_k] = [\sigma_1, \ldots, \sigma_k] \downarrow$, $[\zeta_{k+1}, \ldots, \zeta_n] = [\sigma_{k+1}, \ldots, \sigma_n] \downarrow$, and $\omega = [n, \ldots, 1]$. Then, modulo the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$, one has*

$$G_{(\sigma)} \frac{y_{\sigma_1} \cdots y_{\sigma_k}}{x_1 \cdots x_k} \equiv G_{(\omega)} \, \widehat{\pi}_{\omega\zeta} \, \pi_{\zeta^{-1}\sigma} \,. \tag{4.5.1}$$

Proof. The hypothesis on ζ implies that, with $\mathcal{V}$ the diagram of $v = [n-\zeta_1, \ldots, n-k+1-\zeta_k]$, one has $\pi_{\omega\zeta} = \pi^{\mathcal{V}}$. Thanks to (1.4.4), one has $\pi^{\mathcal{V}} (x_1 \cdots x_k)^{-1} = (x_{v_1+1} \cdots x_{v_k+k})^{-1} \, \widehat{\pi}^{\mathcal{V}}$. Since the factor $(x_1 \cdots x_k)^{-1}$ commutes with $\pi_{\zeta^{-1}\omega\sigma}$ because $\zeta^{-1}\sigma$ belongs to $\mathfrak{S}_{k \times n-k}$, the theorem follows. QED

For example, for $k = 3$, $\sigma = [4, 3, 6, 7, 8, 2, 1, 5]$, one has $\zeta = [6, 4, 3, 8, 7, 5, 2, 1]$, $v = [8, 7, 6] - [6, 4, 3] = [2, 3, 3]$, $\mathcal{V} = \begin{array}{ccc} & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}$, and $\zeta^{-1}\sigma = [2, 3, 1, 5, 4, 7, 8, 6]$ has reduced decomposition $s_1 s_2 s_4 s_6 s_7$. Altogether,

$$\begin{aligned} G_{(\sigma)} \frac{y_4 y_3 y_6}{x_1 x_2 x_3} &\equiv G_{(\omega)} \big(\widehat{\pi}_2 \widehat{\pi}_1 \; \widehat{\pi}_4 \widehat{\pi}_3 \widehat{\pi}_2 \; \widehat{\pi}_5 \widehat{\pi}_4 \widehat{\pi}_3 \big) \big(\pi_1 \pi_2 \pi_4 \pi_6 \pi_7 \big) \\ &= G_{(4,3,6,7,8,2,1,5)} - G_{(4,3,7,6,8,2,1,5)} - G_{(5,3,6,7,8,2,1,4)} + G_{(5,3,7,6,8,2,1,4)} \\ &\qquad - G_{(4,5,6,7,8,2,1,3)} G_{(5,4,6,7,8,2,1,3)} + G_{(4,5,7,6,8,2,1,3)} - G_{(5,4,7,6,8,2,1,3)} \,. \end{aligned}$$

V. Pons [167] shows that the expansion of the right hand side of (4.5.1) in the Grothendieck basis is a signed interval. Lenart and Postnikov [141] give a more general *equivariant K-Chevalley formula* valid for any Weyl group.

The preceding theorem involves products of π_i's and $\widehat{\pi}_j$'s, that one can study using key polynomials rather than Grothendieck polynomials. Let ∇ be an arbitrary product of π_i's and $\widehat{\pi}_j$'s, $i, j < n$. If $G_{(\omega)} \nabla = \sum c_\sigma G_{(\sigma)}$, then $K_\omega \nabla = \sum c_\sigma K_\sigma$, with the same coefficients, since every π_i acts in the same manner on the indices of both families of polynomials. This will allow us to reformulate (4.5.1) in the next statement.

Proposition 4.5.2. *Let $k \le n$, $v \in \mathbb{N}^k$ be antidominant, $\mathcal{V}$ be the v-diagram and σ be a permutation in $\mathfrak{S}_{k \times n-k}$. Then*

$$K_\omega \, \widehat{\pi}^{\mathcal{V}} \, \pi_\sigma = \sum \widehat{K}_\tau \,, \tag{4.5.2}$$

sum over all weights τ in the interval $[\eta, \eta\sigma]$, with $\eta \in \mathbb{N}^n$ permutation of $\omega = [n, \ldots, 1]$ such that $\eta_1 = v_k + k, \ldots, \eta_k = v_1 + 1$, $\eta_{k+1} > \cdots, \eta_n$.

Proof. The weight η is such that $K_\omega\,\widehat{\pi}^{\mathcal{V}} = \widehat{K}_\eta$. The operator π_σ is equal to a sum $\sum_{\nu\le\sigma}\widehat{\pi}_\nu$, where all ν belong to $\mathfrak{S}_{k\times n-k}$. Hence products are reduced and $K_\omega\,\widehat{\pi}^{\mathcal{V}}\,\pi_\sigma = \sum_n u\widehat{K}_{\eta\nu}$. QED

For example, let $k=3$, $v=[1,2,2]$, $\sigma=[3,1,2,5,4]$. Then $\eta=[4,2,1,5,3]$ and

$$\widehat{K}_{54321}\,\widehat{\pi}^{\mathcal{V}}\,\pi_\sigma = \widehat{K}_{54321}\,(\widehat{\pi}_1\widehat{\pi}_3\widehat{\pi}_2\widehat{\pi}_4\widehat{\pi}_3)\,(\pi_2\pi_1\pi_4) = \widehat{K}_{42153}\,(1+\widehat{\pi}_2)(1+\widehat{\pi}_1)(1+\widehat{\pi}_4)$$
$$= \widehat{K}_{42153} + \big(\widehat{K}_{41253}+\widehat{K}_{24153}+\widehat{K}_{42135}\big) + \big(\widehat{K}_{14253}+\widehat{K}_{41235}+\widehat{K}_{24135}\big) + \widehat{K}_{14235}\,.$$

This is also equal to $K_{14235}-K_{15234}-K_{14325}+K_{15324}$, in accordance with

$$G_{(14235)}\frac{y_1y_2y_4}{x_1x_2x_3} = G_{(14235)} - G_{(15234)} - G_{(14325)} + G_{(15324)}\,.$$

4.6 More keys: K^G polynomials

Stability properties of Schubert polynomials can be analyzed by using the isobaric divided differences π_i. Let us show that the operators

$$D_i = (1-x_i^{-1})\pi_i = (x_i-1)\partial_i \tag{4.6.1}$$

play a similar role for what concerns the Grothendieck polynomials.

These operators satisfy the braid relations, being the images of the π_i under the transformation $x_i\to x_i{-}1$. As an operator commuting with multiplication by elements of $\mathfrak{Sym}(x_i,x_{i+1})$, D_i is characterized by

$$1D_i = 1 \quad\&\quad x_{i+1}D_i = 1\,.$$

More generally, $D_\omega = (x_1{-}1)^{n-1}\dots(x_{n-1}{-}1)\partial_\omega = G_\rho(\mathbf{x},\mathbf{1})\,\pi_\omega$ is characterized by the fact that it commutes with multiplication by elements of $\mathfrak{Sym}(\mathbf{x}_n)$ and sends any $x^v : \mathbf{0}\le v\le[0,\dots,n{-}1]$ to 1. Indeed, x^vD_ω may be written $(x^v, G_\rho(\mathbf{x},\mathbf{1}))^\pi$, and Formula 2.9.5 tells that $(x^v, G_\rho(\mathbf{x},\mathbf{y})) = y^{v\omega}$.

Taking the same starting points as for $G_v(\mathbf{x},\mathbf{1})$, one defines recursively K_v^G polynomials by

$$K_\lambda^G = G_\lambda(\mathbf{x},\mathbf{1}) \text{ when } \lambda \text{ dominant } \& \; K_{vs_i}^G = K_v^G\,D_i \text{ when } v_i\ge v_{i+1}\,. \tag{4.6.2}$$

The operators D_i, combined with multiplication by $G_{1^k}(\mathbf{x},\mathbf{1})$, can be used to generate recursively the Grothendieck polynomials $G_v(\mathbf{x},\mathbf{1})$, or to express them in terms of the basis $\{K_v^G\}$.

Proposition 4.6.1. *Given $v\in\mathbb{N}^n$. If $0\notin v$, then*

$$G_v(\mathbf{x},\mathbf{1}) = (1{-}x_1^{-1})\dots(1{-}x_n^{-1})G_{v-1^n}(\mathbf{x},\mathbf{1})\,.$$

Otherwise, let k be such that $v_k = 0$ and $v_i > 0$ for $i < k$, let $u = [v_1{-}1, \ldots, v_{k-1}{-}1, v_{k+1}, \ldots, v_n]$*. Then*

$$G_v(\mathbf{x}, \mathbf{1}) = G_u(\mathbf{x}, \mathbf{1})\,(1{-}x_{k-1}^{-1}) \cdots (1{-}x_1^{-1})\, D_{n-1} \cdots D_k$$
$$= G_u(\mathbf{x}, \mathbf{1})\, D_{n-1} \cdots D_k\,(1{-}x_{k-1}^{-1}) \cdots (1{-}x_1^{-1})\,. \quad (4.6.3)$$

Proof. By trivial commutation, one can transform $D_{n-1} \cdots D_k = (1{-}x_{n-1}^{-1})\pi_{n-1} \ldots (1{-}x_k^{-1})\pi_k$ into $(1{-}x_{n-1}^{-1}) \ldots (1{-}x_k^{-1})\pi_{n-1} \ldots \pi_k$. Therefore

$$G_u(\mathbf{x}, \mathbf{1})\,(1{-}x_{n-1}^{-1}) \ldots (1{-}x_k^{-1})\,\pi_{n-1} \ldots \pi_k\,(1{-}x_{k-1}^{-1}) \cdots (1{-}x_1^{-1})$$
$$= G_u(\mathbf{x}, \mathbf{1})\,(1{-}x_{n-1}^{-1}) \ldots (1{-}x_1^{-1})\,\pi_{n-1} \ldots \pi_k$$
$$= G_{u+1^{n-1}}(\mathbf{x}, \mathbf{1})\,\pi_{n-1} \ldots \pi_k = G_v(\mathbf{x}, \mathbf{1})\,,$$

as claimed. QED

With the same notations than in (**??**), if v is vexillary, then u is also vexillary, as well as $u' = u + [1^{k-1}, 0^{n-k}]$. Suppose that $G_{u'}(\mathbf{x}, \mathbf{1}) = K_{u'}^G$. Then

$$G_v(\mathbf{x}, \mathbf{1}) = G_{u+1^{n-1}}(\mathbf{x}, \mathbf{1})\pi_{n-1} \ldots \pi_k$$
$$= G_{u'}(\mathbf{x}, \mathbf{1})(1{-}x_{n-1}^{-1}) \ldots (1{-}x_k^{-1})\pi_{n-1} \ldots \pi_k$$
$$= G_{u'}(\mathbf{x}, \mathbf{1})D_{n-1} \ldots D_k = K_{u'}^G D_{n-1} \ldots D_k = K_v^G\,.$$

By recursion on n this proves

Corollary 4.6.2. *If v is vexillary code, then $G_v(\mathbf{x}, \mathbf{1}) = K_v^G$.*

Notice that the shift of indices $G_v(\mathbf{x}, \mathbf{1}) \to G_{0v}(\mathbf{x}, \mathbf{1})$ may be obtained with the D_i. Indeed, if $v \in \mathbb{N}^n$, then

$$G_v(\mathbf{x}, \mathbf{1})D_n \ldots D_1 = G_v(\mathbf{x}, \mathbf{1})(1{-}x_n^{-1}) \ldots (1{-}x_1^{-1})\pi_n \ldots \pi_1$$
$$= G_{v+1^n}(\mathbf{x}, \mathbf{1})\pi_n \ldots \pi_1 = G_{0v}(\mathbf{x}, \mathbf{1})\,.$$

4.7 Transitions for Grothendieck polynomials

We have seen that multiplication by x_i, in the case of Schubert polynomials, can be used to provide a recursive definition of these polynomials. We are going to show that one still has a transition formula for Grothendieck and key polynomials (and later also Macdonald polynomials).

The case of Grothendieck polynomials is an extension of the case of Schubert polynomials, and is described in [104, Prop. 3]. Since it is proved by a straighforward recursion, let us state the property without proof (caution: in reference [104], one uses the variables $1 - 1/x_i$ instead of x_i).

It is more convenient to use indexing by permutations and write $G_{(\sigma)}$ instead of G_v, if v is the code of σ. In terms of permutations, the maximal transition formula for Schubert polynomials (3.9.4) reads as follows.

Given ζ and its code v, let k be such that $v_i = 0$ for $i > k$ and $v_k > 0$. Let σ be the permutation whose code is $v - [0^{k-1}10^{n-k}]$. Then

$$X_\zeta = (x_k - y_j)X_\sigma + \sum_i X_{\tau_{ji}\sigma}\,, \qquad (4.7.1)$$

sum over all transpositions τ_{ji} such that $\sigma = [\ldots i \ldots j \ldots]$, $\tau_{ji}\sigma = [\ldots j \ldots i \ldots]$ and $\ell(\tau_{ji}\sigma) = \ell(\tau) + 1$.

Order decreasingly the integers i occuring in (4.7.1): $i_m > \cdots > i_1$, and write $(1 - \tau_{ji}) \star G_{(\sigma)}$ for $G_{(\sigma)} - G_{(\tau_{ji}\sigma)}$. With these conventions, one has

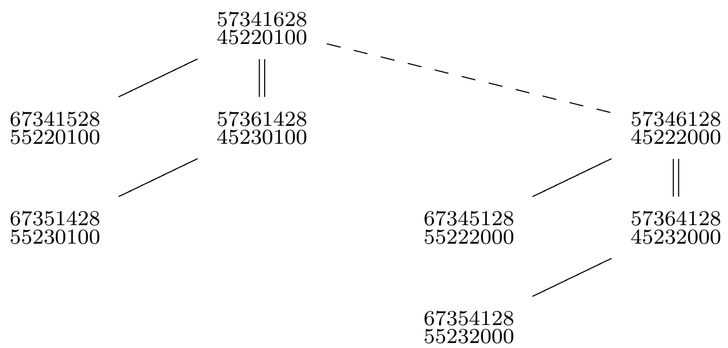
Theorem 4.7.1. *With the conventions of (4.7.1), one has the following transition formula*

$$\left(G_{(\sigma)} - G_{(\zeta)}\right)\frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma)}\,. \qquad (4.7.2)$$

For example, for $\zeta = [5, 7, 3, 4, 1, 8, 2, 6]$, one has $\sigma = [5, 7, 3, 4, 1, 6, 2, 8]$, $k = 6$, $j = 6$, and

$$\left(G_{(57341628)} - G_{(57341826)}\right)\frac{x_6}{y_6} = (1 - \tau_{65}) \star (1 - \tau_{64}) \star (1 - \tau_{61}) \star G_{(57341628)}$$

is equal to the alternating sum of Grothendieck polynomials displayed below (with both indexings) :



Relation (2.6.5) allows to transform transition for G-polynomials to transition for $\widehat{G}$-polynomials.

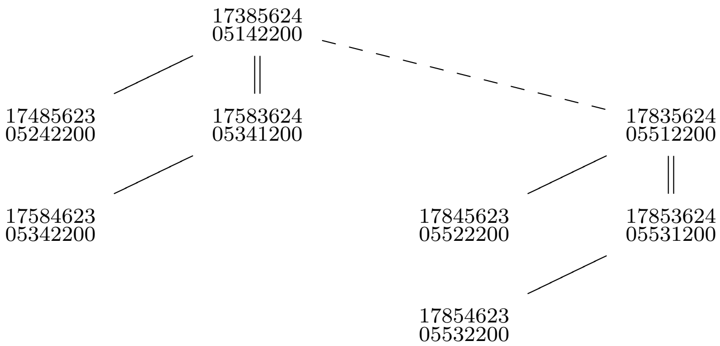
Corollary 4.7.2. *With the conventions of (4.7.1), writing i' for $n{+}1{-}i$, $i = 1, \dots, n$, one has the following transition formula*

$$\left(\widehat{G}_{(\omega\sigma\omega)} + \widehat{G}_{(\omega\zeta\omega)}\right) \frac{x_{k'}}{y_j} = \left(1 + \tau_{j'i'_m}\right) \star \cdots \left(1 + \tau_{j'i'_1}\right) \star \widehat{G}_{(\omega\sigma\omega)} \,. \tag{4.7.3}$$

For example, the transition for $\widehat{G}_{(\omega\zeta\omega)} = \widehat{G}_{[37185624)}$ is the image of the transition for $G_{(\zeta)}$ given above :

$$\left(\widehat{G}_{(17385624)} + \widehat{G}_{(37185624)}\right) \frac{x_3}{y_6} = (1 + \tau_{34}) \star (1 + \tau_{35}) \star (1 + \tau_{38}) \star \widehat{G}_{(17385624)} \,,$$

and can be displayed as



One could in fact extend all transitions of Schubert polynomials, and not only maximal transitions, to transitions of Grothendieck polynomials. This is useful in the case of a permutation $\zeta = \zeta'\zeta''$ belonging to a Young subgroup as in (3.9.3). One has the same property as in (3.9.2). A transition

$$\left(G_{(\sigma)} - G_{(\zeta')}\right) \frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma)}$$

entails the relation

$$\left(G_{(\sigma\zeta'')} - G_{(\zeta)}\right) \frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma\zeta'')} \,. \tag{4.7.4}$$

As a consequence, Grothendieck polynomials satisfy the following factorization property (shown in [99, Prop. 6.7] for the polynomials $G_{(\zeta)}(\mathbf{x}, \mathbf{1})$).

Corollary 4.7.3. *Let ζ belong to a Young subgroup, and $\zeta = \zeta'\zeta''$ its corresponding factorisation. Then*

$$G_{(\zeta)}(\mathbf{x}, \mathbf{y}) = G_{(\zeta')}(\mathbf{x}, \mathbf{y})\, G_{(\zeta'')}(\mathbf{x}, \mathbf{y}) \,. \tag{4.7.5}$$

Using the recursive definition of Grothendieck polynomials to prove factorization would be delicate. For example, $G_{0120}(\mathbf{x},\mathbf{y})$ is a sum of 12 monomials which does not factorize[2]. Its image under π_3 is equal to

$$G_{0101}(\mathbf{x},\mathbf{y}) = G_{01}(\mathbf{x},\mathbf{y})G_{0001}(\mathbf{x},\mathbf{y}) = \left(1 - \frac{y_1y_2}{x_1x_2}\right)\left(1 - \frac{y_1y_2y_3y_4}{x_1x_2x_3x_4}\right).$$

[2]We shall see in (**??**) that it is equal to $S_{222}(\mathbf{x}_3, \mathbf{x}_3 - \mathbf{y}_2, \mathbf{x}_3 - \mathbf{y}_4)/x^{222}$.

4.8 Branching and stable G-polynomials

As in the case of Schubert polynomials, one can use the transition formula (4.7.2) to obtain a transition graph with root a Grothendieck polynomial (indexed by a permutation), vertices being $\pm$ a Grothendieck polynomial, stopping at vexillary permutations.

For example, for $\sigma = [3, 1, 6, 2, 7, 4, 5]$, one has

$$\begin{tikzcd}
 & G_{(3162745)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
G_{(5162347)} & G_{(3165247)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & -G_{(5163247)} & \\
G_{(4163257)} \arrow[d, dash] & G_{(3461257)} & -G_{(4361257)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
G_{(4261357)} \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & -G_{(5341267)} & -G_{(4531267)} & G_{(5431267)} \\
G_{(5241367)} & G_{(4523167)} & -G_{(5421367)} &
\end{tikzcd}$$

The corresponding tree for $X_{3162745}$ is

$$\begin{tikzcd}
 & X_{3162745} \arrow[dl, dash] \arrow[d, dash] \\
X_{5162347} & X_{3165247} \arrow[dl, dash] \arrow[d, dash] \\
X_{4163257} \arrow[d, dash] & X_{3461257} \\
X_{4261357} \arrow[d, dash] \arrow[dr, dash] & \\
X_{5241367} & X_{4523167}
\end{tikzcd}$$

If $v \in \mathbb{N}^n$ is antidominant, then K_v^G is symmetrical in $x_1, \ldots, x_n$, and one has the stability property $K_{0v}^G\Big|_{x_{n+1}=1} = K_v^G$. As for Schubert polynomials, this leads to define the *stable part* of a Grothendieck polynomial[3], for $v \in \mathbb{N}^n$ and $\omega = [n, \ldots, 1]$.

$$\mathcal{S}t(G_v) = G_v(\mathbf{x}, \mathbf{1})D_\omega = G_{0^n v}(\mathbf{x}, \mathbf{1})\Big|_{x_{n+1}=1=\cdots=x_{2n}} . \tag{4.8.1}$$

[3]Contrary to the Schubert case, we eliminate for simplicity the alphabet $\mathbf{y}$.

A transition

$$G_{0^n v}(\mathbf{x},\mathbf{1}) = (1 - x_k^{-1}) G_{0^n v'}(\mathbf{x},\mathbf{1}) + x_k^{-1} \sum G_{0^n u}(\mathbf{x},\mathbf{1})$$

induces the equality

$$\mathcal{S}t(G_v) = \sum \mathcal{S}t(G_u)\,,$$

and therefore, the transition graph is a convenient way of obtaining the stable part of a Grothendieck polynomial.

For example, the above graph shows that the stable part of $G_{(3162745)}$ is equal to

$$\begin{aligned} &\mathcal{S}t(G_{(5162347)}) + \mathcal{S}t(G_{(5241367)}) + \mathcal{S}t(G_{(4521367)}) - \mathcal{S}t(G_{(5421367)}) + \mathcal{S}t(G_{(3461257)}) \\ &\quad - \mathcal{S}t(G_{(5341267)}) - \mathcal{S}t(G_{(4531267)}) + \mathcal{S}t(G_{(5431267)}) - \mathcal{S}t(G_{(5163247)}) \\ &\quad = K^G_{0000124} + K^G_{0000234} + K^G_{0000034} - 2K^G_{0000134} \\ &\qquad + K^G_{0000223} - K^G_{0000224} + K^G_{0000133} - K^G_{0000233}\,. \end{aligned}$$

The terms $\mathcal{S}t(G_{(5421367)})$ and $\mathcal{S}t(G_{(5163247)})$ are both equal to $K^G_{0000134}$, hence a multiplicity 2.

4.9 Transitions for Key polynomials

Key polynomials satisfy a similar transition formula, exhibiting a boolean lattice, except that now one uses weights instead of permutations. The following considerations are drawn from an unpublished manuscript with Lin Hui and Arthur L.B. Yang.

Let $v \in \mathbb{N}^n$, let k be such that $v_i = 0$ for $i > k$ and $v_k > 0$. The leading term x^v of K_v is equal to $x^u x_k$, and we want to describe the difference $K_v - x_k K_u$ as a sum of key polynomials. We can suppose that $v_1 \geq \cdots \geq v_{k-1}$, because $\pi_1, \ldots, \pi_{k-2}$ commute with multiplication by x_k.

Let us compute an example :

$$
x_6 K_{543103} = \quad
\begin{array}{ccccc}
 & K_{543104} & & & \\
 & \| & & & \\
-K_{544103} & -K_{543401} & & & -K_{543140} \\
 & & & & \| \\
K_{544301} & & & K_{544130} & K_{543410} \\
 & & & & \\
 & & & -K_{544310} &
\end{array}
$$

Using the same notation as above for operations on indices, one may rewrite the preceding identity into

$$x_6 K_{543103} = (1 - \tau_{43}) \star (1 - \tau_{41}) \star (1 - \tau_{40}) \star K_{543104} \,.$$

We have used transpositions of values τ_{4i}, ignoring the leftmost 4. However, this example is not generic enough. What to do when values i are repeated?

Let us take a bigger example, which, this time, will pass the test of genericity. Let $v = [5, 4, 3, 3, 1, 1, 1, 0, 5]$. We have to compute

$$K_{5,4,3,3,1,1,1,0,4}\, x_9 = K_{5,4,4,3,3,1,1,1,0}\, \pi_3 \ldots \pi_8\, x_9 \,.$$

Noticing, by the Leibnitz' commutations (1.4.3), that

$$\pi_3 \ldots \pi_8\, x_9 = x_3 \partial_3 x_4 \partial_4 x_5 \partial_5 x_6 \partial_6 x_7 \partial_7 x_8 \partial_8\, x_9 = x_3\, \widehat{\pi}_3 \widehat{\pi}_4 \widehat{\pi}_5 \widehat{\pi}_6 \widehat{\pi}_7 \widehat{\pi}_8 \,,$$

one obtains that $K_{5,4,3,3,1,1,1,0,4}\, x_9 = \widehat{K}_{5,4,3,3,1,1,1,0,4}$. The general case is similar and given in the following statement.

Lemma 4.9.1. *Let $v \in \mathbb{N}^n$ be such that $v_1 \geq \cdots \geq v_{n-1}$, $v_n \neq 0$, and let $u = [\ldots, v_{n-1}, v_n - 1]$. Then*

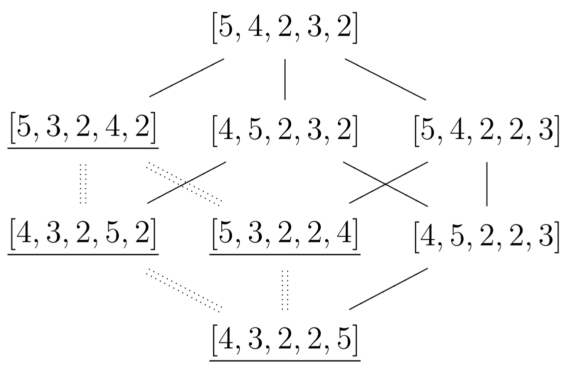
$$K_u\, x_n = \widehat{K}_v \,. \tag{4.9.1}$$

Expanding $\widehat{K}_v$ in terms of K_u (which means taking the Ehresmann-Bruhat interval), one obtains the transition for key polynomials in that case. Let us show the evolution of the transition under successive applications of π_i, $i \neq n{-}1$.

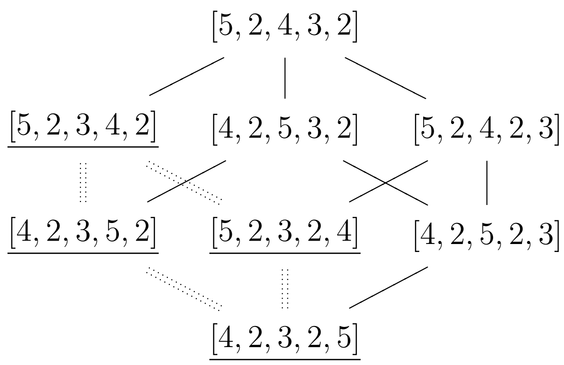
We begin with the transition for $K_{4,3,2,2,5}$:

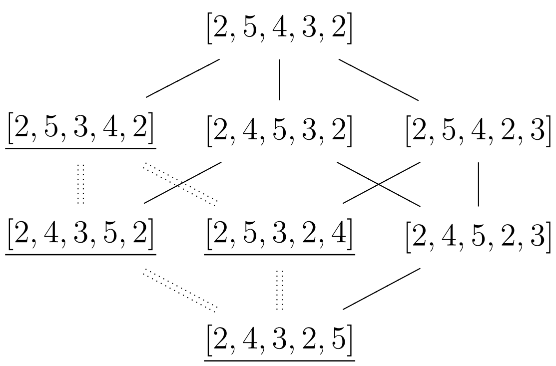
$$
\begin{aligned}
&K_{4,3,2,2,5} - K_{4,3,2,2,4}\, x_5 = K_{4,3,2,2,5} - \widehat{K}_{4,3,2,2,5} \\
&= \big(K_{5,3,2,2,4}+K_{4,5,2,2,3}+K_{4,3,2,5,2}\big) - \big(K_{5,4,2,2,3}+K_{5,3,2,4,2}+K_{4,5,2,3,2}\big) + \big(K_{5,4,2,3,2}\big)\,,
\end{aligned}
$$

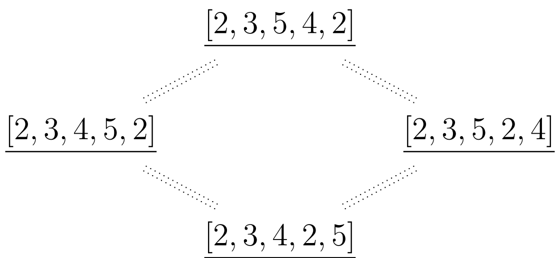
that we display as a boolean lattice (forgetting signs), writing the starting element as the bottom element



Applying π_2, then π_1, then again π_2, one obtains the transitions for $K_{2,4,3,2,5}$ and $K_{2,3,4,2,5}$:







The terms which are not underlined cancel two by two at the last stage, because $(K_{\bullet ji\bullet\bullet} - K_{\bullet ji\bullet\bullet})\,\pi_2 = 0$.

To write the general transition, we need to introduce, for each pair of integers i, j, an operator $\tau_{i,j}$ on linear combinations of K_u, defined[4] by

$$K_{\ldots u_i \ldots u_j \ldots} \star \tau_{i,j} = K_{\ldots u_j \ldots u_i \ldots}\,.$$

Then, one has the following transition formula, similar to the one for Grothendieck polynomials.

Theorem 4.9.2. *Let $v \in \mathbb{N}^n$, such that $v_n > 0$, and $u = [v_1, \ldots, v_{n-1}, v_n - 1]$. Let $i_1 < \cdots < i_r < n$ be the places i such that v_i is strictly maximal among the values $\{v_j : i \le j < n,\, v_j < v_n\}$. Then*

$$K_u\, x_n = K_v \star (1 - \tau_{i_1 n}) \cdots (1 - \tau_{i_r n})\,. \tag{4.9.2}$$

Proof. When $v_1 \ge \cdots \ge v_{n-1}$, the statement comes from rewriting the expansion of $\widehat{K}_v$ in (4.9.1) in terms of the operators τ_{in}.

Given any k such that $v_k > v_{k+1}$, one has $K_u\, x_n\, \pi_k = K_{us_k}\, x_n$. On the other hand, the product of the RHS of (4.9.2) is obtained by replacing v by vs_k and exchanging k and k+1 in the indices of the operators $\tau_{i,n}$, except one has the double factor $(1 - \tau_{k,n})(1 - \tau_{k+1,n})$. In that case the factor $(1 - \tau_{k,n})$ disappears, and this corresponds to the pairs $K_w - K_{ws_k}$ which vanish under π_k. QED

The four examples above must be rewritten

$$\begin{aligned}
K_{43224}\, x_5 &= K_{43225} \star (1-\tau_{15})(1-\tau_{25})(1-\tau_{45}) \\
K_{42324}\, x_5 = K_{43224}\, x_5\, \pi_2 &= K_{42325} \star (1-\tau_{15})(1-\tau_{35})(1-\tau_{45}) \\
K_{24324}\, x_5 = K_{42324}\, x_5\, \pi_1 &= K_{24325} \star (1-\tau_{25})(1-\tau_{35})(1-\tau_{45}) \\
K_{23424}\, x_5 = K_{42324}\, x_5\, \pi_2 &= K_{23425} \star (1-\tau_{35})(1-\tau_{45})\,.
\end{aligned}$$

If $v \in \mathbb{N}^n$ is a vexillary code such that $v_n \neq 0$ and there exists $i : v_i < v_n$, then $Y_v(\mathbf{x}, \mathbf{0})$ and K_v satisfy the same transition :

$$Y_v(\mathbf{x}, \mathbf{0}) = x_k Y_{v'}(\mathbf{x}, \mathbf{0}) + Y_u(\mathbf{x}, \mathbf{0}) \quad \& \quad K_v = x_k K_{v'} + K_u\,,$$

with v' and u vexillary (cf. [124, Lemma 3.10]). Therefore, one has the following property, which is a special case of the expansion of a Schubert polynomial in terms of keys given in (7.3.2).

Lemma 4.9.3. *If v is a vexillary code, then*

$$Y_v(\mathbf{x}, \mathbf{0}) = K_v\,. \tag{4.9.3}$$

For example, there are 23 Schubert polynomials $Y_v(\mathbf{x}, \mathbf{0})$, $v \le [3, 2, 1, 0]$, which coincide with the key polynomial of the same index, while $Y_{1010}(\mathbf{x}, \mathbf{0}) = x_1(x_1{+}x_2{+}x_3)$ is different from $K_{1010} = x_1(x_2{+}x_3)$.

[4] If needed, u is transformed into $u, 0, 0, \ldots$

4.10 Vexillary polynomials

We have already stated that vexillary Schubert and key polynomials have a determinantal expression. This property is also satisfied by Grothendieck polynomials, and we collect together these three families in the next theorem.

First, dominant polynomials can be written as *multi-Schur* functions. Let v be dominant, $u = v\omega$, $k = v_1$. Then

$$\begin{aligned} Y_v &= S_u(\mathbf{x}_n - \mathbf{y}_{v_n}, \ldots, \mathbf{x}_1 - \mathbf{y}_{v_1}) \\ G_v &= (x_1 \cdots x_n)^{-k} S_{k^n}(\mathbf{x}_n - \mathbf{y}_{v_n}, \ldots, \mathbf{x}_1 - \mathbf{y}_{v_1}) \\ K_v &= S_u(\mathbf{x}_n, \ldots, x_1) \end{aligned}$$

For example, for $v = [6, 3, 1]$, one has

$$Y_{631} = S_{136}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) = \begin{vmatrix} S_1(\mathbf{x}_3 - \mathbf{y}_1) & S_4(\mathbf{x}_2 - \mathbf{y}_3) & S_8(\mathbf{x}_1 - \mathbf{y}_6) \\ S_0(\mathbf{x}_3 - \mathbf{y}_1) & S_3(\mathbf{x}_2 - \mathbf{y}_3) & S_7(\mathbf{x}_1 - \mathbf{y}_6) \\ 0 & S_2(\mathbf{x}_2 - \mathbf{y}_3) & S_6(\mathbf{x}_1 - \mathbf{y}_6) \end{vmatrix},$$

$$G_{631} = (x_1 x_2 x_3)^{-6} S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6),$$
$$K_{631} = S_{136}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1).$$

As we already saw, the action of ∂_i or π_i on a determinant of complete functions $S_k(\mathbf{x}_p - \mathbf{y}_q)$ is straightforward if only one column or one row is not invariant under the transposition of x_i, x_{i+1}. In that case, one has to transform this row or column, following the rules $S_k(\mathbf{x}_i - \mathbf{y})\partial_i = S_{k-1}(\mathbf{x}_{i+1} - \mathbf{y})$, $S_k(\mathbf{x}_i - \mathbf{y})\pi_i = S_k(\mathbf{x}_{i+1} - \mathbf{y})$.

For example,

$$\begin{aligned} &Y_{631} \xrightarrow{\partial_2} Y_{612} = S_{126}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) \xrightarrow{\partial_1} Y_{152} \\ &\quad = S_{125}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6) \xrightarrow{\partial_2} Y_{124} = S_{124}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_3 - \mathbf{y}_6), \end{aligned}$$

$$\begin{aligned} &G_{631}(x_1 x_2 x_3)^6 \xrightarrow{\pi_2} = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) \xrightarrow{\pi_1} \\ &\qquad = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6) \xrightarrow{\pi_2} = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_3 - \mathbf{y}_6). \end{aligned}$$

On the other hand, $Y_{631}\partial_1 = S_{135}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6)$ and we cannot proceed so easily with ∂_2, since two columns involve x_2 and not x_3.

When v is vexillary, we have already used the property that there exists at least one sequence of operators ∂_i or π_i respectively, starting from a dominant case, such that at each step, only one column is transformed by the operator

To describe the missing determinants in the Grothendieck case, we have to follow the same recursion than for Schubert, but with different flags. To any $v \in \mathbb{N}^n$, let us associate the two following flags of alphabets. Let w be the sequence $w_i := \max(j : j \geq i, v_j \geq v_i$. Then v^x is the decreasing reordering of w. Let now u be the element of $\mathbb{N}^n$ obtained by decreasingly reordering v according to the rule $[\ldots i, j \ldots] \to [\ldots j{+}1, i \ldots]$ whenever $i < j$. Then v^y is set to be the increasing reordering of u.

Theorem 4.10.1. *Let $v \in \mathbb{N}^n$ be vexillary, v^x, v^y be the two vectors defined above, $k = \max(v^y)$. Then*

$$\begin{aligned} Y_v &= S_{v\uparrow}(\mathbf{x}_{v_1^x} - \mathbf{y}_{v_1^y}, \ldots, \mathbf{x}_{v_n^x} - \mathbf{y}_{v_n^y})\,, && (4.10.1)\\ G_v &= S_{k^n}(\mathbf{x}_{v_1^x} - \mathbf{y}_{v_1^y}, \ldots, \mathbf{x}_{v_n^x} - \mathbf{y}_{v_n^y})\,(x_1 \cdots x_n)^{-k}\,, && (4.10.2)\\ K_v &= S_{v\uparrow}(\mathbf{x}_{v_1^x}, \ldots, \mathbf{x}_{v_n^x})\,. && (4.10.3) \end{aligned}$$

In particular, when v is vexillary, then $K_v = Y_v(\mathbf{x}, \mathbf{0})$.

For example, for $v = [3,5,4,0,2]$, one has $w = [3,2,3,5,5]$, which reorders into $v^x = [5,5,3,3,2]$. On the other hand, the chain $v = [3,5,4,0,2] \to [6,3,4,0,2] \to [6,5,3,0,2] \to [6,5,3,3,0]$ gives the second flag $v^y = [0,3,3,5,6]$. Hence, one has

$$\begin{aligned} Y_{35402} &= S_{02345}(\mathbf{x}_5{-}\mathbf{y}_0, \mathbf{x}_5{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_5, \mathbf{x}_2{-}\mathbf{y}_6)\\ G_{35402} &= S_{66666}(\mathbf{x}_5{-}\mathbf{y}_0, \mathbf{x}_5{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_5, \mathbf{x}_2{-}\mathbf{y}_6)(x_1 \ldots x_5)^{-6}\\ K_{35402} &= S_{02345}(\mathbf{x}_5, \mathbf{x}_5, \mathbf{x}_3, \mathbf{x}_3, \mathbf{x}_2)\,. \end{aligned}$$

Property (2.6.5) allows to write from (4.10.2) a determinantal formula for $\widehat{G}_v$ polynomials such that $v\clubsuit$ be vexillary. This condition is in fact equivalent to requiring that v be vexillary, since if a permutation σ avoids the pattern 2143, then $\omega\sigma\omega$ also avoids this pattern, and conversely.

4.11 Grothendieck and Yang-Baxter

One can degenerate Yang-Baxter bases of Hecke algebras into bases of the 0-Hecke algebra, i.e. the algebra generated by $\widehat{\pi}_1, \widehat{\pi}_2, \ldots$. But as in the case of divided differences, instead of taking products of factors of the type $\widehat{\pi}_i + 1/c$, let us take factors $1 + c\widehat{\pi}_i$. Accordingly, given a spectral vector $[y_1, \ldots, y_n]$, one defines recursively a Yang-Baxter basis $\mho^{\widehat{\pi}}_\sigma$, starting from 1 for the identity permutation, by

$$\mho^{\widehat{\pi}}_{\sigma s_i} = \mho^{\widehat{\pi}}_\sigma \left(1 + \left(1 - \frac{y_{\sigma_i}}{y_{\sigma_{i+1}}}\right) \widehat{\pi}_i\right) \qquad \text{for } \sigma_i < \sigma_{i+1}\,. \tag{4.11.1}$$

For example,

$$\begin{aligned}
\mho^{\widehat{\pi}}_{321} &= \left(1 + \left(1 - \frac{y_1}{y_2}\right)\widehat{\pi}_1\right)\left(1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2\right)\left(1 + \left(1 - \frac{y_2}{y_3}\right)\widehat{\pi}_1\right) \\
&= 1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2 + \left(1 - \frac{y_1}{y_2}\right)\left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_1\widehat{\pi}_2 \\
&\quad + \left(1 - \frac{y_2}{y_3}\right)\left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2\widehat{\pi}_1 + \left(1 - \frac{y_1}{y_2}\right)\left(1 - \frac{y_1}{y_3}\right)\left(1 - \frac{y_2}{y_3}\right)\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\,.
\end{aligned}$$

s As in the case of divided differences, the Yang-Baxter coefficients are specialisations of known polynomials. The proof of the next properties is similar to the proof of Theorem 3.5.1, and we can avoid repeating it.

Theorem 4.11.1. *The matrix of change of basis between $\{\mho^{\widehat{\pi}}_\sigma\}$ and $\{\widehat{\pi}_\sigma\}$, and its inverse, have entries which are specializations of Grothendieck polynomials :*

$$\mho^{\widehat{\pi}}_\sigma = \sum_{\nu \le \sigma} \widehat{\pi}_\nu \, G_{(\nu)}(\mathbf{y}^\sigma, \mathbf{y})\,, \tag{4.11.2}$$

$$\widehat{\pi}_\nu \prod_{i<j}\left(1 - \frac{y_i}{y_j}\right) = \sum_{\sigma \le \nu} (-1)^{\ell(\sigma)-\ell(\nu)} \mho^{\widehat{\pi}}_\sigma \, G_{(\nu^{-1}\omega)}(\mathbf{y}^\omega, \mathbf{y}^\sigma)\ . \tag{4.11.3}$$

For example, for $\nu = [2,3,1]$, one has $\nu^{-1}\omega = [2,1,3]$, and the coefficients of the expansion of $\widehat{\pi}_{231}$ are specialisations of the polynomial $G_{(213} = 1 - y_1 x_1^{-1}$. One has

$$\begin{aligned}
\widehat{\pi}_{231} \prod_{i<j\le 3} (1 - y_i y_j^{-1}) &= \mho^{\widehat{\pi}}_{123} G_{(213}(\mathbf{y}^{321}, \mathbf{y}) - \mho^{\widehat{\pi}}_{213} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{213}) \\
&\qquad - \mho^{\widehat{\pi}}_{132} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{132}) + \mho^{\widehat{\pi}}_{231} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{231}) \\
&= \left(1 - \frac{y_1}{y_3}\right)\mho^{\widehat{\pi}}_{123} - \left(1 - \frac{y_2}{y_3}\right)\mho^{\widehat{\pi}}_{213} - \left(1 - \frac{y_1}{y_3}\right)\mho^{\widehat{\pi}}_{132} + \left(1 - \frac{y_2}{y_3}\right)\mho^{\widehat{\pi}}_{231}\,.
\end{aligned}$$

The general properties of Yang-Baxter bases induce properties of specializations of Grothendieck polynomials.

The symmetry (1.8.4) entails

$$\big(G_{(\nu)}(\mathbf{y}^{\sigma},\mathbf{y})\big)^{\clubsuit} = G_{(\omega\nu\omega)}(\mathbf{y}^{\omega\sigma\omega},\mathbf{y}^{\omega})\,, \tag{4.11.4}$$

using the involution $\clubsuit : y_i \to y_{n+1-i}^{-1}$, $i = 1,\dots,n$ introduced in (2.6.4).

Each of the equations (1.8.9) and (1.8.10) gives, after some rewriting,

$$\sum_{\nu} (\text{-}1)^{\ell(\nu)+\ell(\sigma)} G_{(\nu)}(\mathbf{y}^{\sigma},\mathbf{y}) G_{(\nu\omega)}(\mathbf{y}^{\zeta},\mathbf{y}) = \delta_{\sigma,\zeta\omega} \prod_{i<j} \left(1 - \frac{y_i}{y_j}\right)\,, \tag{4.11.5}$$

which is a special instance of formula (2.9.4).

Chapter 5

$G^{\mathbf{1/x}}$ and $\widetilde{G}$ Grothendieck polynomials

In the preceding sections, we have seen that Grothendieck and Schubert polynomials satisfy similar properties. To relate these two families precisely, it is convenient to perform a change of variables in the former. In fact, we shall use two slightly different transformations, in view of different geometrical considerations.

5.1 Grothendieck in terms of Schubert

Denote the image of the Grothendieck polynomial $G_v(\mathbf{x},\mathbf{y})$ under the inversion $x_i \to x_i^{-1}$, $y_i \to y_i$, by $G_v^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})$, and by $\widetilde{G}_v(\mathbf{x},\mathbf{y})$ the image under the transformation[1]

$$x_i \to (1-x_i)^{-1}\ ,\ y_i \to (1-y_i)^{-1},\ i=1,2,\ldots$$

Thus, in the dominant case, for $\lambda \in \mathbb{N}^n$ a partition, one has

$$G_\lambda^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}(y_j - x_i)y_j^{-1} \quad \& \quad \widetilde{G}_\lambda(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}(x_i - y_j)(1-y_j)^{-1}\,,$$

and the other polynomials are generated using respectively the operators

$$\pi_i^{\mathbf{1/x}} = -x_{i+1}\partial_i \quad \& \quad \widetilde{\pi}_i = (1-x_{i+1})\,\partial_i\,, \tag{5.1.1}$$

or the generation in $\mathbf{y}$ seen in (2.2.3), which uses the isobaric divided differences in $\mathbf{y}$ in the first case, or in the indeterminates $y_1 - 1, y_2 - 1, \ldots$ in the second case :

$$\pi_i^{\mathbf{y}} \qquad \& \qquad \pi_i^{\mathbf{y}^-} = (y_i - 1)\partial_i^{\mathbf{y}}\,. \tag{5.1.2}$$

[1] Many authors use the transformation $x_i \to (1-x_i)^{-1}$, $y_i \to (1-y_i)$. This not compatible with simultaneously using $Y_v(\mathbf{x},\mathbf{y})$, but only with $Y_v(\mathbf{x},\mathbf{0})$. In fact, the factor $x+y-xy$, instead of $1-x/y)$ or $(x-y)/1-y)$ that we now take, does not possess the right symmetry in x,y which is imposed by geometry.

For example, the polynomial $G^{\mathbf{1/x}}_{01}(\mathbf{x},\mathbf{y}) = 1 - x_1x_2y_1^{-1}y_2^{-1}$ is the image of $G^{\mathbf{1/x}}_{20}(\mathbf{x},\mathbf{y}) = (1{-}x_1y_1^{-1})(1{-}x_1y_2^{-1})$ under $\pi_1^{\mathbf{1/x}} = -x_2\partial_1$, as well as the image of $G^{\mathbf{1/x}}_{11}(\mathbf{x},\mathbf{y}) = (1{-}x_1y_1^{-1})(1{-}x_2y_1^{-1})$ under $\pi_1^{\mathbf{y}}$.

From the expression of $\widetilde{\pi}_i$, one sees that the basis $\widetilde{G}_v(\mathbf{x},\mathbf{y})$ is triangular in the basis $Y_v(\mathbf{x},\mathbf{y})$, the term of $\widetilde{G}_v(\mathbf{x},\mathbf{y})$ of lowest degree being

$$(1-y_1)^{-u_1}(1-y_2)^{-u_2}\cdots Y_v(\mathbf{x},\mathbf{y})\,,$$

with u the code of the permutation inverse to $\langle v\rangle$.

Any example reveals that the expansion of these new polynomials in the Schubert basis posseses a structure quite willing to uncover itself. For example, $v = [2,0,2,1]$ is the code of the permutation $[3,1,5,4,2]$, the code of the inverse permutation is equal to $u = [1,3,0,1,0]$, and one has

$$(1{-}y_1)(1{-}y_2)^3(1{-}y_4)\widetilde{G}_{2021} = \begin{tikzcd}[column sep=small]
 & Y_{2021} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\frac{1}{y_1-1}Y_{2220} \arrow[d, dash] \arrow[dr, dash] & \frac{1}{y_3-1}Y_{3021} \arrow[dl, dash] \arrow[dr, dash] & \frac{y_1+y_2-2}{(y_1-1)^2}Y_{2121} \arrow[dl, dash] \arrow[d, dash] \\
\frac{1}{(y_1-1)(y_3-1)}Y_{3220} \arrow[dr, dash] & \frac{1}{(y_1-1)^2}Y_{2221} \arrow[d, dash] & \frac{y_1+y_2-2}{(y_1-1)^2(y_3-1)}Y_{3121} \arrow[dl, dash] \\
 & \frac{1}{(y_1-1)^2(y_3-1)}Y_{3221} &
\end{tikzcd}$$

A similar computation gives the expansion of $G^{\mathbf{1/x}}_{2021}(\mathbf{x},\mathbf{y})$ (writing Y_v for $Y_v(\mathbf{x},\mathbf{y})$) :

$$\begin{aligned}
G^{\mathbf{1/x}}_{2021}(\mathbf{x},\mathbf{y}) = &-\frac{1}{y_1{y_2}^3y_4}Y_{2021} - \frac{(y_1+y_2)}{{y_1}^3{y_2}^3y_4}Y_{2121} - \frac{1}{{y_1}^2{y_2}^3y_4}Y_{222} \\
&- \frac{1}{{y_1}^3{y_2}^3y_4}Y_{2221} - \frac{1}{y_1{y_2}^3y_3y_4}Y_{3021} - \frac{(y_1+y_2)}{{y_1}^3{y_2}^3y_3y_4}Y_{3121} \\
&\qquad - \frac{1}{{y_1}^2{y_2}^3y_3y_4}Y_{322} - \frac{1}{{y_1}^3{y_2}^3y_3y_4}Y_{3221}\,.
\end{aligned}$$

On this single example, it appears that the two expansions are identical, up to a minor transformation of coefficients. Indeed, since a Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$ is invariant under a uniform translation $x_i \to x_i+\epsilon$, $y_i \to y_i+\epsilon$, $i = 1,2,\ldots$, the expansion of $\widetilde{G}_v(\mathbf{x},\mathbf{y})$ is obtained from the expansion of $G^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})$ by the change of variables $y_i \to y_i{-}1$ in the coefficients.

Notice however that the polynomials in $\mathbf{x}$ only are different. In the case of $G^{\mathbf{1/x}}_v(\mathbf{x},\mathbf{y})$, one has to specialize $\mathbf{y}$ to $\{1,1,\ldots\}$, while for $\widetilde{G}_v(\mathbf{x},\mathbf{y})$, one sends $\mathbf{y}$ to $\{0,0,\ldots\}$. Thus $G^{\mathbf{1/x}}_{01}(\mathbf{x},\mathbf{y}) = -y_2^{-1}Y_{01}(\mathbf{x},\mathbf{y}) - y_1^{-1}y_2^{-1}Y_{11}(\mathbf{x},\mathbf{y})$ gives

$$G^{\mathbf{1/x}}_{01}(\mathbf{x},\mathbf{1}) = -Y_{01}(\mathbf{x},\mathbf{1}) - Y_{11}(\mathbf{x},\mathbf{1}) = 1 - x_1x_2$$

while $\widetilde{G}_{01}(\mathbf{x},\mathbf{y}) = (1{-}y_2)^{-1}Y_{01}(\mathbf{x},\mathbf{y}) - (1{-}y_1)^{-1}(1{-}y_2)^{-1}Y_{11}(\mathbf{x},\mathbf{y})$ induces

$$\widetilde{G}_{01}(\mathbf{x},\mathbf{0}) = Y_{01}(\mathbf{x},\mathbf{0}) - Y_{11}(\mathbf{x},\mathbf{0}) = x_1 + x_2 - x_1x_2\,.$$

Normalizing for a moment the polynomials in such a way that $\widetilde{\widetilde{G}}_v = Y_v + \ldots$, one renders the matrix of change of basis unitriangular. Here it is, together with its inverse, for $n = 4$, putting $A = (y_1{-}1)^{-1}$, $B = (y_2{-}1)^{-1}$, $C = (y_1{+}y_2{-}2)(1{-}y_1)^{-2}$.

$$\begin{array}{c}
000\\001\\010\\100\\020\\011\\200\\101\\110\\300\\021\\201\\120\\111\\210\\301\\220\\310\\121\\211\\320\\311\\221\\321
\end{array}
\begin{bmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&B&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&AB&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&A&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&A&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&C&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&B&\cdot&A&\cdot&\cdot&\cdot&\cdot&\cdot&AB&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&A&\cdot&C&\cdot&\cdot&\cdot&A^2&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&A&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&B&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&A&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&B&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{bmatrix}$$

For example, the row of index 101 must be read

$$\begin{aligned}\widetilde{\widetilde{G}}_{101} &= (y_1-1)(y_3-1)\widetilde{G}_{101}\\ &= Y_{101} + \frac{1}{y_2-1}Y_{201} + \frac{1}{y_1-1}Y_{111} + \frac{1}{(y_2-1)(y_1-1)}Y_{211}\,,\end{aligned}$$

or equivalently

$$y_1y_3G^{\mathbf{1/x}}_{101} = Y_{101} + y_2^{-1}Y_{201} + y_1^{-1}Y_{111} + y_1^{-1}y_2^{-1}Y_{211}\,.$$

The inverse matrix looks very similar, apart from signs, and different location of the non-zero entries.

$$
\begin{array}{c}
000\\ 001\\ 010\\ 100\\ 020\\ 011\\ 200\\ 101\\ 110\\ 300\\ 021\\ 201\\ 120\\ 111\\ 210\\ 301\\ 220\\ 310\\ 121\\ 211\\ 320\\ 311\\ 221\\ 321
\end{array}
\begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & -B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -A & \cdot & \cdot & \cdot & AB & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -B & \cdot & -A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -A & \cdot & -C & \cdot & \cdot & \cdot & AB & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -A & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & -A & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -B & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
$$

For example, the row of index 021 must be read, with the normalized polynomials $\widetilde{\widetilde{G}}_v$, or the polynomials $G_v^{\mathbf{1/x}}$,

$$
\begin{aligned}
Y_{021} &= \widetilde{\widetilde{G}}_{021} + \frac{1}{1\text{-}y_1}\widetilde{\widetilde{G}}_{22} + \left(\frac{1}{1\text{-}y_1} + \frac{1\text{-}y_2}{(1\text{-}y_1)^2}\right)\widetilde{\widetilde{G}}_{121} + \frac{1}{(1\text{-}y_1)(1\text{-}y_2)}\widetilde{\widetilde{G}}_{221} \\
Y_{021} &= -{y_2}^2 y_3 G^{\mathbf{1/x}}_{021} - y_1 {y_2}^2 G^{\mathbf{1/x}}_{22} - y_1 y_3\,(y_1+y_2)\,G^{\mathbf{1/x}}_{121} - {y_1}^2 y_2 G^{\mathbf{1/x}}_{221}\,.
\end{aligned}
$$

Let us precise and prove the observation of the closeness between the two matrices. Thanks to (2.6.5), the orthogonality relation between the polynomials G_v and $\widehat{G}_u$ can be rewritten as

$$
(G_u x^\rho)^\omega\, y^{-\rho} G_v\, \pi_\omega \in \{\pm 1, 0\}\,. \tag{5.1.3}
$$

Since the operator $x^{\rho\omega}\,\pi_\omega$ is equal to

$$
(x_1 \dots x_n)^{n-1}\partial_\omega = \left(\sum (\text{-}1)^{\ell(\sigma)}\sigma\right)(x_1 \dots x_n)^{n-1}\Delta(\mathbf{x})^{-1}
$$

its image under the change of variables $x_i \to (1\text{-}x_i)^{-1}$ is the operator

$$
\left(\sum (\text{-}1)^{\ell(\sigma)}\sigma\right)\Delta(\mathbf{x})^{-1} = \partial_\omega\,.
$$

Indexing by permutations rather than codes, one rewrites (5.1.3) as

$$G^{\mathbf{1/x}}_{(\sigma)}(\mathbf{x},\mathbf{y})\, G^{\mathbf{1/x}}_{(\zeta)}(\mathbf{x}^\omega,\mathbf{y})\,\partial_\omega = (\text{-}1)^{\ell(\sigma)} y^{-\rho}\delta_{\sigma,\zeta\omega} \tag{5.1.4}$$

$$\widetilde{G}_{(\sigma)}(\mathbf{x},\mathbf{y})\,\widetilde{G}_{(\zeta)}(\mathbf{x}^\omega,\mathbf{y})\,\partial_\omega = (\text{-}1)^{\ell(\sigma)}(y_1\text{-}1)^{n-1}\cdots(y_{n-1}\text{-}1)\delta_{\sigma,\zeta\omega}\,. \tag{5.1.5}$$

Because $(f,g)^\partial = fg\partial_\omega$ is the scalar product used for Schubert polynomials, the preceding equations mean that, with respect to $(\,,\,)^\partial$ and up to normalization, $\{(\text{-}1)^{|v|}G^{\mathbf{1/x}}_v(\mathbf{x}^\omega,\mathbf{y}),\ v\le\rho\}$ is adjoint to $\{G^{\mathbf{1/x}}_v(\mathbf{x},\mathbf{y}),\ v\le\rho\}$, and similarly $\{(\text{-}1)^{|v|}\widetilde{G}_v(\mathbf{x}^\omega,\mathbf{y}),\ v\le\rho\}$ is adjoint to $\{\widetilde{G}_v(\mathbf{x},\mathbf{y}),\ v\le\rho\}$. Thus one has the following symmetry property:

Theorem 5.1.1. *Let n be an integer. Denote $(y\text{-}1)^\rho = (y_1\text{-}1)^{n-1}\cdots(y_{n-1}\text{-}1)$. Then, for any $\zeta\in\mathfrak{S}_n$, one has*

$$X_\zeta(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\big(X_\zeta(\mathbf{x},\mathbf{y})\,,\, y^\rho G^{\mathbf{1/x}}_{(\sigma\omega)}(\mathbf{x}^\omega,\mathbf{y})\big)^\partial\, G^{\mathbf{1/x}}_{(\sigma)}(\mathbf{x},\mathbf{y}) \tag{5.1.6}$$

$$X_\zeta(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\big(X_\zeta(\mathbf{x},\mathbf{y})\,,\,(y\text{-}1)^\rho \widetilde{G}_{(\sigma\omega)}(\mathbf{x}^\omega,\mathbf{y})\big)^\partial\, \widetilde{G}_{(\sigma)}(\mathbf{x},\mathbf{y}) \tag{5.1.7}$$

and the inverse formulas

$$G^{\mathbf{1/x}}_{(\zeta)}(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\big(G^{\mathbf{1/x}}_{(\zeta)}(\mathbf{x},\mathbf{y})\,,\,X_\sigma(\mathbf{x}^\omega,\mathbf{y})\big)^\partial\, X_{\sigma\omega}(\mathbf{x},\mathbf{y}) \tag{5.1.8}$$

$$\widetilde{G}_{(\zeta)}(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\big(\widetilde{G}_{(\zeta)}(\mathbf{x},\mathbf{y})\,,\,X_\sigma(\mathbf{x}^\omega,\mathbf{y})\big)^\partial\, X_{\sigma\omega}(\mathbf{x},\mathbf{y})\,. \tag{5.1.9}$$

For example,

$$\Big(G^{\mathbf{1/x}}_{(1342)}(\mathbf{x},\mathbf{y}), -X_{1432}(\mathbf{x}^\omega,\mathbf{y})\Big)^\partial = (y_1+y_2)y_1^{-2}y_2^{-2}$$

$$\Big(\widetilde{G}_{(1342)}(\mathbf{x},\mathbf{y}), -X_{1432}(\mathbf{x}^\omega,\mathbf{y})\Big)^\partial = (y_1+y_2-2)(y_1-1)^{-2}(y_2-1)^{-2}$$

imply that

$$\begin{aligned}
G^{\mathbf{1/x}}_{(1342)}(\mathbf{x},\mathbf{y}) &= \cdots + (y_1\text{+}y_2)y_1^{-2}y_2^{-2}X_{2341}(\mathbf{x},\mathbf{y}) + \cdots\\
\widetilde{G}_{(1342)}(\mathbf{x},\mathbf{y}) &= \cdots + (y_1\text{+}y_2\text{-}2)(y_1\text{-}1)^{-2}(y_2\text{-}1)^{-2}X_{2341}(\mathbf{x},\mathbf{y}) + \cdots\\
X_{1432}(\mathbf{x},\mathbf{y}) &= \cdots - y_1y_3(y_1\text{+}y_2)\,G^{\mathbf{1/x}}_{(2431)}(\mathbf{x},\mathbf{y}) + \cdots\\
X_{1432}(\mathbf{x},\mathbf{y}) &= \cdots - (y_1\text{-}1)(y_3\text{-}1)(y_1\text{+}y_2\text{-}2)\,\widetilde{G}_{(2431)}(\mathbf{x},\mathbf{y}) + \cdots
\end{aligned}$$

A combinatorial description of the coefficients will be given in a later section.

We have defined the bases $\{G^{\mathbf{1/x}}_v(\mathbf{x},\mathbf{y})\}$ and $\{\widetilde{G}_v(\mathbf{x},\mathbf{y})\}$ using the operators $\pi_i^{\mathbf{1/x}}$ and $\widetilde{\pi}_i = \pi_i^{\mathbf{1/1-x}}$. Taking the operators $\widehat{\pi}_i^{\mathbf{1/x}} = \pi_i^{\mathbf{1/x}}\text{-}1$ and $\widehat{\pi}_i^{\mathbf{1/1-x}} = \pi_i^{\mathbf{1/1-x}}\text{-}1$ instead gives alternating summations which are described in the following proposition.

Proposition 5.1.2. *Given n, $\sigma \in \mathfrak{S}_n$, one has the following identities involving the Ehresmann-Bruhat interval $[\sigma, \omega]$, and the alphabets $\mathbf{1}\text{–}\mathbf{x} = \{1\text{–}x_1, 1\text{–}x_2, \dots\}$ and $\mathbf{1}\text{–}\mathbf{y} = \{1\text{–}y_1, 1\text{–}y_2, \dots\}$.*

$$x^\rho \sum_{\zeta \le \sigma} (-1)^{\ell(\zeta)} G_{(\zeta)}(\mathbf{x}, \mathbf{y}) \quad = \quad y^\rho G_\sigma^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) \tag{5.1.10}$$

$$(1\text{–}x)^{-\rho} \sum_{\zeta \le \sigma} (-1)^{\ell(\zeta)} \widetilde{G}_{(\zeta)}(\mathbf{x}, \mathbf{y}) \quad = \quad (1\text{–}y)^\rho G_\sigma^{\mathbf{1/x}}(\mathbf{1}\text{–}\mathbf{x}, \mathbf{1}\text{–}\mathbf{y}) \tag{5.1.11}$$

$$x^{-\rho} \sum_{\zeta \le \sigma} (-1)^{\ell(\zeta)} G_{(\zeta)}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) \quad = \quad y^{-\rho} G_\sigma(\mathbf{x}, \mathbf{y})\,. \tag{5.1.12}$$

Proof. The three identities result from each other by change of variables, let us consider the first one. Its left-hand side can be written $(-1)^{\ell(\sigma)} G_\omega(\mathbf{x}, \mathbf{y}) \widehat{\pi}_{(\omega\sigma)}(\mathbf{x}, \mathbf{y})$. However, $\pi_i^{\mathbf{1/x}} x^{-\rho} = -x_{i+1} \partial_i x^{-\rho} = -x^{-\rho} \widehat{\pi}_i$. Therefore

$$\begin{aligned}(-1)^{\ell(\sigma)} G_\omega(\mathbf{x}, \mathbf{y}) \widehat{\pi}_{(\omega\sigma)}(\mathbf{x}, \mathbf{y}) &= (-1)^{\ell(\omega)} G_\omega^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) x^{-\rho} y^\rho \widehat{\pi}_{\omega\sigma} \\ &= G_\omega^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) \pi_{\omega\sigma}^{\mathbf{1/x}} y^\rho = G_\sigma^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) y^\rho\,,\end{aligned}$$

which is the required identity. QED

Combining 5.1.10 and 2.6.5, one can also express the adjoint basis $\{\widehat{G}_v(\mathbf{x}, \mathbf{y})\}$ with alternating summations:

$$\sum_{\zeta \le \sigma} (-1)^{\ell(\zeta)} G_{(\zeta)}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) = \widehat{G}_{(\omega\sigma\omega)}(\mathbf{x}^\omega, \mathbf{y})\,. \tag{5.1.13}$$

For example, for $\sigma = [1, 3, 2]$, one has

$$\begin{aligned}&G_{01}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) - G_{11}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) - G_{20}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) + G_{21}^{\mathbf{1/x}}(\mathbf{x}, \mathbf{y}) \\ &= 1 - \frac{x_2 x_1}{y_1 y_2} - \left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_1}\right) - \left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_1}{y_2}\right) + \left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_1}{y_2}\right)\left(1 - \frac{x_2}{y_1}\right) \\ &\qquad = \frac{x_1\,(x_2 x_1 - y_1 y_2)}{{y_1}^2 y_2} = \widehat{G}_{(213)}(\mathbf{x}^\omega, \mathbf{y})\,.\end{aligned}$$

5.2 Monk formula for $G^{\mathbf{1/x}}$ and $\widetilde{G}$ polynomials

We have described in (4.5.1) the product $G_v(x_1 \dots x_k)^{-1}$. After change of variables, this translates into the product $\widetilde{G}_v(1{-}x_1)\dots(1{-}x_k)$.

We are going to refine this result by giving the product by a single variable, instead of by $(1{-}x_1)\dots(1{-}x_k)$. To do so, we need the k-paths introduced in (3.6.3). Recall also that a *hook* is a word or a sequence $z_1 \dots z_r$ such that there exists j: $z_1 > \dots > z_j$, $z_j \le z_{j+1} \le \dots \le z_r$.

Theorem 5.2.1. *Let $\sigma \in \mathfrak{S}_{n+1}$ be such that $\sigma_{n+1} = n{+}1$, let $k : 1 \le k \le n$. Then*

$$x_k G^{\mathbf{1/x}}_{(\sigma)}(\mathbf{x},\mathbf{y}) = \sum (-1)^{\ell} y_{min}\, G^{\mathbf{1/x}}_{(\tau_{a_r,a_{r-1}}\cdots\tau_{a_1 a_0}\sigma)}(\mathbf{x},\mathbf{y}) \tag{5.2.1}$$

$$(1-x_k)\,\widetilde{G}_{(\sigma)}(\mathbf{x},\mathbf{y}) = \sum (-1)^{\ell}(1{-}y_{min})\,\widetilde{G}_{(\tau_{a_r,a_{r-1}}\cdots\tau_{a_1 a_0}\sigma)}(\mathbf{x},\mathbf{y}), \tag{5.2.2}$$

sum over all the k-paths $\mathbf{a} = [a_r, \dots, a_0]$ such that $\mathbf{a}$ be a hook, as well as a subword of $[\sigma_{k+1}, \dots, \sigma_{n+1}, \sigma_1, \dots, \sigma_k]$, with $min = \min(a_r, \dots, a_0)$, $\ell + 1$ being the height of the hook.

Proof. Thanks to Colin Powell, who has considerably lightened the requirements for a proof[2], I shall content myself of sketching the method. The two statements are equivalent by change of variables. Let us take the $\widetilde{G}$-polynomials. One uses a decreasing induction on length, starting with $\sigma = [n, \dots, 1, n{+}1]$, and using the two recursions

$$\widetilde{G}_{(\sigma)} x_k = \widetilde{G}_{(\sigma s_{k-1})}\widetilde{\pi}_{k-1} x_k = \widetilde{G}_{(\sigma s_{k-1})} x_{k-1}\widetilde{\pi}_{k-1} + \widetilde{G}_{(\sigma s_{k-1})}(x_k{-}1)\,,\ \ \sigma_{k-1} < \sigma_k\,,$$

$$\widetilde{G}_{(\sigma)} x_k = \widetilde{G}_{(\sigma s_k)}\widetilde{\pi}_k x_k = \widetilde{G}_{(\sigma s_k)} x_{k+1}\widetilde{\pi}_k + \widetilde{G}_{(\sigma s_k)}(1{-}x_{k+1})\,,\ \ \sigma_k < \sigma_{k+1}\,,$$

which are a direct consequence of Leibnitz' formula. QED

As a small example, let $\sigma = [3,1,5,2,4,6]$, $k = 4$. One has to enumerate 4-paths which are hooks as well as subwords of $[4,6,3,1,5,2]$. One finds the hooks $\boxed{2}$, $\boxed{1}\boxed{2}$, $\begin{array}{|c|}\hline 4\\ \hline 2\\ \hline\end{array}$ and $\begin{array}{|c|c}\hline 4 & \\ \hline 1 & \multicolumn{1}{|c|}{2}\\ \hline\end{array}$, which correspond respectively to the permutations $[3,1,5,2,4,6]$, $[3,2,5,1,4,6]$, $[3,1,5,4,2,6]$ and $[3,2,5,4,1,6]$. Hence one has the two expansions

$$\begin{aligned} x_4 G^{\mathbf{1/x}}_{(31524)}(\mathbf{x},\mathbf{y}) &= y_2 G^{\mathbf{1/x}}_{(31524)}(\mathbf{x},\mathbf{y}) + y_1 G^{\mathbf{1/x}}_{(32514)}(\mathbf{x},\mathbf{y}) \\ &\qquad - y_2 G^{\mathbf{1/x}}_{(31542)}(\mathbf{x},\mathbf{y}) - y_1 G^{\mathbf{1/x}}_{(32541)}(\mathbf{x},\mathbf{y}) \\ (1{-}x_4)\widetilde{G}_{(31524)}(\mathbf{x},\mathbf{y}) &= (1{-}y_2)\widetilde{G}_{(31524)}(\mathbf{x},\mathbf{y}) + (1{-}y_1)\widetilde{G}_{(32514)}(\mathbf{x},\mathbf{y}) \\ &\qquad - (1{-}y_2)\widetilde{G}_{(31542)}(\mathbf{x},\mathbf{y}) - (1{-}y_1)\widetilde{G}_{(32541)}(\mathbf{x},\mathbf{y})\,. \end{aligned}$$

[2] Colin Powell's presentation to the U.N. Security Council, February 5, 2003. http://edition.cnn.com/2003/US/02/05 /sprj.irq.powell.transcript.10.

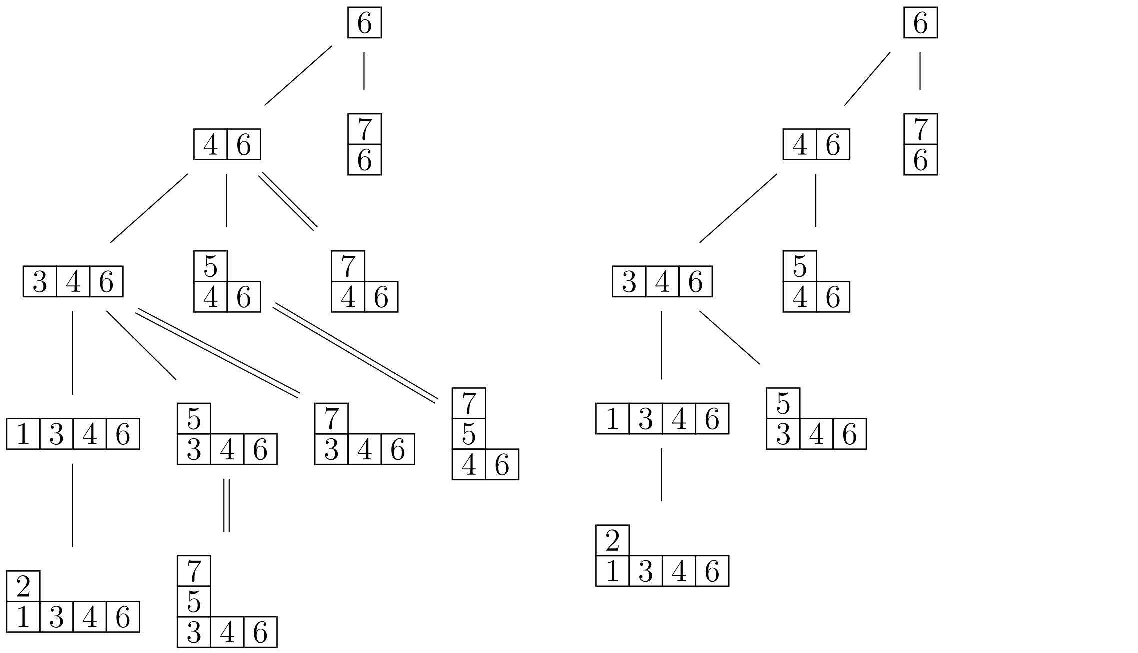
Let us illustrate the recursion for Monk formula on a bigger example. Let $k=4$, $\sigma=[1,3,4,6,2,7,5,8]$ (resp. $\sigma=[1,3,4,6,2,5,7,8]$).

Suppressing the terminal 8 which is a fixed point of all the permutations involved, one has

$$(1\text{-}x_4)\widetilde{G}_{(1346275)} = (1\text{-}y_6)\widetilde{G}_{(1346275)} - (1\text{-}y_6)\widetilde{G}_{(1347265)} + (1\text{-}y_4)\widetilde{G}_{(1364275)}$$
$$-(1\text{-}y_4)\widetilde{G}_{(1365274)}+(1\text{-}y_3)\widetilde{G}_{(1463275)}-(1\text{-}y_4)\widetilde{G}_{(1367245)}+(1\text{-}y_4)\widetilde{G}_{(1367254)}-(1\text{-}y_3)\widetilde{G}_{(1465273)}$$
$$+ (1\text{-}y_1)\widetilde{G}_{(3461275)} - (1\text{-}y_3)\widetilde{G}_{(1467235)} + (1\text{-}y_3)\widetilde{G}_{(1467253)} - (1\text{-}y_1)\widetilde{G}_{(3462175)}\,,$$

$$(1\text{-}x_4)\widetilde{G}_{(1346257)} = (1\text{-}y_6)\widetilde{G}_{(1346257)}-(1\text{-}y_6)\widetilde{G}_{(1347256)}+(1\text{-}y_4)\widetilde{G}_{(1364257)}-(1\text{-}y_4)\widetilde{G}_{(1365247)}$$
$$+ (1\text{-}y_3)\widetilde{G}_{(1463257)} - (1\text{-}y_3)\widetilde{G}_{(1465237)} + (1\text{-}y_1)\widetilde{G}_{(3461257)} - (1\text{-}y_1)\widetilde{G}_{(3462157)}\,.$$

The following two trees describe the cycles $\zeta\sigma^{-1}$ in the preceding expansions $(1\text{-}x_4)\widetilde{G}_\sigma(\mathbf{x},\mathbf{0}) = \sum \pm\widetilde{G}_\zeta(\mathbf{x},\mathbf{0})$, these cycles being hooks which are subwords of $[2,7,5,1,3,4,6]$ when $\sigma=[1,3,4,6,2,7,5]$, and subwords of $[2,5,7,1,3,4,6]$ when $\sigma=[1,3,4,6,2,5,7]$.



For example, the two bottom elements of the left tree must be read as follows. The two permutations $\begin{array}{r}\sigma=1346275\\ 3462175\end{array}$ differ by the cycle $(1\to 3\to 4\to 6\to 2)$, that one writes as a hook finishing by 6: $\begin{array}{|c|ccc}\hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline\end{array}$. The minimum of the cycle is 1, the height of the hook is 2, and therefore the corresponding term in the decomposition of $(1\text{-}x_4)\widetilde{G}_{(sigma)}(\mathbf{x},\mathbf{y})$ is $-(1\text{-}y_1)\widetilde{G}_{(3462175)}(\mathbf{x},\mathbf{y})$.

Similarly, $\begin{array}{r}\sigma=1346275\\ 1467253\end{array}$ differ by the cycle $(3\to 4\to 6\to 7\to 5)$, that one writes $\begin{array}{|c|cc}\hline 7 & & \\ \cline{1-1} 5 & & \\ \hline 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$. The minimum of the cycle is 3, the height of the hook is 3, hence the term $(1{-}y_3)\widetilde{G}_{(1467253)}(\mathbf{x}, \mathbf{y})$.

Lenart[138, Th.3.1] describes the product $x_k \widetilde{G}_v(\mathbf{x}, \mathbf{0})$ in terms of chains in the "k-Bruhat order" and criticizes (4.5.1) for involving cancellations. For our defense, we shall put forward that this is not the same case which is treated in the two formulas. In the former, one multiplies by $1{-}x_k$, in the later, taking into account the change of variables, by $(1{-}x_1) \dots (1 - x_k)$.

The Pieri formula for the products $\widetilde{G}_v(\mathbf{x}, \mathbf{0})\widetilde{G}_{0^i,1^j}(\mathbf{x}, \mathbf{0})$ or $\widetilde{G}_v(\mathbf{x}, \mathbf{0})\widetilde{G}_{0^i,j}(\mathbf{x}, \mathbf{0})$ is given by Lenart and Sottile [140].

5.3 Transition for $G^{\mathbf{1/x}}$ and $\widetilde{G}$ polynomials

By change of variables, one transforms (4.7.2) into a transition formula for $G^{\mathbf{1/x}}$ and $\widetilde{G}$polynomials.

Proposition 5.3.1. *Let $v \in \mathbb{N}^k$ be such that $v_k > 0$, let σ be the permutation of code $v' = [v_1, \ldots, v_{k-1}, v_k{-}1]$. Then, with the conventions of (4.7.2), one has*

$$\left(G^{\mathbf{1/x}}_{v'} - G^{\mathbf{1/x}}_{v}\right)\frac{y_j}{x_k} = (1-\tau_{ji_m})\star\cdots(1-\tau_{ji_1})\star G^{\mathbf{1/x}}_{(\sigma)} \tag{5.3.1}$$

$$\left(\widetilde{G}_{v'} - \widetilde{G}_{v}\right)\frac{1-y_j}{1-x_k} = (1-\tau_{ji_m})\star\cdots(1-\tau_{ji_1})\star \widetilde{G}_{(\sigma)}\,. \tag{5.3.2}$$

These expressions are not a direct corollary of Monk formula for $\widetilde{G}$-polynomials. For example, for $v = [1,0,1,1]$, one has $\sigma = [2,1,4,3,5]$, and, writing at the same time codes and permutations,

$$\begin{aligned}\left(\widetilde{G}_{\substack{10100\\21435}} - \widetilde{G}_{\substack{10110\\21453}}\right)\frac{1{-}y_3}{1{-}x_4} &= (1-\tau_{32})(1-\tau_{31})\star\widetilde{G}_{\substack{10100\\21435}}\\ &= \widetilde{G}_{\substack{10100\\21435}} - \widetilde{G}_{\substack{11100\\23415}} - \widetilde{G}_{\substack{20100\\31425}} + \widetilde{G}_{\substack{21100\\32415}}\,,\end{aligned}$$

while, writing also in a box the hooks appearing in the statement of (5.2.2), one has the Monk formula

$$\begin{aligned}(1{-}x_4)\widetilde{G}_{\substack{10100\\21435}} = (1{-}y_3)\widetilde{G}_{\substack{10100\\21435}}\boxed{3} + (1{-}y_1)\widetilde{G}_{\substack{11100\\23415}}\boxed{13} + (1{-}y_2)\widetilde{G}_{\substack{20100\\31425}}\boxed{23}\\ - (1{-}y_3)\widetilde{G}_{\substack{10110\\21453}}\boxed{\substack{5\\3}} - (1{-}y_1)\widetilde{G}_{\substack{11110\\23451}}\boxed{\substack{5\\13}} - (1{-}y_2)\widetilde{G}_{\substack{20110\\31452}}\boxed{\substack{5\\23}}\,.\end{aligned}$$

5.4 Action of divided differences on $G^{\mathbf{1/x}}$ and $\widetilde{G}$ polynomials

Let us show that the property that, up to normalization, $\{G_v^{\mathbf{1/x}}(\mathbf{x}^\omega,\mathbf{y}),\ v\le\rho\}$ is adjoint to $\{G_v^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}),\ v\le\rho\}$, and similarly $\{\widetilde{G}_v(\mathbf{x}^\omega,\mathbf{y}),\ v\le\rho\}$ is adjoint to $\{\widetilde{G}_v(\mathbf{x},\mathbf{y}),\ v\le\rho\}$, allows to exchange multiplication by x_i with ∂_{n-i}.

Indeed, for any $\sigma,\zeta\in\mathfrak{S}_n$, any $i\le n-1$, let $\zeta'=\zeta s_i$ if $\zeta_i\ge\zeta_{i+1}$ or or $\zeta'=\zeta$ otherwise. Consequently, $G_\zeta^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})(-x_{i+1}\partial_i)=G_{\zeta'}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})$.

Then, according to Theorem 5.1.1, one has

$$\begin{aligned}
y^\rho(-1)^{\ell(\sigma)}\delta_{\zeta',\omega\sigma} &= \Big(\big(G_{(\zeta)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})(-x_{i+1}\partial_i)\big)^\omega, G_{(\sigma)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})\Big)^\partial \\
&= \Big(-\big(G_{(\zeta)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})x_{i+1}\big)^\omega\omega\partial_i\omega, G_{(\sigma)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})\Big)^\partial \\
&= \Big(\big(G_{(\zeta)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})x_{i+1}\big)^\omega, G_{(\sigma)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})\partial_{n-i}\Big)^\partial .
\end{aligned}$$

Therefore, up to reversal of alphabets, multiplication and divided differences are exchanged. Thus, let a $\neg$ *hook* be a word $z_1\ldots z_r$ such that there exists j: $z_1\le\cdots\le z_j$, $z_j>z_{j+1}>\cdots>z_r$. Then Monk formulas (5.2.1) and (5.2.2) translate about the following description of the action of divided differences.

Theorem 5.4.1. *Let $\sigma\in\mathfrak{S}_n$, $k:\ 1\le k\le n{-}1$ be such that $\sigma_k>\sigma_{k+1}$, and $\eta=\sigma s_k$. Then*

$$G_{(\sigma)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})\partial_k = \sum(-1)^\ell\, y_{max}^{-1} G_{(\tau_{a_r,a_{r-1}}\cdots\tau_{a_1a_0}\eta)}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) \tag{5.4.1}$$

$$\widetilde{G}_{(\sigma)}(\mathbf{x},\mathbf{y})\partial_k = \sum(-1)^{\ell-1}(1{-}y_{max})^{-1}\widetilde{G}_{(\tau_{a_r,a_{r-1}}\cdots\tau_{a_1a_0}\eta)}(\mathbf{x},\mathbf{y})\,, \tag{5.4.2}$$

sum over all the k-paths $\mathbf{a}=[a_r,\ldots,a_0]$ such that $\mathbf{a}$ be a $\neg$ hook, as well as a subword of $[\eta_{k+1},\ldots,\eta_{n+1},\eta_1,\ldots,\eta_k]$, with $max=\max(a_r,\ldots,a_0)$, ℓ being the width of the $\neg$ hook.

For example, for $k=3$, $\sigma=[3,1,5,2,6,4]$, then $\eta=[3,1,2,5,6,4]$, and, writing each polynomial together with its $\neg$ hook, one has

$$\begin{aligned}
G_{(3,1,5,2,6,4)}^{\mathbf{1/x}}\,\partial_3 &= -\boxed{2}\; y_2^{-1}G_{(3,1,2,5,6,4)}^{\mathbf{1/x}} - \begin{array}{|c|}\hline 4\\\hline 2\\\hline\end{array}\; y_4^{-1}G_{(3,1,4,5,6,2)}^{\mathbf{1/x}} \\
&\quad + \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\; y_2^{-1}G_{(3,2,1,5,6,4)}^{\mathbf{1/x}} + \begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\; y_4^{-1}G_{(3,4,1,5,6,2)}^{\mathbf{1/x}} \\
&\quad + \begin{array}{|c|c|}\hline 3&4\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\; y_4^{-1}G_{(4,1,3,5,6,2)}^{\mathbf{1/x}} - \begin{array}{|c|c|c|}\hline 1&3&4\\\hline \multicolumn{2}{c|}{}&2\\\cline{3-3}\end{array}\; y_4^{-1}G_{(4,3,1,5,6,2)}^{\mathbf{1/x}}\,,
\end{aligned}$$

$$
\begin{aligned}
\widetilde{G}_{(3,1,5,2,6,4)}\partial_3 = \boxed{2}\,\frac{1}{1-y_2}\widetilde{G}_{(3,1,2,5,6,4)} + \begin{array}{|c|}\hline 4\\\hline 2\\\hline\end{array}\,\frac{1}{1-y_4}\widetilde{G}_{(3,1,4,5,6,2)} \\
- \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,\frac{1}{1-y_2}\widetilde{G}_{(3,2,1,5,6,4)} - \begin{array}{|c|c|}\hline 1&4\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\,\frac{1}{1-y_4}\widetilde{G}_{(3,4,1,5,6,2)} \\
- \begin{array}{|c|c|}\hline 3&4\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\,\frac{1}{1-y_4}\widetilde{G}_{(4,1,3,5,6,2)} + \begin{array}{|c|c|c|}\hline 1&3&4\\\hline \multicolumn{2}{c|}{}&2\\\cline{3-3}\end{array}\,\frac{1}{1-y_4}\widetilde{G}_{(4,3,1,5,6,2)}\,.
\end{aligned}
$$

5.5 Still more keys: $\widetilde{K}^G$ polynomials

When σ is Grassmannian, $X_\sigma(\mathbf{x}, \mathbf{0})$ is equal to a Schur function, as well as to a key polynomial. The polynomial $\widetilde{G}_{(\sigma)}(\mathbf{x}, \mathbf{0})$ is also symmetrical, but not equal to a Schur function, nor to any classical symmetric function. One has to find another family of polynomials which play the role of key polynomials versus Schubert polynomials, and coincide with the polynomials $\widetilde{G}_{(\sigma)}(\mathbf{x}, \mathbf{0})$ in the symmetric case. The polynomials K_v^G seen in (4.6.2) play such a role with respect to $G_{(\sigma)}(\mathbf{x}, \mathbf{1})$. Therefore, we define the polynomials $\widetilde{K}_v^G$, $v \in \mathbb{N}^n$, to be the images of the polynomials K_v^G under the transformation $x_i \to (1-x_i)^{-1}$. In fact, these are the polynomials denoted $KG[v]$ in [104].

The corresponding operators are

$$\widetilde{D}_i = x_i(1 - x_{i+1})\, \partial_i = (1 - x_{i+1})\, \pi_i \,,$$

which can be characterized by

$$1\widetilde{D}_i = 1 \qquad \& \qquad x_{i+1}(1 - x_{i+1})^{-1}\, \widetilde{D}_i = 0$$

and, therefore, are obtained from the isobaric divided differences by the change of variable $x_i \to x_i(1-x_i)^{-1}$. In short, $\widetilde{D}_i = \pi_i^{\mathbf{x}/(\mathbf{1}-\mathbf{x})}$.

In explicit terms,

$$\widetilde{K}_\lambda^G = x^\lambda \text{ when } \lambda \text{ dominant} \quad \& \quad \widetilde{K}_{vs_i}^G = \widetilde{K}_v^G\, \widetilde{D}_i \quad \text{when } v_i \ge v_{i+1}\,. \tag{5.5.1}$$

For example, for the weights which are permutations of $[4, 2, 0]$ these key polynomials expand in the usual key polynomials as follows :

$$\begin{aligned}
\widetilde{K}_{420}^G &= K_{42} \\
\widetilde{K}_{402}^G &= K_{402} - K_{412} \\
\widetilde{K}_{240}^G &= K_{24} - K_{34} \\
\widetilde{K}_{204}^G &= K_{204} - K_{214} - K_{304} + K_{314} \\
\widetilde{K}_{042}^G &= K_{042} + K_{242} - 2K_{142} + K_{341} - K_{34} \\
\widetilde{K}_{024}^G &= K_{024} - K_{234} + 2K_{134} - K_{034} + K_{224} - 2K_{124}
\end{aligned}$$

Notice that the image of $\pi_\omega = x^\rho \partial_\omega$ under the change of variables is

$$\widetilde{D}_\omega = x_1^{n-1} \ldots x_{n-1}\, (1-x_2) \ldots (1-x_n)^{n-1}\, \partial_\omega = (1-x_2) \ldots (1-x_n)^{n-1}\, \pi_\omega \,, \tag{5.5.2}$$

so that

$$\widetilde{K}_{\lambda\omega}^G = x^\lambda (1-x_2) \cdots (1-x_n)^{n-1}\, \pi_\omega \; = x^{\lambda+\rho}(1-x_2) \cdots \partial_\omega = \widetilde{G}_{\lambda\omega}(\mathbf{x}, \mathbf{0})\,. \tag{5.5.3}$$

The following proposition is the image of Proposition 4.6.1 by the transformation $x_i \to (1-x_i)^{-1}$ and shows how to generate the polynomials $\widetilde{G}_v(\mathbf{x}, \mathbf{0})$ using the operators $\widetilde{D}_i$.

Proposition 5.5.1. *Given $v \in \mathbb{N}^n$, let k be such that $v_k = 0$ and $v_i > 0$ for $i < k$. (if no component of v is 0, change $n \to n{+}1$, $v \to [v, 0]$). Let $u = [v_1 - 1, \ldots, v_{k-1} - 1, v_{k+1}, \ldots, v_n]$. Then*

$$\widetilde{G}_v(\mathbf{x}, \mathbf{0}) = \widetilde{G}_u(\mathbf{x}, \mathbf{0})\,(x_{k-1} \cdots x_1)\,\widetilde{D}_{n-1} \cdots \widetilde{D}_k$$
$$= \widetilde{G}_u(\mathbf{x}, \mathbf{0})\,\widetilde{D}_{n-1} \cdots \widetilde{D}_k\,(x_{k-1} \cdots x_1)\,. \quad (5.5.4)$$

For example, if $v = [3, 0, 4]$, then $k = 2$, $u = [3{-}1, 4]$, and

$$\widetilde{G}_{304}(\mathbf{x}, \mathbf{0}) = \widetilde{G}_{24}(\mathbf{x}, \mathbf{0}) x_1 \widetilde{D}_2 = \widetilde{G}_{24}(\mathbf{x}, \mathbf{0}) x_1 (1{-}x_3)\,\pi_2\,.$$

As in the case of Schubert polynomials, the preceding proposition can be used to expand the polynomials $\widetilde{G}_u(\mathbf{x}, \mathbf{0})$ in terms of the $\widetilde{K}^G_v$. For example,

$$\widetilde{G}_{2042}(\mathbf{x}, \mathbf{0}) = \widetilde{K}^G_{2042} + \widetilde{K}^G_{5012} + \widetilde{K}^G_{3041} - \widetilde{K}^G_{3042} - \widetilde{K}^G_{5022} - \widetilde{K}^G_{5031} + \widetilde{K}^G_{5032}\,,$$

the leading terms corresponding to

$$Y_{2042}(\mathbf{x}, \mathbf{0}) = K_{2042} + K_{5012} + K_{3041}\,.$$

The appropriate quadratic form (which is not symmetric) is

$$(f, g)^G = CT\left(f\, g^{\clubsuit} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\,\frac{(1 - x_j)}{(1 - x_i)}\right)\right) = \left(f\, \frac{1}{(1{-}x_1)^{n-1} \ldots (1{-}x_{n-1})}\,,\, g\right)$$

taking the scalar product (,) used for key polynomials, and the involution $\clubsuit$: $x_i \to x^{-1}_{n+1-i}$.

One checks that, with $D_i = (x_i{-}1)\partial_i$, one has

$$\left(f\,\widetilde{D}_i\,,\, g\right)^G = \left(f\,,\, g\, D_{n-i}\right)^G, \quad (5.5.5)$$

the proof of the statement being reduced, as usual, to the case $n = 2$.

This leads to define still another basis,

$$\widehat{\widetilde{K}}{}^G_\lambda = x^\lambda \text{ when } \lambda \text{ dominant } \& \quad \widehat{\widetilde{K}}{}^G_{vs_i} = \widehat{\widetilde{K}}{}^G_v\,\partial_i (x_{i+1}{-}1) \ \text{ when } v_i \ge v_{i+1}\,. \quad (5.5.6)$$

The rest of this section depends on the following lemma, that we leave as an open question for lack of a simple proof.

Lemma 5.5.2. *For any dominant $\lambda \in \mathbb{N}^n$, any $v \in \mathbb{N}^n$, one has*

$$\left(\widetilde{K}^G_v\,,\, x^\lambda\right)^G = \delta_{v, \lambda\omega}\,. \quad (5.5.7)$$

The equations (5.5.5) and (5.5.7) give by a recursion that we already used several times a pair of adjoint bases:

Corollary 5.5.3. *For $u, v \in \mathbb{N}^n$ one has*

$$\big(\widetilde{K}_v^G\,,\ \widehat{\widehat{K}}{}_u^G\big)^G = \delta_{v,u\omega}\,. \tag{5.5.8}$$

For example,

$$\widetilde{K}_{021}^G(1-x_1)^{-2}(1-x_2)^{-1} = K_{021}+K_{22}+K_{301}+K_{031}+K_{23}+K_{041}+2\,K_{401}+K_{32}+\cdots\,,$$

$$\widehat{\widehat{K}}{}_{104}^G = \widehat{K}_{104} - 2\,\widehat{K}_{103} + \widehat{K}_{102} - 2\,\widehat{K}_{301} + \widehat{K}_{201} - \widehat{K}_{13} + \widehat{K}_{12} - \widehat{K}_{31} + \widehat{K}_{21} + \widehat{K}_3$$

and

$$(\widetilde{K}_{021}^G\,,\ \widehat{\widehat{K}}{}_{401}^G)^G = 2(K_{401}, \widehat{K}_{104}) - 2(K_{301}, \widehat{K}_{103}) - (K_{031}, \widehat{K}_{130}) + (K_{021}, \widehat{K}_{120}) = 0\,.$$

5.6 Graßmannian $G^{\mathbf{1/x}}$ and $\widetilde{G}$ polynomials

Graßmannian codes lead to the symmetric world. In that case, $G^{\mathbf{1/x}}$-polynomials are obtained using the symmetriser $\pi_\omega^{\mathbf{1/x}} = (-1)^{\ell(\omega)} x^{0,\ldots,n-1}\partial_\omega$, while one uses $\widetilde{\pi}_\omega = (1-x_2)\ldots(1-x_n)^{n-1}\partial_\omega$ for $\widetilde{G}$-polynomials, and $D_\omega = (1-x_2)\ldots(1-x_n)^{n-1}\pi_\omega = x^\rho \widetilde{\pi}_\omega$ for $\widetilde{K}^G$-polynomials

Explicitly, given a partition λ and $v = \lambda\uparrow$ its reordering, the Graßmannian polynomials of index v are

$$\begin{aligned} G_v^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) &= (-1)^{\ell(\omega)} G_{\lambda+\rho}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) x^{0,\ldots,n-1}\partial_\omega \\ \widetilde{G}_v(\mathbf{x},\mathbf{y}) &= \widetilde{G}_{\lambda+\rho}(\mathbf{x},\mathbf{y})(1-x_2)\ldots(1-x_n)^{n-1}\partial_\omega \\ \widetilde{K}_v^G &= x^\lambda D_\omega = x^{\lambda+\rho}(1-x_2)\ldots(1-x_n)^{n-1}\partial_\omega \end{aligned}$$

Each of these starting points can be written as a flag Schur function. Let μ be the partition conjugate to $\lambda+\rho$ and $c_\lambda = \prod_i (y_i-1)^{\mu_i}$. Then

$$c_\lambda \widetilde{G}_{\lambda+\rho}(\mathbf{x},\mathbf{y}) = Y_{\lambda+\rho}(\mathbf{x},\mathbf{y}) = S_{v_1,v_2+1,\ldots,v_n+n-1}(\mathbf{x}_n-\mathbf{y}_{v_1},\ldots,\mathbf{x}_1-\mathbf{y}_{v_n+n-1})$$

and therefore

$$y^\mu G_{\lambda+\rho}^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) x^{0,\ldots,n-1} = (-1)^{|\lambda+\rho|} S_{v+[(n-1)^n]}(\mathbf{x}_n-\mathbf{y}_{v_1},\ldots,\mathbf{x}_1-\mathbf{y}_{v_n+n-1})$$

$$\begin{aligned} c_\lambda \widetilde{G}_{\lambda+\rho}(\mathbf{x},\mathbf{y})(1-x_2)\ldots(1-x_n)^{n-1} & \\ &= S_{v+[(n-1)^n]}(\mathbf{x}_n-\mathbf{y}_{v_1}-(n-1),\ldots,\mathbf{x}_1-\mathbf{y}_{v_n+n-1}-0) \end{aligned}$$

$$(-1)^{\ell(\omega)} x^{\lambda+\rho}(1-x_2)\ldots(1-x_n)^{n-1} = S_{v+[(n-1)^n]}(\mathbf{x}_n-(n-1),\ldots,\mathbf{x}_1-0)$$

Writing the image under ∂_ω of these functions is immediate, and one obtains the following determinantal expressions of $G_v^{\mathbf{1/x}}$, $\widetilde{G}_v$ and $\widetilde{K}_v^G$.

Proposition 5.6.1. *Let $\lambda \in \mathbb{N}^n$ be dominant, $v = \lambda\uparrow$, $u = v + [0,\ldots,n-1]$, $\rho = [n-1,\ldots,0]$, $\mu = (\lambda+\rho)^\sim$, $c_\lambda = \prod_i (y_i-1)^{\mu_i}$. Then*

$$\begin{aligned} y^\mu G_v^{\mathbf{1/x}}(\mathbf{x},\mathbf{y}) &= (-1)^{|v|} S_{v+\rho}(\mathbf{x}_n-\mathbf{y}_{u_1},\ldots,\mathbf{x}_n-\mathbf{y}_{u_n}) && (5.6.1) \\ c_\lambda \widetilde{G}_v(\mathbf{x},\mathbf{y}) &= S_{v+\rho}(\mathbf{x}_n-\mathbf{y}_{u_1}-(n-1),\ldots,\mathbf{x}_n-\mathbf{y}_{u_n}-0) && (5.6.2) \\ (-1)^{n(n-1)/2}\widetilde{K}_v^G &= S_{v+\rho}(\mathbf{x}_n-(n-1),\ldots,\mathbf{x}_n-0)\,. && (5.6.3) \end{aligned}$$

For example, for $n=3$, $\lambda = [3,1,0]$, one has $v = [0,1,3]$, $v+\rho = [2,2,3]$,

$u=[0,2,5]$, $\mu=[2,2,1,1,1]$ and the determinants

$$y^{22111}G^{\mathbf{1/x}}_{013}(\mathbf{x},\mathbf{y}) = \begin{vmatrix} S_2(\mathbf{x}_3-\mathbf{y}_0) & S_3(\mathbf{x}_3-\mathbf{y}_2) & S_5(\mathbf{x}_3-\mathbf{y}_5) \\ S_1(\mathbf{x}_3-\mathbf{y}_0) & S_2(\mathbf{x}_3-\mathbf{y}_2) & S_4(\mathbf{x}_3-\mathbf{y}_5) \\ S_0(\mathbf{x}_3-\mathbf{y}_0) & S_1(\mathbf{x}_3-\mathbf{y}_2) & S_3(\mathbf{x}_3-\mathbf{y}_5) \end{vmatrix}$$

$$(y_1-1)^2\cdots(y_5-1)\widetilde{G}_{013}(\mathbf{x},\mathbf{y}) = \begin{vmatrix} S_2(\mathbf{x}_3-\mathbf{y}_0-2) & S_3(\mathbf{x}_3-\mathbf{y}_2-1) & S_5(\mathbf{x}_3-\mathbf{y}_5) \\ S_1(\mathbf{x}_3-\mathbf{y}_0-2) & S_2(\mathbf{x}_3-\mathbf{y}_2-1) & S_4(\mathbf{x}_3-\mathbf{y}_5) \\ S_0(\mathbf{x}_3-\mathbf{y}_0-2) & S_1(\mathbf{x}_3-\mathbf{y}_2-1) & S_3(\mathbf{x}_3-\mathbf{y}_5) \end{vmatrix}$$

$$-\widetilde{K}^G_{013} = \begin{vmatrix} S_2(\mathbf{x}_3-2) & S_3(\mathbf{x}_3-1) & S_5(\mathbf{x}_3) \\ S_1(\mathbf{x}_3-2) & S_2(\mathbf{x}_3-1) & S_4(\mathbf{x}_3) \\ S_0(\mathbf{x}_3-2) & S_1(\mathbf{x}_3-1) & S_3(\mathbf{x}_3) \end{vmatrix}.$$

Lenart[137, Th.2.4.] gives the case $\mathbf{y}=\mathbf{0}$ of (5.6.2), that is the case (5.6.3):

$$\widetilde{G}_v(\mathbf{x},\mathbf{0}) = \widetilde{K}^G_v = \pm S_{v+\rho}(\mathbf{x}_n-(n-1),\ldots,\mathbf{x}_n-0)\,.$$

One can expand by linearity such determinants, eliminating[3] the flag $[\ldots,2,1,0]$, or the flag $[\mathbf{y}_{u_1},\ldots,\mathbf{y}_{u_n}]$.

For example, $c_{631}\widetilde{G}_{136}(\mathbf{x},\mathbf{y}) = S_{346}(\mathbf{x}_3-\mathbf{y}_1-2,\mathbf{x}_3-\mathbf{y}_4-1,\mathbf{x}_3-\mathbf{y}_8)$. Writing S_w for $S_w(\mathbf{x}_3-\mathbf{y}_1,\mathbf{x}_3-\mathbf{y}_4,\mathbf{x}_3-\mathbf{y}_8)$, one obtains

$$\begin{aligned} c_{631}\widetilde{G}_{136}(\mathbf{x},\mathbf{y}) &= -S_{346}+2S_{246}+S_{336}-S_{146}-2S_{236}+S_{136} \\ K^G_{136} &= -K_{346}+2K_{246}+K_{336}-K_{146}-2K_{236}+K_{136}\,. \end{aligned}$$

The functions generalizing the complete or elementary symmetric functions are of special interest. For $v=[0^{n-1}k]$, (5.6.2) becomes, with $\lambda=[k+n-1,n-2,\ldots,0]$,

$$c_\lambda\widetilde{G}_v(\mathbf{x},\mathbf{y}) = S_{n-1,n-2,\ldots,1,k}(\mathbf{x}_n-n+1-\mathbf{y}_0,\mathbf{x}_n-n+2-\mathbf{y}_1,\ldots,\mathbf{x}_n-1-\mathbf{y}_{n-2},\mathbf{x}_n-\mathbf{y}_{k+n-1})\,.$$

Using recursively that for any $\mathbf{z}$ one has

$$\begin{aligned} &S_{\ldots,r+1,r,\ldots}(\ldots,\mathbf{z}-1,\mathbf{z}-y_i,\ldots) \\ &\quad= S_{\ldots,r+1,r,\ldots}(\ldots,\mathbf{z},\mathbf{z},\ldots) - S_{\ldots,r,r,\ldots}(\ldots,\mathbf{z},\mathbf{z},\ldots) \\ &\quad- y_iS_{\ldots,r+1,r-1,\ldots}(\ldots,\mathbf{z},\mathbf{z},\ldots) + y_iS_{\ldots,r,r-1,\ldots}(\ldots,\mathbf{z},\mathbf{z},\ldots) \\ &\quad= (y_i-1)S_{\ldots,r,r,\ldots}(\ldots,\mathbf{z},\mathbf{z},\ldots)\,, \end{aligned}$$

one obtains in final that

$$(1-y_1)\ldots(1-y_{k+n-1})\widetilde{G}_{0^{n-1}k}(\mathbf{x},\mathbf{y}) = S_{1,\ldots,1,k}(\mathbf{x}_n-1,\ldots,\mathbf{x}_n-1,\mathbf{x}_n-\mathbf{y}_{k+n-1})\,. \tag{5.6.4}$$

One may separate the two alphabets $\mathbf{x}$ and $\mathbf{y}$, and expand this last determinant as

$$\sum_i\sum_j(-1)^{n-i+j}S_{1^i,k-j}(\mathbf{x}_n)S_{1^j}(\mathbf{y}_{k+n-1})\,.$$

In terms of the Schubert basis, one has

[3] Recall that $S_k(\mathbf{x}-r)=S_k(\mathbf{x})-rS_{k-1}(\mathbf{x})+\binom{r}{2}S_{k-2}(\mathbf{x})-\ldots$

Lemma 5.6.2. *Let $k \geq 0$, $m = k{+}n{-}1$. Then*

$$(-1)^k G^{\mathbf{1/x}}_{0^{n-1}k}(\mathbf{x},\mathbf{y}) = \frac{1}{y_n \cdots y_m} Y_{0^{n-1}k}(\mathbf{x},\mathbf{y}) + \frac{1}{y_{n-1}\cdots y_m} Y_{0^{n-2}1k}(\mathbf{x},\mathbf{y}) + \cdots + \frac{1}{y_1 \cdots y_m} Y_{1^{n-1}k}(\mathbf{x},\mathbf{y})\,. \quad (5.6.5)$$

$$\widetilde{G}_{0^{n-1}k}(\mathbf{x},\mathbf{y}) = \frac{1}{(1{-}y_n)\ldots(1{-}y_m)} Y_{0^{n-1}k}(\mathbf{x},\mathbf{y}) - \frac{1}{(1{-}y_{n-1})\ldots(1{-}y_m)} Y_{0^{n-2}1k}(\mathbf{x},\mathbf{y}) + \cdots + \frac{(-1)^{n-1}}{(1{-}y_1)\ldots(1{-}y_m)} Y_{1^{n-1}k}(\mathbf{x},\mathbf{y})\,. \quad (5.6.6)$$

The case corresponding to elementary symmetric functions, $v = [0^{n-r}1^r]$ is a little more elaborate. To understand it, let us treat more generally the case $v = [0^{n-r}k^r]$, which also comprises the case that we have just disposed of. The determinant (5.6.2) becomes

$$c_\lambda \widetilde{G}_v(\mathbf{x},\mathbf{y}) = S_{n-1,\ldots,r,k+r-1,\ldots,k}\big(\mathbf{x}_n{-}(n{-}1) - \mathbf{y}_0, \ldots, \mathbf{x}_n{-}r - \mathbf{y}_{n-r+1}, \mathbf{x}_n{-}r{-}1 - \mathbf{y}_{n-r+k}, \ldots, \mathbf{x}_n{-}n{+}1 - \mathbf{y}_{n+1-k}\big)\,,$$

with $\lambda = [k{+}n{-}1, \ldots, k{+}n{-}r, n{-}r{-}1, \ldots, 0]$.

As in the case $r = 1$, the determinant can be simplified in the two blocks of columns, factors $(y_i{-}1)$ being extracted, and one obtains, writing $m = k{+}n{-}r$,

$$\prod_{i=1}^{m}(y_i{-}1)^r \widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y}) = S_{r^{n-r},k^r}\big(\underbrace{\mathbf{x}_n{-}r, \ldots, \mathbf{x}_n{-}r}_{n-r}, \underbrace{\mathbf{x}_n{-}\mathbf{y}_m, \ldots, \mathbf{x}_n{-}\mathbf{y}_m}_{r}\big)\,. \quad (5.6.7)$$

Expanding this determinant in the Schubert basis is not straightforward. Let us proceed differently, and use the recursion (5.1.2) in $\mathbf{y}$.

The polynomial $\widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y})$ is the image of

$$\widetilde{G}_{k^n}(\mathbf{x},\mathbf{y}) = Y_{k^n}(\mathbf{x},\mathbf{y})(1{-}y_1)^{-n}\ldots(1{-}y_k)^{-n}$$

under a product of isobaric divided differences in the indeterminates $y_1^- = y_1{-}1, y_2^- = y_2{-}1, \ldots$. Since this product acts on a function which is symmetrical in $y_1, \ldots, y_k$ and symmetrical in $y_{k+1}, \ldots, y_m$, with $m = k{+}n{-}r$, it can be replaced by the maximal symmetrizer $\pi^{\mathbf{y}^-}_{m\ldots 1}$. Concretely,

$$\widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y}) = \frac{Y_{k^n}(\mathbf{x},\mathbf{y})}{(1{-}y_1)^n\ldots(1{-}y_k)^n}(y_1{-}1)^{m-1}\ldots(y_m{-}1)^0\,\partial^{\mathbf{y}}_{m\ldots 1}\,.$$

Taking into account the symmetry in the two blocks of indeterminates y_i, one obtains

$$(y_1{-}1)^r\ldots(y_m{-}1)^r\,\widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y}) = (-1)^{kn} Y_{n^k}(\mathbf{y},\mathbf{x})(y_{k+1}{-}1)^r\ldots(y_m{-}1)^r\left(\partial^{\mathbf{y}}_k\ldots\partial^{\mathbf{y}}_{m-1}\right)\ldots\left(\partial^{\mathbf{y}}_1\ldots\partial^{\mathbf{y}}_{n-r}\right).$$

One can now use (2.14.9), the role of $\mathbf{x}$ and $\mathbf{y}$ having been exchanged, divided differences in the indeterminate y_i being the same as in the $y_i^- = y_i{-}1$. The images of $Y_{k^n}(\mathbf{x},\mathbf{y})$ are Schubert polynomials indexed by antidominant v such that $v \le k^n$. The images of $(y_1{-}1)^r \dots (y_{m-k}{-}1)^r = Y_{(n-r)^r}(\mathbf{y}^-, \mathbf{0})$ being some "skew" Schubert polynomials that will be detailed in the next chapter. For $v \in \mathbb{N}^n$, $u \in \mathbb{N}^n$ antidominant, let

$$v/u = [0^{u_1}, v_1 - u_1, 0^{u_2-u_1}, v_2 - u_2, \dots, 0^{u_n-u_{n-1}}, v_n - u_n]\,.$$

In final, one has the following expansion of $\widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y})$ and, by change of variables, of $G^{\mathbf{1/x}}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y})$.

Proposition 5.6.3. *Given positive integers r, n such $r \le n$, and $k \ge 0$, let $m = k{+}n{-}r$. Then*

$$(\text{-}1)^{kr}(y_1{-}1)^r \dots (y_m{-}1)^r\, \widetilde{G}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y}) = \sum_{v \le k^{n-r}} Y_{v,k^r}(\mathbf{x},\mathbf{y})\, Y_{r^{n-r}/v}(\mathbf{y}^-, \mathbf{0}) \tag{5.6.8}$$

$$(\text{-}1)^{kr} y^{r\dots r} G^{\mathbf{1/x}}_{0^{n-r}k^r}(\mathbf{x},\mathbf{y}) = \sum_{v \le k^{n-r}} Y_{v,k^r}(\mathbf{x},\mathbf{y})\, Y_{r^{n-r}/v}(\mathbf{y}, \mathbf{0})\,. \tag{5.6.9}$$

sum over antidominant v, the polynomials having indices with a negative component being set equal to 0.

For example, for $n = 4, k = 5, r = 2$, writing $Y_v Y_u$ for $Y_v(\mathbf{x},\mathbf{y})Y_u(\mathbf{y},\mathbf{0})$, one has $m = 7$ and

$$y_1^2 \cdots y_7^2\, G^{\mathbf{1/x}}_{0055}(\mathbf{x},\mathbf{y}) =$$

$$\begin{array}{ccccccc} & & & Y_{0255}Y_{200} & & & \\ Y_{0055}Y_{22} & \sim & Y_{0155}Y_{201} & & Y_{1255}Y_{010} & \sim & Y_{2255}Y_{000} \\ & & & Y_{1155}Y_{011} & & & \end{array}$$

Notice that the Schubert polynomials in $\mathbf{y}$ are all the images of Y_{22} under divided differences.

5.7 Dual Graßmannian Grothendieck polynomials

Each of the three sets $\{G_v^{\mathbf{1/x}}(\mathbf{x},\mathbf{y})\}$, $\{\widetilde{G}_v(\mathbf{x},\mathbf{y})\}$, $\{\widetilde{K}_v^G\}$, for $v \in \mathbb{N}^n$ antidominant, constitute a linear basis of $\mathfrak{Sym}(\mathbf{x}_n)$. Recall that the natural scalar product on this space, which orthonormalizes Schur functions, is the restriction of (,). It is therefore natural to consider the bases adjoint to the preceding ones with respect to the usual scalar on symmetric function. Let us determine only the basis adjoint to $\{\widetilde{K}^G_{\lambda\uparrow}\}$.

From (5.5.7), one has, for any pair of partitions in $\mathbb{N}^n$,

$$\left(\widetilde{K}^G_{\mu\uparrow}\prod_{i<j}\frac{1}{(1-x_i)^{n-i}}\,,\,x^\lambda\right)=\left(\widetilde{K}^G_{\mu\uparrow}\,,\,x^\lambda\prod_{i<j}\frac{1}{(1-x_i^{-1})^{i-1}}\right)=\delta_{\lambda,\mu}\,.$$

In the right-hand side, each factor of the type $1/(1-1/x_i)$ is interpreted as the series $1+x_i^{-1}+x_i^{-2}+\cdots$. However, in the expansion, only the terms x^u with $u\geq-\rho$ can give a contribution. Thus one can replace $x^\lambda(1-x_2)^{-1}\cdots(1-x_n)^{-1}$ by

$$\begin{aligned}\spadesuit &= x^\lambda(1+x_2^{-1}+\cdots+x_2^{-\lambda_2-n+2})\cdots\left(1+(n-1)x_n^{-1}\cdots+\binom{n+\lambda_n-2}{\lambda_n}\right)\\ &= x^{-\rho}S_{\lambda_1+n-1}(x_1)S_{\lambda_2+n-2}(x_2+1)\cdots S_{\lambda_n}(x_n+n-1)\,.\end{aligned}$$

Since $\widetilde{K}^G_{\mu\uparrow}$ is symmetrical, one has

$$\left(\widetilde{K}^G_{\mu\uparrow}\,,\,\spadesuit\right)=\left(\widetilde{K}^G_{\mu\uparrow}\,,\,\spadesuit\,\pi_\omega\right)\,.$$

The image of $\spadesuit$ under $\pi_\omega=x^\rho\partial_\omega$ is a multiSchur function, for which we shall follow the terminology used by [3, 14, 93, 182]. For a partition $\lambda\in\mathbb{N}^n$, define the *dual Grothendieck polynomial* $g_\lambda(\mathbf{x}_n)$ to be

$$g_\lambda(\mathbf{x}_n)=\spadesuit\,\pi_\omega=S_{\lambda\uparrow}(\mathbf{x}_n+n\text{-}1,\ldots,x_n+0)\,. \tag{5.7.1}$$

The preceding computations, assuming the validity of (5.5.7), show that the dual Grothendieck polynomials constitute the basis adjoint to $\{\widetilde{K}^G_{\mu\uparrow}=\widetilde{G}_{\mu\uparrow}(\mathbf{x},\mathbf{0})\}$.

Comparing the determinantal expression with the expression of a vexillary Schubert polynomial, one sees that g_λ is equal to a specialization of a Schubert polynomial:

$$g_\lambda(\mathbf{x}_n)=Y_{0^{n-1}\lambda}\big|_{x_{n+1}=1=\cdots=x_{2n-1}} \tag{5.7.2}$$

The expansion of g_λ in the Schur basis is easy to write by multilinearity of the determinant (5.7.1). This amounts to expand $\spadesuit=\sum_u c_u x^u$, and then to formally replace each x^u by the Schur function s_u. For example, for $n=3$, $\lambda=[3,2,1]$, one has

$$\begin{aligned}\spadesuit &= x^{321}\left(1+x_2^{-1}+x_2^{-2}+x_2^{-3}\right)\left(1+2x_3^{-1}\right)\\ &= x^{321}+x^{311}+x^{301}+x^{3,-1,1}+2x^{320}+2x^{310}+2x^{300}+2x^{3,-1,0}\,,\end{aligned}$$

Replacing x by s, dropping exponents, one obtains

$$g_{321}=s_{321}+s_{311}+s_{301}+s_{3,-1,1}+2s_{320}+2s_{310}+2s_{300}+2s_{3,-1,0}\,,$$

and finally, reordering or eliminating indices to keep only partitions,

$$g_{321} = s_{321} + s_{311} + 2s_{320} + 2s_{310} + s_{300} \,.$$

In fact, the combinatorial interpretation of vexillary Schubert polynomials, that we shall see later, gives a better description in terms of tableaux satisfying flag conditions. Lenart [137] used similar tableaux to describe the expansion of the Graßmannian Grothendieck polynomials $\widetilde{G}_{\lambda\uparrow}(\mathbf{x}, \mathbf{0})$ in the Schur basis. Lam and Pylyavskyy [93] call them "elegant fillings", and also give a description of g_λ in terms of reverse plane partitions.

In [117] , one finds several properties of dual Grothendieck polynomials, among which a finite-sum Cauchy identity for an arbitrary integer r:

$$\sum_{\lambda \le r^n} \widetilde{G}_{\lambda\uparrow}(\mathbf{x}_n, \mathbf{0})\, g_\lambda(\mathbf{y}_n) = \sum_{\lambda \le r^n} s_\lambda(\mathbf{x}_n)\, s_\lambda(\mathbf{y}_n) \,. \tag{5.7.3}$$

This formula implies directly the orthogonality property, without assuming (5.5.7).

Chapter 6

Plactic algebra and the module $\mathfrak{Schub}$

6.1 Tableaux

Let $\mathbb{A}$ be a totally ordered alphabet $\mathbb{A} = \{a_1 < a_2 < \ldots\}$, of non commuting letters. We usually take $\mathbb{A} = \{1, 2, \ldots, n\}$. The number of occurences of a given letter a into a word w is denoted $|w|_a$.

Let us repeat the distinction between factors and subwords. A *factor* of w is a word obtained by erasing letters at the beginning and the end of the word w, a *subword* is a word obtained by erasing letters inside the word. It is important not to mistake between these two notions ! We shall also need sometimes to record the position of a subword inside a word. In that case, it will be better to replace erased letters by a black box.

$$\underset{word}{3\,1\,4\,1\,5\,9\,1\,2} \qquad \underset{factor}{\blacksquare\blacksquare 4\,1\,5\,9\,1 \blacksquare} \qquad \underset{subword}{\blacksquare 1\,4 \blacksquare 5 \blacksquare 1\,2}$$

A Young tableau is a labeling of the boxes of the diagram of a partition λ (which is called the *shape* of the tableau) with letters of $\mathbb{A}$, in such way that columns are strictly decreasing from top to bottom, and rows are weakly increasing from left to right. One can replace such an object by its reading[1], that we still call a tableau.

$$\begin{array}{|c|c|c|c|c|c|c|} \cline{1-2} 6 & 8 \\ \cline{1-4} 4 & 5 & 5 & 6 \\ \cline{1-6} 2 & 2 & 3 & 3 & 5 & 7 \\ \hline 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ \hline \end{array} \qquad \Leftrightarrow \qquad 68\ 4556\ 223357\ 1112444\,.$$

One also needs to read (planar) tableaux by columns, still from left to right. The ensuing word is called a *column-tableau*. For the above object, it is 6421 8521 531 632 54 74 4.

[1] Adopting occidental conventions that one reads from left to right, and from top to bottom.

6.2 Strings

How to interpret the space of polynomials in x_1, x_2 of fixed degree, say 5? This is a vector space with basis $\{x^{50}, x^{41}, \ldots, x^{05}\}$, but one can as well replace each $x_1^\alpha x_2^\beta$ with $\underbrace{1\ldots1}_{\alpha}\,\underbrace{2\ldots2}_{\beta}$, and write the *string* of monomials

$$11111 \;\text{——}\; 11112 \;\text{——}\; 11122 \;\text{——}\; 11222 \;\text{——}\; 12222 \;\text{——}\; 22222\,.$$

An homogeneous polynomial $\sum c_v x^v$ in x_1, x_2 may now be considered as a weighted string, attaching a coefficient c_v to each element of the string.

Of course, one can now view the above string as a string of words in the letters $1, 2$. These words look special, because they are increasing and involve only two different letters. However, we show just below how to reduce general words to this case.

Given a totally ordered alphabet $\mathbb{A}$, a word w in $\mathbb{A}^*$, two consecutive letters in the alphabet, say $1, 2$, pair recursively $\cdots 2 \cdots 1 \cdots$ as if they were parentheses. Ignoring the paired letters, and the other letters of the alphabet, we are left with an increasing subword u of w in $1, 2$ that we call the 1-subword (and more generally, for a pair $i, i{+}1$, one has a i-subword).

For example, given the word 1222111211222, we write on the same level the letters in the order that they are paired, and put in boxes the remaining letters.

$$\begin{array}{cccccccccccccl}
1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & \textit{initial word} \\
\hline
 & & & 2 & 1 & & & 2 & 1 & & & & & \\
 & & 2 & & & 1 & & & & & & & & \\
 & 2 & & & & & & & & 1 & & & & \\
\boxed{1} & & & & & & \boxed{1} & & & & \boxed{2} & \boxed{2} & \boxed{2} & 1\textit{-subword}
\end{array}$$

We can now build a string of words, replacing the subword u inside w successively by all the elements of the string of u. By definition, an *i-string* is a sequence of words which differ only by their i-subwords, and such that the sequence of i-subwords is of the type

$$i^\alpha \;\text{——}\; i^{\alpha-1}(i{+}1) \;\text{——}\; i^{\alpha-2}(i{+}1)^2 \;\text{——}\; \cdots \;\text{——}\; (i{+}1)^\alpha\,.$$

Here is, for example, a 2-string :

$$\boxed{2}1\boxed{2}342\boxed{2} \;\text{——}\; \boxed{2}1\boxed{2}342\boxed{3} \;\text{——}\; \boxed{2}1\boxed{3}342\boxed{3} \;\text{——}\; \boxed{3}1\boxed{3}342\boxed{3}$$

One may view the construction of the i-subword of a word w as the reduction of the word to a monomial in x_i, x_{i+1}. Let us show conversely that this allows to lift any degree-preserving operator ϕ on $\mathfrak{Pol}(x_i, x_{i+1})$ to an operator $\widetilde{\phi}$ on the free algebra.

Indeed let w be a word, $i = 1$ and $u = 1^\alpha 2^\beta$ be the 1-subword of w. Then if $\phi(x_1^\alpha x_2 \beta) = \sum c_{\gamma,\delta:\,\gamma+\delta=\alpha+\beta} x_1^\gamma x_2^\delta$, one defines $\widetilde{\phi}(w)$ as the sum $\sum c_{\gamma,\delta}\, w_{\gamma,\delta}$ of all words in the string of w with coefficients $c_{\gamma,\delta}$.

In particular, the lift of the simple transposition s_1 acts by symmetry around the middle of the string, while the lift of π_1 sends w, when $\alpha \geq \beta$, to the sum of all words in the string between w and its image under the lift of s_1.

For example, the image of $w = 1123211112$ under the lift of π_1 is

$$\begin{matrix} w = \\ u = \end{matrix}\ \begin{matrix} 1123211112 \\ 11 \qquad\quad 112 \end{matrix} + \begin{matrix} 1123211122 \\ 11 \qquad\quad 122 \end{matrix} + \begin{matrix} 1123211222 \\ 11 \qquad\quad 222 \end{matrix} + \begin{matrix} 1223211222 \\ 12 \qquad\quad 222 \end{matrix}\ ,$$

in accordance with $x_1^6 x_2^3 x_3\, \pi_1 = x^{331}(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3)$.

We can perform this construction for every pair of consecutive letters in the alphabet, and thus obtain i-strings in the letters $i, i{+}1$ and operators lifting $s_i, \pi_i, \widehat{\pi}_i$ that we shall still denote by the same letters.

In particular, when the words are Young tableaux, their images by $s_i, \pi_i, \widehat{\pi}_i$ are still (sums of) tableaux.

Here is an example of a 3-string of tableaux

$$\begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{3} & \boxed{3} \end{array} \ \text{———}\ \begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{3} & \boxed{4} \end{array} \ \text{———}\ \begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{4} & \boxed{4} \end{array} \ \text{———}\ \begin{array}{lllll} \boxed{4} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{4} & \boxed{4} \end{array}$$

and here is the action of of s_3 and $\widehat{\pi}_3$ on the first element of the string :

$$\begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{3} & \boxed{3} \end{array} \ s_3 = \begin{array}{lllll} \boxed{4} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{4} & \boxed{4} \end{array}$$

$$\begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{3} & \boxed{3} \end{array} \ \widehat{\pi}_3 = \begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{3} & \boxed{4} \end{array} + \begin{array}{lllll} \boxed{3} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{4} & \boxed{4} \end{array} + \begin{array}{lllll} \boxed{4} & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 2 & \boxed{4} & \boxed{4} \end{array}$$

The last two equations lift respectively the identities $x^{2311} x_3^3\, s_3 = x^{2311} x_4^3$, and $x^{2311} x_3^3\, \widehat{\pi}_3 = x^{2311}(x_3^2 + x_3 x_4 + x_4^2) x_4$.

The most elementary operators are the *crystal operators* e_i, f_i, which consist in moving leftwards or rightwards on a i-string (or sending to 0 if not possible). Write a for the letter i and b for the letter $i{+}1$. Then one has :

$$\begin{aligned} e_i(a^\alpha b^\beta) &= a^{\alpha+1} b^{\beta-1} \quad \& \quad e_i(a^\alpha) = 0 \\ f_i(a^\alpha b^\beta) &= a^{\alpha-1} b^{\beta+1} \quad \& \quad f_i(b^\beta) = 0 \end{aligned}$$

6.3 Free key polynomials

Beware that the lifted s_i's satisfy the braid relations according to [118], but not do the π_i's. It is proved[2] in [126] that one can however lift the key polynomials to the free algebra, so that the following definition is consistent (we keep the word "polynomial" for these elements of the free algebra, but distinguish them by the notations $K_v^{\mathcal{F}}, \widehat{K}_v^{\mathcal{F}}$).

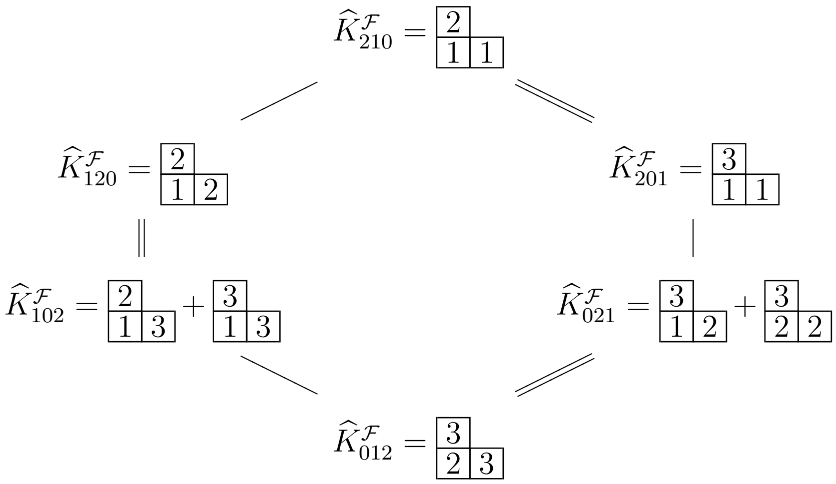
Definition 6.3.1. *The* free key polynomials $K_v^{\mathcal{F}}$, $\widehat{K}_v^{\mathcal{F}}$, *indexed by* $v \in \mathbb{N}^\infty$, *are defined recursively as follows. If* v *is dominant, then*

$$K_v^{\mathcal{F}} = \widehat{K}_v^{\mathcal{F}} = \cdots 3^{v_3} 2^{v_2} 1^{v_1} \ .$$

Otherwise, if v *and* i *are such that* $v_i > v_{i+1}$, *then*

$$K_{vs_i}^{\mathcal{F}} = K_v^{\mathcal{F}} \, \pi_i \quad \& \quad \widehat{K}_{vs_i}^{\mathcal{F}} = \widehat{K}_v^{\mathcal{F}} \, \widehat{\pi}_i \ .$$

We display below the decomposition, in the basis $\widehat{K}_u^{\mathcal{F}}$, of the sum of all tableaux of shape $[2,1]$ on three letters, which is equal to $s_{21}^{\mathcal{F}}(\mathbf{3}) = K_{012}^{\mathcal{F}}$ [3].



Since $\{K_v : v \in \mathbb{N}^n\}$ is a linear basis of $\mathfrak{Pol}(n)$, we have therefore lifted the ring of polynomials into the free algebra, as a free $\mathbb{Z}$-module. Rather than using the free algebra, we will mostly use a quotient algebra, the *plactic algebra*, which is defined in the next section.

Column tableaux belong to our family. Indeed, let $v \in \{0,1\}^n$. Suppose known that $\widehat{K}_v^{\mathcal{F}} = n^{v_n} \cdots 1^{v_1}$, and let i be such that $v_i = 1, v_{i+1} = 0$ (concanating 0 to the right of v if needed). Then the image of $\widehat{K}_v^{\mathcal{F}}$ under $\widehat{\pi}_i$, which by definition is

[2] One directly shows that either Proposition 6.6.1 or Proposition 6.6.2 is compatible with the action of $\widehat{\pi}_i$ or π_i.

[3] The corresponding functions in $\mathfrak{Pol}$ are $Y_{012} = K_{012} = \widehat{K}_{012} + \widehat{K}_{021} + \widehat{K}_{102} + \widehat{K}_{201} + \widehat{K}_{120} + \widehat{K}_{210}$.

$\widehat{K}^{\mathcal{F}}_{vs_i}$, consists of a single column-tableau obtained from $\widehat{K}^{\mathcal{F}}_{v}$ by changing the letter i into $(i{+}1)$. By induction, this proves that any $\widehat{K}^{\mathcal{F}}_{v}\ :\ v\in\{0,1\}^n$ consists of a single column-tableau.

For example,

$$\widehat{K}^{\mathcal{F}}_{111} = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{\widehat{\pi}_3} \widehat{K}^{\mathcal{F}}_{1101} = \begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array} \xrightarrow{\widehat{\pi}_2} \widehat{K}^{\mathcal{F}}_{1011} = \begin{array}{|c|}\hline 4\\\hline 3\\\hline 1\\\hline\end{array} \xrightarrow{\widehat{\pi}_4} \widehat{K}^{\mathcal{F}}_{10101} = \begin{array}{|c|}\hline 5\\\hline 3\\\hline 1\\\hline\end{array} \ \ldots$$

6.4 Embedding of $\mathfrak{Sym}$ into the plactic algebra

By going to the free algebra, we have lost multiplication. In this section, we introduce partial commutations to recover some products.

The combinatorics of the symmetric group makes an extensive use of tableaux, which are an appropriate tool to extend to the non-commutative setting the different bases that we have considered.

Schensted described an algorithm to associate to any word w a tableau $P(w)$. This algorithm, in fact, may be traced back to Robinson. One gives an algebraic formulation of this algorithm by defining, after Knuth [79], the *plactic*[4] *relations*:

$$cab \equiv acb \qquad (a \le b < c)\,, \tag{6.4.1}$$
$$bac \equiv bca \qquad (a < b \le c)\,, \tag{6.4.2}$$

that one can write planarly :

$$\begin{array}{|c|c}\cline{1-1} c & \\\hline a & \multicolumn{1}{c|}{b}\\\hline\end{array} \quad\equiv\quad \begin{array}{|c|c|}\hline a & c\\\hline \multicolumn{1}{c|}{} & b\\\cline{2-2}\end{array} \qquad\qquad (a \le b < c)$$

$$\begin{array}{|c|c}\cline{1-1} b & \\\hline a & \multicolumn{1}{c|}{c}\\\hline\end{array} \quad\equiv\quad \begin{array}{|c|c|}\hline b & c\\\hline \multicolumn{1}{c|}{} & a\\\cline{2-2}\end{array} \qquad\qquad (a < b \le c)$$

Two words are *congruent* iff they differ by a sequence of plactic relations. The *plactic algebra* $\mathfrak{Plac}$ is the quotient of the free algebra under the plactic relations. The plactic algebra is an intermediate quotient between the free algebra $\mathfrak{Free} = \mathbb{Z}[\mathbb{A}^*]$ and the algebra of polynomial :

$$\mathfrak{Free} = \mathbb{Z}[\mathbb{A}^*] \ \twoheadrightarrow\ \mathfrak{Plac} = \mathbb{Z}[\mathbb{A}^*/\equiv] \ \twoheadrightarrow\ \mathfrak{Pol}(\mathbf{x}) = \mathbb{Z}[\mathbf{x}]\ .$$

Let us note *ev* the projection map (called *evaluation*) onto $\mathfrak{Pol}(\mathbf{x})$ (in the free algebra, we shall use the alphabet $\mathbb{A} = \{1,2,\ldots\}$ or $\mathbb{A} = \{a_1,a_2,\ldots\}$; in the commutative case, we stick with $\mathbf{x}$).

We recover now symmetric functions, because of the following property [118]

[4] Terminology chosen by M.P. Schützenberger, as a tribute to *Plate tectonics*. There are indeed ”plates” inside plactic classes, in relation with Kazhdan-Lusztig theory, but the combinatorics of Kazhdan-Lusztig cells is far from being fully understood.

Proposition 6.4.1. *The ring $\mathfrak{Sym}(\mathbf{x})$ is canonically embedded into the plactic algebra, by sending a Schur function $s_\lambda(\mathbf{x})$ to the sum $s_\lambda^{\mathcal{F}}$ of all tableaux of shape λ.*

As a consequence, any algebraic identity in $\mathfrak{Sym}$ has a non-commutative interpretation in the plactic algebra.

For example, the product of two Schur functions

$$s_\lambda(\mathbf{x})s_\mu(\mathbf{x}) = \sum_\nu c_{\lambda,\mu}^\nu \, s_\nu(\mathbf{x})$$

is described by a certain rule due to Littlewood and Richardson. However, the equation

$$s_\lambda^{\mathcal{F}}(\mathbb{A})s_\mu^{\mathcal{F}}(\mathbb{A}) = \sum_\nu c_{\lambda,\mu}^\nu \, s_\nu^{\mathcal{F}}(\mathbb{A})$$

gives more information : the coefficient $c_{\lambda,\mu}^\nu$ is the number of factorizations (in the plactic algebra) of any given tableau of t shape ν into a product of two tableaux of respective shapes λ and μ:

$$t \text{ being of shape } \nu, \text{ one has } \; c_{\lambda,\mu}^\nu = \#\Big((t_1,t_2) : \, t_1 t_2 \equiv t, \mathfrak{sh}(t_1) = \lambda, \, \mathfrak{sh}(t_2) = \mu\Big) \, .$$

The original rule is the case where one takes t a *Yamanouchi tableau*[5], i.e. $t = \cdots 3^{\nu_3}\, 2^{\nu_2}\, 1^{\nu_1} = K_\nu^{\mathcal{F}}$.

For example , to find the multiplicity of s_{6531} in the product $s_{421}s_{422}$, one can start with the Yamanouchi tableau $4^1 3^3 2^5 1^6$, which planarly displays as $\begin{array}{llllll} 4 \\ 3&3&3 \\ 2&2&2&2&2 \\ 1&1&1&1&1&1 \end{array}$,
into a product of two tableaux of respective shapes $[4,2,1]$ and $[4,2,2]$. There are three such products

$$\begin{array}{llll} 3 \\ 2&2 \\ 1&1&2&4 \end{array}\begin{array}{llll} 3&3 \\ 2&2 \\ 1&1&1&1 \end{array} , \quad \begin{array}{llll} 3 \\ 2&4 \\ 1&1&2&2 \end{array}\begin{array}{llll} 3&3 \\ 2&2 \\ 1&1&1&1 \end{array} , \quad \begin{array}{llll} 4 \\ 2&3 \\ 1&1&2&2 \end{array}\begin{array}{llll} 3&3 \\ 2&2 \\ 1&1&1&1 \end{array} ,$$

and this is one way of finding, according to the original rule of Littlewood & Richardson, that the multiplicity $c_{421,422}^{6531}$ is equal to 3. But one could as well have factorized $\begin{array}{llllll} 4 \\ 3&4&4 \\ 2&3&3&4&4 \\ 1&2&2&3&3&4 \end{array}$ into the products

$$\begin{array}{llll} 4 \\ 3&4 \\ 2&3&4&4 \end{array}\begin{array}{llll} 3&4 \\ 2&3 \\ 1&2&3&4 \end{array} , \quad \begin{array}{llll} 4 \\ 3&4 \\ 2&3&4&4 \end{array}\begin{array}{llll} 4&4 \\ 2&3 \\ 1&2&3&3 \end{array} , \quad \begin{array}{llll} 4 \\ 3&4 \\ 2&3&4&4 \end{array}\begin{array}{llll} 4&4 \\ 3&3 \\ 1&2&2&3 \end{array} .$$

[5] A *Yamanouchi word* is a word w such that for every factorization $w = w'w''$, then the right factor is such that $|w''|_1 \geq |w''|_2 \geq |w''|_3 \geq 0$. In particular, for any partition λ, there is only one Yamanouchi word which is a tableau of shape λ, and it is equal to $\ldots 2^{\lambda_2} 1^{\lambda_1}$.

Many determinants occur in the theory of symmetric functions. They give elements of $\mathfrak{Plac}$, once one decides to expand them in some arbitrary order (thanks to the plactic relations, the elements do not depend of the choice of the order).

Take for example the Jacodi-Trudi determinant expressing a Schur function in terms of complete functions. Replace the sum of (unordered) terms $s_\lambda = \sum \pm h_i h_j \cdots h_k$ by the sum $\sum \pm s_i^{\mathcal{F}} s_j^{\mathcal{F}} \cdots s_k^{\mathcal{F}}$. Then, in $\mathfrak{Plac}$, one has

$$s_\lambda^{\mathcal{F}} = \sum \pm s_i^{\mathcal{F}} s_j^{\mathcal{F}} \cdots s_k^{\mathcal{F}} .$$

For example, the determinant

$$s_{2,2,1} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$$

can be expanded as $h_2h_2h_1 - h_2h_3 - h_1h_3h_1 + h_1h_4$ and the element

$$s_2^{\mathcal{F}} s_1^{\mathcal{F}} s_2^{\mathcal{F}} - s_2^{\mathcal{F}} s_3^{\mathcal{F}} - s_1^{\mathcal{F}} s_3^{\mathcal{F}} s_1^{\mathcal{F}} + s_1^{\mathcal{F}} s_4^{\mathcal{F}}$$

is equal to $s_{2,2,1}^{\mathcal{F}}$. All words which are not congruent to a tableau of shape $[2,2,1]$ cancel out.

Tableaux can be interpreted as non-intersecting paths. In that set-up, one recovers properties of (binomial) determinants that Gessel and Viennot[54] obtain by producing an involution which eliminates intersecting paths, instead of having recourse to the plactic relations.

The same function as above is also a determinant of hooks :

$$s_{2,2,1} = \begin{vmatrix} s_{2,1,1} & s_2 \\ s_{1,1,1} & s_1 \end{vmatrix}$$

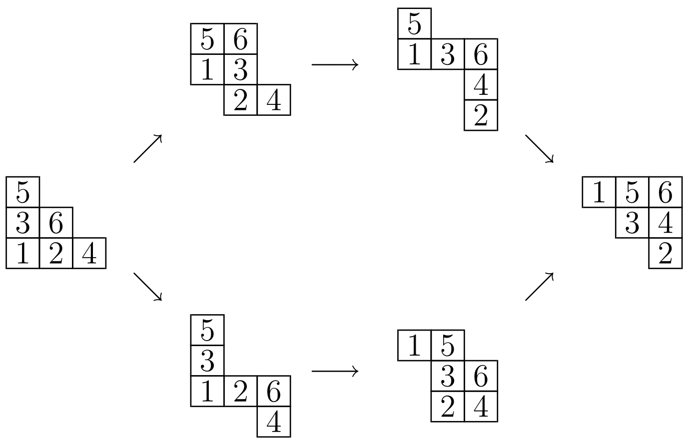
which can be expanded as $s_{2,1,1}s_1 - s_{1,1,1}s_2$. The reader can check that, in four variables,

$$\begin{aligned}
s_{2,1,1}^{\mathcal{F}} s_1^{\mathcal{F}} - s_{1,1,1}^{\mathcal{F}} s_2^{\mathcal{F}} = \Bigg(& \begin{ytableau} 3 \\ 2 \\ 1 & 1 \end{ytableau} + \begin{ytableau} 3 \\ 2 \\ 1 & 2 \end{ytableau} + \begin{ytableau} 3 \\ 2 \\ 1 & 3 \end{ytableau} + \begin{ytableau} 4 \\ 2 \\ 1 & 4 \end{ytableau} + \begin{ytableau} 3 \\ 2 \\ 1 & 4 \end{ytableau} + \begin{ytableau} 4 \\ 2 \\ 1 & 1 \end{ytableau} + \begin{ytableau} 4 \\ 2 \\ 1 & 2 \end{ytableau} + \\
& \begin{ytableau} 4 \\ 3 \\ 1 & 3 \end{ytableau} + \begin{ytableau} 4 \\ 2 \\ 1 & 3 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 1 & 4 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 1 & 1 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 2 & 4 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 2 & 3 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 2 & 2 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau} \Bigg) \left(\boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} \right) \\
& - \left(\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} + \begin{ytableau} 4 \\ 2 \\ 1 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 1 \end{ytableau} + \begin{ytableau} 4 \\ 3 \\ 2 \end{ytableau} \right) \Big(\begin{ytableau} 1 & 1 \end{ytableau} + \begin{ytableau} 2 & 2 \end{ytableau} + \begin{ytableau} 1 & 2 \end{ytableau} + \begin{ytableau} 1 & 3 \end{ytableau} + \begin{ytableau} 3 & 3 \end{ytableau} + \begin{ytableau} 2 & 3 \end{ytableau} \\
& \qquad + \begin{ytableau} 4 & 4 \end{ytableau} + \begin{ytableau} 3 & 4 \end{ytableau} + \begin{ytableau} 2 & 4 \end{ytableau} + \begin{ytableau} 1 & 4 \end{ytableau} \Big)
\end{aligned}$$

is still equal to $s_{2,2,1}^{\mathcal{F}}$, the words which are not congruent to a tableau of shape $[2,2,1]$ eliminating two by two.

6.5 Keys and jeu de taquin on columns

The jeu de taquin on two-columns tableaux produces a *contretableau*[6] with two columns, or conversely. Starting from a tableau with r columns, repeating the jeu de taquin on pairs of adjacent columns, one produces $r!$ objects[7]. Here follows an example of the plactic class of a tableau with three columns of different lengths :



Given a tableau t with r columns, and the $r!$ words $w = w_1 \cdots w_r$ (factorized as a product of columns) obtained by the jeu de taquin, the set of right columns $\mathcal{C}(t) = \{w_r\}$ is totally ordered by inclusion. The *(right) key* of t is defined to be the tableau of the same shape as t with columns in $\mathcal{C}(t)$, or, equivalently, and this is what we shall keep in most cases, to be the exponent of the evaluation of this key-tableau.

$$\begin{gathered} \textit{key} = \textit{set of columns embedded into each other} \\ \Leftrightarrow \textit{tableau congruent to some word of type} \ldots 3^{v_3} 2^{v_2} 1^{v_1} \\ \Leftrightarrow \textit{monomial } x^v = x_1^{v_1} x_2^{v_2} \ldots \\ \Leftrightarrow \textit{weight } v = [v_1, v_2, \ldots]. \end{gathered}$$

For example, for $t = \begin{array}{|c|c|c|} \hline 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array}$, one reads from the preceding hexagon

$$\mathcal{C}(t) = \left\{ \boxed{4}, \begin{array}{|c|} \hline 6 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|} \hline 6 \\ \hline 4 \\ \hline 2 \\ \hline \end{array} \right\} \Leftrightarrow \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 6 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & 4 \\ \hline \end{array} \Leftrightarrow x_2 x_4^3 x_6^2 \Leftrightarrow [0, 1, 0, 3, 0, 2]$$

The fact that one has an action of the symmetric group on the columns of a tableau has for consequence that one can compute the key in several steps, replacing an arbitrary block of left columns by their key.

[6] skew tableau with outer shape a rectangle.

[7] Reading them by columns, they are the only words in the plactic class of the tableau which are products of columns of lengths a permutation of the lengths of the columns of the original tableau.

By taking the set of left columns in the above jeu de taquin, one defines similarly the *left key* of a tableau.

For the preceding tableau $t = 5\ 36\ 124$, the left key is any of the following objects

$$\left\{ \begin{ytableau} 1 \end{ytableau}\,,\ \begin{ytableau} 5 \\ 1 \end{ytableau}\,,\ \begin{ytableau} 5 \\ 3 \\ 1 \end{ytableau} \right\} \Leftrightarrow \begin{ytableau} 5 \\ 3 & 5 \\ 1 & 1 & 1 \end{ytableau} \Leftrightarrow x_1^3 x_3 x_5^2 \Leftrightarrow [3,0,1,0,2]\,.$$

6.6 Keys and keys

According to [126], keys give the following characterization of the elements $\widehat{K}_v^{\mathcal{F}}$.

Proposition 6.6.1. *Let $v \in \mathbb{N}^n$, λ be the reordering of v. Then $\widehat{K}_v^{\mathcal{F}}$ is the sum of all tableaux on the alphabet $\{1,\ldots,n\}$ of shape λ, whose key is equal to v.*

Thus the set of tableaux of shape λ is decomposed into a disjoint union of subsets $\widehat{K}_v^{\mathcal{F}}$, $v\downarrow = \lambda$. Each subset contains a distinguished element t_v, which is the only tableau congruent to a permutation of a Yamanouchi word. One has $t_v \equiv \ldots 3^{v_3} 2^{v_2} 1^{v_1}$, so that $ev(t_v) = x^v$.

The jeu de taquin gives a second action on tableaux, this time by permuting rows. Indeed, in the class of a tableau with two rows, of lengths α, β, there exists a single word which is the product of two rows of lengths β, α, and this element (which is a contretableau) is given by the jeu de taquin :

$$\underset{\textit{tableau}}{\begin{ytableau} 3 & 6 \\ 1 & 2 & 4 & 5 & 7 \end{ytableau}} \longrightarrow \begin{ytableau} 1 & 3 & 6 \\ \none & 2 & 4 & 5 & 7 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 3 & 4 & 6 \\ \none & \none & 2 & 5 & 7 \end{ytableau} \longrightarrow \underset{\textit{contretableau}}{\begin{ytableau} 1 & 3 & 4 & 6 & 7 \\ \none & \none & \none & 2 & 5 \end{ytableau}}$$

Repeating this operation on a tableau t of shape λ,with r rows, one generates $r!$ elements[8]. Given any permutation v of λ, there exists one and only one word in the class of t which is a product $u_1 \cdots u_r$ of rows of respective lengths $v_r, \ldots, v_1$. Let us call this product the *element of shape v* in the class of t.

Here are some such elements in the class of the tableau $4\,34\,23\,112$:

$$\underset{v=[3,2,2,1]}{\begin{ytableau} 4 \\ 3 & 4 \\ 2 & 3 \\ 1 & 1 & 2 \end{ytableau}} \equiv \underset{v=[2,2,3,1]}{\begin{ytableau} 4 \\ 1 & 3 & 4 \\ \none & 2 & 3 \\ \none & 1 & 2 \end{ytableau}} \equiv \underset{v=[2,2,1,3]}{\begin{ytableau} 1 & 4 & 4 \\ \none & \none & 3 \\ \none & \none & 2 & 3 \\ \none & \none & 1 & 2 \end{ytableau}} \equiv \underset{v=[1,2,2,3]}{\begin{ytableau} 1 & 4 & 4 \\ \none & 3 & 3 \\ \none & 2 & 2 \\ \none & \none & 1 \end{ytableau}}\,.$$

The second characterization of $K_v^{\mathcal{F}}$ given in [126] is the following.

Proposition 6.6.2. *Let $\lambda \in \mathbb{N}^n$ be a partition of length r, v be a permutation of it. Let $u_1 < \cdots < u_r$ be the indices of the non-zero components of v. Then $K_v^{\mathcal{F}}$ is the sum of all tableaux of shape λ such that the words congruent to them of shape v satisfy the flag condition $u_1, \ldots, u_r$.*

[8]with repetitions when some rows have equal lengths.

For example, with $\lambda = [4,2,1,0,0]$, $v = [0,2,4,0,1]$, the flag condition is $[2,3,5]$. The tableau

$$\begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 1 & 2 & 2 \\ \hline\end{array} \equiv \begin{array}{cccc|c} 4 & & & & \\ 1 & 2 & 2 & 3 & \\ & & 1 & 2 & \end{array} \quad \begin{array}{c} \textit{flag} \\ 5 \\ 3 \\ 2 \end{array}$$

belongs to $\widehat{K}_{02401}$, but not

$$\begin{array}{|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 1 & 2 & 2 \\ \hline\end{array} \equiv \begin{array}{cccc} 4 & & & \\ 1 & 2 & 2 & \boxed{4} \\ & & 1 & 2 \end{array},$$

because there is a 4 in the middle row of the element of shape $[1,3,2]$.

6.7 vice-tableaux

There is a third characterization of key polynomials which uses another distinguished element in the plactic class of a word w, the *vice-tableau* $\mathcal{V}(w)$.

The word $\mathcal{V}(w)$ can be recursively defined as follows. It is the unique word $a_1 \cdots a_\ell$ in a plactic class $\mathfrak{Pl}(w)$ such that

- a_ℓ is the suffix (i.e. the rightmost letter) of the contretableau in $\mathfrak{Pl}(w)$
- The shape of the tableau congruent to $a_1 \cdots a_{\ell-1}$ is maximum (with respect to the natural order on partitions) among the words in $\mathfrak{Pl}(w)$ having suffix a_ℓ.
- $a_1 \cdots a_{\ell-1}$ is a vice-tableau.

Recall that the inverse operation of "inserting a letter" into a tableau t (*Inverse-Schensted algorithm*) consists in choosing a box at the periphery of t, and finding a pair (t', x) such that t' is a tableau of shape obtained by erasing this box from the shape of t', and $t'x \equiv t$. Thus, the recursive definition of a vice-tableau implies the following algorithm, consisting of a distinguished sequence of Inverse-Schensted operations : at each step, the box, denoted $\blacksquare$, which is erased is the highest which factors out on the right the suffix of the contretableau.

For example

$$\begin{array}{ccccc} 1&3&3&4&5 \\ &&1&2&4 \\ &&&1&2 \end{array} \equiv \begin{array}{ccccc} 1&3&3&4&5 \\ &&1&2&4 \\ &&&\blacksquare&1 \end{array}\ 2 \equiv \underset{\textit{choose}!}{\begin{array}{ccccc} 1&3&3&4&5 \\ &&\blacksquare&2&4 \\ &&&1&1 \end{array}}\ 2 \equiv \begin{array}{ccccc} \blacksquare&1&3&3&5 \\ &&1&2&4 \\ &&&1&2 \end{array}\ 4$$

shows that

$$\mathcal{V}(13345\ 124\ 12) = \mathcal{V}(13345\ 24\ 11) \cdot 2\,.$$

Iterating, one finds a word that one factors into its maximal rows, and that on can display as a skew tableau

$$\mathcal{V}(13345\ 124\ 12) = \begin{array}{ccccc} 3 & 5 & & & \\ 1 & 2 & 3 & 4 & 4 \\ & & 1 & 1 & 2 \end{array}\ , \text{ shape } [0,3,0,5,2]\,.$$

The *shape* of the vice-tableau is the sequence $v_1, v_2, \ldots, v_n$, where v_i is the length of the (single) row ending with a letter i. Empty rows have to be recorded !

The original unpublished characterization of key polynomials is the following.

Proposition 6.7.1. *Given $v \in \mathbb{N}^n$, then $\widehat{K}_v^{\mathcal{F}}$ is the sum, in $\mathfrak{Plac}$, of all vice-tableaux of shape v.*

For example, $\widehat{K}_{01302}^{\mathcal{F}} = \begin{array}{ccc} 5 & 5 & \\ 1 & 3 & 3 \\ & & 2 \end{array} + \begin{array}{ccc} 4 & 5 & \\ 1 & 3 & 3 \\ & & 2 \end{array} + \begin{array}{ccc} 5 & 5 & \\ 3 & 3 & 3 \\ & & 2 \end{array} + \begin{array}{ccc} 4 & 5 & \\ 3 & 3 & 3 \\ & & 2 \end{array} + \begin{array}{ccc} 5 & 5 & \\ 2 & 3 & 3 \\ & & 2 \end{array} + \begin{array}{ccc} 4 & 5 & \\ 2 & 3 & 3 \\ & & 2 \end{array}$

. This sum is congruent to the following sum of tableaux, which is more difficult to identify at first glance : $\begin{array}{ccc} 5 & & \\ 3 & 5 & \\ 1 & 2 & 3 \end{array} + \begin{array}{ccc} 4 & & \\ 3 & 5 & \\ 1 & 2 & 3 \end{array} + \begin{array}{ccc} 5 & & \\ 3 & 5 & \\ 2 & 3 & 3 \end{array} + \begin{array}{ccc} 4 & & \\ 3 & 5 & \\ 2 & 3 & 3 \end{array} + \begin{array}{ccc} 5 & & \\ 3 & 5 & \\ 2 & 2 & 3 \end{array} + \begin{array}{ccc} 4 & & \\ 3 & 5 & \\ 2 & 2 & 3 \end{array}$.

One has a similar notion of *left vice-tableau*, by iterating the operation: factor t into xt' in the plactic monoid, in such a way that x is the first letter of the tableau, and the shapes of t and t' differ by a box which is the lowest possible.

The same word as above gives

$$\begin{array}{ccccc} 3 & 5 & & & \\ 2 & 3 & 4 & & \\ 1 & 1 & 1 & 2 & 4 \end{array} \equiv 3\ \begin{array}{ccccc} 5 & \blacksquare & & & \\ 2 & 3 & 4 & & \\ 1 & 1 & 1 & 2 & 4 \end{array} \equiv \underset{\textit{choose}!}{3\ \begin{array}{ccccc} 3 & 5 & & & \\ 2 & 4 & \blacksquare & & \\ 1 & 1 & 1 & 2 & 4 \end{array}} \equiv 1\ \begin{array}{ccccc} 3 & 5 & & & \\ 2 & 3 & 4 & & \\ 1 & 1 & 2 & 4 & \blacksquare \end{array}$$

and, iterating, the left vice-tableau

$$\begin{array}{cccccc} 3 & 3 & 5 & & & \\ & 2 & 4 & & & \\ & 1 & 1 & 1 & 2 & 4 \end{array}\ .$$

6.8 Ehresmann tableaux

Given a permutation $\sigma \in \mathfrak{S}_n$, the sequence $\{\sigma_1\}, \{\sigma_1, \sigma_2\}, \ldots, \{\sigma_1, \ldots, \sigma_n\}$ is an increasing flag of subsets of $\{1, \ldots, n\}$, and may be interpreted as the sequence of columns of a contretableau, ordering each set decreasingly, or of a tableau, reading the flag from right to left.

$$\sigma = [3,5,1,4,2] \Rightarrow \begin{array}{ccccc} 3 & 5 & 5 & 5 & 5 \\ & 3 & 3 & 4 & 4 \\ & & 1 & 3 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{array} \equiv \begin{array}{ccccc} 5 & & & & \\ 4 & 5 & & & \\ 3 & 4 & 5 & & \\ 2 & 3 & 3 & 5 & \\ 1 & 1 & 1 & 3 & 3 \end{array}\ .$$

These tableaux, that we shall denote $\mathcal{E}(\sigma)$, were used by Ehresmann [35] to describe a cellular decomposition of the flag variety relative to the linear group $GL_n(\mathbb{C})$. Ehresmann described the attachment of cells by introducing an order: $\sigma \le \zeta$ if and only if $\mathcal{E}(\sigma) \le \mathcal{E}(\zeta)$ componentwise. This order on permutations is the same as the one obtained by taking subwords of reduced decompositions, and usually called the *Bruhat order*, or *strong order*[9]. A tableau t on the letters $1,\ldots,n$ of shape $[n,\ldots,1]$ has a key which is some $\mathcal{E}(\sigma)$, and we shall denote this permutation $\clubsuit(t)$. More generally, if t is of arbitrary shape, the set of columns of its key may be completed in the set of columns of an Ehresmann tableau, and we define $\clubsuit(t)$ to be the permutation σ such that $\mathcal{E}(\sigma)$ is (componentwise) minimum among those $\mathcal{E}(\zeta)$ containing the columns of the key.

For example, $\begin{array}{lll} 4 & & \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{array} \equiv \begin{array}{lll} 2 & 4 & 4 \\ 1 & 1 & 3 \\ & & 2 \end{array}$ has key $\begin{array}{lll} 4 & & \\ 3 & 4 & 4 \\ 2 & 2 & 2 \end{array}$, and

$\clubsuit\left(\begin{array}{llll} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 2 & 2 \end{array} \right) = [2,4,3,1]$ is the minimum Ehresmann tableau containing the columns $[4,2], [4,3,2]$.

Similarly, one can take the maximum Ehresmann tableau containing the columns of the left key of a tableau, and one obtains a second permutation $\clubsuit_{left}(t)$.

Given two tableaux t,u of shapes λ,μ, the product $t\,u$ is defined to be *frank* if the shape of the tableau congruent to tu is equal to the sum $\lambda+\mu$. The action of the symmetric group on the columns of a tableau seen above has exhibited frank products of columns.

In the case where t and u are single columns, tu is frank iff tu is a column-tableau (case $\ell(t) \ge \ell(u)$) or a column contretableau (case $\ell(t) \le \ell(u)$). More generally, it is shown in [126, Th.2.8] that tu is frank iff for every permutations $t \equiv t^1 \ldots t^k$, $u \equiv u^1 \ldots u^r$ of the columns of t and u respectively, then the product of two columns $t^k u^1$ is frank. Thus, the test to be frank reduces to the case of pairs of columns.

The notion of frank product is closely related to the Ehresmann-Bruhat order, as shows the following lemma given in [126, Th.2.10].

Lemma 6.8.1. *Given two tableaux t,u of respective shapes λ,μ, then the product tu is frank if and only if $\clubsuit(t) \le \clubsuit_{left}(u)$.*

The order on Ehresmann tableaux is the componentwise order. One could think of avoiding the construction of keys, and directly use the componentwise order on tableaux of a given shape. This structure on tableaux is not related to

[9] Ehresmann did not mention permutations, but was using flags of Plücker coordinates, in other words, was using Ehresmann tableaux. The terminology "Bruhat order" is due to Verma[190], because of the Bruhat decomposition $B\sigma B$, of $GL_n(\mathbb{C})$, B being the Borel subgroup of triangular matrices. I interviewed Bruhat, who, of course, did not claim any paternity about the Bruhat order.

the combinatorics of Schubert, Grothendieck and Key polynomials. It is not even appropriate to characterize the shapes of products of tableaux. For example, the product

$$\begin{array}{lll} 4 & & \\ 2 & 2 & \\ 1 & 1 & 2 \end{array} \quad \begin{array}{lll} 4 & & \\ 2 & 3 & \\ 1 & 1 & 2 \end{array} \quad \equiv \quad \begin{array}{lllll} 4 & 4 & & & \\ 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{array}$$

has shape $[5,5,2] \neq [3,2,1]+[3,2,1]$ though the first tableau (which is eqal to its right key) is componentwise smaller than the second one. In fact, the left key of the second tableau is equal to $\begin{array}{lll} 4 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array}$ and the preceding lemma forbids the shape of the product to be equal to $[6,4,2]$.

6.9 Nilplactic monoid and algebra

The plactic relations

$$ikj \equiv kij \quad \& \quad jki \equiv jik \ , \ i<j<k \, ,$$

are compatible with the braid relations $s_i s_k = s_k s_i$. However, there is a problem in the limit case

$$121 \equiv 211 \quad \& \quad 221 \equiv 212 \, ,$$

while $s_1 s_2 s_1 = s_2 s_1 s_2$. Since $s_2 s_1 s_1$ and $s_2 s_2 s_1$ are not reduced decompositions, one can decide to transform accordingly the plactic relations and define the *nilplactic relations* on reduced words[10] to be

$$ikj \cong kij \quad \& \quad jki \cong jik \quad \& \quad i(i{+}1)i \cong (i{+}1)i(i{+}1) \, , \tag{6.9.1}$$

sending to 0 all non-reduced words.

The *nilplactic monoid* is the quotient of the free monoid under the nilplactic relations, and the *nilplactic algebra* is its associated algebra. The nilplactic monoid is very similar to the plactic monoid. In particular, one has the following analog of Schensted bijection[122, 34].

Proposition 6.9.1. *In each nilplactic class of a word which is a reduced decomposition, there exists a tableau and only one tableau. The words in the class of a tableau t of shape λ are in bijection with the Q-symbols*[11] *of total shape λ.*

[10] reduced when interpreted as products of simple transpositions

[11] Given a word $w = w_1 \dots w_n$, the shapes of the successive tableaux which are congruent to the words $w_1, w_1 w_2, \dots, w_1 \dots w_n$ constitute a flag of shapes which can be encoded by a standard tableau of shape λ. One has a plactic Q-symbol and a nilplactic Q-symbol, depending on the relations that one uses to transform the left factors of a word into a tableau.

The correspondence between the plactic and nilplactic classes can be realized by describing a bijection exchanging the two congruences. I gave such a correspondence with M.P. Schützenberger under the name *plaxification*, but Reiner and Shimozono obtained a much more elegant description. The *plactification* $\mathfrak{Pl}$ is a transformation on words which is defined recursively as follows, starting with the empty word which is exchanged with itself.

$$\Big(w = iw'\ ,\ \mathfrak{Pl}(w') = \eta\Big) \quad \Rightarrow \quad \mathfrak{Pl}(w) = i\,(\eta)^{s_i}\,,$$

where s_i acts as defined in the preceding section.

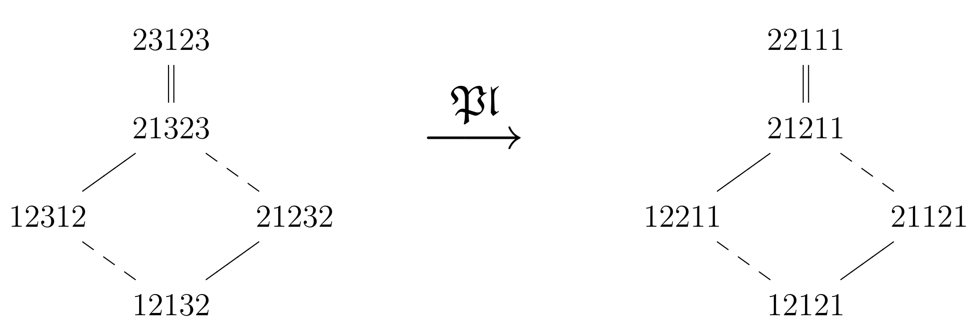
The plactification can be visualized as moving a cursor (here a box) from right to left, a pointed letter acting as a simple transposition s_i on the factor on its right. For example, $\mathfrak{Pl}([3,2,1,3,2,4,3,4]$ is the last word in the following chain

$$\begin{array}{rcl}
[3,2,1,3,2,4,3,\boxed{4}\,] & = & [3,2,1,3,2,4,3,4] \\
[3,2,1,3,2,4,\boxed{3}\,,4] & = & [3,2,1,3,2,4,3,3] \\
[3,2,1,3,2,\boxed{4}\,,3,3] & = & [3,2,1,3,2,4,3,3] \\
[3,2,1,3,\boxed{2}\,,4,3,3] & = & [3,2,1,3,2,4,2,2] \\
[3,2,1,\boxed{3}\,,2,4,2,2] & = & [3,2,1,3,2,3,2,2] \\
[3,2,\boxed{1}\,,3,2,3,2,2] & = & [3,2,1,3,1,3,1,1] \\
[3,\boxed{2}\,,1,3,1,3,1,1] & = & [3,2,1,2,1,2,1,1] \\
[\boxed{3}\,,2,1,2,1,2,1,1] & = & [3,2,1,2,1,2,1,1]
\end{array}$$

Needless to add that the morphism inverse to $\mathfrak{Pl}$ is obtained by moving a cursor from left to right, and acting accordingly on the right factor.

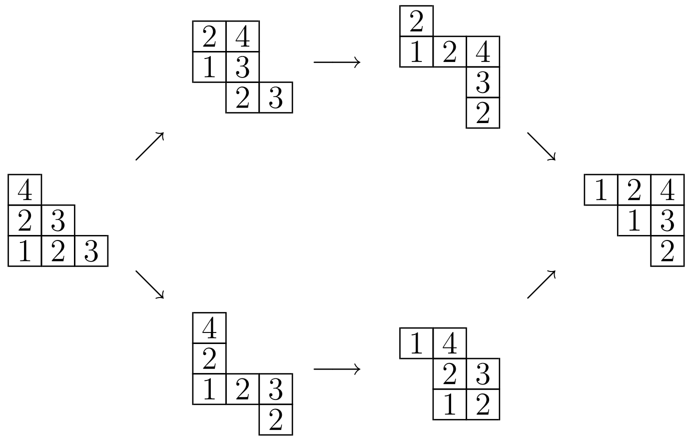
Proposition 6.9.2. *[175] The morphism $\mathfrak{Pl}()$ sends the nilplactic class of a reduced word to a plactic class, preserving the Q-symbol.*

For example, the nilplactic class of $[2,3,1,2,3]$ is sent to the plactic class of $[2,2,1,1,1]$.



One has still an action of the symmetric group on the columns of a nilplactic tableau, so that one can define a *right nilplactic key* or a *left nilplactic key*.

For example, the hexagon

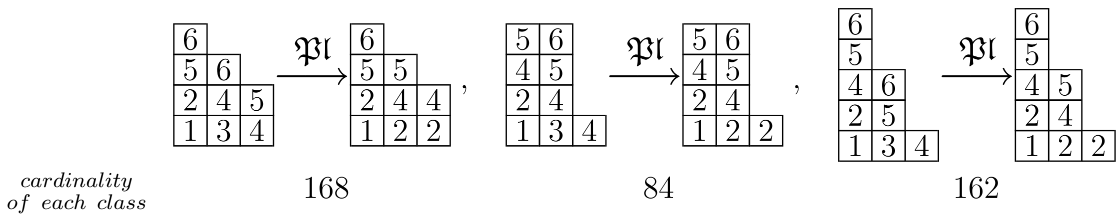


gives the left key $\begin{array}{lll} 4 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array}$ and the right key $\begin{array}{lll} 4 & & \\ 3 & 3 & \\ 2 & 2 & 3 \end{array}$.

Notice that, because the tranpose of a nilplactic tableau is still a nilplactic tableau, one has also two other keys, the bottom key, and the top key, obtained by transposing rows using the niplactic relations.

Since two reduced words which are nilplactically congruent are reduced decompositions of the same permutation, the set of reduced decompositions of a permutation decomposes into a union of nilplactic classes, that one can characterize by the tableau that each of them contain. As a consequence, the number of reduced decompositions of a given permutation is a sum of cardinals of plactic or nilplactic classes[12] The problem of determining the number of reduced decompositions of a permutation has been considered by Stanley [184] who have reformulated it in terms of certain symmetric functions which are now called *Stanley symmetric functions*, and are the stable parts of Schubert polynomials. Edelman and Greene [34] gave a more combinatorial solution in terms of *balanced tableaux*, using also the nilplactic relations. I preferred to use here the point of view of the note [122], supplemented by the plaxification.

For example, there are $414 = 168+84+162$ reduced decompositions of the permutation $[3, 1, 7, 6, 2, 4, 5]$, which regroup into three nilplactic classes. We write below these three tableaux, as well as their images under $\mathfrak{Pl}$.



[12] a plactic or nilplactic class containing a tableau of shape λ has cardinality the number of standard tableaux of shape λ. This number is also the dimension of the irreducible representation of index λ of the symmetric group.

6.10 Lifting $\mathfrak{Pol}$ to a submodule of $\mathfrak{Plac}$

Let us first consider the space of polynomials $\mathfrak{Pol}$ in $x_1, x_2, \ldots, x_n, \ldots$ (no limit on n), as an inductive limit of $\mathfrak{Pol}(\mathbf{x}_n)$, with $\mathbf{x}_n = \{x_1, \ldots, x_n\}$.

We have defined flag-elementary functions $P_v(\mathbf{x})$. There is only one natural way to lift an elementary symmetric function to the free algebra. It must be, in agreement with the definition of free key polynomials, the sum of all strictly decreasing words of a given degree in a totally ordered alphabet, that is to say the sum of all tableaux whose shape is a column of a given length :
$e_k(\mathbf{x}_r) \longleftrightarrow \Lambda^k(\mathbf{r}) =$ sum of all decreasing words of length k in $1, \ldots, r$.

By product, for any $v \in \mathbb{N}^n$, $v \leq [n\text{–}1, \ldots, 0]$, define in the free algebra

$$P_v^{\mathcal{F}} := \Lambda^{v_1}(\mathbf{n\text{–}1})\Lambda^{v_2}(\mathbf{n\text{–}2}) \cdots \Lambda^{v_{n-1}}(\mathbf{1})\Lambda^{v_n}(\mathbf{0})\,.$$

The $\mathbb{Z}$-span of all $P_v^{\mathcal{F}}$ in the quotient algebra $\mathfrak{Plac}$ is denoted $\mathfrak{Schub}$, and is therefore a lift of $\mathfrak{Pol}$. Indeed, every polynomial is lifted to $\mathfrak{Plac}$ by expanding it into the basis $\{P_v\}$, then formally changing every P_v into $P_v^{\mathcal{F}}$.

The word $21 = \Lambda^2(\mathbf{2})$ belongs to $\mathfrak{Schub}$, but not the word 12, because it has the same evaluation. Neither does the word 22 belong to $\mathfrak{Schub}$, because $x_2^2 = P_{1100} - P_{2000} - P_{200}$ lifts into

$$(1{+}2{+}3)(1{+}2) - (21{+}31{+}32) - (1{+}2)1 = 22 + 12 - 21 = \boxed{2\,|\,2} + \boxed{1\,|\,2} - \boxed{\begin{matrix}2\\1\end{matrix}}\,.$$

We have a problem now. We have already lifted $\mathfrak{Pol}$ to the free algebra by defining *free key polynomials*, we must ensure that the two elements corresponding in $\mathfrak{Plac}$ to a polynom $f = \sum c_v P_v = \sum d_u K_u$ coincide.

Let us prove that $\mathfrak{Schub}$ is the image in $\mathfrak{Plac}$ of the product ring

$$\cdots \otimes \mathfrak{Sym}(\mathbf{3}) \otimes \mathfrak{Sym}(\mathbf{2}) \otimes \mathfrak{Sym}(\mathbf{1})\,.$$

First, the lift of a Schur function $s_\lambda(\mathbf{x}_n)$, in terms of $P_v^{\mathcal{F}}$, is given by the ordered expansion of a determinant $\Lambda_u(\mathbf{m}/\mathbf{m-1}/\cdots/\mathbf{n})$, where u is the reordering of $\lambda^\sim$. Since $\Lambda^i(\mathbf{m}) = \Lambda^i(\mathbf{m\text{–}1}) + m\Lambda^{i-1}(\mathbf{m\text{–}1})$, the determinant is the sum of $\Lambda_u(\mathbf{m\text{–}1}/\mathbf{m\text{–}1}/\cdots/\mathbf{n})$ and a determinant with first two rows

$$\begin{vmatrix} \cdots & m\Lambda^i(\mathbf{m\text{–}1}) & \cdots \\ \cdots & \Lambda^i(\mathbf{m\text{–}1}) & \cdots \end{vmatrix}\,.$$

However, all the $\Lambda^i(\mathbf{m\text{–}1})$ commute between themselves in $\mathfrak{Plac}$, and therefore this second determinant is null. Repeating this transformation, one sees that the original determinant is equal to $\Lambda_u(\mathbf{n}/\cdots/\mathbf{n})$, and is therefore equal to the sum $s_\lambda^{\mathcal{F}}(\mathbf{n})$ of all tableaux of shape λ in the alphabet $\mathbf{n} = \{1, \ldots, n\}$.

The intermediate expression $\Lambda_u(\mathbf{n{+}1}/\cdots/\mathbf{n{+}1}/\mathbf{n})$ shows that $s_\lambda^{\mathcal{F}}(\mathbf{n})$ is also equal to a sum $\sum_{\nu,i} \pm s_\nu^{\mathcal{F}}(\mathbf{n{+}1})\Lambda^i(\mathbf{n})$. A product $s_\mu^{\mathcal{F}}(\mathbf{n{+}1})s_\lambda^{\mathcal{F}}(\mathbf{n})$ may therefore be expressed as a sum of products $s_\mu^{\mathcal{F}}(\mathbf{n{+}1})s_\nu^{\mathcal{F}}(\mathbf{n{+}1})\Lambda^i(\mathbf{n})$, then of products $s_\eta^{\mathcal{F}}(\mathbf{n{+}1})\Lambda^i(\mathbf{n})$.

Finally, by right to left transformation, any product $\cdots s_\eta^{\mathcal{F}}(\mathbf{n{+}1})s_\lambda^{\mathcal{F}}(\mathbf{n})s_\nu^{\mathcal{F}}(\mathbf{n{-}1})\cdots$ may be expressed as a linear combination of $P_v^{\mathcal{F}}$.

As a consequence, the elements

$$H_v^{\mathcal{F}} = \cdots s_{v_4}^{\mathcal{F}}(\mathbf{4})s_{v_3}^{\mathcal{F}}(\mathbf{3})s_{v_2}^{\mathcal{F}}(\mathbf{2})s_{v_1}^{\mathcal{F}}(\mathbf{1})$$

constitute a linear basis of $\mathfrak{Schub}$ because they belong to $\mathfrak{Schub}$ and their commutative images are a basis of $\mathfrak{Pol}$.

Let us check that the operators $\pi_i, \widehat{\pi}_i$ preserve $\mathfrak{Schub}$. For any i, any product $f = \cdots s_\lambda^{\mathcal{F}}(\mathbf{i{+}1})s_\mu^{\mathcal{F}}(\mathbf{i})s_\nu^{\mathcal{F}}(\mathbf{i{-}1})\cdots$ is a sum of products Uw, where U is a i-string, and w is a word in the alphabet $1,\ldots,i$. However, for any k, the image of Ui^k under π_i is equal to $Us_k^{\mathcal{F}}(i, i{+}1)$. Therefore the image of Uw, and of f, under π_i belongs to $\mathfrak{Schub}$[13].

Since the dominant free key polynomials $K_\lambda^{\mathcal{F}}$ belong to $\mathfrak{Schub}$, all the $K_v^{\mathcal{F}}$ also belong to $\mathfrak{Schub}$.

In all we have at our disposal four bases of $\mathfrak{Schub}$ *:* $\{P_v^{\mathcal{F}}\}$, $\{H_v^{\mathcal{F}}\}$ *and* $\{K_v^{\mathcal{F}}\}$, $\{\widehat{K}_v^{\mathcal{F}}\}$, *and we can use the operators* π_i, $\widehat{\pi}_i$[14].

The relations between these different bases are exactly the same as at the commutative level. For example, writing the functions together with their expression in terms of tableaux, one has

$$\begin{aligned}
\widehat{K}_{021}^{\mathcal{F}} &= \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \\
&= \left(P_{2100}^{\mathcal{F}} = \left(\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)\left(\boxed{1}+\boxed{2}\right)\right) + \left(P_{3000}^{\mathcal{F}} = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right) \\
&\quad - \left(P_{2010}^{\mathcal{F}} = \left(\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)\boxed{1}\right) + \left(P_{0210}^{\mathcal{F}} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\,\boxed{1}\right) \\
&\quad - \left(P_{1200}^{\mathcal{F}} = \left(\boxed{1}+\boxed{2}+\boxed{3}\right)\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right) \\
&= \left(H_{021}^{\mathcal{F}} = (1+2+3)(11+12+22)\right) - \left(H_{201}^{\mathcal{F}} = (1+2+3)11\right) \\
&\quad - \left(H_{03}^{\mathcal{F}} = 111+112+122+222\right) - \left(H_{12}^{\mathcal{F}} = (11+12+22)\ 1\right) \\
&\quad + \left(H_{21}^{\mathcal{F}} = (1+2)\ 11\right) + \left(H_{3}^{\mathcal{F}} = 111\right).
\end{aligned}$$

[13]As usual, the case ot understand is the case of cardinality 2. The image of $(12)(11)$ under π_1, which is $1211 + 1212 + 2122$ is not equal to $(12)(11\pi_1) = (12)(11 + 12 + 22)$ in $\mathfrak{Plac}$, though 12 is invariant under s_1. The word 12 does not belong to $\mathfrak{Schub}$. On the other hand, $11+12+22$ is a 1-string, and the image of $(11 + 12 + 22)(11)$ under π_1, which is $(111 + 112 + 122 + 222) + (121 + 221) + (0)$, is congruent to $(11 + 12 + 22)(1 + 2) = K_{03}^{\mathcal{F}} + K_{12}^{\mathcal{F}}$ and do belong to $\mathfrak{Schub}$.

[14]But the simple transpositions s_i do not preserve $\mathfrak{Schub}$. The image under s_1 of 11 is 22, which do not belong to $\mathfrak{Schub}$ because the elements of degree 2 in $1, 2$ of $\mathfrak{Schub}$ are linear combinations of $\widehat{K}_2^{\mathcal{F}} = 11, \widehat{K}_{11}^{\mathcal{F}} = 21, \widehat{K}_{02}^{\mathcal{F}} = 12 + 22$.

Let us denote $\mathfrak{Schub}_n$ the lift of $\mathfrak{Pol}(n)$. It has bases $\{\widehat{K}_v^{\mathcal{F}}, v \in \mathbb{N}^n\}$, $\{K_v^{\mathcal{F}}, v \in \mathbb{N}^n\}$ and $\{H_v^{\mathcal{F}}, v \in \mathbb{N}^n\}$, but the finite set $\{P_v^{\mathcal{F}}, v \leq [n-1, \ldots, 0]\}$ generates only a subspace.

Any element f of $\mathfrak{Schub}_n$ is written uniquely as a sum $f = \sum c_v \widehat{K}_v^{\mathcal{F}}$. Instead of characterizing f by its commutative evaluation, one can simply restrict it to the set of tableaux $\{t_v : v \in \mathbb{N}^n\}$. Indeed, c_v is the coefficient of t_v in f. For example, in the above expression, one points out $\begin{ytableau} 3 \\ 2 & 2 \end{ytableau}$, $\begin{ytableau} 3 \\ 2 \end{ytableau}\ \begin{ytableau} 2 \end{ytableau}$ and $\begin{ytableau} 3 \end{ytableau}\ \begin{ytableau} 2 & 2 \end{ytableau}$, to check the multiplicity in $\widehat{K}_{021}^{\mathcal{F}}$, the second tableau $\begin{ytableau} 3 \\ 2 & 2 \end{ytableau}$ is not necessary.

6.11 Allowable products in $\mathfrak{Schub}$

We have just stated that any product of the type $\cdots s^{\mathcal{F}}_{\lambda}(\mathbf{k{+}1}) s^{\mathcal{F}}_{\mu}(\mathbf{k}) s^{\mathcal{F}}_{\nu}(\mathbf{k{-}1}) \cdots$ belongs to $\mathfrak{Schub}$. The weakly decreasing condition on alphabets is necessary. For example, the product

$$s^{\mathcal{F}}_{1}(\mathbf{3}) s^{\mathcal{F}}_{1}(\mathbf{2}) = K^{\mathcal{F}}_{02} + K^{\mathcal{F}}_{011}$$

belongs to $\mathfrak{Schub}$, but

$$s^{\mathcal{F}}_{1}(\mathbf{2}) s^{\mathcal{F}}_{1}(\mathbf{3}) = K^{\mathcal{F}}_{02} + K^{\mathcal{F}}_{11} + \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}$$

does not belong to it.

Let us show that right multliplication by a column $k \dots 1$ is permitted.

Given k, and any $f = \cdots s^{\mathcal{F}}_{\lambda}(\mathbf{k{+}1}) s^{\mathcal{F}}_{\mu}(\mathbf{k}) s^{\mathcal{F}}_{\nu}(\mathbf{k{-}1}) \cdots s^{\mathcal{F}}_{\eta}(\mathbf{1})$, its product on the right by $k \cdots 1$ is equal to

$$\cdots s^{\mathcal{F}}_{\lambda}(\mathbf{k{+}1}) s^{\mathcal{F}}_{\mu}(\mathbf{k})\, (k \cdots 1)\, s^{\mathcal{F}}_{\nu}(\mathbf{k{-}1}) \cdots s^{\mathcal{F}}_{\eta}(\mathbf{1})\,,$$

because all letters not greater than k commute with the column $k \cdots 1$.

By iteration this proves that the product on the right by any $K^{\mathcal{F}}_{\lambda}$, λ partition, preserves $\mathfrak{Schub}$. Thus, the product in $\mathfrak{Pol}$ by a dominant monomial x^{λ}, that we have used many times, lifts to the right multliplication by $K^{\mathcal{F}}_{\lambda}$.

We have seen that column-tableaux belong to $\mathfrak{Schub}$, being equal to some $\widehat{K}^{\mathcal{F}}_{v}$ with $v \in \{1,0\}^*$. Given n, r, the column $(n{+}r) \cdots (n{+}1)$ may be expressed[15] as a linear combination of $P^{\mathcal{F}}_{u,0^n}$. Consequently, for any $v \in \mathbb{N}^n$, the product

$$(n{+}r) \cdots (n{+}1)\, \widehat{K}^{\mathcal{F}}_{v}$$

belongs to $\mathfrak{Schub}$.

For example

$$\begin{aligned} \begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array}\, \widehat{K}^{\mathcal{F}}_{02} &= \big(\Lambda^2(\mathbf{4}) - \Lambda^2(\mathbf{3}) - \Lambda^1(\mathbf{4})\Lambda^1(\mathbf{2}) + \Lambda^1(\mathbf{3})\Lambda^1(\mathbf{2})\big)\big(s^{\mathcal{F}}_{2}(\mathbf{2}) - s^{\mathcal{F}}_{2}(\mathbf{1})\big) \\ &= \begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array} \Big(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \Big) = \begin{array}{|c|c}\hline 4 & \\\cline{1-1} 3 & \\\hline 1 & \multicolumn{1}{c|}{2}\\\hline\end{array} + \begin{array}{|c|c}\hline 4 & \\\cline{1-1} 3 & \\\hline 2 & \multicolumn{1}{c|}{2}\\\hline\end{array} \end{aligned}$$

belongs to $\mathfrak{Schub}$ (it is in fact equal to $\widehat{K}^{\mathcal{F}}_{0211}$).

One can use the fact that products $s^{\mathcal{F}}_{\lambda}(\mathbf{n})\widehat{K}^{\mathcal{F}}_{v}$, $v \in \mathbb{N}^n$, belong to $\mathfrak{Schub}_n$, to generalize the Littlewood-Richardson rule.

---

[15] Explicitely,

$$(n{+}r) \cdots (n{+}1) = \sum_{u \le [r,1\dots r-1]} (-1)^{r-1+\ell(u)} \big(P^{\mathcal{F}}_{u_1,0,u_2-1,\dots,u_r-r+1,0^n} - P^{\mathcal{F}}_{u_1,u_2-1,\dots,u_r-r+1,0^n}\big)$$

sum over all permutations u below (for the Ehresmann-Bruhat order) $[r, 1, \dots, r{-}1]$.

In the case where $v = \mu$ is dominant, one has

$$s_\lambda^{\mathcal{F}}(\mathbf{n})\, t_\mu = \sum\nolimits_t \widehat{K}^{\mathcal{F}}_{\mathfrak{cl}(t\, t_\mu)} ,$$

sum over all tableaux t of shape λ such that $t\, t_\mu$ is congruent to some tableau t_u. Using the operators $\widehat{\pi}_\sigma$, one obtains the product of a Schur function by a key polynomial in terms of tableaux which are in the orbit of Yamanouchi tableaux under the symmetric group.

Lemma 6.11.1. *Let λ be a partition in $\mathbb{N}^n$, $v \in \mathbb{N}^n$, then one has*

$$s_\lambda^{\mathcal{F}}(\mathbf{n}) \widehat{K}^{\mathcal{F}}_v = \sum\nolimits_t \widehat{K}^{\mathcal{F}}_{\mathfrak{cl}(t\, t')} , \tag{6.11.1}$$

sum over all tableaux t of shape λ, $t' \in \widehat{K}^{\mathcal{F}}_v$, such that $t\, t'$ is congruent to some tableau t_u.

For example, restricting the products $K^{\mathcal{F}}_{012}\widehat{K}_v$, with $v\!\downarrow = [2,1,0]$, to the terms which give a key which is a permutation of $[1,2,3]$, one has

$$K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{210} \to \left(\begin{array}{cccc} \boxed{3} & & & \\ \boxed{1} & \boxed{2} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{1} \end{array} \in \widehat{K}^{\mathcal{F}}_{321} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{2} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{1} \end{array} \in \widehat{K}^{\mathcal{F}}_{231} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{1} & \boxed{3} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{1} \end{array} \in \widehat{K}^{\mathcal{F}}_{312} \right) ,$$

$$K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{201} \to \left(\begin{array}{cccc} \boxed{3} & & & \\ \boxed{1} & \boxed{2} & & \\ & & \boxed{3} & \\ & & \boxed{1} & \boxed{1} \end{array} \in \widehat{K}^{\mathcal{F}}_{312} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{3} & & \\ & & \boxed{3} & \\ & & \boxed{1} & \boxed{1} \end{array} \in \widehat{K}^{\mathcal{F}}_{213} \right) ,$$

$$K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{120} \to \left(\begin{array}{cccc} \boxed{2} & & & \\ \boxed{1} & \boxed{3} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{2} \end{array} \in \widehat{K}^{\mathcal{F}}_{231} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{3} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{2} \end{array} \in \widehat{K}^{\mathcal{F}}_{132} \right) ,$$

$$K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{021} \to \left(\begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{3} & & \\ & & \boxed{3} & \\ & & \boxed{1} & \boxed{2} \end{array} \in \widehat{K}^{\mathcal{F}}_{132} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{3} & & \\ & & \boxed{3} & \\ & & \boxed{1} & \boxed{2} \end{array} \in \widehat{K}^{\mathcal{F}}_{123} \right) ,$$

$$K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{102} \to \left(\begin{array}{cccc} \boxed{3} & & & \\ \boxed{2} & \boxed{3} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{3} \end{array} \in \widehat{K}^{\mathcal{F}}_{123} , \begin{array}{cccc} \boxed{3} & & & \\ \boxed{1} & \boxed{3} & & \\ & & \boxed{2} & \\ & & \boxed{1} & \boxed{3} \end{array} \in \widehat{K}^{\mathcal{F}}_{213} \right) ,$$

while $K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{012} = \widehat{K}^{\mathcal{F}}_{024} \to 0$. The sum of all these terms is

$$K^{\mathcal{F}}_{012}K^{\mathcal{F}}_{012} \to \sum\nolimits_{u:\, u\uparrow = 123} 2\widehat{K}^{\mathcal{F}}_u = 2K^{\mathcal{F}}_{123}$$

in accordance with $s_{21}^2 = 2s_{321} + \dots$. The multiplicity 2 can already be read from the first product $K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{210}$, because $K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{210}\pi_{321} = K^{\mathcal{F}}_{012}K^{\mathcal{F}}_{012}$. It is therefore the number of tableaux t of shape $[2,1]$ such that $t\,t_{21}$ be congruent to t_{321}. We are back to the Littlewood-Richardson rule ! Notice that in the product $K^{\mathcal{F}}_{012}\widehat{K}^{\mathcal{F}}_{210}$ one has now ignored the contributions 322 211 and 313 211 because $\widehat{K}^{\mathcal{F}}_{231}\pi_{321} = 0 = \widehat{K}^{\mathcal{F}}_{312}\pi_{321}$.

Linear relations in $\mathfrak{Schub}$ can be checked by examining their commutative image in $\mathfrak{Pol}$, and conversely, any identity in $\mathfrak{Pol}$ has a non-commutative counterpart in $\mathfrak{Schub}$. We already used this property for symmetric polynomials.

For example, we have seen that K_{02301} has a determinantal expression

$$S_{00123}(\mathbf{x}_5,\mathbf{x}_5,\mathbf{x}_5,\mathbf{x}_3,\mathbf{x}_3) = S_{123}(\mathbf{x}_5,\mathbf{x}_3,\mathbf{x}_3)\,,$$

because $[0,2,3,0,1]$ is a vexillary code. Expanding

$$S_{123}(\mathbf{x}_5,\mathbf{x}_3,\mathbf{x}_3) = \begin{vmatrix} K_{00001} & K_{003} & K_{005} \\ 1 & K_{002} & K_{004} \\ 0 & K_{001} & K_{003} \end{vmatrix}$$

in any manner, transforming each K_v into $K^{\mathcal{F}}_v$ after reordering each product into a decreasing flag, one obtains

$$K^{\mathcal{F}}_{02301} \equiv K^{\mathcal{F}}_{00001}K^{\mathcal{F}}_{002}K^{\mathcal{F}}_{003} - K^{\mathcal{F}}_{00001}K^{\mathcal{F}}_{004}K^{\mathcal{F}}_{001} - K^{\mathcal{F}}_{003}K^{\mathcal{F}}_{003} + K^{\mathcal{F}}_{005}K^{\mathcal{F}}_{001}\,.$$

Because $K_{00k} = K_{0k} - x_3K_{0,0,k-1}$, $K_{00k} = K_k - (x_2{+}x_3)K_{0,0,k-1} + x_2x_3K_{0,0,k-2}$, one can transform the preceding determinant into $\begin{vmatrix} x_1+x_4+x_5 & K_3 & K_5 \\ 1 & K_{02} & K_{04} \\ 0 & K_{001} & K_{003} \end{vmatrix}$.

Writing $x_1{+}x_4{+}x_5 = K_{00001} - K_{001} + K_1$, expanding the determinant, and reordering the products, one obtains a second expression

$$\begin{aligned} K^{\mathcal{F}}_{02301} \equiv{} & K^{\mathcal{F}}_{00001}K^{\mathcal{F}}_{003}K^{\mathcal{F}}_{02} - K^{\mathcal{F}}_{00001}K^{\mathcal{F}}_{001}K^{\mathcal{F}}_{04} - K^{\mathcal{F}}_{001}K^{\mathcal{F}}_{003}K^{\mathcal{F}}_{02} + (K^{\mathcal{F}}_{001})^2K^{\mathcal{F}}_{04} \\ & + K^{\mathcal{F}}_{003}K^{\mathcal{F}}_{02}K^{\mathcal{F}}_{1} - K^{\mathcal{F}}_{001}K^{\mathcal{F}}_{04}K^{\mathcal{F}}_{1} - K^{\mathcal{F}}_{003}K^{\mathcal{F}}_{3} + K^{\mathcal{F}}_{001}K^{\mathcal{F}}_{5}\,. \end{aligned}$$

Apart from the vexillary case, we have met other determinantal expressions. The case treated in Th. **??** has for counterpart in $\mathfrak{Schub}$ the following lemma (which is a special case of the description of key polynomials).

Lemma 6.11.2. *Let $v \in \mathbb{N}^n$ be dominant, $u \in \mathbb{N}^n$ be anti-dominant, $v/\!/u = [0^{u_1}, v_1, 0^{u_2-u_1}, v_2, \ldots, 0^{u_n-u_{n-1}}, v_n]$. Then $K^{\mathcal{F}}_{v/\!/u}$ is equal to the sum of all tableaux of shape v satisfying the flag condition $[u_1+1,\ldots,u_n+n]$.*

For example, for $v=[2,1]$, $u=[1,3]$, then $v/\!/u$ is $[0^1,2,0^2,1]$, the flag is $u+[1,2]=[2,5]$ and $K^{\mathcal{F}}_{02001}$ is equal to the sum of all tableaux $\begin{array}{|c|c}\hline c & \\ \hline a & \multicolumn{1}{c|}{b} \\ \hline\end{array}$ such that $b\le 2$, $c\le 5$.

6.12 Generating function of Schubert polynomials in $\mathfrak{Schub}$

According to Cauchy formula (2.10.2), the product $\Omega = \prod_{i,j:\, i+j<n}(y_j+x_i)$ expands a sum $\sum_{\sigma\in\mathfrak{S}_n} X_{\sigma\omega}(\mathbf{y},\mathbf{0})X_\sigma(\mathbf{x},\mathbf{0})$. Thus, one can define the Schubert polynomials in $\mathbf{x}$ as the coefficients in the expansion of Ω in the basis of Schubert polynomials in $\mathbf{y}$. This looks like a circular definition without interest, except that one can easily lift Ω to an element of $\mathfrak{Schub}$, and obtain now the free Schubert polynomials from the commutative ones.

Given n and an alphabet $\mathbb{A} = \{a_1, \ldots, a_n\}$, one defines

$$X_\omega^{\mathcal{F}}(\mathbb{A},\mathbf{x}) = \Big((a_{n-1}-x_1)\ldots(a_1-x_1)\Big)\Big((a_{n-2}-x_2)\ldots(a_1-x_2)\Big)\ldots\Big((a_1-x_{n-1})\Big)\,.$$

Given $\sigma \in \mathfrak{S}_n$, the *free Schubert polynomial* $X_\sigma^{\mathcal{F}}(\mathbb{A},\mathbf{y})$ is defined to be the coefficient of $(-1)^{\ell(\omega\sigma)}X_{\omega\sigma}(\mathbf{x},\mathbf{y})$ in the expansion of $X_\omega^{\mathcal{F}}(\mathbb{A},\mathbf{x})$ in terms of Schubert polynomials in $\mathbf{x},\mathbf{y}$ [16]

Since the kernel $X_\omega^{\mathcal{F}}(\mathbb{A},\mathbf{x})$ belongs to $\mathfrak{Schub}$, the element $X_\sigma^{\mathcal{F}}(\mathbb{A},\mathbf{y})$ coincides with the element defined by expressing $X_\sigma(\mathbf{x},\mathbf{y})$ in any of the bases $K_v(\mathbf{x}), P_v(x), H_v(x)$, and replacing these polynomials by their lift in $\mathfrak{Schub}$.

For example, taking the alphabet $\mathbb{A} = \{\boxed{1},\boxed{2},\ldots,\}$, persevering in reading columnwise, then the coefficient of $X_{213}(\mathbf{x},\mathbf{y})$ in the expansion of

$$X_{321}^{\mathcal{F}}(\mathbb{A},\mathbf{x}) = \begin{array}{ll} \boxed{2}{-}x_1 & \\ \boxed{1}{-}x_1 & \boxed{1}{-}x_2 \end{array},$$

is equal to

$$\begin{array}{|c|c|}\cline{1-1} \cdot & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline \cdot & 1 \\ \hline\end{array} - \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & \cdot \\ \hline\end{array} - (y_1{+}y_2)\begin{array}{|c|c|}\cline{1-1} \cdot & \multicolumn{1}{c}{} \\ \hline \cdot & 1 \\ \hline\end{array} + y_1y_2\begin{array}{|c|c|}\cline{1-1} \cdot & \multicolumn{1}{c}{} \\ \hline \cdot & \cdot \\ \hline\end{array} \equiv \Big(\boxed{1} - y_1\Big)\Big(\boxed{1} - y_2\Big)\,.$$

In the case where σ is dominant, of code λ, the filling of each box of the diagram of λ with the factor $\boxed{i} - y_j$, where (i,j) are the coordinates of the box, still belongs to $\mathfrak{Schub}$, and therefore is equal to $X_\sigma^{\mathcal{F}}(\mathbb{A},\mathbf{y})$. For example,

$$\begin{aligned} X_{3412}^{\mathcal{F}}(\mathbb{A},\mathbf{y}) &= \begin{array}{ll} \boxed{2}{-}y_1 & \boxed{2}{-}y_2 \\ \boxed{1}{-}y_1 & \boxed{1}{-}y_2 \end{array} \\ &\equiv \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array} - (y_1{+}y_2)\left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}\right) + y_1y_2\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}\right) \\ &\qquad + (y_1^2{+}y_1y_2{+}y_2^2)\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} - (y_1^2y_2{+}y_1y_2^2)\left(\boxed{1} + \boxed{2}\right) + y_1^2y_2^2\,. \end{aligned}$$

[16] A function of $\mathbb{A}$, $\mathbf{x}$ may be expanded in terms of functions of $\mathbb{A}$,$\mathbf{x}$,$\mathbf{y}$, one can use simultaneously several Schubert bases $\{X_\sigma(\mathbf{x},\mathbf{z})\}$ of $\mathfrak{Pol}(\mathbf{x})$ with different $\mathbf{z}$.

Free Schubert polynomials $X^{\mathcal{F}}_{\sigma}(\mathbb{A}, \mathbf{y})$ may be recursively obtained by using the divided differences in $\mathbf{y}$[17]. For example

$$
\begin{aligned}
-X^{\mathcal{F}}_{3412}(\mathbb{A},\mathbf{y})\partial_2^{\mathbf{y}} = X^{\mathcal{F}}_{2413}(\mathbb{A},\mathbf{y}) &= \left(\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}\right) - y_1\left(\boxed{1\,|\,1} + \boxed{1\,|\,2} + \boxed{2\,|\,2}\right) \\
&\quad - (y_1+y_2+y_3)\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} + (y_1^2+y_1y_2+y_1y_3)\left(\boxed{1}+\boxed{2}\right) - y_1^2(y_2+y_3) \\
&\equiv \left(\boxed{2} - y_1\right)\left(\boxed{1} - y_1\right)\left(\boxed{2} + \boxed{1} - y_2 - y_3\right) .
\end{aligned}
$$

[17] but defining divided differences in $\mathbb{A}$ which would satisfy the braid relations is not feasible. One can however lift formally the action of divided differences on the basis $\{P_v\}$, as is used in [127, 48].

Chapter 7

Schubert and Grothendieck by keys

7.1 Double keys

In the commutative case, we did not define key polynomials in two sets of variables, contrary to Schubert or Grothendieck polynomials, because there was no "good" candidate. In the free world, it is very easy. Indeed, the operators s_i or π_i on words can be obtained from the operator f_i, which changes, whenever possible, a specific occurence of a letter i in a word w, or a tableau, into i+1. We can act on biletters (i.e. letters with a superscript). Ignoring the superscripts, we point out some biletter $\binom{j}{i}$ as in the case of single letters, and we transform it into $\binom{j+1}{i+1}$. This action does not lift a commutative action.

Noticing that the above transformation preserves the difference between superscript and subscript, we can describe directly the transformations on biwords from the case of words: if a letter i is transformed into a letter k, then in the case of biletters, $\binom{j}{i}$ is transformed into $\binom{j+k-i}{k}$.

Starting with an appropriate word in biletters replacing the word $\cdots 2^{\lambda_2}1^{\lambda_1}$, and extending the action of π_i or $\widehat{\pi}_i$ to biwords, we obtain sums of biwords.

For a partition λ, define

$$\mathbb{K}_\lambda = \widehat{\mathbb{K}}_\lambda = \begin{array}{lllll} \binom{1}{n} & \cdots & \binom{\lambda_n}{n} & & \\ \binom{1}{n-1} & \cdots & \cdots & \binom{\lambda_{n-1}}{n-1} & \\ \vdots & & & \vdots & \\ \binom{1}{1} & \cdots & \cdots & \cdots & \binom{\lambda_1}{1} \end{array}$$

By definition, the elements $\mathbb{K}_v$ and $\widehat{\mathbb{K}}_v$, when v runs over the set of permutations of λ, are all the images of $\mathbb{K}_\lambda$ and $\widehat{\mathbb{K}}_\lambda$ under (reduced) products of π_i's (resp. $\widehat{\pi}_i$'s).

It seems that not much has been gained by passing to biletters. Writing

$$K^{\mathcal{F}}_{102} = \begin{array}{|c|c}\cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \text{instead of}$$

$$\mathbb{K}_{102} = \begin{array}{|c|c}\cline{1-1} \binom{1}{2} \\ \hline \binom{1}{1} & \multicolumn{1}{c|}{\binom{2}{1}} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} \binom{1}{2} \\ \hline \binom{1}{1} & \multicolumn{1}{c|}{\binom{3}{2}} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} \binom{2}{3} \\ \hline \binom{1}{1} & \multicolumn{1}{c|}{\binom{2}{1}} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} \binom{1}{2} \\ \hline \binom{1}{1} & \multicolumn{1}{c|}{\binom{4}{3}} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} \binom{2}{3} \\ \hline \binom{1}{1} & \multicolumn{1}{c|}{\binom{4}{3}} \\ \hline\end{array}$$

is, indeed, immediate. But these new functions possess more properties than in the case of single letters. For example, under the projection

$$\binom{j}{i} \mapsto (x_i - y_j)\,,$$

one obtains polynomials in two sets of variables that we shall still denote $\mathbb{K}_v$.

For example, $K^{\mathcal{F}}_{021} = \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$ is transformed into five tableaux of biletters which compose $\mathbb{K}_{021}$ and become

$$\begin{array}{|c|c}\cline{1-1} x_3-y_2 \\ \hline x_2-y_2 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} x_3-y_2 \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} x_2-y_1 \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_2-y_3} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} x_3-y_2 \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_1-y_2} \\ \hline\end{array} + \begin{array}{|c|c}\cline{1-1} x_2-y_1 \\ \hline x_1-y_1 & \multicolumn{1}{c|}{x_1-y_2} \\ \hline\end{array}\,.$$

We already met polynomials in two sets of variables x,y. The next theorem shows that double key polynomials and Schubert polynomials coincide in the vexillary case.

Theorem 7.1.1. *Let v be a vexillary weight. Then* $\mathbb{K}_v = Y_v(\mathbf{x}, \mathbf{y})$.

Proof. The two polynomials K_v and Y_v satisfy a transition formula involving the same vexillary weights v', u :

$$Y_v(\mathbf{x}, \mathbf{y}) = (x_n - y_j)Y_{v'}(\mathbf{x}, \mathbf{y}) + Y_u(\mathbf{x}, \mathbf{y}) \quad \& \quad K_v = x_n\, K_{v'} + K_u\,.$$

One can suppress terminal zeroes, and therefore suppose that $v_n = k > 0$. In that case $v' = [v_1, \ldots, v_{n-1}, k-1]$. On the other hand, the expansion $K_v = \sum_{w \le v} \widehat{K}_w$ can be cut into two parts, according to whether $w_n = k$ or not. Let $\mathcal{T}$ be the set of tableaux occuring in $K^{\mathcal{F}}_v$ such that the top row of length k ends with n (and there is no n below in the tableau). From the properties of keys and vice-tableaux given in the appendix, one sees that the sum of these tableaux is equal to $\widehat{K}^{\mathcal{F}}_w$. Erasing the pointed occurrence of n in these tableaux, one obtains $K^{\mathcal{F}}_{v'}$, and therefore one has that the x, y-evaluation of $\sum_{t \in \mathcal{T}} t$ is equal to $(x_n - y_j)\widehat{K}_{v'}$, with $j = n + k - \ell$, $(k, \ell$ being the coordinates of the pointed box containing n. Assuming now by induction that $\mathbb{K}_u = Y_u(\mathbf{x}, \mathbf{y})$, $\mathbb{K}_{v'} = Y_{v'}(\mathbf{x}, \mathbf{y})$, one obtains that $\mathbb{K}_v = Y_v(\mathbf{x}, \mathbf{y})$. QED

The function $\mathbb{K}_{021}$ displayed above as a sum of five tableaux is equal to Y_{021}, but the function $\mathbb{K}_{102}$, also displayed above as a sum of five tableaux, is not equal to a Schubert polynomial, because the weight $[1, 0, 2]$ is not vexillary. In fact, one has

$$\mathbb{K}_{102} = Y_{102}(\mathbf{x}, \mathbf{y}) + (y_4 - y_3)Y_2(\mathbf{x}, \mathbf{y}) + (y_2 - y_3)Y_{11}(\mathbf{x}, \mathbf{y}) \\ + (y_3 - y_2)Y_{101}(\mathbf{x}, \mathbf{y}) + (y_2 - y_3)(y_4 - y_3)Y_1(\mathbf{x}, \mathbf{y})\,.$$

The functions $Y_v(\mathbf{x}, \mathbf{y})$ can be characterized by their vanishing properties. For example, $Y_{021}(\mathbf{x}, \mathbf{y})$ vanishes on all specializations corresponding to the permutations in $\mathfrak{S}_4$ of length ≤ 3 different from $[y_1, y_4, y_3, y_2]$. However, the individual tableaux in the expression of $K^{\mathcal{F}}_{021}$ do not necessarily vanish, only their sum does. The following matrix give the non-zero specializations for the tableaux of shape $[2, 1]$ given above, and written here as words in $1, 2, 3$, putting $A = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3)$, $B = (y_2 - y_1)(y_2 - y_4)(y_3 - y_1)$, $D = (y_2 - y_3)(y_2 - y_4)(y_3 - y_1)$, $E = (y_2 - y_1)(y_4 - y_1)(y_4 - y_3)$, $F = (y_2 - y_1)(y_2 - y_4)(y_4 - y_3)$.

	212	322	312	311	211
$[y_2, y_1, y_3, y_4]$	0	$-A$	A	0	0
$[y_2, y_1, y_4, y_3]$	0	$-B$	B	0	0
$[y_3, y_1, y_4, y_2]$	0	$-B$	$B - D$	D	0
$[y_2, y_4, y_1, y_3]$	E	F	$-E - F$	0	0
$[y_3, y_2, y_1, y_4]$	A	0	$-A$	A	$-A$

Let us indicate another manner of using key polynomials to distinguish some interesting elements of $\mathfrak{Pol}(\mathbf{y}) \otimes \mathfrak{Pol}(\mathbf{x})$. In fact, we shall rather use the space $\mathfrak{Free}[\mathbf{y}] = \mathfrak{Pol}(\mathbf{y}) \otimes \mathfrak{Free}$, obtaining the previous case by projection.

For $i = 1, 2, \ldots$, let

$$\Theta_i = 1 \otimes \widehat{\pi}_i + \widehat{\pi}_i^y \otimes s_i$$

acting on $\mathfrak{Free}[\mathbf{y}]$. These operators do not satisfy the braid relations, since the operators $\widehat{\pi}_i$ acting on $\mathfrak{Free}$ do not either.

For any $\in \mathbb{N}^n$, let

$$\mathcal{F}_v = \sum_t K_u(\mathbf{y})\, t \in \mathfrak{Free}[\mathbf{y}]\,,$$

sum over all tableaux t with right key $= v$, u denoting the left key of t.

The following statement shows that the elements $\mathcal{F}_v$ can be generated recursively, in a manner analogous to the generation of the $\widehat{K}^{\mathcal{F}}_v$.

Theorem 7.1.2. *Let $v \in \mathbb{N}^n$, and i be such that $v_i > v_{i+1}$. Then*

$$\mathcal{F}_v \Theta_i = \mathcal{F}_{vs_i}\,. \tag{7.1.1}$$

Proof. To simplify notations, denote $i = 2$, $i+1 = 3$, $v = [\bullet v_2 v_3 \bullet]$. Given a tableau t, let $W(t) = \{w\}$ be the set of columns which occurs as left factors of t in the construction of the left key. If there exists $w \in W(t)$ such that $2 \in w, 3 \notin w$, then the left key u of t is such that $u_2 > u_3$. If on the other hand $2 \notin w, 3 \in w$, then $u_2 < u_3$. Otherwise $u_2 = u3$.

The set of tableaux with right key v decomposes into 2-strings or singletons. Let us write, instead of a tableau, the product of columns which discriminate between 2 and 3 when it is the case, the first column being put in a box, erasing all the other letters as well as the pairs 32 not interfering with the first column. The transformation operates only on the written letters.

The set of tableaux is composed of pieces of the type

- A string

$$\underbrace{\boxed{2}\,22\cdots 22 \to \boxed{2}\,22\cdots 23 \to \cdots \to \boxed{2}\,33\cdots 33}_{key=\ [\bullet\beta\alpha\bullet]} \to \underbrace{\boxed{3}\,33\cdots 33}_{key=\ [\bullet\alpha\beta\bullet]},$$

  with $\beta > \alpha$. The corresponding subsum of $\mathcal{F}_v$ is

$$K^y_{\bullet\beta\alpha\bullet}\boxed{2}\,(2^k + 2^{k-1}3 + \cdots + 3^k) + K^y_{\bullet\alpha\beta\bullet}\boxed{3}\,3^k\,.$$

  Its image under Θ_2 is

$$(K^y_{\bullet\alpha\beta\bullet} - K^y_{\bullet\beta\alpha\bullet})(2^k + \cdots + 3^k)3 + K^y_{\bullet\beta\alpha\bullet}(2^k + \cdots + 3^k)3 \\ + K^y_{\bullet\alpha\beta\bullet}(-(2^k + \cdots + 3^k)3) = 0\,.$$

- A singleton $K^y_{\bullet\beta\alpha\bullet}\boxed{2}\,2^k$. Its image is

$$K^y_{\bullet\alpha\beta\bullet}\boxed{3}\,3^k + K^y_{\bullet\beta\alpha\bullet}\boxed{2}\,\left(2^{k-1}3 + \cdots + 3^k\right)\,.$$

- A singleton $K^y_{\bullet\alpha\beta\bullet}\boxed{3}\,2\,2^k$. Its image is

$$K^y_{\bullet\alpha\beta\bullet}\boxed{3}\,2\left(2^{k-1} + \cdots + 3^{k-1}\right)3\,.$$

- A singleton $K_{\bullet\alpha\alpha\bullet}$, which is sent to 0.

In all the above cases, one has obtained words such that their left key is the index of their coefficient K^y_u. Moreover, the commutative diagram

$$\begin{array}{ccc} \mathfrak{Free}[\mathbf{y}] & \xrightarrow{\Theta_i} & \mathfrak{Free}[\mathbf{y}] \\ {\scriptstyle K^y_u=1}\big\downarrow & & \big\downarrow{\scriptstyle K^y_u=1} \\ \mathfrak{Free} & \xrightarrow{\widehat{\pi}_i} & \mathfrak{Free} \end{array}$$

where the projection sends all key polynomials in $\mathbf{y}$ to 1, shows that all tableaux having right key vs_i have been obtained. QED

For example,

$$\mathcal{F}_{042} = K^y_{042} \begin{smallmatrix} \boxed{3}\boxed{3} & & \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2} \end{smallmatrix} + K^y_{402}\left(\begin{smallmatrix} \boxed{3}\boxed{3} & & \\ \boxed{1}\boxed{2}\boxed{2}\boxed{2} \end{smallmatrix} + \begin{smallmatrix} \boxed{3}\boxed{3} & & \\ \boxed{1}\boxed{1}\boxed{2}\boxed{2} \end{smallmatrix} + \begin{smallmatrix} \boxed{3}\boxed{3} & & \\ \boxed{1}\boxed{1}\boxed{1}\boxed{2} \end{smallmatrix} \right)$$
$$+ K^y_{240} \begin{smallmatrix} \boxed{2}\boxed{3} & & \\ \boxed{1}\boxed{2}\boxed{2}\boxed{2} \end{smallmatrix} + K^y_{420}\left(\begin{smallmatrix} \boxed{2}\boxed{3} & & \\ \boxed{1}\boxed{1}\boxed{2}\boxed{2} \end{smallmatrix} + \begin{smallmatrix} \boxed{2}\boxed{3} & & \\ \boxed{1}\boxed{1}\boxed{1}\boxed{2} \end{smallmatrix} \right) .$$

Indeed, the coefficient of K^y_{402}, for example, is the sum of the three tableaux which have $\begin{smallmatrix} \boxed{3}\boxed{3} & & \\ \boxed{2}\boxed{2}\boxed{2}\boxed{2} \end{smallmatrix}$ as right key and $\begin{smallmatrix} \boxed{2}\boxed{3} & & \\ \boxed{1}\boxed{1}\boxed{1}\boxed{1} \end{smallmatrix}$ as left keys.

The projection on $\mathfrak{Pol}(\mathbf{x})$ of $\mathcal{F}_{042}$ is equal to

$$K^y_{042}\, x^{0420} + K^y_{402}\big(x^{1320} + x^{2220} + x^{3120}\big) + K^y_{240}\, x^{1410} + K^y_{420}\big(x^{2310} + x^{3210}\big) ,$$

while the projection $K^y_v \to 1$ gives the seven tableaux composing $\widehat{K}^{\mathcal{F}}_{042}$.

Notice that the operators Θ_i induce the operators

$$\widetilde{\Theta}_i = 1 \otimes \widehat{\pi}^x_i + \widehat{\pi}^y_i \otimes s^x_i$$

on $\mathfrak{Pol}(\mathbf{y}) \otimes \mathfrak{Pol}(\mathbf{x})$. These new operators satisfy the braid relations. Indeed, there is no difference between computing $f(\mathbf{x})g(\mathbf{x})\widehat{\pi}^x_i = f(\mathbf{x})\big(g(\mathbf{x})\widehat{\pi}^x_i\big) + g(\mathbf{x})\widehat{\pi}^x_i\, f(\mathbf{x})^{s_i}$ and $f(\mathbf{y})g(\mathbf{x})\widetilde{\Theta}_i$, as long as $f(\mathbf{x})$ remains left of $g(\mathbf{x})$ when using Leibnitz' formula for the image of a product under $\widehat{\pi}^x_i = \partial^x_i\, x_{i+1}$.

7.2 Magyar's recursion

In this section, we specialize the alphabet $\mathbf{y}$ to $\mathbf{0}$. The following proposition is due to Magyar [151] (his proof is different) and shows how to generate Schubert polynomials using the isobaric divided differences instead of those of Newton.

Proposition 7.2.1. *Given $v \in \mathbb{N}^n$, let k be such*[1] *that $v_k = 0$ and $v_i > 0$ for $i < k$. Let $u = [v_1 - 1, \ldots, v_{k-1} - 1, v_{k+1}, \ldots, v_n]$. Then*

$$Y_v = Y_u\, \pi_{n-1} \cdots \pi_k\, (x_{k-1} \cdots x_1) = Y_u\, (x_{k-1} \cdots x_1)\, \pi_{n-1} \cdots \pi_k\,. \tag{7.2.1}$$

Proof. $\pi_{n-1} \cdots \pi_k = x_{n-1}\partial_{n-1} \cdots x_k\partial_k$, but the letters can be moved to the left, and therefore

$$Y_u\, \pi_{n-1} \cdots \pi_k\, x_{k-1} \cdots x_1 = Y_u\, x_{n-1} \cdots x_1\, \partial_{n-1} \cdots \partial_k\,.$$

The product of Y_u by the monomial translates, at the level of indices, in the addition of $[1^{n-1}]$. The action of $\partial_{n-1} \cdots x_k\partial_k$ inserts a 0, decreasing by 1 the components on its right, thus producing v. QED

Iterating on Magyar's recursion, one obtains an expression of any Schubert polynomial $Y_v(\mathbf{x}, \mathbf{0})$ as the image of 1 under products of operators of the type $\pi_{n-1} \cdots \pi_k\, (x_{k-1} \cdots x_1)$. For example, one has

$$1 \xrightarrow{x_1} Y_1 \xrightarrow{x_2x_1} Y_{21} \xrightarrow{\pi_2\pi_1} Y_{021} \xrightarrow{\pi_3x_2x_1} Y_{1301} \xrightarrow{\pi_4\pi_3\pi_2x_1} Y_{20301}\,.$$

Since multiplication by $a_k \cdots a_1$ preserve the module $\mathfrak{Schub}$, as well as the operators π_i, the preceding proposition produces the lift of Y_v to $\mathfrak{Schub}$.

For example, the proposition gives the chain $Y_{202} = Y_{12}x_1\pi_2$, $Y_{12} = Y_{01}x_2x_1$, $Y_{01} = x_1\pi_1$.

This lifts into $Y_{202} = a_1\, \pi_1\, (a_2a_1)a_1\pi_2$, i.e.

$$\boxed{1} \to \boxed{1} + \boxed{2} \to \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\to \begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|cc} \hline 3 & & \\ \hline 1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}\,.$$

Combining this construction with the rule for multipliying, inside $\mathfrak{Schub}$, a key polynomial by $a_k \cdots a_1$, one obtains the expression of the lift of Schubert polynomials in $\mathfrak{Schub}$ as a sum of $K_v^{\mathcal{F}}$.

For example, supposing known that

$$Y_{2124}^{\mathcal{F}} = K_{2124}^{\mathcal{F}} + K_{3114}^{\mathcal{F}} + K_{5112}^{\mathcal{F}}$$

[1]if no component of v is 0, change $n \to n{+}1$, $v \to [v, 0]$.

then

$$Y^{\mathcal{F}}_{2124} a_2 a_1 = \left(K^{\mathcal{F}}_{3224} + K^{\mathcal{F}}_{3314} + K^{\mathcal{F}}_{5222} + K^{\mathcal{F}}_{3512}\right) + \left(K^{\mathcal{F}}_{4214} + K^{\mathcal{F}}_{5213} + K^{\mathcal{F}}_{4511}\right) + \left(K^{\mathcal{F}}_{6212} + K^{\mathcal{F}}_{6311}\right)$$

and

$$\begin{aligned} Y^{\mathcal{F}}_{32024} &= Y^{\mathcal{F}}_{2124} \pi_4 \pi_3 \, a_2 a_1 = Y^{\mathcal{F}}_{2124} a_2 a_1 \, \pi_4 \pi_3 \\ &= \left(K^{\mathcal{F}}_{32024} + K^{\mathcal{F}}_{33014} + K^{\mathcal{F}}_{52022} + K^{\mathcal{F}}_{35012}\right) + \left(K^{\mathcal{F}}_{42014} + K^{\mathcal{F}}_{52013} + K^{\mathcal{F}}_{45011}\right) + \left(K^{\mathcal{F}}_{62012} + K^{\mathcal{F}}_{63011}\right). \end{aligned}$$

To describe in a non-recursive manner the key-decomposition of a Schubert polynomial, we shall need the nilplactic monoid[2].

[2] In fact, this application was the original motivation, though not stated, to introduce the nilplactic monoid in [122]. Edelman and Greene's motivation [34] for the same monoid was to classify reduced decompositions.

7.3 Schubert by nilplactic keys

Given a tableau t which is a reduced decomposition of a permutation ζ, let k be the integer such that the first column of t is of the type $[k, \ldots, 1]$ or $[\ldots, r, k, \ldots, 1]$, with $r > k{+}1$. Let $\varphi(t)$ be the tableau obtained from t (in the nilplactic monoid) by erasing the factor $[k, \ldots, 1]$ in the first column of t read by columns. Then the code of ζ^{-1} is of the type $[v' + 1^k, 0, v'']$, with $v \in \mathbb{N}^k$, and $\varphi(t)$ is a reduced decomposition of σ, the code of σ^{-1} being $[0, v', v'']$. One can equivalently use the transposed tableaux and erase in their bottom row a maximal factor of the type $i \ldots k$, i being the minimal letter.

For example, the following tableaux

$$\begin{array}{cccc} 5 & & & \\ 4 & & & \\ 2 & 3 & 5 & \\ \boxed{1} & \boxed{2} & \boxed{3} & 6 \end{array} \;\rightarrow\; \begin{array}{cccc} 5 & & & \\ 4 & & & \\ \boxed{2} & \boxed{3} & 5 & 6 \end{array} \;\rightarrow\; \begin{array}{ccc} 5 & & \\ 4 & 5 & 6 \end{array} \;\rightarrow\; \begin{array}{ccc} 5 & & \\ \boxed{4} & \boxed{5} & \boxed{6} \end{array} \;\rightarrow\; \begin{array}{c} 5 \end{array}$$

are reduced decompositions of the permutations $[[3, 6, 2, 1, 5, 7, 4], [1, 3, 6, 2, 5, 7, 4], [1, 2, 3, 6, 5, 7, 4], [1, 2, 3, 6, 5, 7, 4], [1, 2, 3, 4, 6, 5]]$, whose inverses have respective codes $[3, 2, 0, 3, 1]$, $[0, 2, 0, 3, 1]$, $[0, 0, 0, 3, 1]$, $[0, 0, 0, 3, 1]$, $[0, 0, 0, 0, 1]$.

The following lemma relates the Pieri rule for key polynomials and the recursive construction of tableaux which are reduced decompositions.

Lemma 7.3.1. *Let ζ be a permutation of code $[v' + 1^k, 0, v'']$, and σ be of code $[0, v', v'']$. Let t be a tableau which is a reduced decomposition of σ^{-1}, $[0, u]$ be its key (as a weight). Let $\mathcal{F}$ be the set of tableaux T which are reduced decompositions of ζ^{-1} and such that $\varphi(T) = t$. Then*

$$K_u \, x_1 \ldots x_k = \sum_{T, v = Key(T)} K_v \, . \tag{7.3.1}$$

The following theorem, due to [128], shows that the transition between Schubert and key polynomials is given by the enumeration of tableaux which are reduced decompositions.

Theorem 7.3.2. *Let σ be a permutation, $\mathcal{T}(\sigma)$ be the set of tableaux which are reduced decompositions of σ^{-1}, $\mathcal{K}(\sigma)$ be the set of their left nilplactic keys (as weights). Then*

$$X_\sigma(\mathbf{x}, \mathbf{0}) = \sum_{u \in \mathcal{K}(\sigma)} K_u \, . \tag{7.3.2}$$

Proof. The preceding lemma shows that Magyar's recursive definition $Y_{v'v''} \to Y_{[v'+1^k, 0, v'']}$ corresponds to the recursion on reduced decompositions which are tableaux. QED

For example, for $\sigma = [4,3,1,7,5,2,6]$, of code $[3,2,0,3,1]$, one has four tableaux which are reduced decompositions of $\sigma^{-1} = [3,6,2,1,5,7,4]$:

$$
\begin{array}{l}5\\ 4\;5\\ 2\;3\;6\\ 1\;2\;3\end{array}
\qquad
\begin{array}{l}5\\ 4\;5\\ 2\;3\\ 1\;2\;3\;6\end{array}
\qquad
\begin{array}{l}5\\ 4\\ 2\;3\;5\\ 1\;2\;3\;6\end{array}
\qquad
\begin{array}{l}5\\ 4\\ 2\;3\\ 1\;2\;3\;5\;6\end{array}
$$

$$\textit{keys}\quad [3,2,0,3,1] \quad [4,2,0,2,1] \quad [3,4,0,1,1] \quad [5,2,0,1,1]$$

Hence

$$X_{4317526}(\mathbf{x}, \mathbf{0}) = Y_{32031}(\mathbf{x}, \mathbf{0}) = K_{32031} + K_{42021} + K_{52011} + K_{34011}\,.$$

One can replace tableaux which are reduced decompositions by usual tableaux which satisfy the condition to be *peelable*, cf. the work of Reiner and Shimozono[176]. The preceding decomposition of $X_{4317526}(\mathbf{x}, \mathbf{0})$ is now given by the tableaux

$$
\begin{array}{l}5\\ 4\;4\\ 2\;2\;4\\ 1\;1\;1\end{array}\;,\quad
\begin{array}{l}5\\ 4\;4\\ 2\;2\\ 1\;1\;1\;4\end{array}\;,\quad
\begin{array}{l}5\\ 4\\ 2\;2\;4\\ 1\;1\;1\;4\end{array}\;,\quad
\begin{array}{l}5\\ 4\\ 2\;2\\ 1\;1\;1\;4\;4\end{array}\;,
$$

which are the images of the first tableaux under the plaxification map.

7.4 Schubert by words majorised by reduced decompositions

Billey, Jockusch, Stanley show in [9] that one can obtain Schubert polynomials from reduced decompositions without using keys.

Given a reduced decomposition $s_\alpha = s_{\alpha_1} \dots s_{\alpha_r}$, let $\mathcal{R}(s_\alpha)$ be the set of weakly decreasing words $w \in \mathbb{N}^r$ *below* α, i.e. such that $w_1 \le \alpha_1, \dots, w_r \le \alpha_r$, satisfying the constraints that $\alpha_i > \alpha_{i+1}$ implies $w_i > w_{i+1}$. Then Billey, Jockusch, Stanley prove[3]

Theorem 7.4.1. *Let σ be a permutation, $\{s_\alpha\}$ be the set of reduced decompositions of σ^{-1}. Then the commutative image in $\mathfrak{Pol}$ of the sum of all words belonging to the union of the sets $\mathcal{R}(s_\alpha)$ is equal to the Schubert polynomial $X_\sigma(\mathbf{x}, \mathbf{0})$.*

For example, when $\sigma = [3, 1, 6, 2, 4, 5]$, then $\sigma^{-1} = [2, 4, 1, 5, 6, 3]$ has nine reduced decompositions. Six of them

$$s_1 s_3 s_4 s_5 s_2,\ s_1 s_3 s_2 s_4 s_5,\ s_1 s_3 s_4 s_2 s_5,\ s_3 s_4 s_1 s_5 s_2,\ s_3 s_1 s_4 s_5 s_2,\ s_3 s_1 s_4 s_2 s_5$$

give empty sets of words. The other three are such that

$$\mathcal{R}(s_3 s_4 s_5 s_1 s_2) = \{[2, 2, 2, 1, 1], [3, 2, 2, 1, 1], [3, 3, 2, 1, 1], [3, 3, 3, 1, 1]\}$$
$$\mathcal{R}(s_3 s_4 s_1 s_2 s_5) = \{[2, 2, 1, 1, 1], [3, 2, 1, 1, 1], [3, 3, 1, 1, 1]\}$$
$$\mathcal{R}(s_3 s_1 s_2 s_4 s_5) = \{[2, 1, 1, 1, 1], [3, 1, 1, 1, 1]\}\,,$$

and the Schubert polynomial X_{316245} is indeed equal to $x^{3,2,0} + x^{2,3,0} + x^{2,0,3} + x^{3,0,2} + x^{2,1,2} + x^{2,2,1} + x^{3,1,1} + x^{4,0,1} + x^{4,1,0}$.

Billey-Jockusch-Stanley statement is in fact more precise, the Q-symbol of the reduced decomposition can be used to furnish a sum in the plactic algebra which is equal to $X^{\mathcal{F}}_\sigma$.

Reiner and Shimozono [174] show that the preceding decomposition can be refined, grouping reduced decompositions into nilplactic classes, each of them giving the key polynomial appearing in the decomposition of the Schubert polynomial. In the preceding case, there are two classes, the class of $[3, 4, 5, 1, 2] \cong [3, 4, 1, 2, 5]$ which gives

$$K^{\mathcal{F}}_{203} = \begin{array}{|c|c|c|}\hline 3&3\\\hline 1&1&1\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 2&3\\\hline 1&1&1\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 3&3\\\hline 1&1&3\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 2&3\\\hline 1&1&3\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 2&2\\\hline 1&1&3\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 2&2\\\hline 1&1&2\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 2&2\\\hline 1&1&1\\\hline\end{array}$$

and the class of $[3, 1, 2, 4, 5]$ which gives $K^{\mathcal{F}}_{401} = \begin{array}{|c|c|c|c|}\hline 3\\\hline 1&1&1&1\\\hline\end{array} + \begin{array}{|c|c|c|c|}\hline 2\\\hline 1&1&1&1\\\hline\end{array}\,.$

[3]We reverse words compared to their convention, hence we use reduced decompositions of the inverse permutation.

7.5 Product of a Grothendieck polynomial by a dominant monomial

We have described in (4.2.1) products of the type $G_\sigma(\mathbf{x},\mathbf{y})x_1\dots x_k$ by punching diagrams. Let us have recourse to softer methods by using the jeu de taquin to describe more generally the product by a dominant monomial.

The following theorem has been obtained with Fulton [46], and describe products in terms of keys. Given a tableau t, denote by y_t the monomial image of t by $i \to y_i$.

Theorem 7.5.1. *Let σ be a permutation in $\mathfrak{S}_n$.*

- *Let k be an integer, $k \le n$. Then, modulo $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$, one has*

$$G_{(\sigma)}(\mathbf{x},\mathbf{y})x_k \dots x_1 \equiv \sum_{u\in\mathcal{U}(\sigma,k)} y_u \, G_{\maltese(\mathcal{E}(\sigma)u)}(\mathbf{x},\mathbf{y})\,. \tag{7.5.1}$$

  Let $v \in \mathbb{N}$ be the vector of components $v_{n+1-\sigma_i} = 1$ for $i = 1,\dots,k$, and $v_{n+1-\sigma_i} = 0$ for $i = k+1,\dots,n$. Then $\sum_{u\in\mathcal{U}(\sigma,k)} y_u$ is equal to the key polynomial $K_v(\mathbf{y}^\omega)$ in the reversed alphabet $\mathbf{y}^\omega = [y_n,\dots,y_1]$.

- *Let $\lambda \in \mathbb{N}^n$ be a partition, and v be such that $n^{v_1}\dots 1^{v_n}$ be the reordering of $\sigma_1^{\lambda_1}\dots\sigma_n^{\lambda_n}$. Then*

$$G_{(\sigma)}(\mathbf{x},\mathbf{y})\,x^\lambda \equiv \sum_t y_t \, G_{\maltese(\mathcal{E}(\sigma)t)}(\mathbf{x},\mathbf{y})\,, \tag{7.5.2}$$

  sum over all tableaux of shape λ such that the product $\mathcal{E}(\sigma)t$ be frank. Moreover, the sum $\sum_t y_t$ is equal to the key polynomial $K_v(\mathbf{y}^\omega)$.

Proof. The first assertion concerns the same case as in (4.2.1). However, instead of translating punched diagrams in terms of keys, let us rather use the divided differences in $y_1^{-1}, y_2^{-1},\dots$ to prove it by induction.

The starting point is for $\sigma = \omega$, the assertion resulting in that case from $G_{(\omega)}\mathbf{x}_i \equiv y_{n+1-i}G_{(\omega)}$. Suppose the theorem to be true for the pair σ, k, and let i be such that $\ell(s_i\sigma) \le \ell(\sigma)$. Then, according to (2.2.3), one has the recursion

$$G_{s_i\sigma}(\mathbf{x},\mathbf{y})x_k\dots x_1 = G_\sigma(\mathbf{x},\mathbf{y})x_k\dots x_1\,\pi_i^{\mathbf{y}^\vee}\,.$$

The elements in $\mathcal{U}(\sigma,k)$ are of three types :

- u contains $i, i{+}1$ or none of these letters. Then $y_u G_{\maltese(\mathcal{E}(\sigma)u)}\pi_i^{\mathbf{y}^\vee} = G_{\maltese(\mathcal{E}(\sigma)u)}\pi_i^{\mathbf{y}^\vee}\pi_i^{\mathbf{y}^\vee} y_u$.

- pairs of elements $u'iu''$ and $u'(i{+}1)u''$. The corresponding subsum is of the type $y_{u'u''}\big(y_iG_{s_i\zeta} + y_{i+1}G_\zeta\big)$, with $\ell(s_i\zeta) \le \ell(\zeta)$. Thanks to Leinitz'formula, this subsum is equal to $y_{u'u''}G_\zeta y_i\pi_i^{\mathbf{y}^\vee}$, and therefore invariant under $\pi_i^{\mathbf{y}^\vee}$.

- elements $u = u'(i{+}1)u''$, such that $i \notin u''$ and $u'iu'' \notin \mathcal{U}(\sigma, k)$. In that case, $\zeta = \clubsuit(\mathcal{E}(\sigma)u)$ is such that $\ell(s_i\zeta) \leq \ell(\zeta)$, and

$$G_\zeta\, y_u \pi_i^{\mathbf{y}^\vee} = G_\zeta \pi_i^{\mathbf{y}^\vee} y_{s_i u} + G_\zeta\, y_u = G_{s_i\zeta}\, y_{s_i u} + G_\zeta y_u\,.$$

In total, $\mathcal{U}(s_i\sigma, k) \backslash \mathcal{U}(\sigma, k)$ is the image under the exchange of $i, i{+}1$ of the elements of the third type, and for those elements, one has $s_i\zeta = \clubsuit(\mathcal{E}(s_i\sigma)(s_i u))$ and $\zeta = \clubsuit(\mathcal{E}(s_i\sigma)u)$. Hence the summation (7.5.1) is still valid for $s_i\sigma$.

Moreover, when $w \in \{0,1\}^n$, the element $K_w^{\mathcal{F}}$ is the sum of all columns majorized by a fixed column, and its image under the involution $u \to \omega\, u\, \omega$ is the sum of all columns which majorize a given column. The commutative evaluation of this last element is therefore a key polynomial on a reversed alphabet and one checks that the index corresponding to σ, k is the one stated.

The second part of the theorem results from the associativity of keys: $\clubsuit(tt') = \clubsuit\Big(\mathcal{E}\Big(\clubsuit(t)\Big)\,t'\Big)$. QED

For example of (7.5.2), let $\sigma = [4,1,3,5,2]$ and $\lambda = [2,1,1]$. Then $4^2 1^1 3^1 5^0 2^0$ reorders into $5^0 4^2 3^1 2^0 1^1$, so that $v = [0,2,1,0,1]$, and one has to enumerate the tableaux in $K_{02101}^{\mathcal{F}}$, which are

$$\begin{array}{l}\boxed{3}\\\boxed{2}\\\boxed{1}\boxed{1}\end{array},\ \begin{array}{l}\boxed{3}\\\boxed{2}\\\boxed{1}\boxed{2}\end{array},\ \begin{array}{l}\boxed{4}\\\boxed{2}\\\boxed{1}\boxed{1}\end{array},\ \begin{array}{l}\boxed{4}\\\boxed{2}\\\boxed{1}\boxed{2}\end{array},\ \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{1}\boxed{1}\end{array},\ \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{1}\boxed{2}\end{array},\ \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{2}\boxed{2}\end{array},\ \begin{array}{l}\boxed{5}\\\boxed{2}\\\boxed{1}\boxed{1}\end{array},\ \begin{array}{l}\boxed{5}\\\boxed{2}\\\boxed{1}\boxed{2}\end{array},\ \begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{1}\boxed{1}\end{array},\ \begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{1}\boxed{2}\end{array},\ \begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{2}\boxed{2}\end{array}$$

One then has to take the images of these tableaux under $t \to \omega\, t\omega$, and this furnishes the following value of $G_{(41352)}(\mathbf{x}, \mathbf{y})\, x^{21100}$, writing the tableaux t instead of the monomials y_t :

$$\begin{aligned}G_{(41352)}x^{21100} = {}& \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{1}\boxed{5}\end{array} G_{(51432)} + \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{2}\boxed{4}\end{array} G_{(42531)} + \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{3}\boxed{4}\end{array} G_{(43512)} + \begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{2}\boxed{5}\end{array} G_{(52341)}\\
&+ \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{1}\boxed{4}\end{array} G_{(41532)} + \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{3}\boxed{5}\end{array} G_{(53412)} + \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{2}\boxed{4}\end{array} G_{(42351)} + \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{1}\boxed{5}\end{array} G_{(51342)}\\
&+ \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{2}\boxed{5}\end{array} G_{(52341)} + \begin{array}{l}\boxed{5}\\\boxed{4}\\\boxed{2}\boxed{5}\end{array} G_{(52431)} + \begin{array}{l}\boxed{5}\\\boxed{3}\\\boxed{1}\boxed{5}\end{array} G_{(51342)} + \begin{array}{l}\boxed{4}\\\boxed{3}\\\boxed{1}\boxed{4}\end{array} G_{(41352)}\,.\end{aligned}$$

Notice that the product

$$\begin{aligned}K_{41352}x^{211} = {}& K_{62452} + K_{72442} + K_{62632} + K_{72532}\\ &+ K_{64612} + K_{74512} + K_{63451} + K_{73441} + K_{63631} + K_{73531}\end{aligned}$$

has only 10 terms, and thus does not allow to describe the product $G_{(41352)}(\mathbf{x},\mathbf{y})\, x^{21100}$. Indeed, according to (4.4.1), one has to take parts differing by at least $2=\lambda_1$ to relate the two products. The weight $2\times[4,1,3,5,2]=[8,2,6,10,4]$ is appropriate, the product

$$K_{8,2,6,10,4}x^{211} = K_{10,3,7,10,4} + K_{11,3,7,9,4} + K_{12,3,7,8,4} + K_{10,3,11,6,4} + K_{12,3,9,6,4} \\ +K_{10,7,11,2,4}+K_{12,7,9,2,4}+K_{10,5,7,10,2}+K_{11,5,7,9,2}+K_{12,5,7,8,2}+K_{10,5,11,6,2}+K_{12,5,9,6,2}$$

has 12 terms from which one reads the 12 permutations occurring in the product $G_{(41352)}(\mathbf{x},\mathbf{y})\, x^{21100}$.

7.6 ASM and monotone triangles

We have already met a correspondence between Grothendieck polynomials and key polynomials when describing the multiplication by $x_1 \cdots x_k$. In fact, this correspondence was a direct consequence of the commutation relations between the π_i's and the x_i's.

We describe now a more subtle correspondence, by putting an appropriate weight on staircase tableaux. The tableaux which have a non-zero weight are the tableaux of staircase shape which do not contain a subtableau of the type $\begin{array}{|c|c|}\cline{1-1} b & \multicolumn{1}{c}{} \\ \hline a & c \\ \hline\end{array}$, with $c \geq b$.

Equivalently, they are the tableaux of staircase shape with weakly decreasing diagonals, which appear in the literature as *monotone triangles*. These tableaux are in (easy) bijection with *alternating sign matrices* (ASM), but tableaux will fit better in this text.

Let us define a weight on tableaux, as a product of elementary weights on tableaux of shape $[1,2]$. The weight of a subtableau of shape $[1,2]$ on columns $j{-}1$, j and consecutive rows is, for three integers $a<b<c$,

$$\begin{array}{ccccc}
 & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline a & b \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline b & b \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline b & c \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} b & \multicolumn{1}{c}{} \\ \hline a & c \\ \hline\end{array} \; . \\
\varphi^G \textit{ weight} & y_j x_b^{-1} & y_j x_b^{-1} - 1 & 1 & 0
\end{array}$$

By definition, the weight $\varphi^G(T)$ of a staircase tableau T is the product of these elementary weights, on all subtriangles. The image of $\varphi^G(T)$ under the change of variables $x_i \to (1{-}x_i)^{-1}$, $y_j \to (1{-}y_j)^{-1}$ is denoted $\varphi^{\widetilde{G}}(T)$. Explicitely, the elementary weights are now

$$\begin{array}{ccccc}
 & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline a & b \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline b & b \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} c & \multicolumn{1}{c}{} \\ \hline b & c \\ \hline\end{array} & \begin{array}{|c|c|}\cline{1-1} b & \multicolumn{1}{c}{} \\ \hline a & c \\ \hline\end{array} \; . \\
\varphi^{\widetilde{G}} \textit{ weight} & (x_b{-}1)(y_j{-}1)^{-1} & (x_b{-}y_j)(y_j{-}1)^{-1} & 1 & 0
\end{array}$$

For example, pointing out the rightmost box of the elementary tableaux contributing to the weight, the tableau $t = \begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 \\ \hline\end{array}$ has weight

$$\begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \boxed{3} & \multicolumn{1}{c}{} \\ \hline 1 & \boxed{1} & \boxed{2} \\ \hline\end{array} \;\to\; \varphi^G(t) = \begin{array}{lll} \bullet & & \\ \bullet & y_1 x_3^{-1} & \\ \bullet & y_1 x_1^{-1}{-}1 & y_2 x_2^{-1} \end{array} \quad \to\; \varphi^{\widetilde{G}}(t) = \begin{array}{lll} \bullet & & \\ \bullet & \frac{x_3-1}{y_1-1} & \\ \bullet & \frac{x_1-y_1}{y_1-1} & \frac{x_2-1}{y_2-1} \end{array} \; .$$

The following property is proved[4] in [103], by checking its compatibility with transitions. It states that any Grothendieck polynomial is obtained by enumerating all the monotone triangles having a fixed right key, or, equivalently, because

[4] Keys can be defined directly on ASM. For the correspondence with the keys (of tableaux) that we use here, see Aval [2].

the non-monotone tableaux have weight 0, enumerating all the tableaux in some $\widehat{K}^{\mathcal{F}}_{\sigma}$.

Theorem 7.6.1. *Let $\sigma \in \mathfrak{S}_n$. Then*

$$(-1)^{\ell(\sigma)} G_{(\sigma)}(\mathbf{x},\mathbf{y}) = \sum\nolimits_{T \in \widehat{K}^{\mathcal{F}}_{\sigma}} \varphi^{G}(T)\,, \tag{7.6.1}$$

$$(-1)^{\ell(\sigma)} \widetilde{G}_{(\sigma)}(\mathbf{x},\mathbf{y}) = \sum\nolimits_{T \in \widehat{K}^{\mathcal{F}}_{\sigma}} \varphi^{\widetilde{G}}(T)\,. \tag{7.6.2}$$

For example, for $\sigma = [4,2,1,5,3]$, there are fifteen tableaux in $\widehat{K}^{\mathcal{F}}_{\sigma}$. Only 4 of them are monotone triangles, and their respective φ^{G} weights are

$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \hline 4 & 5 \\ \hline 3 & 4 & 5 \\ \hline 2 & 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 & 4 \\ \hline \end{array} \qquad \tfrac{y_3}{x_4}(\tfrac{y_1}{x_2}-1)(\tfrac{y_2}{x_2}-1)(\tfrac{y_1}{x_1}-1)(\tfrac{y_2}{x_1}-1)(\tfrac{y_3}{x_1}-1)$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \hline 4 & 5 \\ \hline 3 & 3 & 5 \\ \hline 2 & 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 & 4 \\ \hline \end{array} \qquad \tfrac{y_3}{x_4}(\tfrac{y_1}{x_3}-1)(\tfrac{y_1}{x_2}-1)(\tfrac{y_2}{x_2}-1)(\tfrac{y_1}{x_1}-1)(\tfrac{y_2}{x_1}-1)(\tfrac{y_3}{x_1}-1)$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \hline 4 & 5 \\ \hline 3 & 4 & 5 \\ \hline 2 & 2 & 4 & 4 \\ \hline 1 & 1 & 1 & 1 & 4 \\ \hline \end{array} \qquad (\tfrac{y_1}{x_2}-1)(\tfrac{y_3}{x_4}-1)(\tfrac{y_1}{x_1}-1)(\tfrac{y_2}{x_1}-1)(\tfrac{y_3}{x_1}-1)$$

$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \hline 4 & 5 \\ \hline 3 & 3 & 5 \\ \hline 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 4 \\ \hline \end{array} \qquad \tfrac{y_3}{x_4}(\tfrac{y_1}{x_3}-1)(\tfrac{y_1}{x_2}-1)(\tfrac{y_1}{x_1}-1)(\tfrac{y_2}{x_1}-1)(\tfrac{y_3}{x_1}-1)$$

Therefore, the Grothendieck polynomial $G_{(42153)}$ is equal to

$$G_{(42153)}(\mathbf{x},\mathbf{y}) = G_{3101}(\mathbf{x},\mathbf{y}) = (1-\frac{y_1}{x_2})(1-\frac{y_1}{x_1})(1-\frac{y_2}{x_1})(1-\frac{y_3}{x_1})(1-\frac{y_1y_2y_3}{x_2x_3x_4})\ ,$$

while the $\varphi^{\widetilde{G}}$-weight of the same four tableaux furnishes

$$\widetilde{G}_{3101}(\mathbf{x},\mathbf{y}) = \frac{(x_2-y_1)(x_1-y_3)(x_1-y_2)(x_1-y_1)}{(1-y_1)^3(1-y_3)^2(1-y_2)^2}(-y_2y_3y_1 + y_2y_3 + y_1y_2 \\ - y_2 + y_3y_1 - y_3 - y_1 - x_4x_3 + x_4 + x_4x_2x_3 - x_4x_2 + x_3 - x_3x_2 + x_2)\,.$$

Theorem 5.1.9 gives for this polynomial the expression

$$\widetilde{G}_{3101}(\mathbf{xy}) = \frac{Y_{3101}(\mathbf{x},\mathbf{y})}{(1-y_2-1)(1-y_3)^2(1-y_1)^2} - \frac{Y_{3111}(\mathbf{x},\mathbf{y})}{(1-y_2)(1-y_1)^3(1-y_3)^2} \\ - \frac{Y_{3201}(\mathbf{x},\mathbf{y})}{(1-y_1)^2(1-y_3)^2(1-y_2)^2} + \frac{Y_{3211}(\mathbf{x},\mathbf{y})}{(1-y_1)^3(1-y_3)^2(1-y_2)^2}\,.$$

The specialisation $\mathbf{z} = \mathbf{0}$ of (2.9.6) gives the value of the alternating sum of Grothendieck polynomials, which is also the sum of the weights of all monotone triangles :

$$\sum_{T \in K^{\mathcal{F}}_{n\ldots 1}} \varphi^G(T) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} G_{(\sigma)}(\mathbf{x}, \mathbf{y}) = y^\rho x^{-\rho} \,. \tag{7.6.3}$$

The specialisation $\mathbf{x} = \mathbf{1}$ of this last formula is due to Bousquet-Mélou & Habsieger [13].

Chapter 8

Generating Functions

8.1 Binary triangles

One can interpret the Cauchy formula for Schubert and Grothendieck polynomials in many different ways, the Cauchy kernel itself may be thought as the generating function of the Schubert and Grothendieck basis. To generalize this kernel into an element of a non-commmutative algebra, one uses planar displays.

Expanding a planar object with n boxes containing a sum $a+b$ of elements belonging to two families means enumerating the 2^n pairs of complementary objects obtained by choosing either a or b in each box. For example, the expansion of $\begin{smallmatrix} & \square+\blacksquare & \\ \square+\blacksquare & & \square+\blacksquare \\ & \square+\blacksquare & \end{smallmatrix}$ is equal to

$$\left(\begin{smallmatrix} & \square & \\ \square & & \square \\ & \square & \end{smallmatrix}, \begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \end{smallmatrix}\right) + \left(\begin{smallmatrix} & \cdot & \\ \square & & \square \\ & \square & \end{smallmatrix}, \begin{smallmatrix} & \blacksquare & \\ \cdot & & \cdot \\ & \cdot & \end{smallmatrix}\right) \left(\begin{smallmatrix} & \square & \\ \square & & \square \\ & \square & \end{smallmatrix}, \begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \end{smallmatrix}\right) + \left(\begin{smallmatrix} & \square & \\ \cdot & & \square \\ & \square & \end{smallmatrix}, \begin{smallmatrix} & \cdot & \\ \blacksquare & & \cdot \\ & \cdot & \end{smallmatrix}\right)$$
$$+ \cdots + \left(\begin{smallmatrix} & \cdot & \\ \cdot & & \square \\ & \cdot & \end{smallmatrix}, \begin{smallmatrix} & \blacksquare & \\ \blacksquare & & \cdot \\ & \blacksquare & \end{smallmatrix}\right) \left(\begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \square & \end{smallmatrix}, \begin{smallmatrix} & \blacksquare & \\ \blacksquare & & \blacksquare \\ & \cdot & \end{smallmatrix}\right) + \left(\begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \end{smallmatrix}, \begin{smallmatrix} & \blacksquare & \\ \blacksquare & & \blacksquare \\ & \blacksquare & \end{smallmatrix}\right),$$

writing a sum instead of a set of pairs. Each pair can afterwards be read in a precise manner so as to furnish a pair of words.

We shall essentially use planar objects of triangular shape. Decomposing a triangle $\begin{smallmatrix} \square+\blacksquare & \\ \square+\blacksquare & \square+\blacksquare \end{smallmatrix}$ can be thought as enumerating binary triangles (i.e. each box is filled with either 0 or 1) of a fixed shape. In other words, the set

$$\begin{smallmatrix} 1 & \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & \\ 0 & 0 \end{smallmatrix}$$

codes the same information as the set of pairs

$$\left(\begin{smallmatrix} \blacksquare & \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \square & \\ \square & \square \end{smallmatrix}\right), \left(\begin{smallmatrix} \square & \\ \blacksquare & \blacksquare \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \\ \square & \square \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} \square & \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \blacksquare & \\ \blacksquare & \blacksquare \end{smallmatrix}\right)$$

used previously.

In the following of this chapter, we shall enumerate binary triangles and put various weights on them.

8.2 Generating function in the nilplactic algebra

Let us first use the nilplactic algebra with generators $v_1, v_2, \ldots, v_{n-1}$ satisfying

$$v_i^2 = 0 \quad , \quad v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1} \tag{8.2.1}$$

$$v_j v_i v_k = v_j v_k v_i \quad , \quad v_i v_k v_j = v_k v_i v_j \quad (i < j < k) \tag{8.2.2}$$

Given a commutative alphabet $\mathbf{x}_n$, define the nilplactic kernel $\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_n)$ to be

$$\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_n) := \begin{array}{cccc} (1+x_1 v_{n-1}) & & & \\ (1+x_1 v_{n-2}) & (1+x_2 v_{n-1}) & & \\ \vdots & \vdots & & \\ (1+x_1 v_1) & (1+x_2 v_2) & \cdots & (1+x_{n-1} v_{n-1}) \end{array} ,$$

reading the kernel by columns (downwards) from left to right.

One still has a crystal structure on the terms in the expansion of $\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_n)$, as we had in the plactic case (cf. **??**).

EXEMPLE

Given a tableau T in $v_1, v_2, \ldots,$ which is a reduced decomposition, denote by $B(T) \in \mathbb{N}^n$ its bottom key (as a weight). Then an analysis of the nilplactic strings similar to the one performed in **??** gives the following expansion.

Theorem 8.2.1. *Given n, the nilplactic kernel expands in the nilplactic algebra as*

$$\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_n) \cong \sum_T K_{B(T)}\, T \,,$$

sum over all tableaux which are reduced decompositions of permutations of $\mathfrak{S}_n$.

For example, for $n = 4$, there are 25 tableaux (all permutations, except $[2,1,4,3]$, have only one tableau as a reduced decomposition), and $\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_4)$ is equal to

$$K_0 + K_1 \boxed{1} + K_{01} \boxed{2} + K_{001} \boxed{3} + K_{11} \boxed{1\,2} + K_{101} \boxed{1\,3} + K_{02} \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array}$$

$$+ K_2 \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} + K_2 \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline \end{array} + K_{011} \boxed{2\,3} + K_{12} \begin{array}{ll} 3 & \\ 1 & 2 \end{array} + K_3 \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} + K_{201} \begin{array}{ll} 2 & \\ 1 & 3 \end{array} + K_{21} \begin{array}{ll} 2 & \\ 1 & 2 \end{array}$$

$$+ K_{021} \begin{array}{ll} 3 & \\ 2 & 3 \end{array} + K_{111} \boxed{1\,2\,3} + K_{211} \begin{array}{lll} 2 & & \\ 1 & 2 & 3 \end{array} + K_{22} \begin{array}{ll} 2 & 3 \\ 1 & 2 \end{array} + K_{301} \begin{array}{ll} 3 & \\ 2 & \\ 1 & 3 \end{array} + K_{31} \begin{array}{ll} 3 & \\ 2 & \\ 1 & 2 \end{array}$$

$$+ K_{121} \begin{array}{lll} 3 & & \\ 1 & 2 & 3 \end{array} + K_{221} \begin{array}{lll} 2 & 3 & \\ 1 & 2 & 3 \end{array} + K_{311} \begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & 3 \end{array} + K_{32} \begin{array}{ll} 3 & \\ 2 & 3 \\ 1 & 2 \end{array} + K_{321} \begin{array}{lll} 3 & & \\ 2 & 3 & \\ 1 & 2 & 3 \end{array} .$$

All key polynomials K_u of index $u \leq \rho$ occur in the expansion of $\Theta^{\mathfrak{N}\mathfrak{P}}(\mathbf{x}_n)$, but with eventual multiplicities (in the example above, K_2 occurs twice).

8.3 Generating function in the NilCoxeter algebra

By projecting the nilplactic algebra to the NilCoxeter algebra, one obtains a generating function where blocks correspond to all the tableaux which are reduced of the same permutation. In fact, this projection gives back the expansion of Schubert polynomials $X_\sigma(\mathbf{x},\mathbf{0})$ in terms of keys seen in (7.3.2)[1].

To recover Schubert polynomials in two alphabets, let us take three alphabets $\mathbf{x},\mathbf{y},\mathbf{z}$ and define

$$\Theta_n^\partial(\mathbf{x},\mathbf{y}) = \begin{array}{llll} 1{+}(x_1{-}y_{n-1})\partial_{n-1}^z & & & \\ 1{+}(x_1{-}y_{n-2})\partial_{n-2}^z & 1{+}(x_2{-}y_{n-2})\partial_{n-1}^z & & \\ \vdots & \vdots & & \\ 1{+}(x_1{-}y_1)\partial_1^z & 1{+}(x_2{-}y_1)\partial_2^z & \ldots & 1{+}(x_{n-1}{-}y_1)\partial_{n-1}^z \end{array} .$$

The expansion of such kernel is equal to a sum $\sum_{\sigma\in\mathfrak{S}_n} c_\sigma(\mathbf{x},\mathbf{y})\partial_\sigma^z$ that one can determine by making it act on the monomial $z^{n-1,\ldots,1,0}$, but we shall rather take the function $\mathbb{X}_\omega(z\omega,\mathbf{y})$. Indeed, we have already made this computation in (1.7.2), up to minor changes including a reversal of alphabet. One has obtained, say for $n=4$, the identity (reading the display by columns)

$$\begin{array}{lll} \boxed{z_4{-}y_1} & \boxed{z_4{-}y_2} & \boxed{z_4{-}y_3} \\ \boxed{z_3{-}y_1} & \boxed{z_3{-}y_2} & \\ \boxed{z_2{-}y_1} & & \end{array} \quad \begin{array}{lll} \boxed{1{+}(x_1{-}y_3)\partial_3^z} & & \\ \boxed{1{+}(x_1{-}y_2)\partial_2^z} & \boxed{1{+}(x_2{-}y_2)\partial_3^z} & \\ \boxed{1{+}(x_1{-}y_1)\partial_1^z} & \boxed{1{+}(x_1{-}y_2)\partial_3^z} & \boxed{1{+}(x_1{-}y_3)\partial_3^z} \end{array}$$

$$= \begin{array}{lll} \boxed{z_4{-}x_3} & \boxed{z_4{-}x_2} & \boxed{z_4{-}x_1} \\ \boxed{z_3{-}x_2} & \boxed{z_3{-}x_1} & \\ \boxed{z_2{-}x_1} & & \end{array} ,$$

using only that

$$(z_{i+1}-y)(1+(x-y)\partial_i^z)) = (z_{i+1}-y)-(x-y) = z_{i+1}-x\,.$$

More generally, for any n, one has

$$X_\omega(\mathbf{z}^\omega,\mathbf{y})\,\Theta_n^\partial(\mathbf{x},\mathbf{y}) = X_\omega(\mathbf{z}^\omega,\mathbf{x})\,. \tag{8.3.1}$$

Using the Cauchy formula

$$X_\omega(\mathbf{z}^\omega,\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_n} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x})X_{\sigma\omega}(z^\omega,\mathbf{y})\,,$$

[1] left and bottom keys are exchanged, because one takes the reduced decompositions of σ in one case, and of σ^{-1} in the other.

and comparing with

$$X_\omega(\mathbf{z}^\omega,\mathbf{y})c_\sigma(\mathbf{x},\mathbf{y})\partial^z_\sigma = \sum (-1)^{\ell(\sigma)} c_\sigma(\mathbf{x},\mathbf{y}) X_{\omega(\omega\sigma\omega)}(\mathbf{z}^\omega,\mathbf{y})\,,$$

one finds that

$$c_\sigma(\mathbf{x},\mathbf{y}) = (-1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x}) = X_\sigma(\mathbf{x},\mathbf{y})\,.$$

In final, one has obtained that $\Theta^\partial_n(\mathbf{x},\mathbf{y})$ is a generating function of Schubert polynomials.

Theorem 8.3.1. *Let n be a positive integer. Then*

$$\Theta^\partial_n(\mathbf{x},\mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n} X_\sigma(\mathbf{x},\mathbf{y})\,\partial^z_\sigma\,. \tag{8.3.2}$$

For example, for $\sigma=[3,1,5,2,4]$, there are 5 configurations which contribute to $X_{31524}(\mathbf{x},\mathbf{y})$. Grouping reduced decompositions according to their nilplactic class, one has

$$X_{31524}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{llll} x_1{-}y_4 & & & \\ \bullet & \bullet & & \\ x_1{-}y_2 & x_2{-}y_2 & \bullet & \\ x_1{-}y_1 & \bullet & \bullet & \bullet \end{array} + \begin{array}{llll} x_1{-}y_4 & & & \\ \bullet & \bullet & & \\ x_1{-}y_2 & \bullet & \bullet & \\ x_1{-}y_1 & \bullet & x_3{-}y_1 & \bullet \end{array}\right) \begin{array}{|c|c} \hline 4 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} +$$

$$\left(\begin{array}{llll} \bullet & & & \\ \bullet & x_2{-}y_3 & & \\ x_1{-}y_2 & x_2{-}y_2 & \bullet & \\ x_1{-}y_1 & \bullet & \bullet & \bullet \end{array} + \begin{array}{llll} \bullet & & & \\ \bullet & x_2{-}y_3 & & \\ x_1{-}y_2 & \bullet & \bullet & \\ x_1{-}y_1 & \bullet & x_3{-}y_1 & \bullet \end{array} + \begin{array}{llll} \bullet & & & \\ \bullet & \bullet & & \\ x_1{-}y_2 & \bullet & x_3{-}y_2 & \\ x_1{-}y_1 & \bullet & x_3{-}y_1 & \bullet \end{array}\right) \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\,.$$

This theorem is given by Fomin and Kirillov in [39, 38], and is also a corollary of the Hopf decomposition of Schubert polynomials given in [122].

The proof of Fomin and Kirillov is very simple, it consists in noticing that $\Theta^\partial_n\partial^z_i = \Theta^\partial_n\partial_i$. This property instantly allows to characterize the behaviour of coefficients with respect to the divided differences in $\mathbf{x}$, and to recognize these coefficients to be Schubert polynomials. We have already several times encountered cases where the action of divided differences is exchanged with another operation (which is here, multiplication in the algebra generated by the ∂^z_i's).

The Cauchy formula (2.10.2) translates into the following multiplicative property of generating functions

$$\Theta^\partial(\mathbf{x},\mathbf{y}) = \Theta^\partial(\mathbf{u},\mathbf{y})\,\Theta^\partial(\mathbf{x},\mathbf{u})\,, \tag{8.3.3}$$

that Fomin and Kirillov prove directly by using the Yang-Baxter equation.

8.4 Generating function in the 0-Hecke algebra

The preceding considerations can easily be adapted to the Grothendieck world. This time, one has to use the 0-Hecke algebra, with generators $\widehat{\pi}_1^z, \ldots, \widehat{\pi}_{n-1}^z$, instead of the 00-Hecke algebra (NilCoxeter algebra).

Let

$$\Theta_n^{\widehat{\pi}}(\mathbf{x}, \mathbf{y}) = \begin{array}{llll} 1{+}(1{-}\frac{y_{n-1}}{x_1})\widehat{\pi}_{n-1}^z & & & \\ 1{+}(1{-}\frac{y_{n-2}}{x_1})\widehat{\pi}_{n-2}^z & 1{+}(1{-}\frac{y_{n-2}}{x_2})\widehat{\pi}_{n-1}^z & & \\ \vdots & \vdots & & \\ 1{+}(1{-}\frac{y_1}{x_1})\widehat{\pi}_1^z & 1{+}(1{-}\frac{y_1}{x_2})\widehat{\pi}_2^z & \ldots & 1{+}(1{-}\frac{y_1}{x_{n-1}})\widehat{\pi}_{n-1}^z \end{array} .$$

Using that

$$\left(1 - \frac{z_{i+1}}{y}\right)\left(1 + (1 - \frac{y}{x}\widehat{\pi}_i^z)\right) = 1 - \frac{z_{i+1}}{x} ,$$

starting with $\Theta^G(\mathbf{z}, \mathbf{y}) = \prod_{1\le i<j\le n}(1 - z_j y_i^{-1})$, one obtains

$$\Theta^G(\mathbf{z}, \mathbf{y})\Theta_n^{\widehat{\pi}} = \Theta^G(\mathbf{z}, \mathbf{x}) . \tag{8.4.1}$$

Comparing with the Cauchy formula (2.9.4), one obtains the following generating function of Grothendieck polynomials.

Theorem 8.4.1. *Let n be a positive integer. Then*

$$\Theta_n^{\widehat{\pi}}(\mathbf{x}, \mathbf{y}) = \sum_{\sigma\in\mathfrak{S}_n} G_{(\sigma)}(\mathbf{x}, \mathbf{y})\, \widehat{\pi}_\sigma^z . \tag{8.4.2}$$

For example, for $n = 3$, one has

$$\begin{array}{ll} \boxed{1 + (1{-}y_2x_1^{-1})\widehat{\pi}_2^z} & \\ \boxed{1 + (1{-}y_1x_1^{-1})\widehat{\pi}_1^z} & \boxed{1 + (1{-}y_1x_2^{-1})\widehat{\pi}_2^z} \end{array}$$

$$\begin{aligned} &= 1 + (1{-}y_1x_1^{-1})\widehat{\pi}_1^z + (1{-}y_1y_2(x_1x_2)^{-1})\widehat{\pi}_2^z + (1{-}y_1x_1^{-1})(1{-}y_1x_2^{-1})\widehat{\pi}_1^z\widehat{\pi}_2^z \\ &\quad + (1{-}y_2x_1^{-1})(1{-}y_1x_1^{-1})\widehat{\pi}_2^z\widehat{\pi}_1^z + (1{-}y_2x_1^{-1})(1{-}y_1x_1^{-1})(1{-}y_1x_2^{-1})\widehat{\pi}_2^z\widehat{\pi}_1^z\widehat{\pi}_2^z \\ &= 1 + G_{(213)}\widehat{\pi}_1^z + G_{(132)}\widehat{\pi}_2^z + G_{(231)}\widehat{\pi}_{231}^z + G_{(312)}\widehat{\pi}_{312}^z + G_{(321)}\widehat{\pi}_{321}^z . \end{aligned}$$

This generating function has been obtained by Fomin and Kirillov in [38], but is also a corollary of the Hopf decomposition of Grothendieck polynomials given in [122].

The generating function $\widetilde{\Theta}_n^{\widehat{\pi}}(\mathbf{x}, \mathbf{y})$ of $\widetilde{G}$-polynomials is obtained by taking a kernel with factors of the type $(x_i - y_j)(1 - y_j)^{-1}$ instead of $1 - y_j x_i^{-1}$. For example,

$$\left(1 + \frac{x_1{-}y_2}{1{-}y_2}\widehat{\pi}_2^z\right)\left(1 + \frac{x_1{-}y_1}{1{-}y_2}\widehat{\pi}_1^z\right)\left(1 + \frac{x_2{-}y_1}{1{-}y_1}\widehat{\pi}_2^z\right) = \sum_{\sigma\in\mathfrak{S}_3} \widetilde{G}_{(\sigma)}(\mathbf{x}, \mathbf{y})\, \widehat{\pi}_\sigma^z$$

8.5 Hopf decomposition of Schubert and Grothendieck polynomials

Any polynomial in $x_1, \ldots, x_n$ can be filtered according to the powers of x_1. In other words, one can use that $\mathfrak{Pol}(\mathbf{x}_n) \simeq \mathfrak{Pol}(x_1) \otimes \mathfrak{Pol}(x_2, \ldots, x_n)$ and compare expansions in the natural bases of the two spaces.

In the case of symmetric functions, the isomorphism

$$\mathfrak{Sym}(\mathbf{x}_n) \simeq \mathfrak{Sym}(x_1) \otimes \mathfrak{Sym}(x_2, \ldots, x_n)$$

reveals the existence of a Hopf structure on $\mathfrak{Sym}(\mathbf{x}_n)$. I kept the same terminology with M.P. Schützenberger for what concerns polynomials in the Schubert or Grothendieck bases. In the case of Macdonald polynomials, one filters according to x_n instead.

From the generating functions (8.3.2) and (8.4.2) one deduces the following Hopf decompositions given in [122].

Theorem 8.5.1. *Let σ be a permutation in $\mathfrak{S}_n$, v be its code, $\mathbf{x}^+ = \{x_2, \ldots, x_n\}$. Then*

$$Y_v(\mathbf{x}_n, \mathbf{0}) = \sum_{w_\partial, u} x_1^{|v|-|u|} Y_u(\mathbf{x}^+, \mathbf{0}) \,, \tag{8.5.1}$$

sum over all words w_∂ in the expansion of $(1+\partial_{n-1}) \cdots (1+\partial_1)$, all $u \le [n-2, \ldots, 0]$ such that $\partial_\sigma = w_\partial \partial_\zeta$, with ζ of code $[0, u]$.

Similarly

$$\widetilde{G}_v(\mathbf{x}_n, \mathbf{0}) = \sum_{w_{\widehat{\pi}}, u} (-1)^{|u|+\ell(w_{\widehat{\pi}}-|v|} x_1^{|u|+\ell(w_{\widehat{\pi}})} \widetilde{G}_u(\mathbf{x}^+, \mathbf{0}) \tag{8.5.2}$$

$$G_v^{\mathbf{1}/\mathbf{x}}(\mathbf{x}_n, \mathbf{1}) = \sum_{w_{\widehat{\pi}}, u} (-1)^{|u|+\ell(w_{\widehat{\pi}}-|v|)} (1-x_1)^{|u|+\ell(w_{\widehat{\pi}})} G_u^{\mathbf{1}/\mathbf{x}}(\mathbf{x}^+, \mathbf{1}) \tag{8.5.3}$$

sum over all words $w_{\widehat{\pi}}$ in the expansion of $(1+\widehat{\pi}_{n-1}) \cdots (1+\widehat{\pi}_1)$, all $u \le [n-2, \ldots, 0]$ such that $\widehat{\pi}_\sigma = \pm w_\pi \widehat{\pi}_\zeta$, with ζ of code $[0, u]$.

Statements (8.5.1) and (8.5.2) are strictly equivalent to (8.3.2) and (8.4.2) (adding the second alphabet $\mathbf{y}$ is no problem), though, we agree, much less elegant.

The two expansions (8.5.1) and (8.5.2) involve the same set $\{u\}$, but in the second case, each u may correspond to several words $w_{\widehat{\pi}}$. For example, figuring the words in the expansion, one has

$$Y_{021}(\mathbf{x}, \mathbf{0}) = Y_{21}(\mathbf{x}^+, \mathbf{0}) + (x_1\partial_2) Y_{20}(\mathbf{x}^+, \mathbf{0}) + (x_1\partial_3) Y_{11}(\mathbf{x}^+, \mathbf{0}) + (x_1^2\partial_3\partial_2) Y_{01}(\mathbf{x}^+, \mathbf{0}) \,,$$

$$\begin{aligned}\widetilde{G}_{021}(\mathbf{x}, \mathbf{0}) = (1-x_1\widehat{\pi}_3-x_1\widehat{\pi}_2+x_1^2\widehat{\pi}_3\widehat{\pi}_2) \widetilde{G}_{21}(\mathbf{x}^+, \mathbf{0}) + (x_1\widehat{\pi}_2-x_1^2\widehat{\pi}_3\widehat{\pi}_2) \widetilde{G}_{20}(\mathbf{x}^+, \mathbf{0}) \\ + (x_1\widehat{\pi}_3-x_1^2\widehat{\pi}_3\widehat{\pi}_2) \widetilde{G}_{11}(\mathbf{x}^+, \mathbf{0}) + (x_1^2\widehat{\pi}_3\widehat{\pi}_2) \widetilde{G}_{01}(\mathbf{x}^+, \mathbf{0}) \,,\end{aligned}$$

the final expansion being

$$\widetilde{G}_{021}(\mathbf{x},\mathbf{0}) = (1{-}x_1)^2\widetilde{G}_{21}(\mathbf{x}^+,\mathbf{0}) + x_1(1{-}x_1)\widetilde{G}_{20}(\mathbf{x}^+,\mathbf{0}) \\ + x_1(1{-}x_1)\widetilde{G}_{11}(\mathbf{x}^+,\mathbf{0}) + x_1^2\widetilde{G}_{01}(\mathbf{x}^+,\mathbf{0})\,.$$

For a given pair v, u, one checks that, keeping only the terms contributing to the Hopf decomposition, the product $(1{+}x_1\widehat{\pi}_{n-1})\cdots(1{+}x_1\widehat{\pi}_1)$ simplifies into a product of factors of the type $1, x_1\widehat{\pi}_j$ or $1{+}x_1\widehat{\pi}_j$. Therefore, the total coefficient of $\widetilde{G}_u(\mathbf{x}^+,\mathbf{0})$ is equal to some power of x_1 multiplied by a power of $(1{-}x_1)$. Thus, Example 2.6 of [122] reads

$$\widetilde{G}_{14101}(\mathbf{x},\mathbf{0}) = {a_1}^4\widetilde{G}_{1101}(\mathbf{x}^+,\mathbf{0}) + {a_1}^3b_1\widetilde{G}_{2101}(\mathbf{x}^+,\mathbf{0}) + {a_1}^2b_1\widetilde{G}_{41}(\mathbf{x}^+,\mathbf{0}) \\ + {a_1}^2b_1\widetilde{G}_{3101}(\mathbf{x}^+,\mathbf{0}) + a_1{b_1}^2\widetilde{G}_{4101}(\mathbf{x}^+,\mathbf{0})\,,$$

putting $a_i = x_i$, $b_i = 1{-}x_i$.

Iterating the Hopf decomposition, one obtains an expression of $\widetilde{G}_v(\mathbf{x},\mathbf{0})$ as a positive polynomial in $a_1, b_1, a_2, b_2, \ldots$. Interchanging a and b gives the expression of $G_v^{\mathbf{1}/\mathbf{x}}(\mathbf{x},\mathbf{1})$.

For example,

$$\begin{aligned} \widetilde{G}_{202}(\mathbf{x},\mathbf{0}) &= a_3a_2b_2b_1{a_1}^2 + a_3b_2{a_1}^3 + {a_2}^2b_1{a_1}^2 + a_2{a_1}^3 + b_2b_1{a_1}^2{a_3}^2 \\ G_{202}^{\mathbf{1}/\mathbf{x}}(\mathbf{x},\mathbf{1}) &= b_3b_2a_2a_1{b_1}^2 + b_3a_2{b_1}^3 + {b_2}^2a_1{b_1}^2 + b_2{b_1}^3 + a_2a_1{b_1}^2{b_3}^2\,. \end{aligned}$$

Notice that the specialization $b_i = 1$ in the expression of $\widetilde{G}_v(\mathbf{x},\mathbf{0})$ gives the Schubert polynomial $Y_v(\mathbf{x},\mathbf{0})$. We have not investigated the properties of these polynomials.

8.6 Generating function of $\widetilde{G}$-polynomials

We shall use simultaneoulsy several weights on 0–1 triangles T of shape $[1, \ldots, n-1]$: A triangle T gives rise to a monomial $\phi_x(T)$, a product $\phi_y(T)$ of factors $-y_i$, a product $\phi_\pi(T)$ of π_i's and a product $\phi_\partial(T)$ if ∂_i's, as follows :

$$\begin{aligned}
\phi_x(T) &= \prod x_i^{1-T_{ij}} \\
\phi_y(T) &= \prod (-y_i)^{1-T_{ij}} \quad \& \quad \phi_{y-1}(T) = \prod (1-y_i)^{1-T_{ij}} \\
\phi_\pi(T) &= \prod \pi_{n-i+j-1}^{T_{ij}} \\
\phi_\partial(T) &= \prod \partial_{n-i}^{1-T_{ij}} ,
\end{aligned}$$

reading the successive columns from left to right for $\phi_\pi(T)$, and reading by successive rows, each row from right to left, for $\phi_\partial(T)$ (we use matrix coordinates).

$$\begin{array}{llll}
\textit{Reading order} & \begin{array}{ccc} 1 & & \\ 2 & 4 & \\ 3 & 5 & 6 \end{array} & & \begin{array}{ccc} 1 & & \\ 3 & 2 & \\ 6 & 5 & 4 \end{array} \\
 & & \text{for } \phi_\pi , & & \text{for } \phi_\partial . \\
\textit{filling} & \begin{array}{ccc} \pi_3 & & \\ \pi_2 & \pi_3 & \\ \pi_1 & \pi_2 & \pi_3 \end{array} & & \begin{array}{ccc} \partial_3 & & \\ \partial_3 & \partial_2 & \\ \partial_3 & \partial_2 & \partial_1 \end{array}
\end{array}$$

For example, one has

$$\text{for} \quad T = \begin{array}{|c|c|c|} \hline 0 & \multicolumn{2}{c}{} \\ \hline 1 & 0 & \multicolumn{1}{c}{} \\ \hline 0 & 1 & 1 \\ \hline \end{array} \; , \qquad \begin{array}{|c|c|c|} \hline \bullet & \multicolumn{2}{c}{} \\ \hline \pi_2 & \bullet & \multicolumn{1}{c}{} \\ \hline \bullet & \pi_2 & \pi_3 \\ \hline \end{array} \qquad \& \qquad \begin{array}{|c|c|c|} \hline \partial_3 & \multicolumn{2}{c}{} \\ \hline \bullet & \partial_2 & \multicolumn{1}{c}{} \\ \hline \partial_3 & \bullet & \bullet \\ \hline \end{array} \; ,$$

$$\Phi_\pi(T) = \pi_2\pi_2\pi_3 \qquad \Phi_\partial(T) = \partial_3\partial_2\partial_3$$

$$\begin{array}{|c|c|c|} \hline x_3 & \multicolumn{2}{c}{} \\ \hline \bullet & x_2 & \multicolumn{1}{c}{} \\ \hline x_1 & \bullet & \bullet \\ \hline \end{array} \;\Rightarrow \Phi_x(T) = x_3x_2x_1 \;\&\; \Phi_y(T) = (1-y_3)(1-y_2)(1-y_1)\,.$$

Let $\zeta(T)$ be the permutation such that $\phi_\pi(T) = \pi_\zeta$, and $\sigma(T)$ be the permutation such that $\phi_\partial(T) = \partial_\sigma$ (if $\phi_\partial(T)$ is not 0).

Theorem 8.6.1. *Given n and $\zeta, \sigma \in \mathfrak{S}_n$, let $\mathcal{T}(\sigma)$ be the set of 0-1 triangles such that $\sigma(T) = \sigma$. Then*

$$X_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{T \in \mathcal{T}(\sigma)} \phi_y(T)\, \widetilde{G}_{(\zeta\omega)}(\mathbf{x}, \mathbf{y}) \tag{8.6.1}$$

$$X_{\sigma^{-1}}(\mathbf{x}, \mathbf{0}) = \sum_{T \in \mathcal{T}(\sigma)} \phi_x(T)\,. \tag{8.6.2}$$

Proof.

For example, one has eight triangles such that $\sigma(T) = [2,5,1,4,3]$. We list the resulting triangles in π_i, together with $\phi_x(T)\phi_\partial(T)$ and $\widetilde{G}_v$, with v code of $\zeta(T)\omega$.

$$\begin{array}{llll}
\begin{array}{|c|c|c|c|}\hline \bullet \\ \cline{1-1} \pi_3 & \bullet \\ \cline{1-2} \pi_2 & \pi_3 & \pi_4 \\ \cline{1-3} \bullet & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \bullet \\ \cline{1-1} \pi_3 & \bullet \\ \cline{1-2} \bullet & \pi_3 & \pi_4 \\ \cline{1-3} \pi_1 & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \pi_4 \\ \cline{1-1} \bullet & \pi_4 \\ \cline{1-2} \bullet & \bullet & \pi_4 \\ \cline{1-3} \pi_1 & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \bullet \\ \cline{1-1} \pi_3 & \pi_4 \\ \cline{1-2} \bullet & \bullet & \pi_4 \\ \cline{1-3} \pi_1 & \pi_2 & \bullet & \bullet \\ \hline\end{array} \\
x_4x_3x_1^3\,[4,3,1,2,4] & x_4x_3x_2x_1^2\,[4,3,4,1,2] & x_3x_2^2x_1^2\,[4,3,4,1,2] & x_4x_2^2x_1^2\,[43412] \\
\widetilde{G}_{1311} & \widetilde{G}_{2301} & \widetilde{G}_{3301} & \widetilde{G}_{2301}
\end{array}$$

$$\begin{array}{llll}
\begin{array}{|c|c|c|c|}\hline \pi_4 \\ \cline{1-1} \bullet & \bullet \\ \cline{1-2} \pi_2 & \bullet & \pi_4 \\ \cline{1-3} \pi_1 & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \bullet \\ \cline{1-1} \pi_3 & \pi_4 \\ \cline{1-2} \pi_2 & \bullet & \pi_4 \\ \cline{1-3} \bullet & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \bullet \\ \cline{1-1} \bullet & \bullet \\ \cline{1-2} \pi_2 & \pi_3 & \pi_4 \\ \cline{1-3} \pi_1 & \pi_2 & \bullet & \bullet \\ \hline\end{array} &
\begin{array}{|c|c|c|c|}\hline \pi_4 \\ \cline{1-1} \bullet & \pi_4 \\ \cline{1-2} \pi_2 & \bullet & \pi_4 \\ \cline{1-3} \bullet & \pi_2 & \bullet & \bullet \\ \hline\end{array} \\
x_3^2x_2x_1^2\,[3,4,3,1,2] & x_4x_2x_1^3\,[4,3,1,2,4] & x_4x_3^2x_1^2\,[4,3,4,1,2] & x_3x_2x_1^3\,[4,3,1,2,4] \\
\widetilde{G}_{33} & \widetilde{G}_{2311} & \widetilde{G}_{1301} & \widetilde{G}_{3311}
\end{array}$$

The eight words $[4,3,1,2,4], \ldots, [4,3,1,2,4]$ are reduced decompositions of the permutation $[2,5,1,4,3]$, whose code is $[1,3,0,1,0]$. The above enumeration implies that

$$\begin{aligned}
Y_{1301} = (1{-}y_1)^3(1{-}y_4)(1{-}y_3)\widetilde{G}_{1311} + (1{-}y_2)^2(1{-}y_1)^2(1{-}y_3)\widetilde{G}_{3301} \\
+ (1{-}y_1)^3(1{-}y_3)(1{-}y_2)\widetilde{G}_{3311} + (1{-}y_1)^2(1{-}y_4)(1{-}y_2)(2{-}y_2{-}y_3)\widetilde{G}_{2301} \\
+ (1{-}y_3)^2(1{-}y_1)^2(1{-}y_4)\widetilde{G}_{1301} + (1{-}y_2)(1{-}y_1)^3(1{-}y_4)\widetilde{G}_{2311} \\
+ (1{-}y_1)^2(1{-}y_3)^2(1{-}y_2)\widetilde{G}_{33}
\end{aligned}$$

and that

$$\begin{aligned}
X_{[2,5,1,4,3]^{-1}}(\mathbf{x},\mathbf{0}) = X_{31542}(\mathbf{x},\mathbf{0}) = Y_{2,0,2,1}(\mathbf{x},\mathbf{0}) = x_4x_1^2x_3^2 + x_4x_3x_2x_1^2 \\
+ x_4x_2^2x_1^2 + x_4x_3x_1^3 + x_4x_1^3x_2 + x_3^2x_1^2x_2 + x_3x_2^2x_1^2 + x_3x_1^3x_2\,.
\end{aligned}$$

Lenart[137, Th.2.16] gives the decomposition of a Schur function in terms of $\widetilde{G}$-polynomials (see also [107, Prop.1]).

Chapter 9

Key polynomials for type B, C, D

9.1 K^B, K^C, K^D

For each type $\heartsuit = B, C, D$, we are going to define two families of key polynomials, indexed by elements of $\mathbb{Z}^n$, using the divided differences $\pi_i^\heartsuit$ or $\widehat{\pi}_i^\heartsuit$, and modifying the indices using $s_i^\heartsuit$.

In more details, in type $\heartsuit = B, C$, we start with all dominant monomials $x^v : v_1 \geq \cdots \geq v_n \geq 0$ and put

$$x^v = K_v^\heartsuit = \widehat{K}_v^\heartsuit .$$

The other polynomials are defined recursively by

$$K_v^\heartsuit \pi_i = K_{v\,s_i}^\heartsuit \ \& \ \widehat{K}_v^\heartsuit \widehat{\pi}_i = \widehat{K}_{v\,s_i}^\heartsuit \,, \text{ when } v_i > v_{i+1} \,, \, i < n \,, \tag{9.1.1}$$

$$K_v^\heartsuit \pi_n^\heartsuit = K_{v\,s_n^\heartsuit}^\heartsuit \ \& \ \widehat{K}_v^\heartsuit \widehat{\pi}_n^\heartsuit = \widehat{K}_{v\,s_n^\heartsuit}^\heartsuit \,, \text{ when } v_n > 0 \,, \text{ for } \heartsuit = B, C \,. \tag{9.1.2}$$

In type D, we would not obtain enough elements to span the space of polynomials. To the set of dominant monomials $\{x^v\}$ used in types A, B, C, we have to add all x^u, with $u = [v_1, \ldots, v_{n-1}, -v_n]$. In short, let us call *D-dominant* the vectors $[v_1, \ldots, v_{n-1}, \pm v_n]$, $v_1 \geq \cdots \geq v_n \geq 0$.

We start with

$$x^v = K_v^D = \widehat{K}_v^D \quad \text{when } v \text{ is } D\text{-dominant}$$

and define recursively the other polynomials by

$$K_v^D \pi_i = K_{v\,s_i}^D \ \& \ \widehat{K}_v^D \widehat{\pi}_i = \widehat{K}_{v\,s_i}^D \,, \text{ when } v_i > v_{i+1} \,, \, i < n \,, \tag{9.1.3}$$

$$K_v^D \pi_n^D = K_{v\,s_n^D}^D \ \& \ \widehat{K}_v^D \widehat{\pi}_n^D = \widehat{K}_{v\,s_n^D}^D \,, \ v_{n-1} + v_n > 0 \,. \tag{9.1.4}$$

The definition is consistent since the operators satisfy the braid relations, and since the hypotheses used in the recursive steps insure that length increase.

Notice that, when $v \in \mathbb{N}^n$, then all $K_v^\heartsuit$ (resp. $\widehat{K}_v^\heartsuit$), $\heartsuit = A, B, C, D$ coincide with each other, since the exceptional generators s_n^B or s_n^D are not used.

On vectors in $\mathbb{Z}^n$, put the following lexicographic order : $u >_L v$ if there exists $i : u_1 = v_1, \dots, u_{i-1} = v_{i-1}$ and $u_i < v_i$. From the explicit action of the divided differences, one sees that each $K_v^\heartsuit$, $\widehat{K}_v^\heartsuit$ has dominant term x^v. In fact, one has used the operators $\pi_i^\heartsuit$ (resp. $\widehat{\pi}_i^\heartsuit$) to generate the polynomials $K_v^\heartsuit$ (resp. $hK_v^\heartsuit$), and the action of $s_i^\heartsuit$ on the indices of the key polynomials. Therefore, one has the following theorem.

Theorem 9.1.1. *The sets* $\{K_v^\heartsuit : v \in \mathbb{Z}^n\}$, $\{\widehat{K}_v^\heartsuit : v \in \mathbb{Z}^n\}$ *constitute six bases of* $\mathfrak{Pol}(\mathbf{x}_n^\pm)$*, which are triangular in the basis of monomials with respect to the lexicographic order.*

For example,

$$\widehat{K}^B_{-1,-2,1} = x^{-1,-2,1} + x^{-1,-1,0} + x^{-1,-1,1} + x^{-1,0,-1} + x^{-1,0,0} + x^{-1,0,1} + x^{0,-2,0} + x^{0,-2,1} + x^{0,-1,-1} + 2\,x^{0,-1,0} + 2\,x^{0,-1,1} + x^{0,0,-1} + 2\,x^{0,0,0} + x^{0,0,1}\,,$$

$$\widehat{K}^C_{-1,-2,1} = x^{-1,-2,1} + x^{-1,-1,0} + x^{-1,0,-1} + x^{-1,0,1} + x^{0,-2,0} + x^{0,-1,-1} + x^{0,-1,1} + x^{0,0,0}\,,$$

$$\widehat{K}^D_{-1,-2,1} = x^{-1,-2,1} + x^{-1,-1,0} + x^{-1,0,-1} + x^{0,-2,0} + x^{0,-1,-1} + x^{0,-1,1} + x^{0,0,0}\,.$$

To my knowledge, the relations between the bases for different types have not been investigated. Continuing with the preceding example, one has

$$\begin{aligned}
\widehat{K}^B_{-1,-2,1} &= \widehat{K}^C_{-1,-2,1} + \widehat{K}^C_{-1,-1,1} + \widehat{K}^C_{0,-2,1} + \widehat{K}^C_{0,-1,1} \\
&= \widehat{K}^D_{-1,-2,1} + \widehat{K}^D_{-1,-1,1} + \widehat{K}^D_{-1,0,1} + \widehat{K}^D_{0,-2,1} + \widehat{K}^D_{0,-1,1} \\
\widehat{K}^C_{-1,-2,1} &= \widehat{K}^B_{-1,-2,1} - \widehat{K}^B_{-1,-1,1} + \widehat{K}^B_{-1,0,1} - \widehat{K}^B_{0,-2,1} + \widehat{K}^B_{0,-1,1} \\
&= \widehat{K}^D_{-1,-2,1} + \widehat{K}^D_{-1,0,1} \\
\widehat{K}^D_{-1,-2,1} &= \widehat{K}^B_{-1,-2,1} - \widehat{K}^B_{-1,-1,1} - \widehat{K}^B_{0,-2,1} + \widehat{K}^B_{0,-1,1} \\
&= \widehat{K}^C_{-1,-2,1} - \widehat{K}^C_{-1,0,1}\,.
\end{aligned}$$

On the other hand, for a given type $\heartsuit$, the relations between $K^\heartsuit$ and $\widehat{K}^\heartsuit$ are given by the Bruhat order[1], thanks to Lemma 1.10.4.

Lemma 9.1.2. *For any type* $\heartsuit$*, given any weight* v*, one has the following relation*

$$K_v^\heartsuit = \sum_{u \le v} \widehat{K}_u^\heartsuit \tag{9.1.5}$$

[1]One defines the $\heartsuit$-Bruhat order on elements of $\mathbb{Z}^n$ which are in the orbit of a dominant weight, by generating the elements of the orbit by successive application of simple transpositions. For example, for $n = 3$, type B or C, one has the chain $x^{321} < x^{312} < x^{31\overline{2}} < x^{3\overline{2}1} < \cdots$. The *length* is also defined as the minimum length of a sequence of simple transpositions which reorder the weight into a dominant weight. Notice that to order the elements of the group, wich are denoted by the same vectors, one uses the same "Hasse diagram", but starts with $[1, 2, 3]$ instead of $[3, 2, 1]$.

For example,

$$K^B_{0,-2,0} = \widehat{K}^B_{0,-2,0} + \widehat{K}^B_{0,0,-2} + \widehat{K}^B_{0,0,2} + \widehat{K}^B_{0,2,0} + \widehat{K}^B_{2,0,0}\,,$$

but

$$\widehat{K}^B_{0,-2,0} = K^B_{0,-2,0} - K^B_{0,0,-2}\,.$$

The full Bruhat interval does not occur in the second formula, because the Moëbius function takes values in $0, 1, -1$ in the case of the orbit of $[2, 0, 0]$, contrary to the case of the orbit of $[3, 2, 1]$, which is the case of the group itself (Lemma 1.10.4).

9.2 Scalar products for type B, C, D

Let $\Delta^\heartsuit$ be the denominator of Weyl character formula. Weyl defined a scalar product on characters by taking a constant term involving the square of $\Delta^\heartsuit$. This is not appropriate in the case of key polynomials, because they are not invariant under the associated group, contrary to characters. As in the case of the non-symmetric Cauchy kernel, the solution is to take only half of the factors of the symmetric kernel, here, to take only $\Delta^\heartsuit$.

Definition 9.2.1. *Let $\rho^B = [n-\frac{1}{2}, \dots, 2-\frac{1}{2}, 1-\frac{1}{2}]$, $\rho^C = [n, \dots, 2, 1]$, $\rho^D = \rho^A = [n\text{–}1, \dots, 1, 0]$. For $\heartsuit = B, C, D$ let*

$$\Omega^\heartsuit = x^{\rho^\heartsuit} \Delta^\heartsuit .$$

Let moreover $\epsilon = (-1)^n$ when $\heartsuit = B, C$, and $\epsilon = 1$ when $\heartsuit = D$. Then for any pair of Laurent polynomials f, g in $x_1, \dots, x_n$, define

$$(f, g)^\heartsuit = CT\left(\epsilon\, f g\, x^{\rho^\heartsuit} \Delta^\heartsuit\right) . \tag{9.2.1}$$

For example, taking $n = 2$, one has

$$\begin{aligned}
(f,g)^B &= CT\Big(fg x_1^{3/2} x_2^{1/2} (x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})(x_1 - x_2)(1 - \frac{1}{x_1 x_2})\Big) \\
&= CT\Big(fg\,(1 - x_1)(1 - x_2)(1 - x_1/x_2)(1 - x_1 x_2)\Big), \\
(f,g)^C &= CT\left(fg\, x_1^2 x_2\,(x_1 - \frac{1}{x_1})(x_2 - \frac{1}{x_2})(x_1 - x_2)(1 - \frac{1}{x_1 x_2})\right), \\
&= CT\Big(fg\,(1 - x_1^2)(1 - x_2^2)(1 - x_1/x_2)(1 - x_1 x_2)\Big), \\
(f,g)^D &= CT\Big(fg\, x_1\,(x_1 - x_2)(1 - \frac{1}{x_1 x_2})\Big) = CT\Big(fg\,(1 - x_1/x_2)(1 - x_1 x_2)\Big) .
\end{aligned}$$

In the language of root systems, the different kernels are products over all the positive roots.

To define the scalar product $(\ ,\ )^\heartsuit$, one could as well give the finite list of all monomials x^v such that $(x^v\,,1)^\heartsuit \neq 0$.

For example, for type C_2, we need only the list $(x^v, 1)^C = 1$ for $v = [0, 0], [-1, -3], [-3, 1], [-4, -2$ and $(x^v, 1)^C = -1$ for $v = [0, -2], [-1, 1], [-3, -3], [-4, 0]$.

In fact, the scalar product $(\ ,\ )^\heartsuit$ is related to the maximal symmetrizer $\pi_{w_0}^\heartsuit$, as shows the next property.

Lemma 9.2.2. *Let $\heartsuit = B, C, D$. Then $(x^v, 1)^\heartsuit$ takes values in $\{0, 1, \text{–}1\}$. The kernel $\Omega_n^\heartsuit$ expands as*

$$\Omega_n^\heartsuit = \sum_{v \in \mathbb{Z}^n} (x^v, 1)^\heartsuit x^{-v} .$$

Moreover, $x^v \pi_{w_0}^\heartsuit = \pm 1$ if and only if $x^v \pi_{w_0}^\heartsuit = (x^v, 1)^\heartsuit$.

Proof. That x^{-v} occurs with multiplicity c in the expansion of $x^{\rho^\heartsuit}\Delta^\heartsuit$ is is another way of stating that $(x^v, 1)^\heartsuit = c$. This proves the first statement.

The $\heartsuit$-Vandermonde expands as a sum $\sum_u \pm x^u$, over all signed permutations of $\rho^\heartsuit$. Therefore, x^{-v} occurs in the expansion of $\Omega^\heartsuit$ if and only if $v + \rho^\heartsuit$ is a signed permutation of $\rho^\heartsuit$. On the other hand, for any v, $x^v \pi^\heartsuit_{w_0}$ is equal to 0, or there exists a dominant weight λ such that $x^v \pi^\heartsuit_{w_0} = \pm x^\lambda \pi^\heartsuit_{w_0}$. This last function is equal to ± 1 if and only if $\lambda = [0, \dots, 0]$, which exactly means that $v + \rho^\heartsuit$ is a signed permutation of $\rho^\heartsuit$. QED

The crucial property of the scalar products $(\,,\,)^\heartsuit$ is their compatibility with the operators generating the key polynomials.

Proposition 9.2.3. *Write $\pi_n = \pi^\heartsuit_n$, $\widehat{\pi}_n = \widehat{\pi}^\heartsuit_n$, for $\heartsuit = B, C, D$. Then the operators π_i and $\widehat{\pi}_i$ ($1 \le i \le n$) are self-adjoint with respect to $(\,,\,)^\heartsuit$, i.e. for every pair of Laurent polynomials f, g, one has*

$$\big(f\pi_i\,,\,g\big)^\heartsuit = \big(f\,,\,g\pi_i\big)^\heartsuit, \quad \big(f\widehat{\pi}_i\,,\,g\big)^\heartsuit = \big(f\,,\,g\widehat{\pi}_i\big)^\heartsuit.$$

Proof. For $i = 1, \dots, n{-}1$, the proof is similar to the case of type A, except that we start with fg instead of $f(x_1, \dots, x_n)g(x_n^{-1}, \dots, x_1^{-1})$. In the case $i = n$, $\heartsuit = B, C$, one first computes the constant term with respect to x_n and writes

$$(f\,,\,g)^\heartsuit = CT\Big(CT_{x_n}\Big(fg\,\frac{x_n}{1+\beta x_n}(x_n - {x_n}^{-1})\spadesuit\Big)\Big),$$

where $\spadesuit$ is a function invariant under $s_n = s^\heartsuit_n$ and $\beta = 1$ for $\heartsuit = B$ and $\beta = 0$ for $\heartsuit = C$. Therefore, to evaluate $(f\widehat{\pi}_n\,,\,g)^\heartsuit - (f\,,\,g\widehat{\pi}_n)^\heartsuit = (f\pi_n\,,\,g)^\heartsuit - (f\,,\,g\pi_n)^\heartsuit$ one can first compute

$$CT_{x_n}\Big(\,(f\widehat{\pi}_n g - g\widehat{\pi}_n f)\,\frac{x_n}{1+\beta x_n}(x_n - x_n^{-1})\spadesuit\Big) = CT_{x_n}\Big((g^{s_n}f - f^{s_n}g)\spadesuit\Big)$$

which is null, because the function under parentheses is alternating under s_n.

Similarly, for $\heartsuit = D$, neglecting a function invariant under $s_n = s^\heartsuit_n$, to determine $(f\widehat{\pi}_n\,,\,g)^D - (f\,,\,g\widehat{\pi}_n)^D = (f\pi_n\,,\,g)^D - (f\,,\,g\pi_n)^D$, one can first compute

$$CT_{x_{n-1},x_n}\Big((f\widehat{\pi}_n g - g\widehat{\pi}_n f)(1 - x_{n-1}x_n)\Big) = CT_{x_{n-1},x_n}\Big(f^{s_n}g - g^{s_n}f\Big)$$

which is also null, because the function $f^{s_n}g - g^{s_n}f$ is alternating under s_n. QED

9.3 Adjointness

Lemma 2.5.1 evidently extends to the case of $\pi_n^\heartsuit$, $\heartsuit = B, C, D$. For example, for $\heartsuit = C$, the two equations

$$(K_{1\,\bar5\,2}\,,\, \widehat{K}_{\bar1\,5\,2}) = 0 \quad \& \quad (K_{1\,\bar5\,\bar2}\,,\, \widehat{K}_{\bar1\,5\,2}) = 1$$

transfer into

$$(K_{1\,\bar5\,2}\,,\, \widehat{K}_{\bar1\,5\,\bar2}) = 1 \quad \& \quad (K_{1\,\bar5\,\bar2}\,,\, \widehat{K}_{\bar1\,5\,\bar2}) = 0\,.$$

To see that, one needs only write

$$K_{1\,\bar5\,2} = f_1 + f_2 x_3^{-1}\,,\ \widehat{K}_{\bar1\,5\,2} = g_1 + g_2 x_3^{-1}\,,$$

with f_1, f_2, g_1, g_2 invariant under s_3^C. Then, the statement becomes straightforward after expliciting $K_{1\,\bar5\,\bar2} = f_1$, $\widehat{K}_{\bar1\,5\,\bar2} = -g_2 x_3^{-1}$. Once more, computations take place in a two-dimensional space only.

Since the key polynomials $\widehat{K}_u^\heartsuit$ stem from the dominant monomials, we need only explicit the scalar products of all the $K_v^\heartsuit$ with all dominant monomials, to know all the $(K_v^\heartsuit, \widehat{K}_u^\heartsuit)^\heartsuit$.

We refer to [44] to check the following lemma.

Lemma 9.3.1. *Given n, let $\heartsuit = B, C, D$. Let v be $\heartsuit$-dominant (i.e. v is a partition λ, or, in type D, to the set of partitions λ one adds the weights $[\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n]$). Then for all $u \in \mathbb{Z}^n$, one has*

$$(K_u^\heartsuit, \widehat{K}_v^\heartsuit) = 0 \quad \textit{except} \quad (K_{-v}^\heartsuit, \widehat{K}_v^\heartsuit) = 1\,. \tag{9.3.1}$$

Thanks to Lemma 2.5.1 and Lemma 11.1.2, one determines all scalar products between the two bases of key polynomials, for each type, and one concludes :

Theorem 9.3.2. *Let $u, v \in \mathbb{Z}^n$, and $\heartsuit = B, C, D$. Then*

$$(K_u^\heartsuit, \widehat{K}_v^\heartsuit) = 0 \quad \textit{except} \quad (K_{-v}^\heartsuit, \widehat{K}_v^\heartsuit) = 1\,. \tag{9.3.2}$$

Contrary to type A, we do not know how to write a kernel involving *all* the elements of the two adjoint bases. Nevertheless, there are non-symmetric kernels generalizing Littlewood's kernels for types B, C, D. We refer to [44] for the proof of the next theorem.

Theorem 9.3.3. *Let*

$$\begin{aligned}
\Omega^B &:= \frac{\prod_{1\le i<j\le n}(1-x_i x_j)\prod_{i=1}^n(1+x_i)}{\prod_{i,j=1}^n(1-x_i y_j)\ \prod_{i=1}^n\prod_{j=i}^n(1-x_i/y_j)},\\
\Omega^C &:= \frac{\prod_{1\le i<j\le n}(1-x_i x_j)}{\prod_{i,j=1}^n(1-x_i y_j)\ \prod_{i=1}^n\prod_{j=i}^n(1-x_i/y_j)},\\
\Omega^D &:= \frac{\prod_{1\le i\le j\le n-1}(1-x_i x_j)}{\prod_{i=1}^{n-1}\prod_{j=1}^n(1-x_i y_j)\ \prod_{i=1}^{n-1}\prod_{j=i}^n(1-x_i/y_j)}.
\end{aligned}$$

Then these kernels decompose as follows

$$\Omega^B = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K^B_{-v}(\mathbf{y})\,, \tag{9.3.3}$$

$$\Omega^C = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K^C_{-v}(\mathbf{y})\,, \tag{9.3.4}$$

$$\Omega^D = \sum_{v\in\mathbb{N}^n:\, v_n=0} \widehat{K}_v(\mathbf{x})\, K^D_{-v}(\mathbf{y})\,, \tag{9.3.5}$$

where x_n is specialized to 0 in the last equation.

For example, for $n=3$, the term of degree 2 (in $\mathbf{x}$) of the C-kernel

$$\frac{(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}{(1-\frac{x_1}{y_1})(1-\frac{x_1}{y_2})(1-\frac{x_1}{y_3})(1-\frac{x_2}{y_2})(1-\frac{x_2}{y_3})(1-\frac{x_3}{y_3})\prod_{i,j=1}^{3}(1-x_iy_j)}$$

is equal to

$$\widehat{K}_{0,0,2}(\mathbf{x}){K^C}_{0,0,-2}(\mathbf{y}) + \widehat{K}_{2,0,0}(\mathbf{x}){K^C}_{-2,0,0}(\mathbf{y}) + \widehat{K}_{0,2,0}(\mathbf{x}){K^C}_{0,-2,0}(\mathbf{y})$$
$$+ \widehat{K}_{1,0,1}(\mathbf{x}){K^C}_{-1,0,-1}(\mathbf{y}) + \widehat{K}_{0,1,1}(\mathbf{x}){K^C}_{0,-1,-1}(\mathbf{y}) + \widehat{K}_{1,1,0}(\mathbf{x}){K^C}_{-1,-1,0}(\mathbf{y})\,.$$

Remember that we have also stated in Th. 2.15.2, for what concerns type A, that

$$\Omega^A = \frac{1}{\prod_{i+j\le n+1}(1-x_iy_j)} = \sum_{v\in\mathbb{N}^n} \widehat{K}_v(\mathbf{x})\, K_{v\omega}(\mathbf{y})\,.$$

9.4 Symplectic and orthogonal Schur functions

For $v \in \mathbb{N}^n$ antidominant, the functions $K_v^{\heartsuit}$ are symmetrical in $x_1, \ldots, x_n$. As in the usual theory of symmetric functions, it is profitable to define functions independently of the number of variables. One cannot use in the present case projective limits with $x_n \to 0$, because the polynomials $K_v^{\heartsuit}$ are in general Laurent polynomials. Thus, one must find polynomials which by specialization of some variables to $x_1^{-1}, \ldots, x_n^{-1}$ gibe back the $K_v^{\heartsuit}$.

Following Littlewood, one defines, for $\lambda \in \mathfrak{Part}$, the *orthognal Schur function* $\mathcal{O}_\lambda(\mathbf{z})$ and the *symplectic Schur function* $Sp_\lambda(\mathbf{z})$ by the following generating functions, using a second alphabet $\mathbf{y}$:

$$\frac{\prod_{i \le j}(1 - y_i y_j)}{\prod_i \prod_j (1 - y_i z_j)} = \sum_{\lambda \in \mathfrak{Part}} s_\lambda(\mathbf{y})\, \mathcal{O}_\lambda(\mathbf{z}) \tag{9.4.1}$$

$$\frac{\prod_{i < j}(1 - y_i y_j)}{\prod_i \prod_j (1 - y_i z_j)} = \sum_{\lambda \in \mathfrak{Part}} s_\lambda(\mathbf{y})\, Sp_\lambda(\mathbf{z}) \,. \tag{9.4.2}$$

Using that $\prod_i \prod_j (1-y_i z_j)^{-1}$ is a reproducing kernel, one can rewrite $\mathcal{O}_\lambda(\mathbf{z})$ and $Sp_\lambda(\mathbf{z})$ with the help of the operators adjoint to multiplication by $\prod_{i\le j}(1-y_iy_j)$ or $\prod_{i<j}(1-y_iy_j)$ respectively. Explicitly, using the Frobenius notation for partitions, the above formulas become

$$\mathcal{O}_\lambda(\mathbf{z}) = \sum_{\mu=(\alpha+1^r|\alpha)} (-1)^{|\mu|/2}\, s_{\lambda/\mu}(\mathbf{z}) \tag{9.4.3}$$

$$Sp_\lambda(\mathbf{z}) = \sum_{\mu=(\alpha|\alpha+1^r)} (-1)^{|\mu|/2}\, s_{\lambda/\mu}(\mathbf{z}) \,. \tag{9.4.4}$$

In particular one has, for $r \ge 2$,

$$\mathcal{O}_{1^r} = s_{1^r} \ , \ \mathcal{O}_r = s_r - s_{r-2} \ , \ Sp_{1^r} = s_{1^r} - s_{1^{r-2}} \ , \ Sp_r = s_r \,. \tag{9.4.5}$$

In fact, one can avoid decomposing partitions according to their diagonal hooks of their diagrams, and write, for $\lambda \in \mathbb{N}^n$, using (1.6.4) and (1.6.2),

$$\mathcal{O}_\lambda(\mathbf{z}) = \sum_{\epsilon=[\epsilon_1,\ldots,\epsilon_n]\in\{0,1\}^n} (-1)^{|\epsilon|} s_{\lambda/[2\epsilon_1, 4\epsilon_2, \ldots, 2n\epsilon_n]}(\mathbf{z}) \tag{9.4.6}$$

$$Sp_\lambda(\mathbf{z}) = \sum_{\epsilon=[\epsilon_1,\ldots,\epsilon_n]\in\{0,1\}^n} s_{\lambda/[0\epsilon_1, 2\epsilon_2, \ldots, (2n-2)\epsilon_n]}(\mathbf{z}) \,. \tag{9.4.7}$$

For example, writing the non-zero terms only, one has

$$\begin{aligned}
\mathcal{O}_{332} &= s_{332/000} - s_{332/200} - s_{332/040} + s_{332/240} \\
&= \begin{vmatrix} s_3 & s_4 & s_5 \\ s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 \end{vmatrix} - \begin{vmatrix} s_1 & s_4 & s_5 \\ s_0 & s_3 & s_4 \\ 0 & s_1 & s_2 \end{vmatrix} - \begin{vmatrix} s_3 & s_0 & s_5 \\ s_2 & 0 & s_4 \\ s_0 & 0 & s_2 \end{vmatrix} + \begin{vmatrix} s_1 & s_0 & s_5 \\ s_0 & 0 & s_4 \\ 0 & 0 & s_2 \end{vmatrix} \\
&= s_{332/000} - s_{332/200} - s_{332/310} - s_{332/330} \\
&= s_{332} - s_{33} - s_{321} + s_{31} + s_{22} - s_2 \,,
\end{aligned}$$

$$
\begin{aligned}
Sp_{332} &= s_{332/000} + s_{332/020} + s_{332/004} + s_{332/024} \\
&= \begin{vmatrix} s_3 & s_4 & s_5 \\ s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 \end{vmatrix} + \begin{vmatrix} s_3 & s_2 & s_5 \\ s_2 & s_1 & s_4 \\ s_0 & 0 & s_2 \end{vmatrix} + \begin{vmatrix} s_3 & s_4 & s_1 \\ s_2 & s_3 & s_0 \\ s_0 & s_1 & 0 \end{vmatrix} + \begin{vmatrix} s_3 & s_2 & s_1 \\ s_2 & s_1 & s_0 \\ s_0 & 0 & 0 \end{vmatrix} \\
&= s_{332/000} - s_{332/110} + s_{332/211} - s_{332/222} \\
&= s_{332} - s_{222} - s_{321} + s_{211} + s_{22} - s_{11}\,.
\end{aligned}
$$

The sums (9.4.6) and (9.4.7) are the expansions of single determinants of ccomplete functions, due to Weyl ([193, th.7.9.A], [193, th.7.8.E]):

$$
\mathcal{O}_\lambda = \det\Big(s_{\lambda_i+j-i} - s_{\lambda_i-j-i}\Big)_{i,j=1\ldots n} \tag{9.4.8}
$$

$$
Sp_\lambda = \frac{1}{2}\det\Big(s_{\lambda_i+1-i} + s_{\lambda_i-j-i+2}\Big)_{i,j=1\ldots n} \tag{9.4.9}
$$

(the first column is divisible by 2, and simplifies with the outside factor).

For example,

$$
\mathcal{O}_{\lambda_1\lambda_2\lambda_3} = \begin{vmatrix} s_{\lambda_1} - s_{\lambda_1-2} & s_{\lambda_1+1} - s_{\lambda_1+1-4} & s_{\lambda_1+2} - s_{\lambda_1+2-6} \\ s_{\lambda_2-1} - s_{\lambda_2-1-2} & s_{\lambda_2} - s_{\lambda_2-4} & s_{\lambda_2+1} - s_{\lambda_2+1-6} \\ s_{\lambda_3-2} - s_{\lambda_3-2} & s_{\lambda_3-1} - s_{\lambda_3-1-4} & s_{\lambda_3} - s_{\lambda_3-6} \end{vmatrix}
$$

$$
Sp_{\lambda_1\lambda_2\lambda_3} = \begin{vmatrix} s_{\lambda_1} & s_{\lambda_1+1} + s_{\lambda_1+1-2} & s_{\lambda_1+2} + s_{\lambda_1+2-4} \\ s_{\lambda_2-1} & s_{\lambda_2} + s_{\lambda_2-2} & s_{\lambda_2+1} + s_{\lambda_2+1-4} \\ s_{\lambda_3-2} & s_{\lambda_3-1} + s_{\lambda_3-1-2} & s_{\lambda_3} + s_{\lambda_3-4} \end{vmatrix}.
$$

Notice that (9.4.3) and (9.4.4) are exchanged by transposing partitions. In other words, the expansion of $\mathcal{O}_\lambda$ in terms of Schur functions is obtained by transposing partitions in the expansion of $Sp_{\lambda^\sim}$.

One extends the definition of orthogonal and symplectic Schur functions to indices in $\mathbb{N}^n$ by requiring the same reordering rules as for Schur functions:

$$
\mathcal{O}_{\ldots,v_i,v_{i+1},\ldots} = -\mathcal{O}_{\ldots,v_{i+1}-1,v_i+1,\ldots} \quad \& \quad Sp_{\ldots,v_i,v_{i+1},\ldots} = -Sp_{\ldots,v_{i+1}-1,v_i+1,\ldots}\,.
$$

All linear operators on the space of symmetric functions with basis the Schur functions extend to the spaces with bases orthogonal or symplectic Schur functions, by just a formal substitution ${}_\lambda{\to}\mathcal{O}_\lambda$ or $s_\lambda \to Sp_\lambda$. In particular, the notations $\mathcal{O}_{\lambda/\mu}$ and $Sp_{\lambda/\mu}$ make sense, even in the absence of a determinantal expression:

$$
s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu \Rightarrow \mathcal{O}_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu}\mathcal{O}_\nu \quad \& \quad Sp_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} Sp_\nu\,.
$$

Formula (9.4.3) can be written with the operator $D_{\sigma_1(-S^2}$ adjoint to multiplication by $\sigma_1(-S^2) = \sum S^i(-S^2)$:

$$
\mathcal{O}_\lambda = D_{\sigma_1(-S^2)}\, s_\lambda\,. \tag{9.4.10}
$$

Similarly, (9.4.4) uses the operator adjoint to multiplication by $\sigma_1(-\Lambda^2)$:

$$
Sp_\lambda = D_{\sigma_1(-\Lambda^2)}\, s_\lambda\,. \tag{9.4.11}
$$

The inverse operators are respectively $D_{\sigma_1(S^2)}$ and $D_{\sigma_1(\Lambda^2)}$. Therefore, using Litlewood's formulas (1.6.7) and (1.6.8), one has

$$s_\lambda = D_{\sigma_1(S^2)}\,\mathcal{O}_\lambda = \sum_{\mu\, even\, rows} \mathcal{O}_{\lambda/\mu} \tag{9.4.12}$$

$$s_\lambda = D_{\sigma_1(\Lambda^2)}\,Sp_\lambda = \sum_{\mu\, even\, columns} Sp_{\lambda/\mu} \tag{9.4.13}$$

For example,

$$\begin{aligned} s_{433} &= \mathcal{O}_{433} + \mathcal{O}_{433/2} + \mathcal{O}_{433/4} + \mathcal{O}_{433/22} + \mathcal{O}_{433/42} + \mathcal{O}_{433/222} + \mathcal{O}_{433/422} \\ &= \mathcal{O}_{433} + \mathcal{O}_{431} + \mathcal{O}_{332} + \mathcal{O}_{33} + O_{321} + \mathcal{O}_{411} + \mathcal{O}_{31} + \mathcal{O}_{211} + \mathcal{O}_{11}\ . \end{aligned}$$

$$\begin{aligned} s_{433} &= Sp_{433} + Sp_{433/11} + Sp_{433/22} + Sp_{433/33} \\ &= Sp_{433} + Sp_{332} + Sp_{422} + Sp_{321} + Sp_{411} + Sp_{31} + Sp_4\,. \end{aligned}$$

The Jacobi-Trudi like determinant $\left|Sp_{\lambda_i+j-i}\right|$ is not equal to Sp_λ when $\ell(\lambda) > 1$, but to s_λ, since $Sp_r = s_r$. However, the determinantal expression of a Schur function in terms of hooks extends to the symplectic and orthogonal Schur functions, as shown by Abramsky, Jahn and King.

Lemma 9.4.1. *Let* $(\alpha|\beta)$ *be the Frobenius decomposition of a partition* λ*, with* $\alpha, \beta \in \mathbb{N}^r$*. Then*

$$\mathcal{O}_\lambda = \det(\mathcal{O}_{(\alpha_i|\beta_j)})_{i,j=1\dots r} \quad \& \quad Sp_\lambda = \det(Sp_{(\alpha_i|\beta_j)})_{i,j=1\dots r}\,.$$

The proof is straightforward, multiplying the Weyl determinants by the matrix $\left[(-1)^{j-i} s_{1^{j-i}}\right]$. The functions $\mathcal{O}_{(\alpha_i|\beta_j)}$, and $Sp_{(\alpha_i|\beta_j)}$ being explicit sums of hook Schur functions, one has just to recognize in them the entries of the matrices obtained by multiplication, up to reordering. For example, for $\lambda = [7,4,2] = \big([6,2]\,|\,[2,1]\big)$, one has

$$\mathcal{O}_{742} = \begin{vmatrix} s_7 - s_5 & s_8 - s_4 & s_9 - s_3 \\ s_3 - s_1 & s_4 - 1 & s_5 \\ 1 & s_1 & s_2 \end{vmatrix} \begin{vmatrix} 1 & \text{-}s_1 & s_{11} \\ 0 & 1 & \text{-}s_1 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} \mathcal{O}_7 & -\mathcal{O}_{71} & \mathcal{O}_{711} \\ \mathcal{O}_3 & -\mathcal{O}_{31} & \mathcal{O}_{311} \\ \mathcal{O}_0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -\mathcal{O}_{(6|1)} & \mathcal{O}_{(6|2)} \\ -\mathcal{O}_{(2|1)} & \mathcal{O}_{(2|2)} \end{vmatrix}\,.$$

$$Sp_{742} = \begin{vmatrix} s_7 & s_8 + s_6 & s_9 + s_5 \\ s_3 & s_4 + s_2 & s_5 + s_1 \\ 1 & s_1 & s_2 \end{vmatrix} \begin{vmatrix} 1 & -s_1 & s_{11} \\ 0 & 1 & -s_1 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} Sp_7 & -Sp_{71} & Sp_{711} \\ Sp_3 & -Sp_{31} & Sp_{311} \\ Sp_0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -Sp_{(6|1)} & Sp_{(6|2)} \\ -Sp_{(2|1)} & Sp_{(2|2)} \end{vmatrix}\,.$$

In λ-ring notation, Littlewood's definitions (9.4.1), (9.4.2) read

$$\sigma_1(\mathbf{x}\mathbf{y}-S^2(\mathbf{x})) = \sum s_\lambda(\mathbf{x})\mathcal{O}_\lambda(\mathbf{y}) \tag{9.4.14}$$

$$\sigma_1(\mathbf{x}\mathbf{y}-\Lambda^2(\mathbf{x})) = \sum s_\lambda(\mathbf{x})Sp_\lambda(\mathbf{y})\,. \tag{9.4.15}$$

Changing $\mathbf{y}\to\mathbf{y}_r$, taking a finite alphabet $\mathbf{x}_n$, these equations become[2]

$$\sigma_1(-\mathbf{x}_n\mathbf{y}_r-S^2(\mathbf{x}_n)) = \prod_{i=1}^{n}\prod_{j=1}^{r}(1-x_iy_j)\prod_{i\le j\le n}(1-x_ix_j)$$
$$= \sum(-1)^{|\lambda|}s_\lambda(\mathbf{x}_n)Sp_{\lambda^\sim}(\mathbf{y}_r) \tag{9.4.16}$$

$$\sigma_1(-\mathbf{x}\mathbf{y}-\Lambda^2(\mathbf{x})) = \prod_{i=1}^{n}\prod_{j=1}^{r}(1-x_iy_j)\prod_{i<j\le n}(1-x_ix_j)$$
$$= \sum(-1)^{|\lambda|}s_\lambda(\mathbf{x})\mathcal{O}_{\lambda^\sim}(\mathbf{y})\,. \tag{9.4.17}$$

Changing $\mathbf{x}\to\mathbf{x}^\vee$, multiplying by the appropriate power of $x_1\ldots x_n$, one obtains

$$\prod_{i=1}^{n}\prod_{j=1}^{r}(x_i-y_j)\prod_{i\le j\le n}(1-x_i^{-1}x_j^{-1})$$
$$= (x_1\ldots x_n)^{-n-1}\sum_{\lambda\subseteq(r+n+1)^n}(-1)^{|\lambda|}s_{(r+n+1)^n/\lambda}(\mathbf{x}_n)Sp_{\lambda^\sim}(\mathbf{y}_r) \tag{9.4.18}$$

$$\prod_{i=1}^{n}\prod_{j=1}^{r}(x_i-y_j)\prod_{i<j\le n}(1-x_i^{-1}x_j^{-1})$$
$$= (x_1\ldots x_n)^{-n+1}\sum_{\lambda\subseteq(r+n-1)^n}(-1)^{|\lambda|}s_{(r+n)^n/\lambda}(\mathbf{x}_n)\mathcal{O}_{\lambda^\sim}(\mathbf{y}_r)\,. \tag{9.4.19}$$

[2] Cauchy formula for the expansion of $\sigma_1(-\mathbf{x}_n\mathbf{y}_r)$ involves only the partitions $\lambda\subseteq r^n$. In the present case, since orthogonal and symplectic Schur functions $\mathcal{O}_\mu(\mathbf{y}_r)$, $Sp_\mu(\mathbf{y}_r)$ do not necessarily vanish for $\ell(\mu)>r$, one has extra terms. For example, for $n=2$, $r=1$, putting $s_\lambda=s_\lambda(x_1,x_2)$, $\mathcal{O}_\lambda=\mathcal{O}_\lambda(y_1)$, one has

$$(1-x_1y_1)(1-x_2y_1)(1-x_1x_2) = 1-s_1\mathcal{O}_1+s_2\mathcal{O}_{11}+s_{11}\mathcal{O}_2-s_{21}\mathcal{O}_{21}+s_{22}\mathcal{O}_{22}$$
$$= 1-s_1y_1+s_20+s_{11}(y_1^2-1)-s_{21}(-y_1)+s_{22}(-y_1^2)\,.$$

One can eliminate the terms $Sp_\mu(\mathbf{y}_{r)}$, $\mathcal{O}_\mu(\mathbf{y}_r)$ such that $\ell(\mu) > r$. Indeed, for $\lambda \subseteq (r{+}\alpha)^n$, one has $r \geq r - \lambda_1 \geq -\alpha$ and

$$(x_1 \dots x_n)^{-\alpha} s_{(r+\alpha)^n}(\mathbf{x}_n) = x^{r-\lambda_n,\dots,r-\lambda_1}\pi_{n\dots 1}\,.$$

Symmetrizing under $\pi_{N\dots 1}$, with $N \geq n{+}\alpha$, annihilates all $x^{r-\lambda_n,\dots,r-\lambda_1}$ with $r{-}\lambda_1 < 0$. In final, rewriting $\prod_{i=1}^n \prod_{j=1}^r (x_i{-}y_j) = Y_{r^n}(\mathbf{x}, \mathbf{y})$, one obtains :

Lemma 9.4.2.

$$Y_{r^n}(\mathbf{x}, \mathbf{y}) \prod_{i\leq j\leq n} (1{-}x_i^{-1}x_j^{-1})\, \pi_{N\dots 1} = \sum_{\lambda\subseteq r^n} (-1)^{|\lambda|} s_{r^n/\lambda}(\mathbf{x}_n) Sp_{\lambda^\sim}(\mathbf{y}_r) \quad (9.4.20)$$

$$Y_{r^n}(\mathbf{x}, \mathbf{y}) \prod_{i< j\leq n} (1{-}x_i^{-1}x_j^{-1})\, \pi_{N\dots 1} = \sum_{\lambda\subseteq r^n} (-1)^{|\lambda|} s_{r^n/\lambda}(\mathbf{x}_n) \mathcal{O}_{\lambda^\sim}(\mathbf{y}_r)\,, \quad (9.4.21)$$

with $N \geq 2n{+}1$, in the first case, $N \geq 2n{-}1$ in the second.

9.5 Maximal key polynomials

The image of each operator $\pi^{\heartsuit}_{w_0}$ is the space of polynomials invariant under the action of the group of type $\heartsuit$. In particular, the images of dominant monomials are given by the Weyl character formula (1.12.1). In our present notations, they are the functions, corresponding to all partitions $\lambda \in \mathbb{N}^n$, $K_{\lambda\omega}(\mathbf{x})$ in type A, $K^{\heartsuit}_{-\lambda}(\mathbf{x})$ in the other types, adding also the functions $K_{-\lambda_1,\ldots,-\lambda_{n-1},\lambda_n}$ in type D.

We shall recognize in these functions the symplectic and orthogonal Schur functions introduced in the preceding section by symmetrizing the kernels $\Omega^{\heartsuit}$ using the operator π_ω (acting only on $\mathbf{x}$).

Indeed, all $K_v(\mathbf{x})$ are sent to 0, except in the case $v = \lambda$ dominant. The RHS of (9.3.3, 9.3.4, 9.3.5) become

$$\sum_\lambda K_{\lambda\omega}(\mathbf{x})\, K^{\heartsuit}_{-\lambda}(\mathbf{y}) \,.$$

On the other hand, all left-hand sides are symmetric functions multiplied $\prod_{i=1}^n \prod_{j=i}^n (1 - x_i/y_j)^{-1}$ in type B, C, and $\prod_{i=1}^{n-1} \prod_{j=i}^n (1 - x_i/y_j)^{-1}$ in type D. Writing

$$\prod_{i=1}^{n} \prod_{j=i}^{n} \frac{1}{1 - x_i/y_j} = \frac{\prod_{i=1..n} \prod_{j<i} 1 - x_i/y_j}{\prod_{i=1...n} \prod_{j=1..n} 1 - x_i/y_j} \,,$$

one has a denominator which is symmetrical, and a numerator which is a sum of monomials x^v, with $v \le [0, 1, \ldots, n-1]$. These monomials are annihilated by π_ω, except $x^{0\ldots0}$, and therefore the image of the numerator is 1. The same property is true for type D, and in final one obtains the following identities (still with $x_n = 0$ in the last equation) :

$$\widetilde{\Omega}^B = \frac{\prod_{1\le i<j\le n}(1 - x_i x_j) \prod_{i=1}^n (1 + x_i)}{\prod_{i,j=1}^n (1 - x_i y_j)(1 - x_i/y_j)} = \sum_\lambda s_\lambda(\mathbf{x})\, K^B_{-\lambda}(\mathbf{y}), \tag{9.5.1}$$

$$\widetilde{\Omega}^C = \frac{\prod_{1\le i<j\le n}(1 - x_i x_j)}{\prod_{i,j=1}^n (1 - x_i y_j)(1 - x_i/y_j)} = \sum_\lambda s_\lambda(\mathbf{x})\, K^C_{-\lambda}(\mathbf{y}), \tag{9.5.2}$$

$$\widetilde{\Omega}^D = \frac{\prod_{1\le i\le j\le n}(1 - x_i x_j)}{\prod_{i,j=1}^n (1 - x_i y_j)(1 - x_i/y_j)} = \sum_{\lambda:\, \lambda_n=0} s_\lambda(\mathbf{x})\, K^D_{-\lambda}(\mathbf{y}) \,. \tag{9.5.3}$$

Comparing with Littlewood's generating functions (9.4.1) and (9.4.2), putting $\mathbf{y}^+ = \{y_1, \ldots, y_n, y_n^{-1}, \ldots, y_1^{-1}\}$, one obtains

$$K^B_{-\lambda}(\mathbf{y}) = \mathcal{O}_\lambda(\mathbf{y}^+ {+} 1) \;\&\; K^C_{-\lambda}(\mathbf{y}) = Sp_\lambda(\mathbf{y}^+) \;\&\; K^D_{-\lambda}(\mathbf{y}) = \mathcal{O}_\lambda(\mathbf{y}^+) \,, \tag{9.5.4}$$

with $\lambda_n = 0$ in type D.

In particular, when $\lambda_n = 0$, one passes from type D to type B by "adding 1 to the alphabet". As in the theory of Schur functions, this means enumerating all partitions differing by an *horizontal strip* from a given one. Instead of using (9.5.4), let us have recourse to Weyl's determinants to check the following property.

Lemma 9.5.1. *Let $\lambda \in \mathbb{N}^n$ be a partition such that $\lambda_n = 0$. Then*

$$K^B_{-\lambda}(\mathbf{x}) = \sum_{\mu:\, \lambda/\mu\, horizontal} K^D_{-\mu}(\mathbf{x}) \tag{9.5.5}$$

$$K^D_{-\lambda}(\mathbf{x}) = \sum_{\mu:\, \lambda/\mu\, vertical} (-1)^{|\lambda/\mu|} K^B_{-\mu}(\mathbf{x}) \tag{9.5.6}$$

Proof. We use Weyl's determinants (1.12.4, 1.12.6). Let us show the proof on an example, taking $\lambda = [5,2,0]$. Weyl determinant for type B is then

$$\Big| x^{7+1/2} - x^{-7-1/2} \ ,\ x^{3+1/2} - x^{-3-1/2} \ ,\ x^{1/2} - x^{-1/2} \Big|_{x=x_1,x_2,x_3} .$$

Dividing each row by $(x^{1/2} - x^{-1/2})$, one obtains

$$\Big| x^7 + x^6 + \cdots + x^{-7} \ ,\ x^3 + \cdots + x^{-3} \ ,\ 1 \Big| .$$

Subtracting each column to the preceding one transforms the determinant into

$$\Big| (x^7{+}x^{-7}) + \cdots (x^4{+}x^{-4}) \ ,\ (x^3{+}x^{-3}) + (x^2{+}x^{-2}) + (x^1{+}x^{-1}) \ ,\ 1 \Big| .$$

The factor $\prod_x (\sqrt{x} - \sqrt{x})$ being equal to Δ^B/Δ^D, one reads from the preceding determinant that

$$K^B_{-5,-2,0}(\mathbf{x}) = \sum_\mu K^D_{-\mu}(\mathbf{x}) ,$$

sum over all partitions μ such that $5 \geq \mu_1 \geq 2 \geq \mu_2 \geq 0 \geq \mu_3 \geq 0$, which is just another way of describing horizontal strips. The second formula results formally from the first one. QED

In detail, one has

$$\begin{aligned} K^B_{-5,-2,0} = {} & K^D_{-5,-2,0} + K^D_{-5,-1,0} + K^D_{-4,-2,0} + K^D_{-5,0,0} + K^D_{-3,-2,0} + K^D_{-4,-1,0} \\ & + K^D_{-3,-1,0} + K^D_{-2,-2,0} + K^D_{-4,0,0} + K^D_{-3,0,0} + K^D_{-2,-1,0} + K^D_{-2,0,0} , \end{aligned}$$

$$K^D_{-5,-2,0} = K^B_{-5,-2,0} - K^B_{-5,-1,0} - K^B_{-4,-2,0} + K^B_{-4,-1,0} .$$

It is easy to extend (9.5.5) and (9.5.6) to the case of partitions with last part $\neq 0$. Indeed, (1.10.6) shows that

$$\begin{aligned} x^\lambda (1 + s^C_n)\, \pi^D_{w_0} = {} & x^\lambda (1 + s^C_n) x^{n-1,\ldots,0} \frac{1}{2} (1 + s^C_1) \ldots (1 + s^C_n)\, \partial^\bullet_\omega \\ & x^{\lambda + [n-1,\ldots,0]} (1 + s^C_1) \ldots (1 + s^C_n)\, \partial^\bullet_\omega \\ & = \begin{vmatrix} x_1^{\lambda_1+n-1} + x_1^{-\lambda_1-n+1} & \cdots & x_1^{\lambda_n} + x_1^{-\lambda_n} \\ \vdots & & \vdots \\ x_n^{\lambda_1+n-1} + x_n^{-\lambda_1-n+1} & \cdots & x_n^{\lambda_n} + x_n^{-\lambda_n} \end{vmatrix} \frac{1}{\Delta(x^\bullet)} . \end{aligned}$$

Hence, Weyl's determinant describes the sum $K^D_{-\lambda} + K^D_{-\lambda s^C_n}$, and the preceding computation remains valid, at the cost of replacing $K^D_{-\lambda}$ by

$$\widetilde{K}^D_{-\lambda} = K^D_{-\lambda} + K^D_{-\lambda\, s^C_n} \ \text{ if } \lambda_n \neq 0 \quad , \quad \widetilde{K}^D_{-\lambda} = K^D_{-\lambda} \text{ otherwise } .$$

In final, one has

$$K^B_{-\lambda}(\mathbf{x}) = \sum_{\mu:\, \lambda/\mu\, horizontal} \widetilde{K}^D_{-\mu}(\mathbf{x}) \tag{9.5.7}$$

$$\widetilde{K}^D_{-\lambda}(\mathbf{x}) = \sum_{\mu:\, \lambda/\mu\, vertical} (-1)^{|\lambda/\mu|} K^B_{-\mu}(\mathbf{x}) \tag{9.5.8}$$

For example,

$$K^B_{-2,-2,-1} = \left(K^D_{-2,-2,-1} + K^D_{-2,-1,1}\right) + K^D_{-2,-2,0} + \left(K^D_{-2,-1,-1} + K^D_{-2,-1,1}\right) + K^D_{-2,-1,0}\,.$$

Notice that the determinantal expression of $\widetilde{K}^D_{-\lambda}$ shows that formula (9.5.4) extends to all partitions, without the restriction $\lambda_n = 0$:

$$\widetilde{K}^D_{-\lambda}(\mathbf{x}) = \mathcal{O}_\lambda(\mathbf{x}^+)\,. \tag{9.5.9}$$

Thus,

$$\widetilde{K}^D_{-4,-2}(\mathbf{x}) = K^D_{-4,-2}(\mathbf{x}) + K^D_{-4,2}(\mathbf{x}) \\ = \mathcal{O}_{42}(\mathbf{x}^+) = s_{42}(\mathbf{x}^+) - s_{42/2}(\mathbf{x}^+) + s_{42/31}(\mathbf{x}^+)\,.$$

Type D is also related to type C as shows the next lemma.

Lemma 9.5.2. *Let $\lambda \in \mathbb{N}^n$ be a partition such that $\lambda_n \neq 0$. Then*

$$K^D_{-\lambda}(\mathbf{x}) - K^D_{-\lambda\, s^C_n}(\mathbf{x}) = K^C_{-\lambda+1^n} \prod_{i=1\ldots n} \left(x_i^{-1} - x_i\right)\,. \tag{9.5.10}$$

Proof. The left-hand side is the image of $(-1)^n x^{\lambda_1\ldots\lambda_{n-1}}(x_n^{\lambda_n} - x_n^{-\lambda_n}) = (-1)^n x^\lambda (1 - s^C_n)$ under $\pi^D_{w_0}$. According to (1.10.6),

$$(1 - s^C_n)\pi^D_{w_0} = x^{n-1,\ldots,0}(1 - s^C_1)\ldots(1 - s^C_n)\partial^\bullet_\omega\,,$$

and therefore

$$\begin{aligned} (-1)^n x^\lambda (1 - s^C_n)\pi^D_{w_0} &= (-1)^n x^{\lambda-1^n+[n-1,\ldots,0]}\pi^C_1(x_1 - \frac{1}{x_1})\ldots\pi^C_n(x_n - \frac{1}{x_n})\partial^\bullet_\omega \\ &= x^{\lambda-1^n+[n-1,\ldots,0]}\pi^C_1\ldots\pi^C_n(\frac{1}{x_1} - x_1)\ldots(\frac{1}{x_n} - x_n)\partial^\bullet_\omega \\ &= x^{\lambda-1^n+[n-1,\ldots,0]}\pi^C_1\ldots\pi^C_n\partial^\bullet_\omega(\frac{1}{x_1} - x_1)\ldots(\frac{1}{x_n} - x_n) \\ &= x^{\lambda-1^n}\,\pi^C_{w_0}\prod(x_i^{-1} - x_i)\,. \end{aligned}$$

QED

The two characters $K^D_{-\lambda}(\mathbf{x})$ and $K^D_{-\lambda\, s^C_n}(\mathbf{x})$ come by pair. The following lemma shows that they are in fact exchanged by any s^C_i.

Lemma 9.5.3. *Let λ be a partition in $\mathbb{N}^n$. Then for any $i = 1, \ldots, n$, one has*

$$K^D_{-\lambda}(\mathbf{x})\, s^C_i = K^D_{-\lambda\, s^C_n}(\mathbf{x})\,. \tag{9.5.11}$$

Proof. Given a constant ϵ, $(1+\epsilon s^C_1) \ldots (1+\epsilon s^C_n)$ commutes with $\partial^\bullet_\omega = \left(\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)}\sigma\right)/\Delta(\mathbf{x}^\bullet)$, because it is symmetrical in $x_1, \ldots, x_n$, and because $\Delta(\mathbf{x}^\bullet)$ is invariant under any s^C_i. Hence, using the expression (1.10.6) of $\pi^D_{w_0}$, one has

$$\begin{aligned} 2 s^C_n \pi^D_{w_0} s^C_i &= \Big((1+s^C_1)\ldots(1+s^C_n) - (1-s^C_1)\ldots(1-s^C_n)\Big)\partial^\bullet_\omega\, s^C_i \\ &= \partial^\bullet_\omega\Big((1+s^C_1)\ldots(1+s^C_n) - (1-s^C_1)\ldots(1-s^C_n)\Big)s^C_i \\ &= \Big((1+s^C_1)\ldots(1+s^C_n) + (1-s^C_1)\ldots(1-s^C_n)\Big)\partial^\bullet_\omega \end{aligned}$$

and therefore $s^C_n \pi^D_{w_0} s^C_i = \pi^D_{w_0}$. QED

For example, $K^D_{-2,-1}(\mathbf{x}) = \left(x_2 + \frac{1}{x_1}\right)\left(1 + \frac{x_2}{x_1}\right)\left(x_1^2 + \frac{1}{x_2^2}\right)$ and $K^D_{-2,1}(\mathbf{x}) = \left(\frac{1}{x_2} + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_1 x_2}\right)(x_1^2 + x_2^2)$.

A direct corollary of the expression given in Proposition (1.10.3) of $\pi^C_{w_0}$ by type A-divided differences is the following description of symplectic Schur functions in terms of type A-key polynomials. Indeed, the operator π_ζ appearing in (1.10.3) acts only by reordering the index of $x^\lambda = K_\lambda$.

Proposition 9.5.4. *Let $\lambda \in \mathbb{N}^n$ be a partition, $v = [0, \lambda_n, 0, \lambda_{n-1}, \ldots, 0, \lambda_1]$. Then*

$$Sp_\lambda(\mathbf{x}) = K^C_{-\lambda}(\mathbf{x}) = K_v\Big|_{\mathbf{x}\to\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots\}}\,. \tag{9.5.12}$$

What is remarkable in this formula is that the RHS uses the alphabet $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots\}$ (which corresponds to considering the hyperoctaedral group as a wreath product), while the LHS is generated using the order $x_1, x_2, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}$. The general key polynomials K^C_v, for $v \neq \lambda$ cannot be related to the type A key polynomials in the alphabet $\{x_1, x_1^{-1}, \ldots\}$.

Since key polynomials have a combinatorial interpretation in terms of tableaux, symplectic Schur functions inherits from the above proposition such a combinatorial description, evaluating i in x_i and $\bar{\imath}$ in x_i^{-1}.

Corollary 9.5.5. *Let $\lambda \in \mathbb{N}^n$ be a partition. Then $K^C_{-\lambda}(\mathbf{x})$ is the sum of all contretableaux of shape λ on the alphabet $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$, such that the letters $i, \bar{\imath}$ can occur only in rows $n, n-1, \ldots, n+1-i$.*

For example, for $\lambda = [5, 3, 2]$, the admissible tableaux are those contretableaux in $\{1, \bar{1}, \ldots\}$ of shape $[5, 3, 2]$ which remain tableaux when concatanating on the right the column $[\bar{3}, \bar{2}, \bar{1}]$. Here is one of them :

$$\begin{array}{|c|c|c|c|c|} \hline 1 & \bar{1} & \bar{1} & 3 & 3 \\ \hline \end{array}\quad \begin{array}{|c|} \hline \bar{3} \\ \hline \end{array}$$
$$\begin{array}{ccccc} & & \boxed{1} & \boxed{\bar{1}} & \boxed{2} \\ & & & \boxed{1} & \boxed{\bar{1}} \end{array}\quad \begin{array}{c} \boxed{\bar{2}} \\ \boxed{\bar{1}} \end{array}\quad , \quad \text{evaluation } x_1^{-1} x_2 x_3^2\,.$$

For $\lambda = [1,1,0]$, there are 14 tableaux of shape $[1,1]$ on the alphabet $\{1,\overline{1},2,\overline{2},3,\overline{3}\}$, without 3 in the bottom row : $\begin{array}{|c|}\hline \overline{3}\\ \hline 1\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{3}\\ \hline \overline{1}\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{3}\\ \hline \overline{2}\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{3}\\ \hline 2\\ \hline\end{array} + \begin{array}{|c|}\hline 3\\ \hline \overline{2}\\ \hline\end{array} + \begin{array}{|c|}\hline 3\\ \hline \overline{1}\\ \hline\end{array} + \begin{array}{|c|}\hline 3\\ \hline 1\\ \hline\end{array} + \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array} +$ $\begin{array}{|c|}\hline \overline{2}\\ \hline 2\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{2}\\ \hline 1\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{2}\\ \hline \overline{1}\\ \hline\end{array} + \begin{array}{|c|}\hline 2\\ \hline \overline{1}\\ \hline\end{array} + \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} + \begin{array}{|c|}\hline \overline{1}\\ \hline 1\\ \hline\end{array}$ and the sum of these tableaux evaluates into

$$\begin{aligned} K^C_{-1,-1,0} = Sp_{11}(\mathbf{x}_3) = x_1\,x_2 + \frac{x_2}{x_3} + x_2\,x_3 + \frac{x_2}{x_1} + \frac{1}{x_1\,x_3} + \frac{x_1}{x_3} + 2 + x_1\,x_3 \\ + \frac{x_3}{x_1} + \frac{x_1}{x_2} + \frac{1}{x_2\,x_3} + \frac{x_3}{x_2} + \frac{1}{x_1\,x_2}\,. \end{aligned}$$

9.6 Symmetrizing further

One can easily increase the number of variables in a Schur function by using symmetrizing operators. Indeed, for $\lambda \in \mathbb{N}^r$, $r \le n$, $\omega = [n, \ldots, 1]$, one has $s_\lambda(\mathbf{x}_r)\pi_\omega = s_\lambda(\mathbf{x}_n)$, since $\pi_{[r\ldots 1]}\pi_\omega = \pi_\omega$.

Because of the orientation of the Dynkin graph that we have chosen in types B, C, D, the symmetrization of symplectic and orthogonal Schur functions is not as straightforward.

For example, for $n = 3$, one has

$$K^C_{-4,-2}\,\pi^C_{w_0} = K^C_{-4,-2,0} + K^C_{-4,0,0} + K^C_{-3,-1,0} + K^C_{-2,-2,0} + K^C_{-2,0,0}\,,$$

$$K^B_{-4,-2}\,\pi^B_{w_0} = K^B_{-4,-2,0} + K^B_{-4,-1,0} + K^B_{-4,0,0} + K^B_{-3,-2,0} \\ + 2\,K^B_{-3,-1,0} + K^B_{-3,0,0} + K^B_{-2,-2,0} + K^B_{-2,-1,0} + K^B_{-2,0,0}\,.$$

To describe these symmetrizations, we need the following values, which are a direct consequence of (1.10.1), (1.10.3), (1.10.6).

Lemma 9.6.1. *Given n, and i : $-n{+}1 \le i \le n{-}1$, $i \ne 0$, then for the groups of type $\heartsuit = B_n, C_n, D_n$ one has the vanishing of all $x_n^i\,\pi^\heartsuit_{w_0}$, except*

$$x_n^{-1}\pi^B_{w_0} = -1 = x_n^{-2}\pi^C_{w_0}\,.$$

Passing from $\mathbf{x}_{n-1}$ to $\mathbf{x}_n$ and keeping the same set $\mathbf{y}_{n-1}$, corresponds, for what concerns Littlewood's generating functions[3], to division by

$$\Xi = \prod_{i=1}^{n-1}(1 - y_i x_n)(1 - y_i x_n^{-1}) = \sigma_1\left(-(x_n{+}x_n^{-1})\mathbf{y}_{n-1}\right)\,.$$

Therefore

$$\sum_\lambda s_\lambda(\mathbf{y}_{n-1})K^\heartsuit_{-\lambda}\,\pi^\heartsuit_{w_0} = \left(\Xi\,\pi^\heartsuit_{w_0}\right)\left(\sum_\lambda s_\lambda(\mathbf{y}_{n-1})K^\heartsuit_{-\lambda,0}\right)\,, \tag{9.6.1}$$

sum over all partitions in $\mathbb{N}^{n-1}$ (with $\lambda_{n-1} = 0$ in type $\heartsuit = D$).

To evaluate the factor $\Xi\,\pi^\heartsuit_{w_0}$, we need, according to the preceding lemma, to extract the terms x_n^0, x_n^{-1} and x_n^{-2} of

$$\Xi = \sum_{i,j}(\text{-}1)^j s_{i,i}(x_n{+}x_n^{-1})s_j(x_n{+}x_n^{-1})\,s_{2^i1^j}(\mathbf{y}_{n-1})\,.$$

Thus, the constant term of Ξ is equal to $\sum_{i,j:\,i+2j\le n-1} s_{2^i1^{2j}}(\mathbf{y}_{n-1})$, the coefficient of x_n^{-2} is equal to $\sum_{i,j:\,i+2j\le n-3} s_{2^i1^{2j+2}}(\mathbf{y}_{n-1})$, while the coefficient of x_n^{-1} is $\sum_{0\subseteq\lambda\subseteq 2^{n-1}} s_\lambda(\mathbf{y}_{n-1})$.

Using the scalar product on symmetric functions of $\mathbf{y}_{n-1}$, one transforms as in (9.4.10), (9.4.11), multiplication by its adjoint operation. In final, the preceding computations give the looked for images of $K^\heartsuit_{-\lambda}$ under $\pi^\heartsuit_{w_0}$:

[3] We have exchanged the role of $\mathbf{x}$ and $\mathbf{y}$ compared to (9.5.1), (9.5.2), (9.5.3).

Proposition 9.6.2. *Let n be an integer ≥ 3, $\lambda \in \mathbb{N}^n$ be a partition (with last part $\lambda_{n-1} = 0$ in type D). Then*

$$\mathcal{O}_\lambda(\mathbf{x}^B_{n-1})\, \pi^B_{w_0} = \sum\nolimits_{\mu \subseteq 2^{n-1}} \mathcal{O}_{\lambda/\mu}(\mathbf{x}^B_n) \tag{9.6.2}$$

$$Sp_\lambda(\mathbf{x}^C_{n-1})\, \pi^C_{w_0} = \sum\nolimits_k Sp_{\lambda/2^k}(\mathbf{x}^C_n) \tag{9.6.3}$$

$$\mathcal{O}_\lambda(\mathbf{x}^D_{n-1})\, \pi^D_{w_0} = \sum\nolimits_{\mu \subseteq 2^{n-2},\, |\mu|\, even} \mathcal{O}_{\lambda/\mu}(\mathbf{x}^D_n)\,, \tag{9.6.4}$$

writing $\mathbf{x}^B_n = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1\}$ *and* $\mathbf{x}^C_n = \mathbf{x}^D_n = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$.

The examples on which we started the section may be rewritten

$$\mathcal{O}_{42}(\mathbf{x}^B_2)\pi^B_{w_0} = \mathcal{O}_{42}(\mathbf{x}^B_3) + \mathcal{O}_{42/1}(\mathbf{x}^B_3) + \mathcal{O}_{42/2}(\mathbf{x}^B_3) + \mathcal{O}_{42/11}(\mathbf{x}^B_3) \\ + \mathcal{O}_{42/21}(\mathbf{x}^B_3) + \mathcal{O}_{42/22}(\mathbf{x}^B_3)\,,$$

$$Sp_{42}(\mathbf{x}^C_2)\pi^C_{w_0} = Sp_{42}(\mathbf{x}^C_3) + Sp_{42/2}(\mathbf{x}^C_3) + Sp_{42/22}(\mathbf{x}^C_3)\,,$$

and we complete them, for $n = 4$, by

$$K^D_{-4,-1,0}\pi^D_{w_0} = \mathcal{O}_{42}(\mathbf{x}^D_3)\pi^D_{w_0} = K^D_{-4,-2,0,0} + K^D_{-4,0,0,0} + 2\, K^D_{-3,-1,0,0} \\ + K^D_{-2,-2,0,0} + K^D_{-2,0,0,0} \\ = \mathcal{O}_{42}(\mathbf{x}^D_4) + \mathcal{O}_{42/2}(\mathbf{x}^D_4) + \mathcal{O}_{42/11}(\mathbf{x}^D_4) + \mathcal{O}_{42/22}(\mathbf{x}^D_4)\,.$$

9.7 Finite symplectic Cauchy identity

In the case of symmetric functions, the two Cauchy identities relative to the expansion of the resultant $R(\mathbf{x}_n, \mathbf{y}_m)$ or of $\prod_{i,j}(1-x_iy_j)^{-1}$ are equivalent. This was no more the case in the non-symmetric case. For example, in type A the finite form involves Schubert polynomials (2.10.1), while the other involves Demazure characters (2.15.2).

Hasegawa [62] has given a symplectic generalization of the expansion of the resultant.

Theorem 9.7.1 (Hasegawa). *Let n, m be two positive integers. Then*

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} - y_j - y_j^{-1}) = \sum_{\lambda\subseteq n^m} (-1)^{|\lambda|} Sp_{m^n/\lambda^\sim}(\mathbf{x}_n) Sp_\lambda(\mathbf{y}_m) . \qquad (9.7.1)$$

Proof. Since $x_i^k \pi_i^C = x_i^k + x_i^{k-2} + \cdots + x_i^{-k}$, the image of the Vandermonde in $\mathbf{x}_n \cup \mathbf{y}_m$ by $\pi_1^{xC} \ldots \pi_n^{xC} \pi_1^{yC} \ldots \pi_m^{yC}$ is equal to the Weyl determinant of type D:

$$\begin{vmatrix} 1 & x_i{+}x_i^{-1} & x_i^2{+}x_i^{-2} & \ldots & x_i^{n+m-1}{+}x_i^{-n-m+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_j{+}y_j^{-1} & y_j^2{+}y_j^{-2} & \ldots & y_j^{n+m-1}{+}y_j^{-n-m+1} \end{vmatrix}_{i=1\ldots n,\, j=1\ldots m} = \Delta\left(\mathbf{x}_n^\bullet \cup \mathbf{y}_m^\bullet\right) .$$

However,

$$\Delta(\mathbf{x}_n) s_\lambda(\mathbf{x}_n) \pi_1^{xC} \ldots \pi_n^{xC} = x^{\lambda+\rho} \pi_1^{xC} \ldots \pi_n^{xC} \partial_\omega^x = Sp_\lambda(\mathbf{x}_n) \Delta(\mathbf{x}_n^\bullet) \Delta(\mathbf{x}_n)^{-1} ,$$

and therefore the image of

$$\Delta(\mathbf{x}_n{\cup}\mathbf{y}_m) = \Delta(\mathbf{x}_n)\Delta(\mathbf{y}_m) R(\mathbf{x}_n, \mathbf{y}_m) = \Delta(\mathbf{x}_n)\Delta(\mathbf{y}_m) \sum_{\lambda\subseteq n^m} (-1)^{|\lambda|} s_{m^n/\lambda^\sim}(\mathbf{x}_n) s_\lambda(\mathbf{y}_m)$$

under the product of π_i^C gives the required identity

$$\Delta\left(\mathbf{x}_n^\bullet \cup \mathbf{y}_m^\bullet\right) \Delta(\mathbf{x}_n^\bullet)^{-1} \Delta(\mathbf{y}_m^\bullet)^{-1} = \sum_{\lambda\subseteq n^m} (-1)^{|\lambda|} Sp_{m^n/\lambda^\sim}(\mathbf{x}_n) Sp_\lambda(\mathbf{y}_m) .$$

Hamel and King [61] give a bijective proof of this identity. There is no known analog for the orthogonal Schur functions.

9.8 Rectangles and sums of Schur functions

As in the case of Schur functions, the characters indexed by rectangles (i.e. partitions with all parts equal) play a special role. Instead of type D, we introduce two formal types D^+, D^- such that

$$\pi_i^{D^+} = 1 + s_i^C \quad \& \quad \pi_i^{D^-} = 1 - s_i^C \ , \ i = 1, \ldots, n \, ,$$

and extend the expression of $\pi_{w_0}^B$ and $\pi_{w_0}^C$ to all types $B, C, D^\pm$, taking $\rho = [n{-}1, \ldots, 0]$:

$$\pi_{w_0}^\heartsuit = x^\rho \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, \partial_\omega^\bullet = x^\rho \partial_\omega^\bullet \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, , \tag{9.8.1}$$

Lemma 9.8.1. *For any $r \geq 0$, any type $\heartsuit = B, C, D^\pm$, one has*

$$K_{(-r)^n}^\heartsuit = x^{r\ldots r} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i^{-1} x_j^{-1}} \, \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, . \tag{9.8.2}$$

Proof. The function is equal to

$$x^{r^n+\rho} \partial_\omega^\bullet \, \pi_1^\heartsuit \ldots \pi_n^\heartsuit = x^{r^n+\rho} \partial_\omega \prod_{i<j} \frac{1}{1 - x_i^{-1} x_j^{-1}} \, \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, ,$$

and $x^{r^n+\rho}\partial_\omega = x^{r^n}$ allows to conclude. QED

The following proposition shows that the functions $K_{(-r)^n}^\heartsuit$ can be obtained as determinants of functions for $n = 2$.

Proposition 9.8.2. *Let $n = 2m$ be an even integer, $r \geq m{-}1$. Then for $\heartsuit = B, C, D^\pm$, one has*

$$\left| K_{-r,-r}^\heartsuit(x_i, x_j) \right|_{\substack{i=1\ldots m \\ j=m+1\ldots n}} \frac{1}{\Delta(x_1^\bullet, \ldots, x_m^\bullet)\Delta(x_{m+1}^\bullet, \ldots, x_n^\bullet)}$$

$$= K_{(-r+m-1)^n}^\heartsuit(\mathbf{x}_n) \, . \tag{9.8.3}$$

Proof. The determinant can be written with an alternating sum over $\mathfrak{S}_m$, or with the summation $(m!)^{-1} \sum_{\sigma \in \mathfrak{S}_{m,m}} (\text{-}1)^{\ell(\sigma)} \sigma$ over the Young subgroup $\mathfrak{S}_{m,m}$. Therefore, putting $\omega' = [m, \ldots, 1, n \ldots, m{+}1]$, using (9.8.2) for $n = 2$, one rewrites the left-hand side as

$$K_{-r,-r}^\heartsuit(x_1, x_{m+1}) \ldots K_{-r,-r}^\heartsuit(x_m, x_n) \, \partial_{\omega'}^\bullet (m!)^{-1}$$
$$= x^{r\ldots r} \frac{1}{1 - x_1^{-1} x_{m+1}^{-1}} \cdots \frac{1}{1 - x_m^{-1} x_n^{-1}} \, \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, \partial_{\omega'}^\bullet (m!)^{-1} \, .$$

The divided difference $\partial_{\omega'}^\bullet$ commutes with the product $\pi_1^\heartsuit \ldots \pi_n^\heartsuit$ because each $x_i^\bullet$ does. Thus, the expression becomes

$$x^{r\ldots r} \frac{1}{1 - x_1^{-1} x_{m+1}^{-1}} \cdots \frac{1}{1 - x_m^{-1} x_n^{-1}} \partial_{\omega'} \frac{1}{m!}$$
$$\prod_{i<j \le m} \frac{1}{1 - x_i^{-1} x_j^{-1}} \prod_{m<i<j \le n} \frac{1}{1 - x_i^{-1} x_j^{-1}} \pi_1^\heartsuit \ldots \pi_n^\heartsuit \, \partial_{\omega'}^\bullet \, .$$

Cauchy formula for the determinant $\det\bigl(1/(1-x_i^{-1}x_j^{-1})\bigr)$ allows to compute the action of the divided difference, taking inverse variables $1/x_i$ causing an extra factor $(x_1\dots x_n)^{1-m}$ compared to the usual case. In final the expression becomes equal to

$$(x_1\dots x_n)^{r+1-m}\prod_{1\le i<j\le n}\frac{1}{1-x_i^{-1}x_j^{-1}}\,\pi_1^\heartsuit\dots\pi_n^\heartsuit$$

which is equal, thanks to (9.8.2), to $K^\heartsuit_{(-r+m-1)^n}(\mathbf{x}_n)$. QED

The determinants with entries $K^D_{-r,-r}(x_i,x_j)$ or $K^D_{-r,r}(x_i,x_j)$ can also be described. In fact, the matrices $\bigl[x_i^r x_j^r K^D_{-r,-r}(x_i,x_j)\bigr]_{i=1\dots m,\,j=m+1\dots n}$ and $\bigl[x_i^r x_j^r K^D_{-r,r}(x_i,x_j)\bigr]_{i=1\dots m,\,j=m+1\dots n}$ factorize into

$$\begin{bmatrix}1 & \cdots & x_i^{2r}\end{bmatrix}_{i=1\dots m}\begin{bmatrix}1\\ \vdots\\ x_j^{2r}\end{bmatrix}_{j=m+1\dots n} \quad \& \quad \begin{bmatrix}1 & \cdots & x_i^{2r}\end{bmatrix}_{i=1\dots m}\begin{bmatrix}x_j^{2r}\\ \vdots\\ 1\end{bmatrix}_{j=m+1\dots n}$$

respectively. From this one obtains

$$\det\bigl(K^D_{-r,-r}(x_i,x_j)\bigr)_{i=1\dots m,\,j=m+1\dots n}=\frac{\Delta(x_1,\dots,x_m)\Delta(x_{m+1},\dots,x_n)}{x^{r\dots r}}\times \sum_{\lambda\subseteq(2r-m+1)^m}s_\lambda(x_1,\dots,x_m)s_\lambda(x_{m+1},\dots,x_n) \tag{9.8.4}$$

$$\det\bigl(K^D_{-r,r}(x_i,x_j)\bigr)_{i=1\dots m,\,j=m+1\dots n} = (-1)^{\binom{m}{2}}\frac{\Delta(x_1,\dots,x_m)\Delta(x_{m+1},\dots,x_n)}{x^{r\dots r}}s_{(2r-m+1)^m}(x_1,\dots,x_n)\,. \tag{9.8.5}$$

The determinants (9.8.3) occur in the computation of Pfaffians. According to [110, Th. 4.1], given $n=2m$, given indeterminates $a_1,\dots,a_n$, g_{ij}, $i,j=1..n$, with $g_{ij}=g_{ji}$, then the Pfaffian $\mathfrak{Pfaff}\bigl((a_i-a_j)g_{ij}\bigr)$ is obtained, up to a scalar, as the image under the alternating sum of permutations, acting on a_i and g_{ij} simultaneously, of

$$(a_1-a_{m+1})\dots(a_m-a_n)\ \det\bigl(g_{ij}\bigr)_{i=1\dots m,\,j=m+1\dots n}\,.$$

Taking $g_{ij}=K^\heartsuit(x_i,x_j)$, one has a summation where the symmetric function $K^\heartsuit_{(-r+m-1)^n}(\mathbf{x}_n)$ occurs as a common factor. The initial case is for $r=m-1$ the determinant being equal to $\Delta(x_1^\bullet,\dots,x_m^\bullet)\Delta(x_{m+1}^\bullet,\dots,x_n^\bullet)$. Up to a minor change, this case corresponds to the Pffafian $\mathfrak{Pfaff}\left(\frac{a_i-a_j}{1-x_ix_j}\right)$ considered by Sundquist[189]. Indeed, for $g_{ij}=(1-x_ix_j)^{-1}$, the determinant $\det\bigl(g_{ij}\bigr)_{i=1\dots m,\,j=m+1\dots n}$ is equal, thanks to Cauchy again, to

$$\frac{\Delta(x_1^\bullet,\dots,x_m^\bullet)\Delta(x_{m+1}^\bullet,\dots,x_n^\bullet)}{\prod_{1\le i<j\le n}1-x_ix_j}\,x^{m-1,\dots,m-1}\,.$$

Proposition 9.8.3. *Let $n = 2m$ be an even integer, $r \geq m-1$. Let $\mathfrak{P}_n = \mathfrak{Pfaff}\left(\frac{a_i-a_j}{1-x_ix_j}\right)\prod_{i<j}(1-x_ix_j)\Delta(\mathbf{x}_n)^{-1}$. Then for $\heartsuit = B, C, D^{\pm}$, one has*

$$\mathfrak{Pfaff}\big((a_i-a_j)\, K^{\heartsuit}_{-r,-r}(x_i,x_j)\big) = K^{\heartsuit}_{(-r+m-1)^n}(\mathbf{x}_n)\,\mathfrak{P}_n\,. \tag{9.8.6}$$

Specializing $a_i \to x_i$, $i = 1\ldots n$, and using that $\mathcal{P}_n$ becomes equal[4] to $\Delta(\mathbf{x}_n)\prod_{i<j}(1-x_ix_j)^{-1}$, one obtains

Corollary 9.8.4. *Given $n = 2m$, $r \geq m-1$, one has, for type $\heartsuit = B, C, D^{\pm}$*

$$\mathfrak{Pfaff}\big((x_i-x_j)\, K^{\heartsuit}_{-r,-r}(x_i,x_j)\big) = K^{\heartsuit}_{(-r+m-1)^n}(\mathbf{x}_n)\,\Delta(\mathbf{x}_n)\, x^{1-m,\ldots,1-m}\,. \tag{9.8.7}$$

For example, for $n = 4$, $r = 2$, $\heartsuit = C$, one has

$$K^{C}_{-2,-2}(x_i,x_j) = s_{22}(x_i + x_j + x_i^{-1} + x_j^{-1}) - s_{11}(x_i + x_j + x_i^{-1} + x_j^{-1}) := f(i,j)\,,$$

and

$$\mathfrak{Pfaff}((x_i - x_j)f(i,j))_{1\leq i<j\leq 4} = K^{C}_{-1,-1,-1,-1}(\mathbf{x}_4)\Delta(\mathbf{x}_4)/x^{1111}\,,$$

with $K^{C}_{-1,-1,-1,-1}(\mathbf{x}_4) = 2 + \sum_v x^v$, sum over all exponents: $v_i \in \{0,1,-1\}$, $\sum |v_i| = 2$ or 4.

The elementary functions $K^{\heartsuit}_{-r,-r}$ can be written in terms of Schur functions of x_1, x_2. From Weyl's determinants, one finds that

$$x^{rr}\, K^{B}_{-r,-r} = \sum_{\lambda\subseteq[2r,2r]} s_\lambda\,, \tag{9.8.8}$$

$$x^{rr}\, K^{C}_{-r,-r} = \sum_{\lambda\subseteq[2r,2r],\,\lambda\, even} s_\lambda\,, \tag{9.8.9}$$

$$x^{rr}\, K^{D^+}_{-r,-r} = \sum_{i=0\ldots r} s_{rr} + s_{(2r)}\,. \tag{9.8.10}$$

$$x^{rr}\, K^{D^-}_{-r,-r} = \sum_{i=0\ldots r} s_{rr} - s_{(2r)}\,. \tag{9.8.11}$$

One remarks that the first sum is the sum of all minors of order 2 of the matrix $\big[s_{j-i}(\mathbf{x}_2)\big]_{i\leq 2, j\leq 2r+2}$. This indicates that symplectic and orthogonal characters can be used to describe some sums of Schur functions, as first shown by Macdonald [146, p. 83]. The idea to use Pfaffians is due to Stembridge [185].

Indeed, given a matrix M of order $2m \times N$, with $N > 2m$, then, according to [92, **?**] the sum of all minors of order $2m$ of M is equal to the Pfaffian $\mathfrak{Pfaff}(z_{ij})$, denoting z_{ij} the sum of all minors of order 2 taken on rows i, j.

[4] The value of $\mathfrak{Pfaff}((x_i - x_j)(1 - x_ix_j)^{-1})$ has been obtained by many authors, among which [92, 185] .

Since $x^{rr}(x_1 - x_2)K^B_{-r,-r}(x_1, x_2)$ is a sum of minors, (9.8.7) gives the first statement[5], due to Macdonald [146, p. 83] in the next theorem. More elaborate summations on minors lift the restriction that n be even as in (9.8.7). They also give the second statement, due to Stembridge [185], and the last two which are due to Okada [156] (going back to type D instead of using $D^{\pm}$).

Proposition 9.8.5. *Let n, r be positive integers. Then*

$$K^B_{(-r)^n} = x^{-r,\ldots,-r} \sum_{\lambda \subseteq (2r)^n} s_\lambda(\mathbf{x}_n) \tag{9.8.12}$$

$$K^C_{(-r)^n} = x^{-r,\ldots,-r} \sum_{\lambda \subseteq (2r)^n,\, even} s_\lambda(\mathbf{x}_n) \tag{9.8.13}$$

$$K^D_{(-r)^n} = x^{-r,\ldots,-r} \sum_{\lambda \subseteq (2r)^n,\, even\, cols} s_\lambda(\mathbf{x}_n) \tag{9.8.14}$$

$$K^D_{(-r)^(n-1),r} = x^{-r,\ldots,-r} \sum_{\lambda \subseteq (2r)^(n-1),\, even\, cols} s_{2r,\lambda}(\mathbf{x}_n) \tag{9.8.15}$$

Remark To evaluate the Pfaffians above, we have only used appropriate factorizations of $\pi^{\heartsuit}_{w_0}$. But to pass from these evaluations to sums of Schur functions, we had recourse to theorems on sums of minors. One can bypass this step by following the action on sums of Schur functions of the $\pi^{\heartsuit}_i$ operators.

For example, $K^B_{-2,-2,-2}$ is equal to

$$x^{222}\pi^B_{w_0} = x^{222}(\pi^B_3\pi_2\pi^B_3\pi_2)(\pi_1\pi_2\pi^B_3\pi_2\pi_1) = K^B_{2,-2,-2}(\pi_1\pi_2\pi^B_3\pi_2\pi_1)\,.$$

By induction on n, one knows that $K^B_{2,-2,-2} = x_1^2/x^{022}\sum_{\lambda\subseteq 44} s_\lambda(x_2, x_3)$, and thus

$$K^B_{2,-2,-2}\pi_1\pi_2 = K_{-2,-2,2} = x^{-2,-2,-2}\sum_{\lambda\subseteq 44} s_{4,\lambda}(x_1, x_2, x_3)\,.$$

It remains to show that the image of this sum under $\pi^B_3\pi_2\pi_1$, which is $K^b_{-2,-2,-2}$, is equal to $x^{-2,-2,-2}\sum_{\mu\subseteq 444} s_\mu(x_1, x_2, x_3)$. This is done using (1.11.5), but not totally straightforward since cancellations occur.

[5] Macdonald takes r to be a half-integer, and thus $(2r)^n$ can be any rectangle of width n. We have avoided using square roots of variables to handle only polynomials, but the computation of the Pfaffian is still valid in this more general case. We have also restricted n to be even, but it is well known how to adapt the theory of Paffians to matrices of odd order.

Chapter 10

Macdonald polynomials

There is an abundant literature about Macdonald polynomials, we shall restrict ourselves to properties of the type encountered for Schubert, Grothendieck, key polynomials: recursive generation, multiplication by a single variable, transition formula, Hopf decomposition, etc. In that respect, there is a strong similarity between Schubert polynomials and Macdonald polynomials.

To make connections with literature easier, we specialize the parameters $t_1 \to t$, $t_2 \to -1$, though keeping t_1, t_2 would reveal more symmetry.

10.1 Interpolation Macdonald polynomials

We have at our disposal three bases of $\mathfrak{Pol}(\mathbf{x}_n)$, $\{Y_v,\, v \in \mathbb{N}^n\}$, $\{\widetilde{G}_v,\, v \in \mathbb{N}^n\}$, $\{K_v,\, v \in \mathbb{N}^n\}$, we want to add a fourth one, $\{M_v,\, v \in \mathbb{N}^n\}$, which relates easily with the usual symmetric or nonsymmetric Macdonald polynomials.

This basis, the *interpolation Macdonald polynomials*, has been defined by Sahi and Knop. It can be defined, up to normalization, by the vanishing in certain interpolation points, exactly as Schubert polynomials. These points have coordinates of the type $q^i t^j$, for Schubert we are using points with coordinates a permutation of independent parameters $y_1, y_2, \ldots$.

The underlying group is the affine symmetric group, instead of only the symmetric group. In consequence, though using vectors of n components, it will be convenient to consider these vectors as the n first components of an infinite vector, as we do in [102].

For what concerns the indexing of Macdonald polynomials, $v \in \mathbb{N}^n$ is extended to $v \in \mathbb{N}^\infty$ such that $v_{i+rn} = v_i + r$, $r \in \mathbb{Z}$.

Similarly, we shall use an infinite set of indeterminates $x_i : i \in \mathbb{Z}$, such that $x_{i+rn} = q^r x_i$. Now, apart from the simple transpositions s_i, $0 < i < n$ (which transpose x_{i+rn} and x_{i+1+rn}, resp. v_{i+rn} and v_{i+1+rn}, for all r at the same time), we also have a translation $\tau : x_i \to x_{i+1}$, $v_i \to v_{i+1}$, and its inverse $\bar{\tau} = \tau^{-1}$, that one can also write

$$\begin{array}{rcl}[x_1,\ldots,x_{n-1},x_n] & \xrightarrow{\tau} & [x_2,\ldots,x_n,qx_1]\,,\\ [v_1,\ldots,v_{n-1},v_n] & \xrightarrow{\tau} & [v_2,\ldots,v_n,v_1+1]\,.\end{array}$$

Let moreover $s_0 := \tau s_1 \bar\tau = \bar\tau s_{n-1}\tau$. This is the extra generator such that $\{s_0, s_1, \ldots, s_{n-1}\}$ generates the affine symmetric group.

The interpolation points have also an interpretation as *spectral vectors* (relative to the *Cherednik elements*). We shall keep this last terminology, though not using the Cherednik elements. Given a dominant $\lambda \in \mathbb{N}^n$, the spectral vector $\langle\lambda\rangle$ is $[t^{n-1}q^{\lambda_1},\ldots,t^0q^{\lambda_n}]$. For a general $v \in \mathbb{N}^n$, such that $\lambda = v\downarrow$, and $\sigma \in \mathfrak{S}_n$, of minimal length, such that $v = \lambda\sigma$, one defines $\langle v\rangle = \langle\lambda\rangle\sigma$. When needed, $\langle v\rangle$ can be thought as infinite, by putting $\langle v\rangle_{n+i} = q\,\langle v\rangle_i$.

For example, for $v = [2,0,6,2]$, one has $v = [6,2,2,0]\,s_1s_3s_2$, $\langle 6,2,2,0\rangle = [q^6t^3, q^2t^2, q^2t, 1]$, $\langle 2,0,6,2\rangle = \langle 6,2,2,0\rangle\, s_1s_3s_2 = [q^2t^2, 1, q^6t^3, q^2t]$. Moreover, v must be looked at as the prefix of $[2,0,6,2,\ 3,1,7,3,\ 4,2,8,4,\ldots]$.

We need to generalize the inversions of a permutation. We define recursively $\Bbbk(v)$ by

$$\Bbbk(v\tau) = \Bbbk(v) \quad \& \quad \Bbbk\,(vs_i) = \Bbbk(v)\,(t\gamma-1)(\gamma-1)^{-1}$$

when $v_i < v_{i+1}$, with $\gamma = \langle v\rangle_{i+1}\langle v\rangle_i^{-1}$, starting with $\Bbbk([0\ldots 0]) = 1$.

Contrary to the case of Schubert polynomials, the lexicographic order is no more convenient. We have to combine orders on partitions and on permutations. Recall that the *natural order* on partitions is defined as follows. For $\lambda, \mu \in \mathfrak{Part}$, then $\lambda \geq_{\mathfrak{Part}} \mu$ iff for any $i:\ 1 \leq i \leq n$, $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$.

Given $v \in \mathbb{N}^n$, denote $\lambda(v,i)$ the decreasing reordering of $v_1,\ldots,v_i$. Then, for two permutations u, v of the same element of $\mathbb{N}^n$, $u >_{\mathfrak{S}} v$ iff for any i, $\lambda(u,i) \geq \lambda(v,i)$ *componentwise*.

We can now set : $u > v$ iff

$$|u| > |v| \text{ or } \Big(|u| = |v| \ \&\ \lambda(u,n) >_{\mathfrak{Part}} \lambda(v,n)\Big) \text{ or } \Big(\lambda(u,n) = \lambda(v,n)\ \&\ u >_{\mathfrak{S}} v\Big)\,.$$

For example $[4,0,0] > [0,0,4] > [2,2,0] > [2,0,2] > [1,2,1] > [3,0,0]$ is a chain with respect to this order.

The *leading term* of a polynomial is the restriction of the polynomial to its maximal elements with respect to this order, used by Knop.

Definition 10.1.1. *Given* $v \in \mathbb{N}^n$*, then the* interpolation Macdonald polynomial M_v *is the only polynomial of degree* $|v|$ *such that*

$$M_v(\langle u\rangle) \;=\; 0\,,\ u \neq v,\ |u| \leq |v| \qquad (10.1.1)$$

$$\textit{The leading term is} \qquad x^v q^{-\sum_i \binom{v_i}{2}}\,. \qquad (10.1.2)$$

10.2 Recursive generation of Macdonald polynomials

As for Schubert polynomials, the existence and unicity is proved by extracting from the vanishing conditions a recursion $M_v \to M_{vs_i}$, together with a recursion $M_v \to M_{v\tau}$.

In the first case, one essentially needs to change the conditions
$M_v(\langle vs_i\rangle) = 0 \,\&\, M_v(\langle v\rangle) \neq 0$ into $M_{vs_i}(\langle vs_i\rangle) \neq 0 \,\&\, M_{vs_i}(\langle v\rangle) = 0$
and this is done by using the operator $T_i + c$, where c is a specific constant furnished by the following lemma.

Lemma 10.2.1. *Let $\alpha, \beta, \beta \neq \alpha, t\alpha$, $F(x_i, x_{i+1})$ be such that $F(\alpha,\beta) \neq 0$, $F(\beta,\alpha) = 0$. Then $G(x_i, x_{i+1}) := F(x_i, x_{i+1}) \left(T_i + \frac{t-1}{\beta/\alpha - 1}\right)$ is such that $G(\alpha,\beta) = 0$, $G(\beta,\alpha) \neq 0$.*

Proof. Write $F = f + x_{i+1} g$, with $f, g \in \mathfrak{Sym}(x_i, x_{i+1})$. Then $G = (t+c)f + (x_i + cx_{i+1})g$. The hypothesis is that

$$f(\alpha,\beta) + \beta g(\alpha,\beta) \neq 0\,, \quad f(\alpha,\beta) + \alpha g(\alpha,\beta) = 0$$

The vanishing of $G(\alpha,\beta)$ requires that $c = (t-1)(\beta/\alpha - 1)^{-1}$, in which case $G(\beta,\alpha) = (\beta\alpha^{-1} - t)(\beta\alpha^{-1} - 1)^{-1} F(\alpha,\beta) \neq 0$. QED

Of course, if $F(\alpha,\beta) = 0 = F(\beta,\alpha)$, then $G(x_i, x_{i+1})(T_i + c)$, for any constant c, is such that $G(\alpha,\beta) = 0 = G(\beta,\alpha)$.

From this remark and Lemma 10.2.1, one deduces that if F satisfies (10.1.1) then $G = F\left(T_i + (t-1)(\gamma-1)^{-1}\right)$, with $\beta = \langle v\rangle_{i+1}$, $\alpha = \langle v\rangle_i$, $\gamma = \beta/\alpha$, also satisfies (10.1.1).

Vanishing conditions propagate under translation: $f(\langle u\rangle) = 0$ implies $g(\langle u\tau\rangle) = 0$, with $g = f\bar\tau$. But the vectors $u\tau$ are exactly those w such that $w_n \neq 0$. Therefore, if $f(\langle u\rangle) = 0$ for all $u : |u| \leq |v|$, $u \neq v$, then $g(\langle w\rangle) = 0$ for all $w : |w| \leq |v\tau|$, $w \neq w$, $w_n \neq 0$. Since $\langle w\rangle_n = 1$ when $w_n = 0$, the polynomial $M_v \bar\tau(x_n - 1)$ satisfies the vanishing conditions (10.1.1) for the index $v\tau$, and the coefficient of $x^{v\tau}$ is equal to the coefficient of x^v in M_v, divided by q.

Finally, one has the following recursive definition of Macdonald polynomials, due to Knop [77] (who reverses the alphabet $\mathbf{x}_n$, compared to the present definition):

Theorem 10.2.2. *The Macdonald polynomials satisfy the recursions*

$$M_{vs_i} = M_v \left(T_i + \frac{t-1}{\langle v\rangle_{i+1}\langle v\rangle_i^{-1} - 1}\right)\ , \ \textit{if} \quad v_i < v_{i+1}\,, \tag{10.2.1}$$

and

$$M_{v\tau} = M_v\, \bar\tau\, (x_n - 1)\ , \tag{10.2.2}$$

starting with $M_{0\ldots 0} = 1$.

One could have chosen to normalize Macdonald polynomials by specifying the value $\|v\| := M_v(\langle v\rangle)$. The theorem implies that, when $v_i < v_{i+1}$, $\|vs_i\|/\|v\| = (\gamma - t)/(\gamma - 1)$, with $\gamma = \langle v\rangle_{i+1}\langle v\rangle_i^{-1}$. Moreover, $\|v\tau\|/\|v\| = q\langle v\rangle_1 - 1$, and this suffices to determine $\|v\|$ starting from 1.

Notice that (10.2.1) means that the linear span $\mathcal{V}_\lambda$ of the M_v, with v such that $v\!\downarrow= \lambda$, is a space of representation of the Hecke algebra with a Yang-Baxter basis M_v (generated from $M_{v\uparrow}$, taking the spectral vector $\langle v\!\uparrow\rangle$).

When λ has equal parts, then the space $\mathcal{V}_\lambda$ is not of dimension $n!$, but the construction is still valid! Indeed, if $v_i = v_{i+1}$, then $\langle v\rangle_{i+1} = t\langle v\rangle_i$, M_v is symmetrical in x_i, x_{i+1} and $M_v\big(T_i + (t{-}1)/(t{-}1)\big) = M_v(t+1)$.

The matrices representing $T_1, \ldots, T_{n-1}$ in the Macdonald are easy to write, thanks to (10.2.1). More generally, the matrix representing any element $\hbar$ of the Hecke algebra is easy to describe. According to the vanishing conditions, its entries are

$$M_v\hbar\big(\langle u\rangle\big)\, \|u\|^{-1} .$$

We can keep the matrices and specialize the Macdonald polynomials, or replace them by simpler polynomials. They are many ways to compute in the space $\mathcal{V}_\lambda$.

Indeed, suppose that there exist $v \in \mathbb{N}^n$, μ dominant, a constant C, and a function $f(x_1, \ldots, x_n)$ such that

$$f\big(\langle u\rangle\big) = C\, M_v\big(\langle u\rangle\big) \quad \forall u :\, u\!\downarrow= \mu .$$

Then f can be extended to a family $\{f_w : w\!\downarrow= v\!\downarrow\}$, such that

$$f_{ws_i} = \left(T_i + \frac{t-1}{\langle w\rangle_{i+1}\langle w\rangle_i^{-1} - 1}\right) \;, \text{ if } \quad w_i < w_{i+1} ,$$

and

$$f_w\big(\langle u\rangle\big) = C\, M_w\big(\langle u\rangle\big) .$$

Notice that one can generate the Macdonald polynomials in $\mathcal{V}_\lambda$ starting from any of them, say M_v, using operators $T_i + \frac{t-1}{\gamma-1}$, or $\left(T_i + \frac{t-1}{\gamma^{-1}-1}\right)\frac{(\gamma-1)^2}{(t\gamma-1)(\gamma-t)}$. We can use the same operators, starting from f, to generate the f_w, f being renamed f_v on this occasion.

For example, take f such that $f\big(\langle\mu\rangle\big) = 1$ and that $f\big(\langle u\rangle\big) = 0$ for all u such that $u\!\downarrow= \mu$, $u \neq \mu$. Then $f_w\big(\langle u\rangle\big) = 0$ if $u \neq w$, and $f_w\big(\langle u\rangle\big) = \|w\|\,\|\mu\|^{-1}$.

We shall describe in the next section simple polynomials which allow to evaluate $M_v\big(\langle u\rangle\big)$ when $|u| = [v| + 1$.

It is shown in [102] that one can generalize Macdonald polynomials by using other affine operations than (10.2.2). In particular, taking two parameters a, b, one defines polynomials $M_v(\mathbf{x}, a, b)$ by using both (10.2.1) and

$$M_{v\tau}(\mathbf{x}, a, b) = M_v(\mathbf{x}, a, b)\, \bar\tau\, \frac{x_n - a}{1 - bx_n} . \tag{10.2.3}$$

These polynomials are related to the BC_n-symmetric polynomials of Rains [170] in the symmetric case. The constants appearing in connection with the polynomials $M_v(\mathbf{x}, a, b)$ give a better understanding of the constants related to the usual polynomials $M_v = M_v(\mathbf{x}, 1, 0)$.

10.3 A baby kernel

Much of the theory of Schubert polynomials can be recovered from the study of the kernel $\prod_{i+j\le n}(y_i - x_j)$. We cannot expect a finite kernel for Macdonald polynomials, nevertheless a "baby kernel" similar to the kernel for Schubert polynomials will already provide properties of Macdonald polynomials.

Define this kernel as

$$\boxdot(\mathbf{x},\mathbf{y}) = \prod_{1\le i<j\le n} (x_j - y_i)(x_i - ty_j)\,.$$

Then the action of the Hecke algebra on this element is easily described :

Proposition 10.3.1. *Let $v \in \mathbb{N}^n$. Then, with $\gamma = y_{i+1}/y_i$, one has*

$$\boxdot\,(\mathbf{x},\mathbf{y}) \left(T_i + \frac{t-1}{\gamma-1}\right) = \frac{t\gamma-1}{\gamma-1}\,\boxdot\,(\mathbf{x},\mathbf{y}^{s_i})\,. \tag{10.3.1}$$

As a consequence, one has the symmetry

$$\boxdot\,(\mathbf{x},\mathbf{y})\,\mathbb{U}_i^{\mathbf{x}} = \boxdot(\mathbf{x},\mathbf{y})\,\mathbb{U}_i^{\mathbf{y}}\,, \tag{10.3.2}$$

Proof. Modulo a factor symmetrical in x_i, x_{i+1},

$$\begin{aligned}\boxdot\,(\mathbf{x},\mathbf{y}) &= (x_i - ty_{i+1})(x_{i+1} - y_i)\\ &= \big(x_i x_{i+1} + ty_i y_{i+1} - y_i x_i - y_i x_{i+1}\big) + \big(y_i - y_{i+1})x_{i+1}\,.\end{aligned}$$

At this stage, we need only know that $1T_i = t$, $x_{i+1}T_i = x_i$ to conclude for the first equation.

Recall that $\mathbb{U}_i^{\mathbf{x}} = T_i + 1 = (tx_i - x_{i+1})\partial_i$, $\mathbb{U}_i^{\mathbf{y}} = (ty_i - y_{i+1})\partial_i^{\mathbf{y}}$. Writing

$$T_i + 1 = T_i + \frac{t-1}{y_{i+1}y_i^{-1} - 1} + \frac{y_{i+1} - ty_i}{y_{i+1} - y_i}\,,$$

one sees that the second statement is a rewriting of the first one, since

$$\begin{aligned}\boxdot(\mathbf{x},\mathbf{y})\,\square_i^y &= \boxdot(\mathbf{x},\mathbf{y})(ty_i - y_{i+1})\partial_i^y\\ &= \boxdot(\mathbf{x},\mathbf{y})\,\frac{ty_i - y_{i+1}}{y_i - y_{i+1}} + \boxdot(\mathbf{x},\mathbf{y}^{s_i})\,\frac{ty_{i+1} - y_i}{y_{i+1} - y_i}\,.\end{aligned}$$

QED

As a consequence, the space generated by the action of $\mathcal{H}_n$ on $\boxdot(\mathbf{x},\mathbf{y})$ coincides with the span of $\boxdot(\mathbf{x},\mathbf{y}^\sigma)\,:\,\sigma\in\mathfrak{S}_n$ (with rational functions in $y_1,\dots,y_n$ as coefficients). Putting $\cap(\mathbf{y}^\sigma)=\prod_{ji\in\sigma}(ty_j-y_i)(y_j-y_i)^{-1}$, then Eq.10.3.1 tells us that $\{\cap(\mathbf{y}^\sigma)\boxdot(\mathbf{x},\mathbf{y}^\sigma)\,:\,\sigma\in\mathfrak{S}_n\}$ is a Yang-Baxter basis with spectral vector $\mathbf{y}$.

Let now $w\in\mathbb{N}^n$ be regular anti-dominant, i.e. $0\le w_1<\dots<w_n$. Specializing $\mathbf{y}=\langle w\rangle$, one sees from (10.3.1) that $\{M_v\,:\,v\uparrow= w\}$ and $\{\cap(\langle v\rangle)\boxdot(\mathbf{x},\langle v\rangle)\}$ are two Yang-Baxter bases with the same spectral vector.

To study M_v, we need one more function $F(x,v)$, which belongs to $\mathfrak{Sym}(\mathbf{x})$ and depends only upon $\lambda:v\downarrow$. Let $\mathcal{F}(\lambda):=\{t^{n-i}q^{\lambda_i-j}, i=1\dots n,\ j=1\dots\lambda_i-\lambda_{i+1}-1\}$, putting $\lambda_{n+1}=0$ for the careful reader. Define

$$F(\mathbf{x},v)=\frac{R(\mathbf{x},\mathcal{F}(v^+))}{R(\mathbf{x},t^nq+\dots+t^nq^k)}\,,$$

with $k=max(v)$, and $R(A,B)=\prod_{a\in A}\prod_{b\in B}(a-b)$ as before.

We moreover notice that, if v has equal components, then the specialization $\boxdot(\mathbf{x},v)$ of $\boxdot(\mathbf{x},\mathbf{y})$ in $\mathbf{y}=\langle v\rangle$ has a factor which is symmetrical in $\mathbf{x}$, namely $R(\mathbf{x},\langle v\rangle\cap\langle v0\rangle)$, i.e. the product of all $R(\mathbf{x},t^iq^j)$ such that both t^iq^j and $t^{i-1}q^j$ are components of $\langle v\rangle$.

For example, if $v=[6,1,3,6,0,6,0,3]$, then, as sets, $\langle 6,1,3,6,0,6,0,3\rangle=\{t^7q^6,t^2q^1,t^4q^3,t^6q^6,t^1q^0,t^5q^6,t^0q^0,t^3q^3\}$, $\langle v0\rangle=\langle 6,1,3,6,0,6,0,3,0\rangle=\{t^8q^6,t^3q^1,t^5q^3,t^7q^6,t^2q^0,t^6q^6,t^1q^0,t^4q^3,t^0q^0\}$ and the intersection is $\{t^7q^6,t^6q^6,t^4q^3,t^1q^0\}$.

Define $\widetilde{\boxdot}(\mathbf{x},v)$ to be the quotient of $\boxdot(\mathbf{x},v)$ by the symmetrical factor.

Following [134] let, for $u\in\mathbb{N}^n$, and $\mu=u\downarrow$,

$$E_u(t,1)=(t{-}1)^{-1}\prod_{i=1}^{n}\prod_{j=0}^{\mu_i-1}(t-q^jt^{i-n})$$

and denote $F(u,v)$, $\widetilde{\boxdot}(u,v)$ the respective specializations in $\mathbf{x}=\langle u\rangle$ of $F(\mathbf{x},v)$, $\widetilde{\boxdot}(\mathbf{x},v)$. We can replace $\cap(\langle v\rangle)$ by $\Cap(v)$, since $\Cap(\langle vs_i\rangle)=\Cap(\langle v\rangle)(t\gamma-1)(\gamma-1)^{-1}$ when $v_i<v_{i+1}$, with $\gamma=\langle v\rangle_{i+1}\langle v\rangle_i^{-1}$.

Proposition 10.3.2. *For u,v such that $|u|=|v|+1$, then*

$$M_v\big(\langle u\rangle\big)=(-qt^{n-1})^{|v|}F(u,v)\,E_u(t,1)\,\Cap\,(v)\,\widetilde{\boxdot}(u,v)\,.\qquad(10.3.3)$$

Proof. The proposition is compatible with $v\to vs_i$, thanks to (10.3.1). We have to check the behaviour of each function with respect to $(u,v)\to(u\tau,v\tau)$, but this presents no difficulty. The only specializations which are missing for $M_{v\tau}$, knowing those of M_v, are $M_{v\tau}(\langle u\rangle)$, $u:\ u_n=0$, such u having no predecessor under τ. But in that case $M_{v\tau}(\langle u\rangle)=0$ since $\langle u\rangle_n=1$. On the other hand, if $v\tau$ has a zero part, then $\widetilde{\boxdot}(u,v)=0$; if not then $t^0q^0\in\mathcal{F}(v\tau)$ and $F(u,v)=0$. Therefore, the proposition is true for $v\tau$ and all the permutations of $u\tau$. This suffices to make it valid for any w permutation of $v\tau$, and any permutation of $u\tau$, and therefore the first part of the proposition is proved by induction on $|v|$. The case $|u| = |v|$ is treated in a similar manner, the slight difference pertaining to the extra factor $t-1$ occuring indeed for $v = [0, \ldots, 0]$. QED

For $|u| - |v| > 1$, the specialization $M_v\big(\langle u\rangle\big)$ does not, in general, factor into products of the type $(t^i q^j - 1)^{\pm 1}$. Therefore the function $F(u,v)\widetilde{\boxdot}(u,v)$ cannot qualify to approximate $M_v\big(\langle u\rangle\big)$. We conjecture however that

$$M_v\big(\langle u\rangle\big) \neq 0 \quad \text{iff} \quad F(u,v)\,\widetilde{\boxdot}(u,v) \neq 0\,. \tag{10.3.4}$$

Knop has shown that $u^+ \not\supseteq v^+$ implies the nullity of $M_v\big(\langle u\rangle\big)$. It is a pure combinatorial problem that we leave to the reader, to check that $u^+ \not\supseteq v^+$ implies the nullity of the explicit function $F(u,v)\widetilde{\boxdot}(u,v)$.

The product that we have written in (10.3.3) is not optimal, since $F(u,v)$ has terms in denominator which can cancel with other terms. We have given a more compact expression elsewhere, that we shall not use in this text.

The only reduced evaluation that we shall need is given in the next proposition (the proof, checking the compatibility with respect to $(u,v) \to (u\tau, v\tau)$ being omitted).

Proposition 10.3.3. *Given $u \in \mathbb{N}^n$, let $k = max(u)$, i be the leftmost position of k in u. Let $v = [\ldots, u_{i-1}, k-1, u_{i+1}, \ldots]$, and β be such that $\langle v\rangle_i = q^{k-1}t^\beta$. Then*

$$M_v\big(\langle u\rangle\big)\|u\|^{-1} = t^{-\beta}(t^{n-1-\beta}q - 1)^{-1} \tag{10.3.5}$$

10.4 Multiplication by an indeterminate

Given any polynomial $f(\mathbf{x})$ of degree 1, then $f(\mathbf{x})M_v(\mathbf{x})$ vanishes on all $u : |u| \leq |v|$, $u \neq v$, and therefore

$$\big(f(\mathbf{x}) - f(\langle v\rangle)\big)\, M_v(\mathbf{x}) = \sum_{u:\, |u| = |v|+1} c_v^u M_u\,.$$

The structure constants c_v^u are determined by specializing the equation in every u, allowing to rewrite it as

$$\big(f(\mathbf{x}) - f(\langle v\rangle)\big)\, M_v(\mathbf{x}) = \sum_{u:\, |u| = |v|+1} \big(f(\langle u\rangle) - f(\langle v\rangle)\big)\, \frac{M_v(\langle u\rangle)}{\|u\|}\, M_u(\mathbf{x})\,.$$

Taking $f(\mathbf{x}) = x_1 + \cdots + x_n$ is sufficient to see all non-zero specializations $M_v(\langle u\rangle)$ occur, since $f(\langle u\rangle) \neq f(\langle v\rangle)$ when $|u| \neq |v|$.

Denote $\mathbb{U}$, $\mathbb{V}$ the sum of components of $\langle u\rangle$, $\langle v\rangle$. Using (10.3.3), one gets :

Theorem 10.4.1.

$$(x_1+\cdots+x_n-\mathbb{V})\,M_v(\mathbf{x})$$
$$=(-qt^{n-1})^{|v|}\sum_{u\colon |u|=|v|+1}(\mathbb{U}\text{-}\mathbb{V})\frac{\Cap(v)}{\|u\|}\,F(u,v)E_u(t,1)\,\widetilde{\boxdot}(u,v)\,M_u(\mathbf{x}) \quad (10.4.1)$$

$$\left(\frac{x_i}{\langle v\rangle_i}-1\right)(-qt^{n-1})^{-|v|}\,M_v(\mathbf{x})$$
$$=\sum_{u\colon \langle u\rangle_i\neq\langle v\rangle_i}\left(\frac{\langle u\rangle_i}{\langle v\rangle_i}-1\right)\frac{\Cap(v)}{\|u\|}\,F(u,v)E_u(t,1)\,\widetilde{\boxdot}(u,v)\,M_u(\mathbf{x}) \quad (10.4.2)$$

For example, for $i=1$, $v=[1,0]$, $u=[0,2]$, then $\langle u\rangle_1/\langle v\rangle_1-1=1/(tq)-1$, $\Cap([1,0])=(qt^2-1)(tq-1)^{-1}$, $||0,2||=t\,(q-1)\,(q^2t-1)$, $F([0,2],[1,0])=\big(tq(-q+t)(t^2q-1)\big)^{-1}$, $E_{02}(t,1)=t-q$, $\widetilde{\boxdot}([0,2],[1,0])=-tq(t-1)(q-1)$ and the coefficient of M_{02} in the product $(x_1/tq-1)M_{10}$ is $(1/t-1)(tq^2-1)^{-1}$.

One can write more compactly the coefficients occurring in the preceding two formulas, in particular with the help of the functions $E_{u/v}(a,b)$ studied in [134]. The important property of these coefficients is that they are products of factors of the type $(t^iq^j-1)^{\pm1}$, as in the case of many of the constants appearing in the theory of symmetric Macdonald polynomials.

Computing an example, one sees a structure emerge on the set of $u:\ |u|=|v|+1,\ M_v(\langle u\rangle\neq 0$ (call such u the *successors* of v; here $v=[5,0,2]$) :

$$\begin{tikzcd}[arrows=dash]
503 \arrow[r, dashed] & 206 \arrow[dr] & & & \\
& & 026 \arrow[r, equal] & 062 \arrow[r] & 602 \\
512 \arrow[r, equal] & 521 \arrow[ur, dashed] & & &
\end{tikzcd},$$

edges being the simple transpositions s_0 $--$, s_1 ——, s_2 $=\!=$.

The statement generalizing the preceding figure (equivalent to the description of Knop [78], and which results from easy-to-prove combinatorial properties of the function $F(u,v)E_u(t,1)$, is

Proposition 10.4.2. *Let $u,v\in\mathbb{N}^n$, $|u|=|v|{+}1$. Then u is a successor of v iff there exist $k\in\{0,\ldots,n{-}1\}$, and a subword σ of $s_{n-1}\cdots s_1$ such that*

$$u\tau^k=v\tau^{k+1}\,\sigma\,.$$

For example, the above figure decomposes into the (overlapping) strings $[0,2,6]\sigma$, $[2,6,1]\sigma\tau^{-1}$, $[6,1,3]\sigma\tau^{-2}$, with $\sigma\in\{1,s_2,s_1,s_2s_1\}$.

Baratta [4] has also obtained a degree-1 Pieri formula for Macdonald polynomials.

10.5 Transitions

As for Schubert polynomials, choosing an appropriate i for the product $x_i M_v(\mathbf{x})$ provides a recursion on the Macdonald basis, that we shall still call a *transition*.

Proposition 10.5.1. *Given $u \in \mathbb{N}^n$, let $k = max(u)$, i be the leftmost position of k in u. Let $v = [\dots, u_{i-1}, k-1, u_{i+1}, \dots]$, and β be such that $\langle v \rangle_i = q^{k-1}t^\beta$. Then*

$$M_u(\mathbf{x}) = (x_i q^{-k+1} - t^\beta) M_v(\mathbf{x}) + t^\beta \sum_w \frac{M_v(\langle w \rangle)}{\|w\|} \left(1 - \frac{\langle w \rangle_i}{\langle v \rangle_i}\right) M_w(\mathbf{x}) \,, \tag{10.5.1}$$

summed over the successors w of v such that $\langle w \rangle_i \neq \langle v \rangle_i$, and $w \neq u$. Moreover, for such w, one has $w < u$.

Proof. We have evaluated $M_v(\langle w \rangle)$ in Prop. 10.3.3. There remains only to check the statement about the order, that we skip. QED

Notice that the exponent β is equal to

$$n - 1 - \#(j : j > i, u_j = k) - \#(j : j < i, u_j = k{-}1) \,. \tag{10.5.2}$$

In other words, representing u by a diagram of boxes of coordinates $(1,0), \dots, (1, u_1{-}1), \dots, (n, 0), \dots, (n, u_n{-}1)$, then β is equal to the number of points $(k, j), j > i$ and $(k{-}1, j), j < i$ which are not occupied by a box.

One can iterate the transition formula. This gives a canonical decomposition of any Macdonald polynomial into sums of products of "shifted monomials" $\prod (x_i q^{-a} - t^b)$, the specialization $t = 0$ of these monomials being of degree $|v|$.

For example, writing ij for a factor $t^i q^j - 1$, starting with $u = [2,0,2]$, one has $v = [1,0,2]$, $\langle v \rangle = [tq, 1, t^2q^2]$ and the following sequence of transitions :

$$\begin{aligned}
M_{202} &= \left(x_1 q^{-1} - t\right) M_{102} + \frac{10}{22} M_{022} \,, \\
M_{022} &= \left(x_2 q^{-1} - t\right) M_{012} + tq \frac{10 \cdot 10}{11 \cdot 21} M_{121} + \frac{10 \cdot 31}{21 \cdot 21} M_{112} \,, \\
M_{121} &= \left(x_2 q^{-1} - t\right) M_{111} + \frac{10}{21} M_{112} \,, \\
M_{112} &= \left(x_3 q^{-1} - 1\right) M_{111} \,,
\end{aligned}$$

leading to polynomials of degree 3 that one assumes to be known by induction on the degree.

To reduce the size of the output, let us represent each factor $x_j / q^{i-1} - t^\beta$ by a black square in the Cartesian plane (row i, column j) (β is determined by i, j, according to (10.5.2)). Then the final outcome of the transitions for M_{202} is

$$\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \blacksquare & \cdot \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{10 \cdot 10}{11 \cdot 22} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{10 \cdot 10}{11 \cdot 22} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \cdot \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix} \frac{10}{11} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \cdot & \blacksquare & \blacksquare \end{smallmatrix} \frac{10}{11} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \blacksquare & \blacksquare \\ \cdot & \blacksquare & \blacksquare \end{smallmatrix} \frac{10}{22} + \begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \blacksquare & \cdot & \blacksquare \end{smallmatrix}$$

with leading term $\begin{smallmatrix} \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \blacksquare & \cdot & \blacksquare \end{smallmatrix} = (x_1 q^{-1} - t)(x_1 - t)(x_3 q^{-1} - t)(x_3 - 1)$.

Haglund, Haiman, Loehr [59] give a combinatorial formula for the component of degree $|u|$ of M_u, which involves, in general, another enumeration than the one by transition. Still another decomposition is furnished, in the symmetric and non-homogeneous case, by Okounkov [159, 160, 161].

10.6 Symmetric Macdonald polynomials

In the space V_λ, which has basis $\{M_v : v{\downarrow}=\lambda\}$, one can build another basis $\widehat{M}_v$, still starting with $\widehat{M}_{\lambda\uparrow} = M_{\lambda\uparrow}$, but using the spectral vector $[0, 1, \ldots, t^{n-1}]$ this time.

In the case $n = 2$, we have already used $\mathbb{U}_i = T_i + 1 = (tx_i - x_{i+1})\partial_i$, wich sends polynomials onto polynomials symmetrical in x_i, x_{i+1}. This shows that $\widehat{M}_\lambda$ is symmetrical in every pair of consecutive variables, hence symmetrical in $\mathbf{x}$. However, in the space V_λ, there is only one symmetrical polynomial (up to a scalar): being invariant under each $\mathbb{U}_i/(1+t)$ determines the expansion in the basis M_v, once knowing one coefficient. This polynomial is the *symmetrical Macdonald polynomial* of index λ, its component of degree $|\lambda|$ being the original Macdonald polynomial [**?**].

Thus $\widehat{M}_\lambda = \widehat{M}_{\lambda\uparrow}\mathbb{U}_\omega$ is symmetrical, and moreover every $M_v\mathbb{U}_\omega$ is symmetrical and proportional to the symmetric Macdonald polynomial belonging to the space V_λ. As a consequence, one can study the symmetric polynomial by just using $\mathbb{U}_\omega$. There are other methods, in particular some operators on symmetric functions which are described in the book of Macdonald.

As a side remark, let us determine the image of $\boxdot(\mathbf{x}, \mathbf{y})$ under $\mathbb{U}_\omega$. From (10.3.2), one obtains that this polynomial $G_n(\mathbf{x}, \mathbf{y})$ is also symmetrical in $\mathbf{y}$. Writing the recursion $G_n \to G_{n+1}$, one realizes that $G_n(\mathbf{x}, \mathbf{y})$ is equal to the Gaudin-Izergin-Korepin function :

$$\frac{R(\mathbf{x}, \mathbf{y})R(\mathbf{x}, t\mathbf{y})}{\Delta(\mathbf{x})\Delta(\mathbf{y})} \det\left(\frac{1}{x_i - y_j)(x_i - ty_j)}\right) .$$

The relevance of this last function to the theory of Macdonald polynomials has been pointed out by Warnaar [192], as well as physicists [70, 109].

10.7 Macdonald polynomials versus Key polynomials

The generation of Macdonald polynomials involves operators of the type

$$T_i + \frac{t-1}{\gamma-1} = \pi_i(t-1) - s_i + \frac{t-1}{\gamma-1}\,,$$

with some γ's, which are rational functions in q, t, given by a spectral vector.

The limits $t = 0$ or $t = \infty$ clearly must have special properties. To study them, it is better to take the Hecke relation $(T_i - t_1)(T_i - t_2) = 0$ instead of $(T_i - t)(T_i + 1) = 0$, and transform spectral vectors by $t \to -t_1/t_2$, without changing the affine induction.

Let us still denote, in this section, by M_v the homogeneous Macdonald polynomials with parameters t_1, t_2, q. The operators to use in the recursion are now

$$T_i + \frac{t_1+t_2}{\gamma-1} = \pi_i(t_1+t_2) - t_2 s_i + \frac{t_1+t_2}{\gamma-1}\,.$$

The first specialisation that we shall consider is $t_1 = 0, t_2 = -1$. Let us denote $\widetilde{M}_v$ the specialisation $t_1 = 0, t_2 = -1$ of the normalized polynomial $M_v/coeff(M_v, x^v)$.

In that case, the constant $\gamma = \langle v\rangle_{i+1}\langle v\rangle_i^{-1}$ used in the recursion $M_v \to M_{vs_i}$, when $v_i < v_{i+1}$, is of the type $q^{v_{i+1}-v_i}(-t_1/t_2)^\alpha$, with $\alpha > 1$, and tends towards 0. The operator $T_i + (t_1+t_2)(\gamma-1)^{-1}$ tends towards $-\pi_i + s_i + 1$, which sends 1 to 1 and $x_i + 1$ to $x_i + x_{i+1}$. Thus this last operator is a divided difference π_{x_{i+1},x_i} for the reversed alphabet. Therefore,

$$\widetilde{M}_{vs_i} = \widetilde{M}_v\, \pi_{x_{i+1},x_i}\,. \tag{10.7.1}$$

Suppose that we know that $M_v = N_v + q\,\star$, with $N_v = K_{v\omega}(\mathbf{x}^\omega)$, and $\star \in \mathbb{C}[q](\mathbf{x}_n)$. Then (10.7.1) shows that, modulo q, M_{vs_i} is still a key polynomial for the reversed alphabet $\mathbf{x}_n^\omega$.

Let v be antidominant. The affine operation does not imply t_1, t_2. The predecessor of M_v under the affine operation is M_u, with $u = [v_n - 1, v_1, \ldots, v_{n-1}]$. One has

$$\widetilde{M}_u = N_u + q\sum c^u_w x^w = x^u + \sum x^{u'} + q\sum c^u_w x^w\,,$$

sum over monomials $x^{u'}$ such that $u'_1 < u_1$ (because, by induction, N_u is a key polynomial) and monomials x^w such that $w_1 \le u_1$, with coefficients which are polynomials in q.

Therefore

$$\widetilde{M}_v = q^{u_1}\left(q^{-u_1}x^v + \sum q^{-u'_1}x^{u'\tau} + q\sum c^u_w q^{-w_1}x^{w\tau}\right)\,,$$

and the term of degree 0 in q is the single monomial x^v.

In conclusion, one has the following specialization property due to B. Ion [66].

Theorem 10.7.1. *Let $v \in \mathbb{N}$. Then the normalized Macdonald polynomial $M_v/(coeff(M_v, x^v)$ specializes, for $t_1 = 0, t_2 = -1, q = 0$ into the key polynomial*

$$K_{v\omega}(\mathbf{x}_n^\omega)$$

for the reversed alphabet $\mathbf{x}_n^\omega = \{x_n, \dots, x_1\}$.

One could hope that the specialization $t_1 = 1, t_2 = 0$ be treated exactly in the same manner. Let us still denote $\widetilde{M}_v$ the specialization $t_1 = 1, t_2 = 0$ of the normalized Macdonald polynomials. One computes

$$\begin{tikzcd}[column sep=tiny]
& \widetilde{M}_{012} = x^{012} \arrow[dl, dash] \arrow[dr, equal] & \\
\widetilde{M}_{102} = x^{102} + x^{111} + \frac{1}{q}x^{012} \arrow[d, equal] & & \widetilde{M}_{021} = x^{021} + x^{111} + \frac{1}{q}x^{012} \arrow[d, dash] \\
\widetilde{M}_{120} = x^{120} + \frac{1}{q}x^{111} + \frac{1}{q}x^{021} \arrow[dr, dash] & & \widetilde{M}_{201} = x^{120} + \frac{1}{q}x^{111} + \frac{1}{q}x^{102} \arrow[dl, equal] \\
& \widetilde{M}_{210} = x^{210} + \frac{1}{q}\star &
\end{tikzcd}$$

There is no way that one can obtain $\widetilde{M}_{021}$ from $\widetilde{M}_{012}$ using an operator involving only x_2, x_3 !

However, one can read the hexagon upwards. The space $\{M_v : v \uparrow\}$ can be generated starting from M_{210}, the arrows $M_v \to M_{vs_i}$ being invertible when t_1, t_2, q remain generic. Indeed, for any i, any $\gamma \neq 0, 1$, one has

$$\left(T_i + \frac{t_1+t_2}{\gamma-1}\right)\left(T_i + \frac{t_1+t_2}{\gamma^{-1}-1}\right) = \left(t_1 + \frac{t_1+t_2}{\gamma-1}\right)\left(t_1 + \frac{t_1+t_2}{\gamma^{-1}-1}\right) = -\frac{(t_1\gamma+t_2)(t_1+t_2\gamma)}{(\gamma-1)^2}\,.$$

Taking into account that we use the normalized polynomials $M_v/coeff(M_v, x^v)$, we have the recursion

$$\frac{1}{coeff} M_v \frac{(t_1\gamma + t_2)(t_1 + t_2\gamma)}{t_2(\gamma-1)^2}\left(T_i + \frac{t_1+t_2}{\gamma-1}\right) = c\, M_{vs_i}\,,$$

when $v_i > v_{i+1}$, with $\gamma = \langle v\rangle_{i+1}\langle v\rangle_i^{-1} = q^{-\alpha}\left(\frac{-t_2}{t_1}\right)^\beta$, $\alpha, \beta > 0$, and some constant $c \neq 0$. The extra factor $-1/t_2$ is due to the fact that $x_{i+1}T_i = -t_2 x_i$. The limit $t_1 = 1, t_2 = 0$ of $\frac{(t_1\gamma+t_2)(t_1+t_2\gamma)}{t_2(\gamma-1)^2}$ is 1 when $\beta > 1$ and $1 - q^{-\alpha}$ when $\beta = 1$, while $T_i + (t_1 + t_2)(\gamma - 1)^{-1}$ specializes into $\pi_i - 1 = \widehat{\pi}_i$. Therefore, up to a possible factor $1 - q^{-\alpha}$, one has

$$\widetilde{M}_{vs_i} = \boxed{(1 - q^{-\alpha})}\,\widetilde{M}_v\, \widehat{\pi}_i\,.$$

Hence, if $\widetilde{M}_v = \widehat{K}_v + q^{-1}(\star)$, then $\widetilde{M}_{vs_i} = \widehat{K}_{vs_i} + q^{-1}(\star\star)$, with $\star$ and $\star\star$ polynomials in q^{-1}.

It remains to determine $\widetilde{M}_v$ for v dominant. If $v_n > 0$, then $M_v = M_u\Phi$, with $u = v\tau^{-1} = [v_n - 1, v_1, \ldots, v_{n-1}]$. However, $\widetilde{M}_u = x^u + \sum c^u_w x^w$, with $w_1 \geq u_1$. By induction on the degree $|u|$, one can suppose that $\widetilde{M}_u = \widehat{K}_u$ modulo q^{-1}. Since $\widehat{K}_u = x^u + \sum x^w$ with $w_1 > u_1$, one has $\widetilde{M}_u\Phi = q^{-u_1}x^v + q^{-u_1-1}(\star)$ and therefore $\widetilde{M}_v = x^v + q^{-1}(\star)$ as needed.

In the case where $v_n = 0$, one has recourse to another ingredient. For any $u \in \mathbb{N}^{n-1}$, one has $\widetilde{M}_{u0} = \widetilde{M}_u + x_n q^{-1}(\star)$. Assuming that for $u = [v_1, \ldots, v_{n-1}]$ dominant, one has $\widetilde{M}_u = x^u + q^{-1}(\star)$, this implies that $\widetilde{M}_{u0} = x^u + q^{-1}(\star\star)$.

In final, one has the following specialization theorem due to B.Ion [66].

Theorem 10.7.2. *Let $v \in \mathbb{N}$. Then the limit $t_1 = 1, t_2 = 0, q = \infty$ of the normalized Macdonald polynomial $M_v/coeff(M_v, x^v)$ is equal to the key polynomial $\widehat{K}_v$.*

In both limits $(t_1 = 1, t_2 = 0)$ and $(t_1 = 0, t_2 = -1)$, we have eliminated q by sending it to ∞ or 0. Sanderson [180] shows that the limit $t = 0$ of the usual normalized homogeneous nonsymmetric Macdonald polynomial is equal to an *affine Demazure character* (for us, on the reversed alphabet $\mathbf{x}^\omega$). It would be interesting to further develop the combinatorics of these affine Demazure characters. In fact, they are all the polynomials generated from the polynomial 1 using Φ and the divided differences π_{x_i+1,x_i}, no other ingredient is needed. Since π_i acts on key polynomials by sorting indices, one needs only to describe the polynomials $\widetilde{M}_v$ (equal to $q^{||v||}$ times the specialization $t = 0$ of the usual homogeneous Macdonald polynomial) for v antidominant. The polynomials for v dominant are symmetrical, since they are obtained using a maximal product of π_{x_i+1,x_i}, and in fact are equal to the *Hall-Littlewood polynomials*[1].

For example, using key polynomials in the reversed alphabet $\{x_3, x_2, x_1\}$, one has

$$\widetilde{M}_{013} = K_{310} + qK_{220} + qK_{211} + q^2K_{112}$$

from which one obtains, by sorting indices,

$$\widetilde{M}_{310} = K_{013} + qK_{022} + qK_{112} + q^2K_{112}\,.$$

This last polynomial is explained by enumerating all tableaux of evaluation $1123 = 1^2 2^1 3^1$ ($[2, 1, 1]$ is the conjugate of $[3, 1]$) together with their *charge* :

$$q^0 \begin{array}{|c|c}\cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + q^1 \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline\end{array} + q^1 \begin{array}{|c|cc}\cline{1-1} 2 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + q^2 \begin{array}{|c|cc}\cline{1-1} 3 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,.$$

[1] There are several species of Hall-Littlewood polynomials. The relevant one is here $\mathbb{Q}'_\mu = \sum_t q^{c(t)} s_{\lambda(t)}$, sum over all tableaux of evaluation μ, $\lambda(t)$ being the shape of the tableau, but one has to conjugate partitions.

Reading the conjugate shapes, one obtains

$$q^0 s_{31} + q^1 s_{22} + (q^1 + q^2) s_{211} \,,$$

which coincides with $\widetilde{M}_{310}$. A refinement of charge is needed to explain $\widetilde{M}_{013}$.

Notice that one can define nonsymmetric Hall-Littlewood polynomials P_v, $v \in \mathbb{N}^n$ [30], by starting with all dominant monomials and using the spectral vector $[t^{n-1}, \ldots, 1]$ (with the Hecke relations $(T_i - t)(T_i + 1) = 0$).

For example one has $P_{310} = x^{310} = K_{310}$, $P_{130} = K_{13} - tK_{22}$, $P_{103} = \frac{t^2}{t+1} K_{22} - tK_{112} + K_{103} - \frac{t}{t+1} K_{13} + t^2 K_{211} - tK_{202}$, $P_{013} = K_{013} - tK_{022} - tK_{112}$.

The last polynomial is, indeed, the Hall-Littlewood polynomial indexed by the partition $[3, 1]$, and is the specialization $q = 0$ of the symmetric Macdonald polynomial, but, except in the dominant or antidominant case, the polynomials P_v are not specializations of Macdonald polynomials, and are not related to the affine Demazure characters.

Chapter 11

Hall-Littlewood polynomials

Hall-Littlewood polynomials are specializations of Macdonald polynomials. However we shall study them independently in this chapter. This study is part of a joint work with Jennifer Morse.

11.1 From a quadratic form on the Hecke algebra to a quadratic form on polynomials

We have defined in (1.8.5) the quadratic form $(\,,\,)^{\mathcal{H}}$ on the Hecke algebra $\mathcal{H}_n$. Since we can use linear bases of $\mathcal{H}_n$, as we have used $\{\partial_\sigma\}$, $\{\pi_\sigma\}$, $\{\widehat{\pi}_\sigma\}$, to generate bases of $\mathfrak{Pol}(\mathbf{x}_n)$, a natural problem is to find a quadratic form on $\mathfrak{Pol}(\mathbf{x}_n)$ compatible with $(\,,\,)^{\mathcal{H}}$. We propose a t_1t_2 deformation of the form $(\,,\,)$ defined in 2.4.1.

Let

$$\Theta := \prod_{1\le i<j\le n} \frac{1-x_ix_j^{-1}}{1+t_2x_it_1^{-1}x_j^{-1}} \ .$$

Let us use it to define a bilinear form $(\,,\,)_{t_1t_2}$ on $\mathfrak{Pol}$ by

$$(f\,,\,g)_{t_1t_2} = CT\left(f\,g^{\clubsuit}\Theta\right) = CT\left(f\,g^{\clubsuit}\prod_{1\le i<j\le n}(1-\frac{x_i}{x_j})\sum_{k=0}^{\infty}\left(-\frac{t_2}{t_1}\frac{x_i}{x_j}\right)^k\right) \qquad (11.1.1)$$

where $\clubsuit$ is the automorphism, already met, defined by $x_i \to 1/x_{n+1-i}$ for $1\le i\le n$, and where $CT(f\,\Theta)$ means

$$CT(f\,\Theta) := CT_{x_n}\left(CT_{x_{n-1}}\left(\ldots\left(CT_{x_1}\left(f\,\Theta\right)\right)\ldots\right)\right) \ .$$

Lemma 11.1.1. *For $i\le n-1$, the operator T_i is adjoint to T_{n-i} with respect to $(\,,\,)_{t_1t_2}$.*

Proof. Same proof as in the case of π_i and $(\,,\,)$ seen in 2.4.1. As usual, one is reduced to the case of two consecutive variables. QED

As in the case of the form (,), one has to determine the scalar product of two monomials. Since $(x^u, x^v)_{t_1t_2} = (x^{u-v\omega}, 1)_{t_1t_2}$, the answer is provided by the following lemma [1] for which we refer to [30, Lemma 4.2].

Introduce a partial order on elements of $\mathbb{Z}^n$ by using right sums (this orders generalizes the *dominance order* on partitions in $\mathbb{N}^n$):

$$v \geq u \;\Leftrightarrow\; v_n \geq u_n \,\&\, v_{n-1}+v_n \geq u_{n-1}+u_n \,\&\, v_{n-2}+v_{n-1}+v_n \geq u_{n-2}+u_{n-1}+u_n \,\&\, \ldots$$

Lemma 11.1.2. *For any $u \in \mathbb{Z}^n$, then $(x^u, 1)_{t_1t_2} \neq 0$ if and only if $|u| = 0$ and $u \geq [0, \ldots 0]$.*

Proposition 11.1.3. *Let $\lambda, \mu \in \mathbb{N}^n$ be dominant, σ, ζ be two permutations in $\mathfrak{S}_n$. If $\lambda \neq \mu$, then $(x^\lambda T_\sigma \,,\, x^\mu \widehat{T}_\zeta)_{t_1t_2} = 0$.*

If $\lambda = \mu$, and if σ, ζ are of minimum length in their coset modulo the stabilizer of λ, then

$$(x^\lambda T_\sigma \,,\, x^\lambda \widehat{T}_\zeta)_{t_1t_2} \neq 0 \;\Leftrightarrow\; \sigma\omega = \omega(\lambda)\zeta \,, \tag{11.1.2}$$

where $\omega(\lambda)$ is the element of maximal length of the stabilizer of λ. In that case $(x^\lambda T_\sigma \,,\, x^\lambda \widehat{T}_\zeta)_{t_1t_2} = 1$.

Proof. In the case μ, λ different, suppose that $\lambda_1 = \mu_1, \ldots, \lambda_r = \mu_r$, $\lambda_{r+1} < \mu_{r+1}$. Since $\widehat{T}_i$ is adjoint to $\widehat{T}_{n-i}$, the nullity of $(x^\lambda T_\sigma \,,\, x^\lambda \widehat{T}_\zeta)_{t_1t_2}$ results from the nullity of $(x^\lambda \mathcal{H}_n \,,\, x^\mu)_{t_1t_2}$. Each monomial x^u appearing in the expansion of some $x^\lambda T_\sigma$ is such that $u \leq \lambda\omega$. However, $(x^u, x^\mu)_{t_1t_2} \neq 0$ requires that $u \geq \mu\omega$, hence $\lambda\omega \geq \mu\omega$, which is a contradiction.

In the case $\lambda = \mu$, the same reasoning shows that $(x^u, x^\lambda)_{t_1t_2} \neq 0$ only in the case $u = \lambda\omega$. The space $x^\lambda \mathcal{H}_n$ has basis $\{U_v : v{\downarrow} = \lambda\}$, and $x^{\lambda\omega}$ occurs only in the expansion of $U_{\lambda\uparrow}$. Since $(x^{\lambda\omega}, x^\lambda)_{t_1t_2} = 1$, one concludes. QED

When λ is strict, its stabilizer is reduced to the identity, and in that case, for any two permutations,

$$(x^\lambda T_\sigma \,,\, x^\lambda \widehat{T}_\zeta)_{t_1t_2} \neq 0 \;\Leftrightarrow\; \sigma\omega = \zeta \,. \tag{11.1.3}$$

On the other hand, $(T_\sigma, \widehat{T}_\zeta)^{\mathcal{H}}$ is different from 0 if and only if $\sigma = \omega\zeta$. Thus in that case we have a perfect correspondence between the quadratic form on $\mathcal{H}_n$ and the quadratic form on $x^\lambda \mathcal{H}_n$. When λ is not strict, the dimension of the space $x^\lambda \mathcal{H}_n$ is less than $n!$. This explains why we have to use the stabilizer of λ.

11.2 Nonsymmetric Hall-Littlewood polynomials

We have shown in [30] how to use the two adjoint Yang-Baxter bases $\{\nabla_{\omega\sigma}\}$, $\{\mathbb{U}_\sigma\}$ to generate noncommutative Hall-Littlewood polynomials U_v and $\widehat{U}_v$, $v \in \mathbb{N}^n$. Let us recall the construction. Given $v \in \mathbb{Z}^n$, denote $\langle v \rangle$ its standardization.

[1] $t_1 = 1$, $t_2 = -t$ for [30]. By homogeneity, one recovers the case of a general pair t_1, t_2.

When λ is dominant, then

$$U_\lambda = \widehat{U}_\lambda = x^\lambda \,.$$

For v and i such that $v_i > v_{i+1}$, then

$$U_{vs_i} = U_v T_i\big(\langle v\rangle_i - \langle v\rangle_{i+1}\big) \quad \& \quad \widehat{U}_{vs_i} = \widehat{U}_v T_i\big(\langle -v\rangle_i - \langle -v\rangle_{i+1}\big) \tag{11.2.1}$$

For example, for $v = [2,2,0]$, one has $\langle v\rangle = [2,3,1]$, $\langle -v\rangle = [1,2,3]$; for $v = [2,0,2]$, one has $\langle v\rangle = [2,1,3]$, $\langle -v\rangle = [1,3,2]$ and for $v = [0,2,2]$, one has $\langle v\rangle = [1,2,3]$, $\langle -v\rangle = [3,1,2]$. Accordingly

$$x^{220} = U_{220} \xrightarrow{T_2(2)} U_{202} \xrightarrow{T_1(1)} U_{022} \,,$$

$$x^{220} = \widehat{U}_{220} \xrightarrow{T_2(-1)} \widehat{U}_{202} \xrightarrow{T_1(-2)} \widehat{U}_{022} \,.$$

The fact that $\{\nabla_{\omega\sigma}\}$ and $\mathbb{U}_\sigma$ are adjoint bases with respect to $(\ ,\ )^{\mathcal{H}}$ has its counterpart at the level of polynomials.

Theorem 11.2.1. *The two sets of polynomials $\{U_v : v \in \mathbb{N}^n\}$ and $\{\widehat{U}_v : v \in \mathbb{N}^n\}$ are two adjoint bases of $\mathfrak{Pol}$ with respect to the scalar product $(\ ,\ )_{t_1t_2}$. More precisely, they satisfy*

$$(U_v\,,\, \widehat{U}_{u\omega})_{t_1t_2} = \delta_{v,u} \,.$$

Proof. If $u\!\downarrow$ or $v\!\downarrow$ are strict, then the statement results from (11.1.3). One has just to check that $(x^\lambda T_\omega, x^\lambda)_{t_1t_2} = (x^{\lambda\omega}, x^\lambda)_{t_1t_2} = 1$. In the non strict case, one has to replace (11.1.3) by (11.1.2). QED

Notice that, we had met the pairing $\sigma \leftrightarrow \omega\sigma$ for bases of the Hecke algebra, while we have now the pairing $v \leftrightarrow v\omega$.

11.3 Adjoint basis with respect to (,)

Using the quadratic form (,) instead of $(\ ,\ )_{t_1t_2}$, one obtains a basis, denoted $\{V_v\}$, $v \in \mathbb{N}^n$, adjoint to $\{U_v\}$. The transition matrix $V_v \to \widehat{K}_u$ is the transpose of the transition matrix $K_v \to U_u$, and should be investigated.

For example, the rows of the following matrix describe the expansions of V_v,

$v \in \mathbb{N}^n$, $|v| = 3$.

$$\begin{array}{c} \mathtt{300} \\ \mathtt{210} \\ \mathtt{201} \\ \mathtt{120} \\ \mathtt{111} \\ \mathtt{102} \\ \mathtt{030} \\ \mathtt{021} \\ \mathtt{012} \\ \mathtt{003} \end{array} \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{t}{t+1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{t}{t+1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{t}{t+1} & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 0 & t & t\,(t+1) & t^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Row 111 reads

$$V_{111} = \widehat{K}_{111} + t\widehat{K}_{201} + t\widehat{K}_{120} + t\,(t+1)\,\widehat{K}_{210} + t^3\widehat{K}_{300}\ .$$

A bigger example

$$V_{421} = \widehat{K}_{421} + t\widehat{K}_{511} + t\widehat{K}_{430} + t(t{+}1)\widehat{K}_{520} + t^2(t{+}1)\widehat{K}_{610} + t^4\widehat{K}_{700} + t^2\widehat{K}_{601}$$

shows that the charge of tableaux of commutative evaluation $1^4 2^2 3$ explains the terms $\widehat{K}_\mu$, μ dominant, the term $t^2\widehat{K}_{601}$ being apart.

11.4 Symmetric Hall-Littlewood polynomials for types A, B, C, D

For the remainder of this chapter, take $t_1 = 1, t_2 = -t$, and write $(\ ,\ )_t$ for the corresponding specialization of $(\ ,\ )_{t_1 t_2}$.

The polynomials $U_{\lambda\uparrow}$ are symmetrical, being in the image of $\mathbb{U}_\omega$. More precisely, taking into account the stabilizer of λ, the recursive definition of $U_{\lambda\sigma}$ implies

$$U_{\lambda\uparrow} = b_\lambda^{-1} x^\lambda \prod_{i<j\le n} (1 - t x_j x_i^{-1}) \pi_\omega\,, \tag{11.4.1}$$

where $b_\lambda = \prod_i \big((1{-}t)\cdots(1{-}t^{\alpha_i})\big) = 1\mathbb{U}_{\omega(\lambda)}$, writing $\lambda = 0^{\alpha_0}1^{\alpha_1}2^{\alpha_2}\cdots$ in exponential form, $\omega(\lambda)$ being the permutation of maximal length in the stabilizer of λ (we have taken $\lambda \in \mathbb{N}^n$, with eventual terminal 0's, contrary to the usual conventions). This equation is precisely the definition by Littlewood [144] of the *Hall-Littlewood function* P_λ:

$$U_{\lambda\uparrow} = P_\lambda\,.$$

Identifying the pairs $x_j x_i^{-1}$ with the negative roots of the root system of type A, one naturally extends the definition of Hall-Littlewood polynomials to all types.

Let $\mathcal{R}^\heartsuit$ be the set of positive roots of the root system of type $\heartsuit = A, B, C, D$. Then for any $\heartsuit$-dominant weight λ, one defines the *Hall-Littlewood polynomial* $P_\lambda^\heartsuit$ by

$$P_\lambda^\heartsuit = b_\lambda^{-1} \prod_{\alpha\in\mathcal{R}^\heartsuit} (1-te^{-\alpha})\, \pi_{w_0}^\heartsuit\,, \tag{11.4.2}$$

where b_λ is such that the coefficient of x^λ be 1 in $P_\lambda^\heartsuit$.

For example, for $n=3$, one has

$$P_\lambda^B = b_\lambda^{-1} x^\lambda \left(1-\frac{tx_2}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_3}\right)$$
$$\left(1-\frac{tx_3}{x_2}\right)\left(1-\frac{t}{x_2x_3}\right)\left(1-\frac{t}{x_1}\right)\left(1-\frac{t}{x_2}\right)\left(1-\frac{t}{x_3}\right)\pi_{w_0}^B$$

$$P_\lambda^C = b_\lambda^{-1} x^\lambda \left(1-\frac{tx_2}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_3}\right)\left(1-\frac{tx_3}{x_2}\right)$$
$$\left(1-\frac{t}{x_2x_3}\right)\left(1-\frac{t}{{x_1}^2}\right)\left(1-\frac{t}{{x_2}^2}\right)\left(1-\frac{t}{{x_3}^2}\right)\pi_{w_0}^C$$

$$P_\lambda^D = b_\lambda^{-1} x^\lambda \left(1-\frac{tx_2}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_3}\right)$$
$$\left(1-\frac{tx_3}{x_2}\right)\left(1-\frac{t}{x_2x_3}\right)\pi_{w_0}^D\,.$$

Notice that $\prod_{\alpha\in\mathcal{R}^\heartsuit}(1-te^{-\alpha})\,\pi_{w_0}^\heartsuit$ specializes, for $t=1$, to the operator $\sum_{w\in W} w$. This leads to define, for λ dominant, a $\heartsuit$-monomial function $m_\lambda^\heartsuit$ to be the normalized image of x^λ under $\sum_{w\in W} w$.

11.5 Atoms

Instead of using the full set of positive roots, let us delete the simple roots and use the operator

$$\mho^\heartsuit = \prod_{\alpha\in\mathcal{R}\setminus\mathcal{S}} (1-te^{-\alpha})\, \pi_{w_0}^\heartsuit\,. \tag{11.5.1}$$

For $n=3$, these operators are

$$\mho^B = \left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{t}{x_1x_3}\right)\left(1-\frac{t}{x_2x_3}\right)\left(1-\frac{t}{x_1}\right)\left(1-\frac{t}{x_2}\right)\pi_{w_0}^B$$

$$\mho^C = \left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{t}{x_1x_3}\right)\left(1-\frac{t}{x_2x_3}\right)\left(1-\frac{t}{{x_1}^2}\right)\left(1-\frac{t}{{x_2}^2}\right)\pi_{w_0}^C$$

$$\mho^D = \left(1-\frac{tx_3}{x_1}\right)\left(1-\frac{t}{x_1x_2}\right)\left(1-\frac{t}{x_1x_3}\right)\pi^D_{w_0}\,.$$

With these restricted sets of roots, one defines *atoms* $A^\heartsuit_\lambda$, λ dominant, to be

$$A^\heartsuit_\lambda = x^\lambda\,\mho^\heartsuit\,. \tag{11.5.2}$$

We shall show that the functions $A^\heartsuit_\lambda$ and $P^\heartsuit_\lambda$ are related by the Moebius function of the dominance order on dominant weights, but we have first to say a few words about this order.

In type A, lower intervals for the dominant order can be defined by using the expansion of Schur functions in terms of monomials. Indeed, for two partitions, $\nu \le \lambda$ if and only if x^ν occurs in the expansion of s_λ.

One adopts the same definition in type B, C, D. Given two dominant weights, then $\nu \le_\heartsuit \lambda$ if and only if x^ν occurs in the expansion of $x^\lambda \pi^\heartsuit_{w_0}$.

For example, for $n = 3$, and the weight $\lambda = [3,1,1]$, the different sets $\{\nu \le \lambda\}$ are:

$$\text{type } A\ : \{[2,2,1],[3,1,1]\}$$

$$\begin{aligned}\text{type } B\ : \{&[0,0,0],[1,0,0],[1,1,0],[1,1,1],[2,0,0],[2,1,0],[2,1,1],\\ &\qquad\qquad [2,2,0],[2,2,1],[3,0,0],[3,1,0],[3,1,1]\}\end{aligned}$$

$$\text{type } C\ : \{[1,0,0],[1,1,1],[2,1,0],[2,2,1],[3,0,0],[3,1,1]\}$$

$$\text{type } D\ : \{[1,0,0],[1,1,-1],[1,1,1],[2,1,0],[2,2,1],[3,0,0],[3,1,1]\}$$

We have to adapt the definition of $|\lambda|$ and $\mathbf{n}(\lambda)$ to take into account that in type D, for n odd, the last component of a dominant weight may be negative. Thus let $||\lambda|| := \sum|\lambda_i|$, $\mathbf{n}(\lambda) = 0\lambda_1 + \cdots + (n-2)\lambda_{n-1} + (n-1)|\lambda_n|$.

Then the expression of $A^\heartsuit_\lambda$ in terms of $P^\heartsuit_\nu$ by a mere summation over the lower interval of λ, and conversely, the expression of $P^\heartsuit_\lambda$ in terms of $A^\heartsuit_\nu$ is given by a summation involving the Moebius function of the interval.

Theorem 11.5.1. *Let $\heartsuit$ be A, B, C or D, and λ be a dominant weight for this type. Then*

$$A^\heartsuit_\lambda = \sum_{\nu\le_\heartsuit\lambda} t^{k(||\lambda||-||\nu||)+\mathbf{n}(\nu)-\mathbf{n}(\lambda)}\,P^\heartsuit_\nu\,, \tag{11.5.3}$$

$$P^\heartsuit_\lambda = \sum_{\nu\le_\heartsuit\lambda} \mu^\heartsuit(\lambda,\mu)\,t^{k(||\lambda||-||\nu||)+\mathbf{n}(\nu)-\mathbf{n}(\lambda)}\,A^\heartsuit_\nu\,, \tag{11.5.4}$$

were $\mu^\heartsuit(\,,\,)$ is the Moebius function of the dominance order, with $k = 0$ in type A, n in type B, $n - 1/2$ in type C, $n{-}1$ in type D.

For example,

$$
\begin{array}{rcl}
A^B_{311} & = & t^{12}P^B_{000}+t^9P^B_{100}+t^7P^B_{110}+t^6P^B_{111}+t^6P^B_{200}+t^4P^B_{210}+t^3P^B_{211} \\
 & & \qquad +t^2P^B_{220}+tP^B_{221}+t^3P^B_{300}+tP^B_{310}+P^B_{311}\,, \\
P^B_{311} & = & A^B_{311}-tA^B_{211}-tA^B_{310}+t^2A^B_{220}\,. \\
A^C_{311} & = & t^7P^C_{100}+t^5P^C_{111}+t^3P^C_{210}+tP^C_{221}+t^2P^C_{300}+P^C_{311}\,, \\
P^C_{311} & = & A^C_{311}-t^2A^C_{300}-tA^C_{221}+t^3A^C_{210}\,. \\
A^D_{311} & = & t^5P^D_{1,0,0}+t^4P^D_{1,1,-1}+t^4P^D_{1,1,1}+t^2P^D_{2,1,0}+tP^D_{2,2,1}+tP^D_{3,0,0}+P^D_{3,1,1}\,, \\
P^D_{311} & = & A^D_{3,1,1}-tA^D_{3,0,0}-tA^D_{2,2,1}+t^2A^D_{2,1,0}\,. \\
A^D_{3,1,-1} & = & t^5P^D_{1,0,0}+t^4P^D_{1,1,-1}+t^4P^D_{1,1,1}+t^2P^D_{2,1,0}+tP^D_{2,2,-1}+tP^D_{3,0,0}+P^D_{3,1,-1}\,, \\
P^D_{3,1,-1} & = & A^D_{3,1,-1}-tA^D_{3,0,0}-tA^D_{2,2,-1}+t^2A^D_{2,1,0}\,.
\end{array}
$$

Here are the transitions matrices from Atoms to Hall-Littlewood (dominance order), and from $K^{\heartsuit}_{\lambda w_0}$ to Atoms, for $\lambda\in\mathbb{N}^3$ and $|\lambda|=3$ (in type D, there is the extra dominant weight $[1,1,-1]$).

Type B

$$
\left[\begin{array}{ccccccc}
000 & 100 & 110 & 200 & 111 & 210 & 300 \\ \hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
t^3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
t^5 & t^2 & 1 & \cdot & \cdot & \cdot & \cdot \\
t^6 & t^3 & t & 1 & \cdot & \cdot & \cdot \\
t^6 & t^3 & t & \cdot & 1 & \cdot & \cdot \\
t^8 & t^5 & t^3 & t^2 & t^2 & 1 & \cdot \\
t^9 & t^6 & t^4 & t^3 & t^3 & t & 1
\end{array}\right]
\left[\begin{array}{ccccccc}
000 & 100 & 110 & 200 & 111 & 210 & 300 \\ \hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
t{+}t^3 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
t^2{+}t^4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
t^2 & t & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & t^3{+}t^2{+}t & \cdot & \cdot & t & 1 & \cdot \\
\cdot & t^2{+}t^4 & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
$$

Type C

$$
\left[\begin{array}{ccccccc}
000 & 100 & 110 & 200 & 111 & 210 & 300 \\ \hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
t^4 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
t^5 & \cdot & t & 1 & \cdot & \cdot & \cdot \\
\cdot & t^2 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & t^4 & \cdot & \cdot & t^2 & 1 & \cdot \\
\cdot & t^5 & \cdot & \cdot & t^3 & t & 1
\end{array}\right]
\left[\begin{array}{ccccccc}
000 & 100 & 110 & 200 & 111 & 210 & 300 \\ \hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
t^2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
t+t^3 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & t+t^2 & \cdot & \cdot & t & 1 & \cdot \\
\cdot & t+t^3 & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
$$

Type D

$$\begin{array}{c}\begin{matrix} 000 & 100 & 110 & 200 & 111 & 11\bar{1} & 210 & 300 \end{matrix}\\ \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t^3 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ t^4 & \cdot & t & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & t & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & t & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & t^3 & \cdot & \cdot & t^2 & t^2 & 1 & \cdot \\ \cdot & t^4 & \cdot & \cdot & t^3 & t^3 & t & 1 \end{bmatrix}\end{array} \quad \begin{array}{c}\begin{matrix} 000 & 100 & 110 & 200 & 111 & 11\bar{1} & 210 & 300 \end{matrix}\\ \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t{+}t^2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ t^2 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & t & \cdot & \cdot & t & t & 1 & \cdot \\ \cdot & t^2 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}\end{array}$$

The following property of the specialization in $t = 1$ of the functions $A_\lambda^\heartsuit$ has been obtained by Postnikov [168].

Corollary 11.5.2.

$$A_\lambda^\heartsuit\Big|_{t=1} = \sum_{\nu \leq_\heartsuit \lambda} m_\nu^\heartsuit$$

For example, $A^D_{2,1,-1}\big|_{t=1} = x^{0,0,0}+x^{2,0,0}+x^{0,2,0}+x^{1,1,0}+x^{1,0,-1}+x^{1,-1,0}+x^{-2,0,0}+x^{0,0,-2}+x^{0,-2,0}+x^{-2,-1,-1}+x^{0,0,2}+x^{0,-1,-1}+x^{0,1,1}+x^{1,0,1}+x^{-1,0,-1}+x^{-1,-1,0}+x^{-1,-1,-2}+x^{-1,2,1}+x^{-1,1,2}+x^{-2,1,1}+x^{0,1,-1}+x^{0,-1,1}+x^{2,1,-1}+x^{1,2,-1}+x^{2,-1,1}+x^{1,-1,2}+x^{1,-2,1}+x^{-1,-2,-1}+x^{1,1,-2}+x^{-1,1,0}+x^{-1,0,1}$ has indeed no multiplicity.

11.6 Q'-Hall-Littlewood functions

By definition the Q'-Hall-Littlewood functions are the symmetric functions such that

$$(P_\lambda, Q'_\mu) = \delta_{\lambda,\mu}\,.$$

We have defined a basis $\{V_v\}$ which is adjoint to $\{U_v\}$. Since $P_\lambda = U_{\lambda\uparrow}$, one has

$$(P_\lambda, V_\mu) = \delta_{\lambda,\mu} = (P_\lambda, V_\mu \pi_\omega)\,.$$

Hence

$$Q'_\mu = V_\mu\, \pi_\omega\,. \tag{11.6.1}$$

From the expansion of V_μ in the basis $\{\widehat{K}_v\}$, one obtains the expansion of Q'_μ in the basis of Schur functions, since $\widehat{K}_v \pi_\omega = 0$ if v is not dominant, and $\widehat{K}_\lambda \pi_\omega = s_\lambda$ if λ is a partition.

This expansion is positive, and coefficients have been interpreted in terms of *charge* of tableaux. Denote by $\mathfrak{Tab}^\mu$ the set of tableaux of evaluation μ. Then one has [119]:

Proposition 11.6.1. *Let μ be a partition. For a tableau T, denotes $\lambda(T)$ its shape, and $\mathfrak{c}(T)$ its charge. Then*

$$Q'_\mu = \sum\nolimits_{T \in \mathfrak{Tab}^\mu} t^{\mathfrak{c}(T)} s_{\lambda(T)} \,. \tag{11.6.2}$$

The set $\mathfrak{Tab}^\mu$ has a structure of rank poset given by the cyclage [118, 116]. If $\nu \le \mu$ with respect to the dominance order, then $\mathfrak{Tab}^\nu$ is canonically isomorphic to a subposet of $\mathfrak{Tab}^\mu$. The complement in $\mathfrak{Tab}^\mu$ of all the subposets isomorphic to $\mathfrak{Tab}^\nu : \nu < \mu$ is a poset called *atom*[2] and denoted $\mathcal{A}(\mu)$ [98].

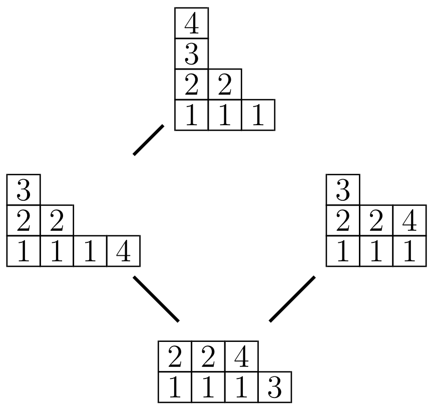
Using the Möbius function $(-1)^{\langle \mu,\nu \rangle}$ of the lattice of partition to define functions Q''_μ, one has [98]

Proposition 11.6.2. *Let μ be a partition. Then*

$$Q''_\mu := \sum_{\nu \le \mu} (-1)^{\langle \mu,\nu \rangle} t^{\mathbf{n}(\nu) - \mathbf{n}(\mu)} \, Q'_\nu = \sum_{T \in \mathcal{A}(\mu)} t^{\mathfrak{c}(T)} s_{\lambda(T)} \,. \tag{11.6.3}$$

For example, for $\mu = [3,2,1,1]$, the atom is

$\mathcal{A}([3,2,1,1]) =$



and this gives

$$Q''_{3211} = Q'_{3211} - tQ'_{322} - tQ'_{322} + 0Q'_{331} + t^3 Q'_{421} = s_{3211} + ts_{421} + ts_{331} + t^2 s_{43} \,.$$

In summary, one has

Proposition 11.6.3. *With respect to $(\ ,\ )$, $\{Q'_\lambda\}$ is the basis adjoint to $\{P_\lambda\}$ and $\{Q''_\lambda\}$ is the basis adjoint to $\{A_\lambda\}$.*

Both scalar products (s_λ, Q'_μ) and (s_λ, Q''_μ) are equal to sums $\sum t^{\mathfrak{c}(T)}$ over subsets of tableaux of shape λ, tableaux of weight μ in the first case, tableaux in $\mathcal{A}(\mu)$ in the second case.

[2] In [98], one rather takes the image of $\mathcal{A}(\mu)$ into the set of standard tableaux, i.e. tableaux with weight $[1,\ldots,1]$.

The following matrices record the scalar products (s_λ, Q''_μ). Rows must be read as the expansion of Schur functions in the basis A_λ, columns give the expansion of the functions Q''_μ in the Schur basis.

$$
\begin{array}{r|ccc|}
\texttt{3} & 1 & \cdot & \cdot \\
\texttt{21} & \cdot & 1 & t \\
\texttt{111} & \cdot & \cdot & 1
\end{array}
\qquad\qquad
\begin{array}{r|ccccc|}
\texttt{4} & 1 & \cdot & \cdot & \cdot & \cdot \\
\texttt{31} & \cdot & 1 & \cdot & t & t^3 \\
\texttt{22} & \cdot & \cdot & 1 & \cdot & t^2 \\
\texttt{211} & \cdot & \cdot & \cdot & 1 & t^2+t \\
\texttt{1111} & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
$$

$$
\begin{array}{r|ccccccc|}
\texttt{5} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\texttt{41} & \cdot & 1 & \cdot & t & \cdot & t^3 & t^6 \\
\texttt{32} & \cdot & \cdot & 1 & \cdot & t & t^2 & t^5+t^4 \\
\texttt{311} & \cdot & \cdot & \cdot & 1 & \cdot & t^2+t & t^5+t^4+t^3 \\
\texttt{221} & \cdot & \cdot & \cdot & \cdot & 1 & t & t^3+t^2+t^4 \\
\texttt{2111} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & t^3+t^2+t \\
\texttt{11111} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
$$

$$
\left[\begin{array}{ccccccccccc}
6 & 51 & 42 & 411 & 33 & 321 & 3111 & 222 & 2211 & 21111 & 111111 \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & t & \cdot & \cdot & t^3 & \cdot & \cdot & t^6 & t^{1\cdot} \\
\cdot & \cdot & 1 & \cdot & \cdot & t & t^2 & t^2 & \cdot & t^5+t^4 & t^9+t^8+t^7 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & t^2+t & \cdot & \cdot & t^5+t^4+t^3 & t^9+t^8+t^7+t^6 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & t^2 & t^4 & t^8+t^6 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & t & t & t & t^4+2\,t^3+t^2 & t^8+2\,t^7+2\,t^6+2\,t^5+t^4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & t^3+t^2+t & t^7+t^6+2\,t^5+t^4+t^3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & t^2 & t^6+t^5+t^3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & t^2+t & 2\,t^4+t^5+t^6+t^3+t^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & t^4+t^3+t^2+t \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]
$$

Chapter 12

Kazhdan-Lusztig bases

We have already used the Hecke algebra $\mathcal{H}_n$ to generate bases of polynomials (Macdonald polynomials, Hall-Littlewood polynomials). Kazhdan and Lusztig have defined a linear basis $\{C_w : w \in \mathfrak{S}_n\}$ with which we shall build still another linear basis of $\mathfrak{Pol}(\mathbf{x}_n)$ in this chapter.

12.1 Basis of the Hecke algebra

We take the Hecke algebra of type A with algebraic generators satisfying the Hecke relations $(T_i - t_1)(T_i - t_2) = 0$. It has a linear basis $\{T_w : w \in \mathfrak{S}_n\}$. Many families of interesting elements in the group algebra of $\mathfrak{S}_n$ are globally invariant under the inversion of permutations. However $T_{w^{-1}} \neq (T_w)^{-1}$, when w is not the identity, the Hecke algebra has more subtle symmetry properties than the group algebra.

Kazhdan and Lusztig [72] defined a basis which is invariant under the involution

$$\iota : \; T_w \to (T_{w^{-1}})^{-1}\,, \quad t_1 \to -t_2, \quad t_2 \to -t_1\,,$$

and has many interesting properties. In particular this basis gives information about singularities of Schubert varieties and of specializations of Schubert polynomials.

Requiring invariance under ι is not enough to characterize the basis. For example, $\{1, T_1 - t_1\}$ and $\{1, T_1 - t_2\}$ are two bases of $\mathcal{H}_2$ which could be candidate to replace the basis $\{1, s_1\}$ of $\mathbb{C}[\mathfrak{S}_2]$.

One has to complete the condition of being invariant under ι by a condition of "positivity".

Definition 12.1.1. *An element* $\sum c_w T_w$ *is* t-positive *if and only if the coefficients* c_w *belong to the linear span of the monomials* $t_1^\alpha t_2^\beta$ *with* $\alpha > \beta$.

The following theorem is due to Kazhdan and Lusztig [72].

Theorem 12.1.2. *There exist a unique linear basis C_w of $\mathcal{H}_n$, called* Kazhdan-Lusztig basis*, such that*

$$C_w = T_w + \sum_{v<w} P_w^v(t_1, t_2) T_v$$

is invariant under the involution ι, with t-positive coefficients $P_w^v(t_1, t_2)$ (the summation is over the Ehresmann-Bruhat order).

The specializations $P_w^v(-1, t_2)$ are called *Kazhdan-Lusztig polynomials.*

Thus, with the positivity condition, we have discriminated between T_1–t_1 and T_1–t_2, and must have

$$C_i := C_{s_i} = T_i – t_1 = T_i(-1)\,.$$

In length 2, one has, for $i \neq j$,

$$C_{s_i s_j} = C_i C_j = T_i T_j - t_1 T_i - t_1 T_j + t_1^2\,.$$

However,

$$C_1 C_2 C_1 = T_{321} - t_1 T_{231} - t_1 T_{312} + (t_1^2 - t_1 t_2) T_{213} + t_1^2 T_{132} - (t_1^3 - t_1^2 t_2) T_{123}$$

exhibits a violation $t_1 t_2 T_{213}$. In fact, the absence of symmetry in T_1, T_2 is also a good reason to exclude $C_1 C_2 C_1$. This can be repaired by taking

$$C_1 \left(C_2 - \frac{t_1 t_2}{t_1 - t_2} \right) C_1 = \sum_{w \in \mathfrak{S}_3} (-t_1)^{3-\ell(w)} T_w\,,$$

which satisfies all the requirements to be a Kazhdan-Lusztig element.

However,

$$C_1 C_2 C_1 = T_{321} - t_1 T_{231} - t_1 T_{312} + (t_1^2 - t_1 t_2) T_{213} + t_1^2 T_{132} - (t_1^3 - t_1^2 t_2) T_{123}$$

exhibits a violation $t_1 t_2 T_{213}$. In fact, the absence of symmetry in T_1, T_2 is also a good reason to exclude it. This can be repaired by taking

$$C_1 \left(C_2 - \frac{t_1 t_2}{t_1 - t_2} \right) C_1 = \sum_{w \in \mathfrak{S}_3} (-t_1)^{3-\ell(w)} T_w\,,$$

which satisfies all the requirements to be a Kazhdan-Lusztig element.

More generally, once known their existence, the strategy to build recursively the Kazhdan-Lusztig elements is clear. Knowing C_w, given i such that $\ell(ws_i) > \ell(w)$, one computes $f = C_w C_i$. Enumerating permutations $v < w$ by decreasing length, one replaces, for each term of the type $t_1^j t_2^j T_v$, f by $f - t_1^j t_2^j C_v$, and iterate till arriving to the identity permutation. The final value of f is equal to C_{ws_i}. In summary, there exists integers $\mu(v, w)$ such that

$$C_{ws_i} = C_w C_i + \sum \mu(v, w)(-t_1 t_2)^{(\ell(w)-\ell(v)+1)/2} C_v\,, \tag{12.1.1}$$

sum over a certain subset[1] of permutations smaller than w.

For example, the above expression of $C_1C_2C_1$ can be rewritten $C_{321} = C_{231}C_1 + t_1t_2C_1$.

Though the preceding algorithm is very simple to implement, it is unsatisfactory because it does not shed much light over the Kazhdan-Lusztig elements and polynomials.

Notice that all v appearing in (12.1.1) must be such that $\ell(v) > \ell(vs_i)$. Indeed, the image of this equation by right multiplication by C_i is

$$C_{ws_i}C_i = (t_2 - t_1)C_wC_i - \sum \mu(v,w)(-t_1t_2)^{(\ell(w)-\ell(v)+1)/2}C_vC_i\,,$$

and the unicity of the basis implies that for each v appearing in the summation one has $C_vC_i = (t_2 - t_1)C_v$.

For example, for $w = [3,4,5,1,2]$, one has

$$C_{34512}C_2 = C_{35412} - t_1t_2C_{34152} - t_1t_2C_{34215} + t_1^2t_2^2C_{1,4,3,2,5}\,,$$

and all permutations v appearing in the right hand side are such that $v_2 > v_3$.

12.2 Duality

We have introduced in (1.8.5) a quadratic form such that $\{\widehat{T}_{\omega\sigma}\}$ is the basis adjoint to $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$. Thus, the coefficients $P_w^v(t_1,t_2)$ may be expressed as

$$P_w^v(t_1,t_2) = \left(C_w\,,\,\widehat{T}_{\omega v}\right)^{\mathcal{H}}\,.$$

Taking any total order compatible with the Bruhat order, one has the property that the transition matrix between $\{\mathcal{C}_w\}$ and $\{T_w\}$ is unitriangular. It is natural to invert it, here it is for $n = 3$ (read by rows):

	123	132	213	312	231	321
123	1	0	0	0	0	0
132	t_1	1	0	0	0	0
213	t_1	0	1	0	0	0
312	${t_1}^2$	t_1	t_1	1	0	0
231	${t_1}^2$	t_1	t_1	0	1	0
321	${t_1}^3$	${t_1}^2$	${t_1}^2$	t_1	t_1	1

For this small example, the inverse matrix is obtained by just changing t_1 into $-t_1$. We refer to [65, Prop. 7.13] for the next proposition which describes the inverse of the matrix of Kazhdan-Lusztig polynomials for general n.

[1] Kazhdan and Lusztig characterize this subset in terms of a graph which, even in type A, presents much mystery.

Proposition 12.2.1. *For any pair of permutations: ν, ζ such that $\nu \le \zeta$, one has*

$$\sum_{z:\,\nu\le z\le\zeta} \left(C_z\,,\,\widehat{T}_{\omega\nu}\right)^{\mathcal{H}} \left(\left(C_{\omega z}\,,\,\widehat{T}_{\zeta}\right)^{\mathcal{H}}\right)^{t_1\to -t_1, t_2\to -t_2} = \delta_{\nu,\zeta}\,. \tag{12.2.1}$$

Equation 12.2.1 can be rewritten, using the KL involution on the second scalar product, as

$$\sum_{z:\,\nu\le z\le\zeta} \left(C_z\,,\,\widehat{T}_{\omega\nu}\right)^{\mathcal{H}} \left(\left(C_{\omega z}\,,\,T_{\zeta}\right)^{\mathcal{H}}\right)^{t_1\leftrightarrow t_2} = \delta_{\nu,\zeta}\,. \tag{12.2.2}$$

Define, for any $\zeta \in \mathfrak{S}_n$, $\widetilde{C}_\zeta$ to be the image of C_ζ under the exchange of t_1 and t_2. Then, since $\{\widehat{T}_{\omega\nu}\}$ is the basis adjoint to $\{T_\nu\}$, Eq. 12.2.2 translates into the following duality property.

Proposition 12.2.2. *The basis $\{\widetilde{C}_\zeta : w \in \mathfrak{S}_n\}$ is adjoint to the basis $\{C_\nu\}$, i.e. one has*

$$\left(C_{\omega\nu}\,,\,\widetilde{C}_\zeta\right)^{\mathcal{H}} = \delta_{\nu,\zeta}\,. \tag{12.2.3}$$

Notice that (12.2.3) is a statement about products $C_\nu \widetilde{C}_{\zeta^{-1}}$, while (12.2.2) involves sums of products of KL-polynomials. For example, one has $C_{45123} = C_3C_2C_1C_4C_3C_2$, $\widetilde{C}_{34512} = \widetilde{C}_2\widetilde{C}_1\widetilde{C}_3\widetilde{C}_2\widetilde{C}_4\widetilde{C}_3$, and $\left(C_{45123}\,,\,\widetilde{C}_{34512}\right)^{\mathcal{H}} = 0$, because the coefficient of T_{54321} in the expansion of $(C_3C_2C_1C_4C_3C_2)\left(\widetilde{C}_3\widetilde{C}_4\widetilde{C}_2\widetilde{C}_3\widetilde{C}_1\widetilde{C}_2\right)$ is null.

Relation 12.1.1 can be used to describe the regular representation of the Hecke algebra, as well as its irreducible representations [72]. The matrix M^ω representing T_ω, i.e. describing the multiplication by T_ω in the KL-basis, has special properties which should be investigated. Its entries are

$$\left[M^\omega\right]_{\zeta,\nu} = \left(C_\zeta T_\omega\,,\,\widetilde{C}_{\omega\nu}\right)^{\mathcal{H}}.$$

The last column of this matrix is the list of the coefficients of C_ω in the products $C_\zeta T_\omega$. One has

$$\left(C_\zeta T_\omega\,,\,\widetilde{C}_{1\ldots n}\right)^{\mathcal{H}} = \left(C_\zeta\,,\,T_\omega\right)^{\mathcal{H}} = \left(\left(C_\zeta\,,\,\widehat{T}_\omega\right)^{\mathcal{H}}\right)^{t_1\leftrightarrow -t_2},$$

using the KL-involution for the last equality. The last expression shows that the coefficient of C_ω in the product $C_\zeta T_\omega$ is equal to the image under the exchange $t_1 \leftrightarrow -t_2$ of the coefficient of $T_{1\ldots n}$ in C_ζ. In other words, the last column of M^ω furnishes the Kazhdan-Lusztig polynomials in the identity. For example,

$$\begin{aligned} C_{3412}T_{4321} &= (t_2^4 - t_1t_2^3)C_{4321} + \ldots \\ C_{3412} &= (t_1^4 - t_1^3t_2)T_{1234} + \ldots \end{aligned}$$

The inverse of the matrix M^ω are described by the following proposition.

Proposition 12.2.3. *Let N be the matrix with entries*

$$N_{\zeta,\nu} = (\text{-}t_2)^{-\ell(\zeta)} \big(C_\zeta T_\omega\,,\, \widetilde{C}_{\omega\nu}\big) t_1^{-\ell(\omega\nu)}\,.$$

Then the inverse of N is the image of N under the exchange $t_1 \leftrightarrow \text{-}t_2$, and N possesses the symmetry

$$N_{\omega\nu,\omega\zeta} = (\text{-}1)^{\ell(\zeta)+\ell(\omega\nu)} \Big[N_{\zeta,\nu}\Big]^{t_1\leftrightarrow -t_2}\,.$$

Proof. The entries of N^{-1} are obtained by taking the adjoint bases in the scalar products:

$$\big[N^{-1}\big]_{\nu,\zeta} = (\text{-}t_2)^{\ell(\zeta)} \Big(\widetilde{C}_{\omega\zeta}\widehat{T}_\omega(\text{-}t_1t_2)^{-\ell(\omega)}, C_\nu\Big)\, t_1^{\ell(\omega\nu)}\,.$$

Exchanging the role of ζ, ν, one has

$$\big[N^{-1}\big]_{\zeta,\nu} = (\text{-}t_2)^{-\ell(\omega\zeta)} \Big(\widetilde{C}_{\omega\nu}\widehat{T}_\omega, C_\zeta\Big)\, t_1^{-\ell(\zeta)}\,.$$

Using the KL-involution to transform the scalar product, one obtains

$$\begin{aligned}\big[N^{-1}\big]_{\zeta,\nu} &= (\text{-}t_2)^{-\ell(\omega\zeta)} \Big(\widetilde{C}_{\omega\nu}T_\omega, C_\zeta\Big)^{t_1\leftrightarrow -t_2} t_1^{-\ell(\zeta)} \\ &= \Big[t_1^{-\ell(\omega\nu)}\Big(\widetilde{C}_{\omega\nu}, C_\zeta T_\omega\Big)(\text{-}t_2)^{-\ell(\zeta)}\Big]^{t_1\leftrightarrow -t_2}\,,\end{aligned}$$

and this proves the required property about N^{-1}. The second statement results from the fact that $\widetilde{C}_w$ is obtained from C_w by the exchange of t_1, t_2; the scalar products $\big(C_\zeta T_\omega\,,\,\widetilde{C}_{\omega\nu}\big)^{\mathcal{H}}$ are homogeneous polynomials in t_1, t_2, so that taking their image under $t_1 \leftrightarrow \text{-}t_2$ or by $t_1 \leftrightarrow t_2$ just introduces an eventual sign that we took into account. QED

For $n = 3$, the matrices M and N are

$$\begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 312 & 231 & 321 \end{array} \\ \begin{bmatrix} {t_1}^3 & {t_1}^2 & {t_1}^2 & t_1 & t_1 & 1 \\ 0 & 0 & 0 & t_1t_2 & 0 & t_2 \\ 0 & 0 & 0 & 0 & t_1t_2 & t_2 \\ 0 & \text{-}{t_1}^2{t_2}^2 & 0 & 0 & 0 & {t_2}^2 \\ 0 & 0 & \text{-}{t_1}^2{t_2}^2 & 0 & 0 & {t_2}^2 \\ 0 & 0 & 0 & 0 & 0 & {t_2}^3 \end{bmatrix} \end{array} \quad \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 312 & 231 & 321 \end{array} \\ \begin{bmatrix} {t_1}^{-3} & -\frac{1}{t_2{t_1}^3} & -\frac{1}{t_2{t_1}^3} & \frac{1}{{t_2}^2{t_1}^3} & \frac{1}{{t_2}^2{t_1}^3} & -\frac{1}{{t_1}^3{t_2}^3} \\ 0 & 0 & 0 & -\frac{1}{{t_1}^2{t_2}^2} & 0 & \frac{1}{{t_2}^3{t_1}^2} \\ 0 & 0 & 0 & 0 & -\frac{1}{{t_1}^2{t_2}^2} & \frac{1}{{t_2}^3{t_1}^2} \\ 0 & \frac{1}{t_1t_2} & 0 & 0 & 0 & -\frac{1}{{t_2}^3t_1} \\ 0 & 0 & \frac{1}{t_1t_2} & 0 & 0 & -\frac{1}{{t_2}^3t_1} \\ 0 & 0 & 0 & 0 & 0 & {t_2}^{-3} \end{bmatrix} \end{array}$$

12.3 Peeling out canonical factors

Let us recall that, according to 1.9.12, the quasi-idempotent $\sum_{w\in\mathfrak{S}_n}(\text{-}t_1)^{\ell(\omega w)}T_w$ can be written as a product of factors of the type

$$T_i(\text{-}k) = T_i + \frac{t_1+t_2}{(\text{-}t_2/t_1)^k - 1} = C_i - t_1t_2\frac{[k\text{-}1]}{[k]} =: C_i(k\text{-}1)\,,$$

with $[k] = t_1^{k-1} - t_1^{k-2}t_2 + \cdots + (\text{-}t_2)^{k-1}$ and $[-k] = t_2^{k-1} - t_1 t_2^{k-1} + \cdots + (\text{-}t_1)^{k-1}$ for $k > 0$. The last expression is invariant under the KL involution, and therefore, for the permutation of maximal length of the symmetric group, one recognizes an element already met several times,

$$C_\omega = \sum_{w \in \mathfrak{S}_n} (\text{-}t_1)^{\ell(\omega w)} T_w = \nabla_\omega \,. \tag{12.3.1}$$

In other words, all Kazhdan-Lusztig polynomials $P_\omega^w(-1, t_2)$ are equal to 1. By exchange of t_1, t_2, one has

$$\widetilde{C}_\omega = \sum_{w \in \mathfrak{S}_n} (\text{-}t_2)^{\ell(\omega w)} T_w = \mathbb{U}_\omega \,. \tag{12.3.2}$$

More generally, given a Young subgroup $\mathfrak{S}_{a \times b \times \cdots}$, let $\omega_{a \times b \times \cdots}$ be the permutation of maximal length of this group. Then, by direct product, one has

$$C_{\omega_{a \times b \times \cdots}} = \sum_{w \in \mathfrak{S}_{a \times b \times \cdots}} (\text{-}t_1)^{\ell(\omega_{a \times b \times \cdots} w)} T_w \,.$$

Under certain conditions on w, one can factor out from C_w some $C_{\omega_{a \times b \times \cdots}}$. Let us just give a case that we shall need in the sequel. Given $m, r : 1 \leq m < r \leq n$, denote by $\square_{m,r,n} = C_{\omega_{1^m, r-m, 1^{n-r}}}$, and by $\widetilde{\square}_{m,r,n}$ the idempotent $\square_{m,r,n}[2]^{-1} \cdots [r\text{-}m\text{+}1]^{-1}$.

Lemma 12.3.1. *Let $w \in \mathfrak{S}_n$, $m, r : 1 \leq m < r \leq n$ be such that $w_m > w_{m+1} > \cdots > w_r$. Then there exists $h \in \mathcal{H}_n$ such that*

$$C_w = h \, \square_{m,r,n} \,. \tag{12.3.3}$$

Proof. The preceding lemma has shown that C_w is invariant under right multiplication by the idempotent $C_i/(t_2\text{-}t_1)$ for each $i = m, \ldots, r\text{-}1$. Therefore, it must be invariant under the idempotent $\widetilde{\square}_{m,r,n}$.

Thus one can write

$$C_w = \left(\sum_u Q_w^u(t_1, t_2) T_u \right) \square_{m,r,n} \,,$$

sum over permutations u which are of minimum length in their coset $u\mathfrak{S}_{1^m, r-m, 1^{n-r}}$. The coefficients $Q_w^u(t_1, t_2)$ must be polynomials in t_1, t_2, and not rational functions, and invariant under ι, otherwise the RHS would not be a Kazhdan-Lusztig element. QED

The left factor h is not necessarily a Kazhdan-Lusztig element. For example, one has the factorizations

$$C_{321} = \left(T_2 T_1 - t_1 T_1 + t_1^2\right) C_2 = \left((T_2 - t_1)(T_1 - \frac{t_1^2}{t_1 - t_2}) \right) C_2 \,,$$

but the first left factor is not invariant under ι, being different from C_{312}, and the second left factor does not have polynomial coefficients.

Corollary 12.3.2. *Let $w \in \mathfrak{S}_n$ be such that there exists $m < n : w_n = n$, and $w_m > w_{m+1} > \cdots > w_n$. Let $v = [w_1, \ldots, w_{m-1}, w_{m+1}, \ldots, w_n]$. Then*

$$C_w = C_v\big(T_{n-1} \ldots T_m - t_1 T_{n-1} \ldots T_{m+1} + \cdots + (-t_1)^{n-m}\big) \\ = C_v T_{n-1}(m-n) \cdots T_m(-1)\,. \quad (12.3.4)$$

Proof. From the hypothesis on w, there exists $h \in \mathcal{H}_n$ such that $C_v = h\,\square_{m,n-1}$, $C_w = h\,\square_{m,n}$. The result follows from
$\square_{m,n} = \square_{m,n-1} T_{n-1}(m-n) \cdots T_m(-1)$. QED

If the preceding corollary can be applied to w, or w^{-1}, or $\omega w \omega$, or $\omega w^{-1} \omega$, say that the permutation is *peelable.* allows to peel right or left factors from C_w. In the contrary case, and if w is not the identity, say that w is *irreducible.*

For example, let $w = [4, 1, 7, 6, 2, 3, 5]$. Then 1 is the first valley, one extracts $T_1(-1)$ from the right, and obtains $w' = [1, 4, 7, 6, 2, 3, 5]$. Taking the inverse $w" = [1, 5, 6, 2, 7, 4, 3]$, one sees that 7 is the last peak, and this allows the factorization of $T_5(-1)T_6(-2)$. One is left with $[1, 4, 6, 5, 2, 3, 7]$. Erasing the fixed points 1 and 7, one sees that the ensuing permutation $[4, 5, 1, 3, 2]$ cannot be reduced any more. The corollary has given

$$C_{4176235} = T_5(-1)T_6(-2)\, C_{1465237}\, T_1(-1) = (T_5T_6 - t_1T_6 + t_1^2)C_{1465237}(T_1 - t_1)\,.$$

In fact, $C_{4176235}$ factorizes totally, but this requires more work to be proved !

$$C_{4176235} = T_5(-1)T_6(2) \Big(T_3(-1)T_2(-1)T_5(-1)T_4(-2)T_3(-2)T_5(-1)T_4(-1)\Big)\, T_1(-1)\,.$$

Let us call *totally reducible* a permutation such that there exists a chain of reductions leading to the identity permutation.

12.4 Non-singular permutations

Let us call *non-singular*[2] a permutation w such that

$$C_w = \sum_{v \le w} (-t_1)^{\ell(w) - \ell(v)} T_v\,,$$

i.e. such that all $P^v_w(-1, t_2)$, $v \le w$, are equal to 1.

Lakshmibai and Sandhya [90] have proved that a Schubert variety w is non-singular if and only if the indexing permutation avoids the patterns 3412 and 4231. Notice that in that case $w\omega$ avoids 2143 and 1324, the first condition being that $w\omega$ be vexillary.

The following proposition [100] shows that the two notions, being non-singular, or being totally reducible, coincide.

[2]They are the permutations such that the corresponding Schubert varieties are non-singular. The relations between Kazhdan-Lusztig polynomials and Schubert varieties are explained in [73].

Proposition 12.4.1. *A permutation w is such that all $P_w^v(-1,t_2)$ are equal to* 1 *when $v \le w$ iff it is totally reducible.*

Proof. With the hypotheses of the last corollary, the set $\{P_w^u(-1,t_2),\ u \in \mathfrak{S}_n\}$ coincides with the set $\{P_v^u(-1,t_2),\ u \in \mathfrak{S}_n\}$. Therefore, if v is non-singular, then w is so. The same reasoning is valid when the reduction applies to w^{-1}, or $\omega w \omega$, or $\omega w^{-1}\omega$, instead of w.

Conversely, let w be irreducible and such that n is not a fixed point. If w avoids the pattern 3412, then one checks that w contains a subword of the type $[\ldots n \ldots b \ldots c \ldots a]$, with $a = w_n$, $a < b < c$. If w avoids the pattern 4231, then one checks that w contains a subword of the type $[\ldots c \ldots n \ldots a \ldots b]$, with $b = w_n$, $a < b < c$. QED

Thus, non-singularity can be controlled by looking for patterns 3412 and 4231, or testing recursively a condition on n and 1 inside w.

In the non-singular case, the preceding proposition gives a factorization [100] of C_w, and by specialization, of the Poincaré polynomial of the interval $[1,w]$.

For example, $w = [4,1,6,5,3,2]$ is non-singular, C_w factorizes as

$$C_{416532} = \Big(\Big(T_2(-1)T_3(-2)\big(T_2(-1)T_1(-1)\big)\Big)T_4(-2)T_3(-1)\Big)T_5(-3)T_4(-2)T_3(-1)\,.$$

Sending $T_i \to t_2$ transforms $T_i(-k)$ into $-[k{+}1]/[k]$. Therefore the image of $C_w = \sum_{v\le w}(-t_1)^{\ell(w)-\ell(v)}T_v$, using the preceding factorisation, is equal to

$$-\left(\left([2]\frac{[3]}{[2]}\big([2]^2\big)\right)\frac{[3]}{[2]}[2]\right)\frac{[4]}{[3]}\frac{[3]}{[2]}[2] = -[2]^2[3]^2[4]\,.$$

Notice that we thus recover the fact that C_ω factorizes into simple factors1.9.10:

$$C_{54321} = C_1(0)C_2(1)C_1(0)C_3(2)C_2(1)C_1(0)C_4(3)C_3(2)C_2(1)C_1(0)\,.$$

The factorization of the Poincaré polynomial in the non-singular case is due to Carrell and Peterson [18]. The specialisation $T_i = t_2$ of C_w has a geometrical interpretation in terms of some sophisticated cohomology theories [73]. Notice that, from a combinatorial point of view, using Kazhdan-Lusztig elements instead of intervals with respect to the Ehresman-Bruhat order regularizes the specialization. For example, the Poincaré polynomial for $w = [3,4,1,2]$ is equal to $(t_2 - t_1)({t_2}^3 - 3{t_2}^2 t_1 + 2t_2{t_1}^2 - {t_1}^3)$, while C_{3412} specializes to $(t_2 - t_1)^4$.

12.5 Kazhdan-Lusztig polynomial bases

We have used the pair $\{\nabla_{\omega\sigma}\}$, $\{\mathbb{U}_\sigma\}$ of adjoint bases of $\mathcal{H}_n$ to generate a pair $\{U_v\}, \{\widehat{U}_{v\omega}\}$ of adjoint bases of $\mathfrak{Pol}(\mathbf{x}_n)$. We shall see that the same construction works when starting from the pair of Kazhdan-Lusztig elements $\{C_{\omega\sigma}\}, \{\widetilde{C}_\sigma\}$. Take $t_1 = 1, t_2 = -t$ in this section.

As in the case of nonsymmetric Hall-Littlewood polynomials, some care is needed when indices have equal components. *Standardization* provides the link between elements of $\mathbb{N}^n$ and permutations. We have already used $v \to \langle v\rangle$ the standardization[3] , reading from left to right, by increasing values. We need a second one, the standardization by decreasing values, reading from right to left, that we denote $\langle\langle v\rangle\rangle$. For example,

$$
\begin{array}{c|cccccccc}
v = & 2 & 0 & 3 & 2 & 0 & 2 & 3 & 0 \\
\hline
 & & & 2 & & & & 1 & \\
 & 5 & & & 4 & & 3 & & \\
 & & 8 & & & 7 & & & 6 \\
\hline
\langle\langle v\rangle\rangle = & 5 & 8 & 2 & 4 & 7 & 3 & 1 & 6
\end{array}
$$

Then, for any $v \in \mathbb{N}^n$, with $\lambda = v\!\downarrow$, one defines

$$\widetilde{C}^x_v = \frac{x^\lambda}{b_\lambda}\, \widetilde{C}_{\langle\langle v\rangle\rangle}\,, \tag{12.5.1}$$

and

$$\mathcal{C}^x_v = \frac{x^\lambda}{b_\lambda}\, \mathcal{C}_{\langle -v\rangle}\,, \tag{12.5.2}$$

where the constant b_λ has been defined in (**??**).

For example, for $v = [2,2,0], [2,0,2], [0,2,2]$, one has $\langle\langle v\rangle\rangle = [2,1,3]$, $[2,3,1]$, $[3,2,1]$, and $\langle -v\rangle = [1,2,3], [1,3,2], [3,1,2]$. Therefore

$$\widetilde{C}^x_{220} = \frac{x^{220}}{1+t}\, \widetilde{C}_{213} = x^{220}\ , \quad \widetilde{C}^x_{202} = \frac{x^{220}}{1+t}\, \widetilde{C}_{231}\ , \quad \widetilde{C}^x_{022} = \frac{x^{220}}{1+t}\, \widetilde{C}_{321}\,,$$

$$\mathcal{C}^x_{220} = x^{220}\ , \quad \mathcal{C}^x_{202} = x^{220}\,\mathcal{C}_{132}\ , \quad \mathcal{C}^x_{022} = x^{220}\,\mathcal{C}_{312}\,.$$

Using the explicit values of the Kazhdan-Lusztig elements, one finds

$$\widetilde{C}^x_{220} = \frac{x^{220}}{1+t}\, T_1(1) = x^{220} = U_{220}\,,\ \widetilde{C}^x_{202} = \frac{x^{220}}{1+t}\, T_1(1)T_2(1) = U_{202} + \frac{t}{1+t}U_{220}\,,$$
$$\widetilde{C}^x_{022} = \frac{x^{220}}{1+t}\, T_1(1)T_2(2)T_1(1) = U_{022}\,,$$

[3]We also standardize $-v$, that is, we standardize v from left to right by decreasing values.

$$\mathcal{C}^x_{220} = x^{220} = \widehat{U}_{220}\ , \quad \mathcal{C}^x_{202} = x^{220}T_2(-1) = \widehat{U}_{202}\ ,$$
$$\mathcal{C}^x_{022} = x^{220}T_2(-1)(T_1-1) = \widehat{U}_{022} - \frac{t}{1+t}\,\widehat{U}_{202}\ .$$

In the case where $v = \lambda\uparrow$ is antidominant, then $\widetilde{C}^x_v = x^\lambda b_\lambda^{-1}\,\widetilde{C}_\omega = x^\lambda b_\lambda^{-1}\,\mathbb{U}_\omega$, and therefore is equal to the Hall-Littlewood polynomial $P_\lambda(x_1,\ldots,x_n)$. When λ is strict (i.e. all parts are different), then

$$\mathcal{C}^x_v = x^\lambda\,\mathcal{C}_\omega = x^\lambda\,\nabla_\omega = x^\lambda\,\partial_\omega \prod_{i<j}(x_j - tx_i) = s_{\lambda-\rho}(\mathbf{x}_n)\prod_{i<j}(x_j - tx_i)\ .$$

The same proof as for the bases $\{U_v\}$, $\{\widehat{U}_{v\omega}\}$ (Theorem **??**) gives the following duality between $\{\mathcal{C}^x_v\}$, $\{\widetilde{C}^x_{v\omega}\}$:

Theorem 12.5.1. *The two sets of polynomials* $\{\mathcal{C}^x_v : v\in\mathbb{N}^n\}$ *and* $\{\widetilde{C}^x_v : v\in\mathbb{N}^n\}$ *are two adjoint bases of* $\mathfrak{Pol}$ *with respect to the scalar product* $(\ ,\ )_t$*. More precisely, they satisfy*

$$(\mathcal{C}^x_v\ ,\ \widetilde{C}^x_{u\omega})_t = \delta_{v,u}\ .$$

12.6 Kazhdan-Lusztig and Hall-Littlewood

We have already seen that $\widetilde{C}^x_v = U_v$ when v is antidominant. In fact, one may consider the Kazhdan-Lusztig basis to be a deformation of the Hall-Littlewood basis.

Let $\lambda \in \mathbb{N}^n$ be a partition. Then the $\mathcal{H}_n$-module $x^\lambda \mathcal{H}_n$ has bases $\{U_v\}$, $\{\widehat{U}_v\}$, $\{\mathcal{C}^x_v\}$, $\{\widetilde{C}^x_v\}$, where v varies over all permutations of λ. The transition matrices between these different bases seem to present some interest.

Here is the transition matrix $\widetilde{C}^x_v \to U_v$ for $\lambda = [4,2,2,0]$, the rows of which describe the expansion of the successive $\widetilde{C}^x_v$:

$$
\begin{array}{r|cccccccccccc|}
4220 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4202 & \frac{t}{t+1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4022 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2420 & \frac{t}{t+1} & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2402 & \frac{t^2}{(t+1)^2} & \frac{t}{t+1} & \cdot & \frac{t}{t+1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2240 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2204 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{t(t+1)}{t^2+t+1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
2042 & \frac{t^2}{t+1} & \cdot & \frac{t}{t+1} & \cdot & \frac{t(t+1)}{t^2+t+1} & \frac{t}{t+1} & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
2024 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{t^2}{t^2+t+1} & \frac{t}{t+1} & \cdot & 1 & \cdot & \cdot & \cdot \\
0422 & \cdot & \cdot & \frac{t(t+1)}{t^2+t+1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
0242 & \cdot & \cdot & \frac{t^2}{t^2+t+1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{t}{t+1} & 1 & \cdot \\
0224 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
$$

From the recursions defining the different bases, one sees that the transition matrices do not depend on the parts of λ, but only on multiplicities. The same matrix as above is obtained for $\lambda = [2,1,1,0]$.

An example with bigger multiplicities:

$$
\begin{aligned}
\widetilde{C}^x_{02202} = U_{0,2,2,0,2} + \frac{U_{2,0,2,0,2}t}{t+1} + \frac{t^2 U_{2,2,0,0,2}}{t^2+t+1} + \frac{(t+1)\,tU_{0,2,2,2,0}}{t^2+t+1} \\
+ \frac{t^2 U_{2,0,2,2,0}}{t^2+t+1} + \frac{t^3\,(t+1)\,U_{2,2,0,2,0}}{(t^2+t+1)^2} + \frac{t^3 U_{2,2,2,0,0}}{t^2+1}\,.
\end{aligned}
$$

12.7 Using key polynomials

We have seen that,, then C_w, for a nonsingular permutation w, is a sum over an interval for the Ehresmann-Bruhat order. On the other hand, a key polynomial K_v is also a sum of polynomials $\widehat{K}_u$ over an interval for the same order. It is thus natural to try to relate Kazhdan-Lusztig elements to key polynomials.

Let, for this section only, $\rho = [1, \ldots, n]$, and let V_ρ be the linear span of monomials of exponents $w \in \mathfrak{S}_n$. Notice that $x^\rho T_w \cap V_\rho = (-t_2)^{\ell(w)} x^w$ Hence one has

$$x^\rho C_w \cap V_\rho = \sum_v (-t_1)^{\ell(w)-\ell(v)} P_w^v(t_1, t_2)(-t_2)^{\ell(v)} x^v \,.$$

Define a linear morphism φ from V_ρ to $\mathfrak{Pol}(\mathbf{x}_n)$ by

$$\varphi(x^v) = (-t_2)^{\ell(v)} x^{v\omega} \text{ if } v \text{ is a permutation },\ \varphi(x^v) = 0 \text{ otherwise}, \qquad (12.7.1)$$

with $\varphi(t_1) = -1$, $\varphi(t_2) = t_2$. The preceding equation reads now

$$\varphi(x^\rho C_w) = \sum_{v \le w} P_w^v(-1, t_2) x^{v\omega} \,. \qquad (12.7.2)$$

In particular, when w is non-singular, the image of the Kazhdan-Lusztig element reduces to a single key polynomial:

$$\varphi(x^\rho C_w) = \sum_{v \ge w\omega} x^v = K_{w\omega} \cap V_\rho \qquad (12.7.3)$$

For $n = 4$ there are only two singular permutations, which satisfy

$$\begin{array}{rcl} C_{3412} = C_2C_3C_1C_2 \quad , & \varphi(x^{1234}C_{3412}) & \equiv \; K_{2143} + t_2K_{4231} \\ C_{4231} = C_3C_2C_1(1)C_2C_3 \quad , & \varphi(x^{1234}C_{4231}) & \equiv \; K_{1324} + t_2K_{3412}\,. \end{array}$$

For $n = 5$, there are 32 singular permutations. Taking into account symmetries, eliminating those permutations coming from $[3, 4, 1, 2]$ or $[4, 2, 3, 1]$, one is left with only 8 permutations to study. We give a factorisation of C_w, writing C_i^+, C_i^{++} instead of $C_i(1), C_i(2)$. Only C_{45312} does not factorize into simple elements $C_i(k)$. It can however be written

$$C_{45312} = C_2C_3^+C_2 \left(C_4^+C_1^+ - \frac{t_1^3 t_2^3}{(t_2 - t_1)^2(t_2^2 - t_1t_2 + t_1^2)} \right) C_2C_3^+C_2 \,.$$

C_{45123}	$C_3C_2C_1C_4C_3C_2$	$K_{32154}+t_2K_{52341}+t_2K_{53142}+t_2K_{35241}-t_2K_{53241}$
C_{53412}	$C_4C_3C_2^+C_3C_4C_1^+C_2C_3$	$K_{21435}+t_2K_{41523}+t_2K_{42315}+t_2K_{24513}-t_2K_{42513}$
C_{52341}	$C_4C_3C_2C_1^+C_2C_3C_4$	$K_{14325}+t_2K_{45123}+t_2K_{34512}+t_2^2K_{45312}$
C_{45132}	$C_3C_4C_2^+C_3^+C_4C_1C_2$	$K_{23154} + t_2K_{25341}$
C_{35142}	$C_2C_1C_4C_3^+C_4C_2$	$K_{24153} + t_2K_{45231}$
C_{52431}	$C_1C_2C_3^+C_4^{++}C_2C_3^+C_2C_1$	$K_{13425} + t_2K_{34512}$
C_{54231}	$C_4C_3^+C_4C_2^+C_3C_1^{++}C_2^+C_3C_4$	$K_{13245} + t_2K_{34125}$
C_{45312}	*see above*	$K_{21354} + t_2^2K_{52341}$

From the table, by summation of the coefficients in the last column, one obtains the Kazhdan-Lusztig polynomials $P_w^{12345}(-1, t_2)$:

$$P_{45123}^{12345} = P_{53412}^{12345} = 1 + 2t_2,\ P_{52341}^{12345} = 1 + 2t_2 + t_2^2\,,$$
$$P_{45132}^{12345} = P_{35142}^{12345} = P_{52431}^{12345} = P_{54231}^{12345} = 1 + t_2,\ P_{45312}^{12345} = 1 + t_2^2\,.$$

The permutation $[4, 5, 3, 1, 2]$ belongs to the family $\{[n-1, n, n-2, \ldots, 3, 1, 2]\}$ which give rise to the Kazhdan-Lusztig polynomials equal to $1 + t_2^{n-1}$. These permutations are used by Polo [166] to build arbitrary Kazhdan-Lusztig polynomials.

12.8 Parabolic Kazhdan-Lusztig polynomials

Taking the action of Kazhdan-Lusztig elements on a weight space V_λ, when λ has repeated parts, produce polynomials with coefficients which, in general, are linear combinations of Kazhdan-Lusztig polynomials (see Deodhar [29]).

Let $\lambda \in \mathbb{N}^n$be a partition, $v = \lambda\uparrow$. Since $(x_i x_{i+1})^k C_i = 0$, then $x^v C_\sigma = 0$ if there exists $i : v_i = v_{i+1}$ and $\ell(s_i\sigma) < \ell(\sigma)$. To avoid this nullity, given v, one has to take any σ such that for any $i : v_i = v_{i+1}$, then σ contains the subword $i, i{+}1$.

One has

$$x^{12}C_1 = x^{12}(T_{21} - t_1 T_{12}) = -t_2 x^{21} - t_1 x^{12}\,.$$

To identify the action of C_1 on x^{12} to the expression of C_1 in the T_σ basis, one has to normalize monomials by length[4]:

$$-t_2 x^{21} - t_1 x^{12} \to -t_2\big((\text{-}t_2)^{-1}x^{21}\big) - t_1\big((\text{-}t_2)^{-0}x^{12}\big)\,.$$

More generally, given a partition λ, and an element in $x^v\mathcal{H}_n$ with $v = \lambda\uparrow$, one defines ψ_λ to be the morphism

$$\sum_{u:\, u\uparrow = v} c_u x^u + \sum_{w:\, w\uparrow \neq v} c_w x^w \to \sum c_u\Big|_{t_1=1} (\text{-}t_2)^{-\ell(u)} x^u\,.$$

One checks that for any $\sigma \in \mathfrak{S}_n$ one has

$$\psi_\lambda\left(x^v T_\sigma\right) = x^{v\sigma}\,, \tag{12.8.1}$$

so that the action of the Hecke algebra on x^v projects on the usual action of the symmetric group.

Given a partition $\lambda \in \mathbb{N}^n$, $v = \lambda\uparrow$ and $\sigma \in \mathfrak{S}_n$ such that $x^v C_\sigma \neq 0$, let $w = v\sigma$. Then

$$\psi_\lambda\left(x^v C_\sigma\right) = x^w + \sum_u (\text{-}1)^{\ell(w)-\ell(u)} P^u_{\lambda,\sigma}(t_2) x^u$$

and the polynomials $P^u_{\lambda,\sigma}(t_2)x^u$ are called *parabolic Kazhdan-Lusztig polynomials*. The next proposition relates these polynomials to the usual Kazhdan-Lusztig polynomials.

Given a composition $\alpha = [\alpha_1, \ldots, \alpha_k]$, let $\beta = 0^{\alpha_1}1^{\alpha_2}\ldots(k\text{-}1)^{\alpha_k}$. The projection of $\mathfrak{S}_n$ onto $\mathfrak{S}_\alpha\backslash\mathfrak{S}_n$ can be identified with the morphism

$$\mathfrak{S}_n \ni \sigma \xrightarrow{p_\beta} \beta_{\sigma_1}\ldots\beta_{\sigma_n}$$

from the symmetric group to words which are permuted of β.

[4] the *length* of an exponent u is defined to be the number of pairs such that $u_i > u_j$,$i < j$, by extension of the case of a permutation.

Proposition 12.8.1. *Given a partition $\lambda \in \mathbb{N}^n$, $v = \lambda\!\uparrow$ and $\sigma \in \mathfrak{S}_n$ such that $x^v C_\sigma \neq 0$, let $w = v\sigma$. Then*

$$P^u_{\lambda,\sigma}(t_2) = \sum_{\nu \in \mathfrak{S}_n :\, p_v(\nu) = u} (-1)^{\ell(\sigma)-\ell(\nu)} P^\nu_\sigma(-1, t_2)\,. \tag{12.8.2}$$

Proof. The statement is a direct consequence of (12.8.1). QED

For example, let $\sigma = [3, 4, 1, 2]$, $v = [0, 0, 1, 1]$. The correspondence between $x^v C_{3412} = x^{1100} - x^{1010} + t_2 x^{0101} - t_2 x^{0011}$ and the expansion of C_{3412} in the T_w basis is shown in the following enumeration (writing ν instead of T_ν):

$$\frac{\begin{matrix} & 2143 & \\ -1243 & & -2143 \\ & (1-t_2)1234 & \end{matrix}}{-t_2 x^{0011}} + \frac{\begin{matrix} & -2413 & \\ 2314 & & 1423 \\ & (t_2-1)1234 & \end{matrix}}{t_2 x^{0101}} + \frac{\begin{matrix} 1432 \\ -1342 \end{matrix}}{0x^{0110}} + \frac{\begin{matrix} 3214 \\ -3124 \end{matrix}}{0x^{1010}} + \frac{-3142}{-x^{1010}} + \frac{3412}{x^{1100}}$$

12.9 Graßmannian case

The Kazhdan-Lusztig polynomials corresponding to Schubert subvarieties of Graßmannians have been described in terms of increasing labelling of trees [120, 100]. The relevant permutations are the *coGraßmannian* permutations, i.e. $w \in \mathfrak{S}_n$ issuch that there exists $r: w_1 > \cdots > w_r;\ w_{r+1} > \cdots > w_n$.

Let $\mathcal{G}(r,n)$ be the module $C_{r\ldots 1\,n\ldots r+1}\mathcal{H}_n$. It has a linear basis $\{C_w\}$, where the w are coGraßmannian with a *rise* in r. Recall that, as an operator on polynomials,

$$C_{r\ldots 1n\ldots r+1} = \partial_{r\ldots 1n\ldots r+1}\Delta^{t_2t_1}(x_1,\ldots,x_r)\Delta^{t_2t_1}(x_{r+1},\ldots,x_n)$$

where $\Delta^{t_2t_1}(x_1,\ldots,x_r) = \prod_{1\le i<j\le r}(t_2x_i + t_1x_j)$.

In particular, putting $\rho_{r,n-r} = [r{-}1,\ldots,1,0,n{-}r,\ldots,1,0]$, one has

$$x^{\rho_{r,n-r}}C_{r\ldots 1n\ldots r+1} = \Delta^{t_2t_1}(x_1,\ldots,x_r)\Delta^{t_2t_1}(x_{r+1},\ldots,x_n)\,.$$

The space $\Delta^{t_2t_1}(x_1,\ldots,x_r)\Delta^{t_2t_1}(x_{r+1},\ldots,x_{2r})\mathcal{H}_{2r}$ is in fact a representation of the Temperley-Lieb algebra[5], and together with its Kazhdan-Lusztig basis, has been the object of numerous articles in the physics literature [43, 56, 70]. It has also a basis of Macdonald polynomials degenerated in $q = -(t_2/t_1)^3$. The relations between the Kazhdan-Lusztig basis and the Macdonald basis are described in [57].

Instead of computing in the space $C_{r\ldots 1n\ldots r+1}\mathcal{H}_n$, let us show that one obtains more simply the same Kazhdan-Lusztig polnomials using the dual basis $\{\widetilde{C}_v : v\!\downarrow = 1^r0^{n-r}\}$. The elementary elements that one has to use are $\widetilde{C}_i = \widetilde{C}_i(0) = T_i - t_2$, and their shifted versions:

$$\begin{aligned}
\widetilde{C}_i^{+} = \widetilde{C}_i(1) &= T_i + \frac{t_1+t_2}{(\text{-}t_1/t_2)^2-1} = \widetilde{C}_i - \frac{t_1t_2}{[2]}\\
\widetilde{C}_i^{++} = \widetilde{C}_i(2) &= T_i + \frac{t_1+t_2}{(\text{-}t_1/t_2)^3-1} = \widetilde{C}_i - \frac{t_1t_2[2]}{[3]}\\
\cdots \quad &\quad \cdots\\
\widetilde{C}_i(k) &= T_i + \frac{t_1+t_2}{(\text{-}t_1/t_2)^{k+1}-1} = \widetilde{C}_i - \frac{t_1t_2[k]}{[k{+}1]}
\end{aligned}$$

The description of the dual basis is made easier by interpreting the indices v as describing the border of the diagram of a partition. The correspondence between Graßmannian permutations w (with descent in r), $v : v\!\downarrow = 1^r0^{n-r}$ and partitions λ is

$$w \leftrightarrow v = [0^{w_1-1},1,0^{w_2-w_1-1},1,\ldots,0^{n-w_r}] \leftrightarrow \lambda = [w_r{-}r,\ldots,w_1-1]\,.$$

Given a partition $\lambda \in \mathbb{N}^r$, label recursively the boxes of the diagram of λ (using matrix conventions) as follows. Corners have label $\ell = 0$. Erase them. The new corners have labels $\ell = 1$. &c. Iterate till exhausting all boxes.

$$\begin{matrix} \ & \ & \ & 0\\ \ & 0 & & \\ 0 & & & \end{matrix}\ ,\quad \begin{matrix} \ & \ & 1 & 0\\ 1 & 0 & & \\ 0 & & & \end{matrix}\ ,\quad \begin{matrix} \ & 2 & 1 & 0\\ 1 & 0 & & \\ 0 & & & \end{matrix}\ ,\quad \begin{matrix} 3 & 2 & 1 & 0\\ 1 & 0 & & \\ 0 & & & \end{matrix}\ .$$

[5] quotient of $\mathcal{H}_n$ by the relations $T_i(1)T_{i\pm1}(2)T_i(1) = 0$.

Let $\widetilde{C}_{r,\lambda}$ be the product of the elements $\widetilde{C}_{r+j-i}(\ell(\square))$, i, j being the coordinates of the box $\square$, reading the boxes of the diagram of λ by successive rows. With $r = 3$, the preceding diagram gives $\widetilde{C}_{3,[421]} =$

$$\begin{array}{|c|c|c|c|} \hline \widetilde{C}_3(3) & \widetilde{C}_4(2) & \widetilde{C}_5(1) & \widetilde{C}_6(0) \\ \hline \widetilde{C}_2(1) & \widetilde{C}_3(0) & \multicolumn{2}{c}{} \\ \cline{1-2} \widetilde{C}_1(0) & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \Rightarrow \widetilde{C}_3(3)\widetilde{C}_4(2)\widetilde{C}_1(1)\widetilde{C}_6(0)\widetilde{C}_2(1)\widetilde{C}_3(0)\widetilde{C}_1(0)\,.$$

The following theorem is given in [76]

Theorem 12.9.1. *Let $r < n$ be two integers. Then*

$$\{x_1 \cdots x_r \widetilde{C}_{r,\lambda} : \lambda \subseteq (n-r)^r\}$$

coincides with the dual basis $\{\widetilde{C}^x_v : v \downarrow = 1^r 0^{n-r}\}$. The coefficients of the elements of the basis in the basis of monomials are the Kazhdan-Lusztig polynomials corresponding to pairs of coGraßmannian permutations.

For example, for $r = 2$, $n = 4$, the space $x_1 x_2 \mathcal{H}_4$ is 6-dimensional, with basis

$$\{x^{11},\ x^{11}\widetilde{C}_2,\ x^{11}\widetilde{C}_2^+\widetilde{C}_3,\ x^{11}\widetilde{C}_2^+\widetilde{C}_1,\ x^{11}\widetilde{C}_2^+\widetilde{C}_3\widetilde{C}_1,\ x^{11}\widetilde{C}_2^{++}\widetilde{C}_3^+\widetilde{C}_1^+\widetilde{C}_2\}\,.$$

The element $\widetilde{C}_{3,[421]} = \widetilde{C}^x_{0101001}$ has coefficient in x^{111} equal to $(t_1 - t_2)^2$. This implies that the Kazhdan-Lusztig polynomial $P^{1234567}_{6531742}$ is equal to $(1+t)^2$, the Graßmannian permutation corresponding to $[4, 2, 1]$ being $w = [2, 4, 7, 1, 3, 5, 6]$.

12.10 Dual basis and key polynomials

In a preceding section, we have used that $x^{1\dots n}T_w \cap V_{1\dots n} = (-t_2)^{\ell(w)}x^w$ to relate the Kazhdan-Lusztig basis to key polynomials.

On the other hand, the space $x_1 \cdots x_r \mathcal{H}_n$ has linear basis the monomials with exponent $v : v\!\downarrow = 1^r 0^{n-r}$, and thus has linear basis $\{K_v\}$ as well as $\{\widehat{K}_v = x^v\}$. In that case we can directly use the key polynomials without having to pass to a quotient space.

Under the specialization t_1 to 1 and t_2 to 0, the action of $\widetilde{C}_1$ becomes: $x^{10}\widetilde{C}_1 = t_1(x_1+x_2)$ is sent to x_1+x_2, $x^{01}\widetilde{C}_1 = -t_2(x_1+x_2)$ is sent to 0, and $x^{ii}\widetilde{C}_1 = (t_1-t_2)x^{ii}$ is sent to x^{ii}. In other words, $\widetilde{C}_1$ acts like π_1. Thus the dual Kazhdan-Lusztig basis in the weight space $V_{1^r0^{n-r}}$ may be considered as a deformation of the basis of key polynomials.

For example, for $r = 2$, $n = 5$, the Kazhdan-Lusztig basis is indexed by partitions contained in $[3,3]$ and has the following expansion in terms of the key polynomials K_{11}, K_{101}, K_{1001}, K_{10001}, K_{011}, K_{0101}, K_{0011}, K_{01001}, K_{00101}, K_{00011} :

$$\begin{array}{c|cccccccccc|}
[\,] & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
[1] & \cdot & t_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
[2] & \cdot & \cdot & {t_1}^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
[3] & \cdot & \cdot & \cdot & {t_1}^3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
[1,1] & \cdot & \cdot & \cdot & \cdot & {t_1}^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
[2,1] & -{t_1}^2 t_2 & \cdot & \cdot & \cdot & \cdot & {t_1}^3 & \cdot & \cdot & \cdot & \cdot \\
[2,2] & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {t_1}^4 & \cdot & \cdot & \cdot \\
[3,1] & -{t_1}^3 t_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {t_1}^4 & \cdot & \cdot \\
[3,2] & \cdot & \cdot & \cdot & \cdot & -t_2\, {t_1}^4 & \cdot & \cdot & \cdot & {t_1}^5 & \cdot \\
[3,3] & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {t_1}^6
\end{array}$$

Row [3,2], for example, has to be read $\widetilde{C}_{2,[32]} = t_1^5 K_{00101} - t_2 t_1^4 K_{011}$.

The Kazhdan-Lusztig polynomials $P_w^v(-1, t_2)$ themselves are equal to the image under $t_1 \to 1, t_2 \to -t_2$ of the coefficients in the basis $\widehat{K}_v$. For example, the expansion

$$\widetilde{C}_{3,[321]} = t_1^6 K_{010101} - t_2 t_1^5 K_{0111} - t_2 t_1^5 K_{110001} + -t_2^2 t_1^4 K_{1101}$$

furnishes the Kazhdan-Lusztig polynomials 1, $1+t_2$, $1+2t_2+t_2^2$, the expansion of $\widetilde{C}_{3,[321]} = \widetilde{C}^x_{010101}$ in the basis of monomials $x^v = \widehat{K}_v$ being

${t_1}^4 (t_1{-}t_2)^2 x_{111000} + {t_1}^4 (t_1{-}t_2)^2 x_{110100} + {t_1}^5 (t_1{-}t_2)\, x_{011100} + {t_1}^5 (t_1{-}t_2)\, x_{101100} + {t_1}^5 (t_1{-}t_2)\, x_{110010} + {t_1}^6 x_{101010} + {t_1}^6 x_{011010} + {t_1}^5 (t_1{-}t_2)\, x_{110001} + {t_1}^6 x_{101001} + {t_1}^6 x_{011001} + {t_1}^6 x_{100110} + {t_1}^6 x_{010110} + {t_1}^6 x_{010101} + {t_1}^6 x_{100101}$.

The rule to read Kazhdan-Lusztig polynomials from a partition is given in [120, 100].

Chapter 13

Complements

13.1 The symmetric group

13.1.1 Permutohedron

Given $v \in \mathbb{N}^n$, one generates a directed graph $\mathcal{I}(v)$ by iterating sorting operations $u \to us_i$ if u_i if $v_i > v_{i+1}$. This graph is in fact a rank lattice, with extremal elements v and $v\!\uparrow$. The rank $\ell(u)$ is the number of inversions (as for permutations, an inversion is a subword ji with $j > i$).

The set $\mathcal{I}(v)$ can be generated recursively by using a restricted shuffle Ⓢ defined as follows. For $v \in \mathbb{N}^n$, $k \in \mathbb{N}$, let i be such that $v_i \cdots v_n$ be the maximal left factor of v such that $v_j > k$, $j = i, \ldots, n$. Then

$$v Ⓢ k = \{[v_1, \ldots, v_{i-1}, k, v_i, \ldots, v_n], [v_1, \ldots, v_i, k, v_{i+1}, \ldots, v_n], \ldots, [v_1, \ldots, v_n, k]\} .$$

It is clear that

$$\mathcal{I}(v) = v_1 Ⓢ v_2 Ⓢ \cdots Ⓢ v_n .$$

When $v = [n, \ldots, 1]$, the poset $\mathcal{I}(v)$ is called the (right) *permutohedron*, and the underlying order on the symmetric group is called, unfortunately [11], the (right) *weak order*.

Another approach to intervals for the weak order is to replace sets of elements of $\mathfrak{S}_n$ by sums in the group algebra $\mathbb{Z}[\mathfrak{S}_n]$ or in the algebra $\mathbb{Z}[\partial_1, \ldots, \partial_{n-1}]$.

Let $\square_n = \sum_{\sigma\in\mathfrak{S}_n} \partial_\sigma$. We have already used that

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_n} \sigma &= \left(\sum_{\sigma\in\mathfrak{S}_{n-1}} \sigma\right) (1 + s_{n-1} + s_{n-1}s_{n-2} + \cdots + s_{n-1}\cdots s_1) && (13.1.1)\\ &= (1 + s_{n-1} + s_{n-2}s_{n-1} + \cdots + s_1\cdots s_{n-1}) \sum_{\sigma\in\mathfrak{S}_{n-1}} \sigma && (13.1.2) \end{aligned}$$

Correspondingly, one has

$$\begin{aligned} \square_n &= \square_{n-1} (1 + \partial_{n-1} + \partial_{n-1}\partial_{n-2} + \cdots + \partial_{n-1}\cdots\partial_1) && (13.1.3)\\ &= (1 + \partial_{n-1} + \partial_{n-2}s_{n-1} + \cdots + \partial_1\cdots\partial_{n-1}) \square_{n-1} . && (13.1.4) \end{aligned}$$

Factorizing further 13.1.1 requires using the Yang-Baxter relations **??**. This is in fact easier in $\mathbb{Z}[\partial_1, \ldots, \partial_{n-1}]$. Indeed

$$\begin{aligned} &1 + \partial_{n-1} + \partial_{n-1}\partial_{n-2} + \cdots + \partial_{n-1}\cdots\partial_1 \\ &\qquad = (1 + \partial_{n-1})(1 + \partial_{n-1}\partial_{n-2}) \cdots (1 + \partial_{n-1}\partial_{n-2}\cdots\partial_1) \end{aligned}$$

because all products $(\partial_{n-1}\cdots\partial_i)(\partial_{n-1}\cdots\partial_j)$ vanish. Therefore, one has

$$\begin{aligned} \square_n &= \square_{n-1} (1 + \partial_{n-1})(1 + \partial_{n-1}\partial_{n-2}) \cdots (1 + \partial_{n-1}\partial_{n-2}\cdots\partial_1) && (13.1.5)\\ &= (1 + \partial_1\cdots\partial_{n-1}) \cdots (1 + \partial_{n-2}\partial_{n-1})(1 + \partial_{n-1}) \square_{n-1} . && (13.1.6) \end{aligned}$$

The inverse of $1+\partial_{n-1}\cdots\partial_i$ being $1-\partial_{n-1}\cdots\partial_i$, the element $\square_n$ has an inverse Ω_n which is equal to

$$\Omega_n = (1-\partial_{n-1}\partial_{n-2}\cdots\partial_1)\cdots(1-\partial_{n-1})\,\Omega_{n-1} \tag{13.1.7}$$
$$= \Omega_{n-1}\,(1-\partial_{n-1})\cdots(1-\partial_1\cdots\partial_{n-1})\,. \tag{13.1.8}$$

One can in fact check by induction on n that

$$\Omega_n = \sum_{k=1}^{n} (\text{-}1)^{n-k} \sum_{v\in\mathbb{N}_+^k,\,|v|=n} \partial_{\omega_v}\,, \tag{13.1.9}$$

sum over the maximal elements of all the Young subgroups of $\mathfrak{S}_n$. This expression encodes the Möbius function of the permutohedron [11, Cor. 3.2.8].

Thus,

$$\Omega_3 = \partial_{123} - \partial_{213} - \partial_{132} + \partial_{321}$$
$$= 1 - \partial_1 - \partial_2 + \partial_2\partial_1\partial_2 = (1-\partial_1)(1-\partial_2)(1-\partial_1\partial_2)\,.$$

The Grothendieck polynomials are a deformation of Schubert polynomial: one obtains $\widetilde{G}_v(\mathbf{x},\mathbf{y})$ from $Y_v(\mathbf{x},\mathbf{y})$ by adding terms of degree $> |v|$. Intervals in the permutohedron furnish another deformation, but this time adding terms of lower degree. Define $L_\sigma(\mathbf{x},\mathbf{y}) = X_\sigma(\mathbf{x},\mathbf{y})\,\square_n$, $\sigma\in\mathfrak{S}_n$.

Divided difference in $\mathbf{y}$ commute with divided differences in $\mathbf{x}$, and therefore, when $\ell(s_i\sigma) < \ell(\sigma)$,

$$L_{s_i\sigma}(\mathbf{x},\mathbf{y}) = -X_\sigma(\mathbf{x},\mathbf{y})\partial_i^{\mathbf{y}}\,\square_n = -L_\sigma(\mathbf{x},\mathbf{y})\partial_i^{\mathbf{y}}\,.$$

In other words, the basis $\{L_\sigma(\mathbf{x},\mathbf{y})\}$ is generated from $L_\omega(\mathbf{x},\mathbf{y})$ by using the divided differences in $\mathbf{y}$. Taking intervals of the left permutohedron, one would obtain a basis $\{X_\sigma \sum_\zeta (\text{-}1)^{\ell(\zeta)}\partial_\zeta^{\mathbf{y}} \,:\, \sigma\in\mathfrak{S}_n\}$ generated from its top element by using divided differences in $\mathbf{x}$.

Since $\square_n\Omega_n = 1$, the relations (2.6.6)

$$\Big(X_\sigma(\mathbf{x},\mathbf{y})\,,\,X_\zeta(\mathbf{x}^\omega,\mathbf{y})\Big)^\partial = (\text{-}1)^{\ell(\zeta)}\delta_{\sigma,\zeta\omega}$$

are equivalent to

$$\Big(X_\sigma(\mathbf{x},\mathbf{y})\square_n\,,\,X_\zeta(\mathbf{x}^\omega,\mathbf{y})\Omega_n\Big)^\partial = (\text{-}1)^{\ell(\zeta)}\delta_{\sigma,\zeta\omega}\,, \tag{13.1.10}$$

and therefore, the basis adjoint to $\{L_\sigma(\mathbf{x},\mathbf{y})\}$, with respect to $(\ ,\ )^\partial$, is $\{(\text{-}1)^{\ell(\zeta)}X_\zeta(\mathbf{x}^\omega,\mathbf{y})\Omega_n\}$.

13.1.2 Rothe diagram

A permutation σ can be represented by a matrix $M(\sigma)$, which describes its action on the vector space with basis $1, 2, \ldots, n$. Explicitly, $M(\sigma)$ has entries 1 in positions $[i, \sigma_i]$, and 0 elsewhere (taking the usual coordinates of matrices, not the Cartesian plane).

Rothe[177] found in 1800 a graphical display of the inversions of σ, starting from $M(\sigma)$ (though, of course, matrices had still to wait 50 years to appear), which leads to many combinatorial properties of permutations.

For each pair of 1's in $M(\sigma)$ in relative position $\begin{smallmatrix}0 & \cdots & 1\\ \vdots & & \\ 1 & & \end{smallmatrix}$ write a box $\square$ at the intersection of the top row and left column containing these entries, thus obtaining $\begin{smallmatrix}\square & \cdots & 1\\ \vdots & & \\ 1 & & \end{smallmatrix}$.

The planar set of such boxes is called the *Rothe diagram* of σ. The list of the number of boxes in the successive rows is the code of the permutation, One can also read the canonical reduced decomposition of σ (defined in section 1.1) from the Rothe diagram: number boxes in each row by consecutive numbers, starting from the number i in row i. Reading rows from right to left, from top to bottom gives the canonical reduced decomposition.

For example, the code of $\sigma = [4, 2, 6, 5, 8, 1, 3, 7]$ is $[3, 1, 3, 2, 3, 0, 0, 0]$, the canonical reduced decomposition of σ is $(s_3 s_2 s_1)(s_2)(s_5 s_4 s_3)(s_5 s_4)(s_7 s_6 s_5)$, and the numbered Rothe diagram is (the 1's in the matrix representing the permutation are replaced by $\bullet$)

$$
\begin{array}{c|c|cccccccc}
4 & 3 & \boxed{1} & \boxed{2} & \boxed{3} & \bullet & \cdot & \cdot & \cdot & \cdot \\
2 & 1 & \boxed{2} & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & 3 & \boxed{3} & \cdot & \boxed{4} & \cdot & 5 & \bullet & \cdot & \cdot \\
5 & 2 & \boxed{4} & \cdot & \boxed{5} & \cdot & \bullet & \cdot & \cdot & \cdot \\
8 & 3 & \boxed{5} & \cdot & \boxed{6} & \cdot & \cdot & \cdot & \boxed{7} & \bullet \\
1 & 0 & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 0 & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
7 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot \\
\sigma & \mathit{code} & & & & & & & &
\end{array}
$$

To build the Rothe diagram, instead of considering pairs of 1's in the matrix representing a permutation, one can use the fact that there is no box right of a 1 in its row, and no box below a 1 in the same column. The Rothe diagram occupies the places which are not eliminated and which do not contain a 1.

The simplest non trivial Rothe diagram is $\left[\begin{smallmatrix}\square & 1\\ 1 & 0\end{smallmatrix}\right]$. Instead of putting a box, one can use a parameter x, and consider $\left[\begin{smallmatrix}x & 1\\ 1 & 0\end{smallmatrix}\right]$, or more generally, for i: $1 \le i < n$, replace the matrix representing s_i by

$$T_i(x) := \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & x & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

Let r be an integer and $I = [i_1, \ldots, i_r] \in \{1, \ldots, n{-}1\}^r$, such that $s^I := s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of a permutation σ. Define $T_I(x_1, \ldots, x_r)$ to be the product

$$T_I(\mathbf{x}) = T_I(x_1, \ldots, x_r) = T_{i_r}(x_r) \cdots T_{i_1}(x_1)\ .$$

The matrix $T_I(\mathbf{x})$ depends on the choice of the reduced decomposition of σ. When specializing all x_i's to 0, one recovers the matrix representing σ. The combinatorial properties of the matrix $T_I(\mathbf{x})$ are studied in [74]. In particular, removing in the matrix all monomials of degree different from 1, one obtains a *balanced labelling* of the Rothe diagram [37], the concept of being balanced first appearing in the work of Edelman and Greene [34] about reduced decompositions.

For example, for $\sigma = [3, 4, 2, 5, 1]$, the canonical reduced decomposition $[2, 1, 3, 2, 3, 4]$ and the reduced decomposition $[1, 2, 1, 3, 2, 4]$ give the following matrices:

$$\begin{bmatrix} x_2 & x_1 & 1 & 0 & 0\\ x_4 & x_3 & 0 & 1 & 0\\ x_5 & 1 & 0 & 0 & 0\\ x_6 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad \begin{bmatrix} x_1x_3 + x_2 & x_3 & 1 & 0 & 0\\ x_1x_5 + x_4 & x_5 & 0 & 1 & 0\\ x_1 & 1 & 0 & 0 & 0\\ x_6 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

13.1.3 Ehresmann-Bruhat order

The most elementary order is the *inclusion order* of sets. We shall argue in this section that ordering finite sets can be formulated in terms of inclusion of sets. For example, Schubert cells in a Graßmannian are indexed by partitions. The order associated to a cellular decomposition of the Graßmannian corresponds to teh inclusion of diagrams of the corresponding partitions. Ehresmann [35] gave more generally a cellular decomposition of the flag variety. Cells are indexed by permutations (Ehresmann was instead taking the left factors of a permutation, viewed as a word, and considered them as a flag of subsets of $\{1, \ldots, n\}$):

$$\sigma \in \mathfrak{S}_n \to \{\sigma_1\} \subset \{\sigma_1, \sigma_2\} \subset \cdots \subset \{\sigma_1 \ldots, \sigma_n\} = \{1, \ldots, n\} .$$

Writing a set of k integers as a decreasing vector, and subtracting $[k, \ldots, 1]$, one obtains from a permutation a sequence $\lambda_1(\sigma), \lambda_2(\sigma), \ldots, \lambda_n(\sigma)$ of partitions. Ehresmann defined an order on cells by requiring that

$$\sigma \leq \zeta \;\Leftrightarrow\; \lambda_1(\sigma) \subseteq \lambda_1(\zeta),\, \lambda_2(\sigma) \subseteq \lambda_2(\zeta),\, \ldots, \lambda_n(\sigma) \subseteq \lambda_n(\zeta) . \tag{13.1.11}$$

This definition amounts to the componentwise order of the corresponding Ehresmann tableaux, using partitions instead of sets of integers. Equivalently, a permutation gives a sequence of Graßmannian permutations

$$p_1(\sigma), p_2(\sigma), \ldots, p_{n-1}(\sigma) ,$$

where $p_k(\sigma)$ is the permutation of minimal length in the coset $\sigma\mathfrak{S}_{k|n-k}$.

Notice that comparing a permutation σ to a Graßmannian permutation g with descent in k requires only the comparison of $p_k(g)$ and σ. Associating to σ the set $\mathcal{G}(\sigma) = \{g,\, g \leq \sigma\}$ of Graßmannian permutations smaller than it, one can rephrase the Ehresmann order :

$$\sigma \leq \zeta \;\Leftrightarrow\; \mathcal{G}(\sigma) \subseteq \mathcal{G}(\zeta) .$$

One can think of using the same type of construction for any finite ordered set. Given a poset X, find an "optimal" subset B of X such that $X \to 2^B$ be a morphism of posets (2^B is the set of subsets of B, ordered by inclusion, the morphism being $x \to B(x) = \{b \in B,\, b \leq x\}$). Given two subsets C, C' having this property, then the intersection $C \cap C'$ also satisfies it. Therefore, there exists an optimal subset, that is called the *basis* of the order, such that $X \to 2^B$ be a poset morphism.

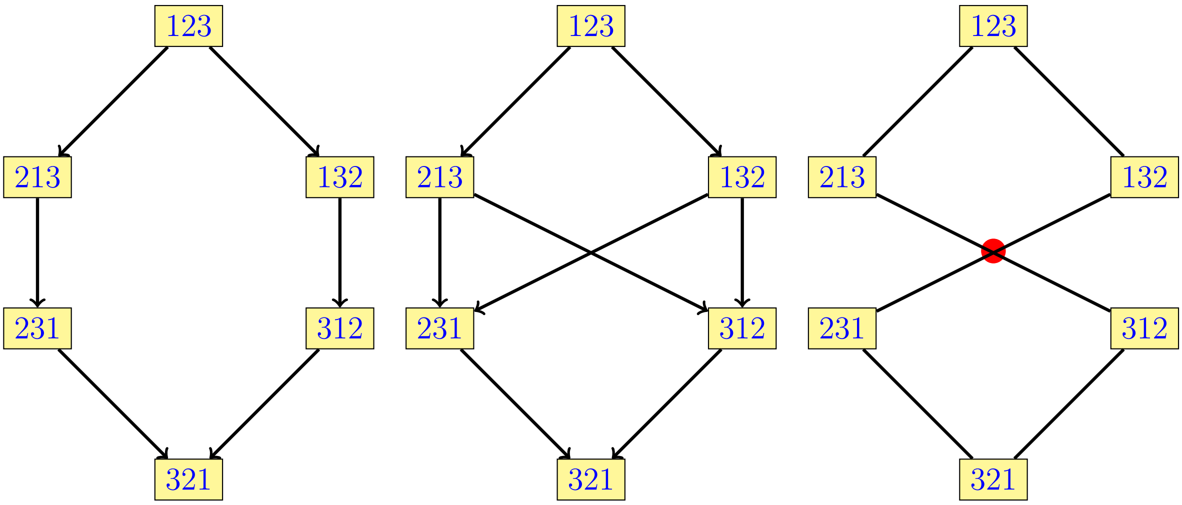
In the case of the symmetric group, the basis is the set of *biGraßmannian permutations* (i.e. permutations whose code is of the type $[0^a b^c 0^d]$, $bc \neq 0$), which is a subset of the set of Graßmannian permutations. Hence, the basis of $\mathfrak{S}_n$ is of cardinality $\binom{n+1}{3}$ and provides an efficient way of coding intervals in the Ehresmann order. Geck and Kim [51] describe the basis of every finite Coxeter group.

As a consequence, the symmetric group $\mathfrak{S}_n$ is embedded into a lattice (obtained by taking unions of $B(\sigma)$), which is called its *enveloping lattice* , or Mac Neille completion. The enveloping lattice, which happens to be distributive, is also easily obtained by replacing permutations by their Ehresmann tableaux, and taking the supremum or infimum of tableaux componentwise: given two tableaux $t = \{t[i,j]\}$, $u = \{u[i,j]\}$ of the same shape, then $t \wedge u$ is the tableau $\{max(t[i,j], u[i,j]\}$. The vertices of the enveloping lattice are, in this interpretation, tableaux of staircase shape made of elementary pieces of the type

$$\begin{array}{|c|c} \hline c & \\ \hline a & \multicolumn{1}{c|}{b} \\ \hline \end{array}\,, \quad \begin{array}{|c|c} \hline c & \\ \hline a & \multicolumn{1}{c|}{a} \\ \hline \end{array}\,, \quad \begin{array}{|c|c} \hline c & \\ \hline a & \multicolumn{1}{c|}{c} \\ \hline \end{array}\,,$$

with $a < b < c \in \{1, \ldots, n\}$ (in other words $\begin{array}{|c|c} \hline b & \\ \hline a & \multicolumn{1}{c|}{c} \\ \hline \end{array}$ is forbidden). These tableaux (also called *monotone triangles*) are in bijection with alternating sign matrices.

For $n = 3$, one has



Permutohedron Ehresmannoedre Enveloping lattice

The enveloping lattice is obtained by adding one element to $\mathfrak{S}_3$, this red element being interpreted as the tableau $\begin{array}{l} 3 \\ 2\,3 \\ 1\,1\,2 \end{array}$, or the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

For $n = 4$, there are $42 - 24 = 18$ tableaux not coming from permutations:

$$\begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,3 \\ 1\,1\,1\,1 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,3 \\ 1\,1\,2\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,3 \\ 1\,1\,1\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,3 \\ 1\,1\,2\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,3 \\ 1\,1\,1\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,3 \\ 2\,2\,3 \\ 1\,1\,1\,2 \end{array}$$

$$\begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,4 \\ 1\,1\,1\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,1\,1\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,1\,2\,2 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,1\,1\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,4 \\ 1\,1\,1\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,2\,2\,3 \end{array}$$

$$\begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,4 \\ 1\,1\,2\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,1\,2\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,3 \\ 1\,1\,1\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,3 \\ 1\,1\,2\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,2\,3 \\ 1\,1\,2\,3 \end{array} \quad \begin{array}{l} 4 \\ 3\,4 \\ 2\,3\,4 \\ 1\,1\,2\,4 \end{array}\,.$$

The symmetric group $\mathfrak{S}_n$, with its generators $\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ is a *Coxeter system* [11]. Given a Coxeter system $(W, \mathcal{S})$, one usually defines a *Bruhat order* on W by having recourse to reduced decompositions. Let $s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of $w \in W$. Then $v \leq w$ with respect to the Bruhat order if and only if there exists a reduced decomposition of v which is a subword of $s_{i_1} \cdots s_{i_k}$.

It is easy to check that two permutations σ, ζ are consecutive with respect to the Bruhat order if and only if $\sigma\zeta^{-1}$ is a transposition. Therefore the Bruhat order for the symmetric group coincides with the Ehresmann order. We shall use the terminology *Ehresmann-Bruhat order* for it, and refering to [11] for more properties.

We have seen in Lemma 1.10.4 that 0-Hecke algebras give an easy way of generating lower intervals for the Ehresmann-Bruhat order on classical Weyl groups. Stembridge [187] gives more generally a short derivation of the Möbius function for the Bruhat orderings of Coxeter groups and their parabolic quotients by using the 0-Hecke algebra.

13.2 t-Schubert polynomials

The spectral vectors for Macdonald polynomials specialize, in $q = 1$, into a permutation of $[t^0, t^1, \ldots, t^{n-1}]$. Let us show that one can similarly generalize the spectral vectors for Schubert polynomials by multiplying the components by a permutation of $[t^0, t^1, \ldots, t^{n-1}]$.

Let $v \in \mathbb{N}^n$ be the code of a permutation σ. Let $\beta \in \mathbb{N}^n$ be such that $\beta_i = \#(\sigma_j < \sigma_i,\, i \neq j)$, that is, β and $[\sigma_1, \ldots, \sigma_n]$ are two copies of the same permutation in $\mathfrak{S}(\{0, 1, \ldots, n{-}1\})$ and $\mathfrak{S}(\{\sigma_1, \ldots, \sigma_n\})$ respectively. Let moreover

$$\langle v^t \rangle = [t^{\beta_1} y_{\sigma_1}, \ldots, t^{\beta_n} y_{\sigma_n}] \, .$$

One defines the *t-Schubert polynomial* $Y_v^t(\mathbf{x})$ by the condtions

$$\begin{cases} Y_v^t\big(\langle u^t \rangle\big) = 0 \text{ for all } u : \, |u| \leq |v|,\, u \neq v \, , \\ Y_v^t(\mathbf{x})\Big|_{t=1} = Y_v(\mathbf{x}, \mathbf{y}) \end{cases} \tag{13.2.1}$$

These polynomials specialize, in $t = 1$, into the usual Schubert polynomials $Y_v(\mathbf{x}, \mathbf{y})$, share many of their properties and constitute still another basis of $\mathfrak{Pol}(\mathbf{x}_n)$.

Proposition 13.2.1. *The family $\{Y_v^t(\mathbf{x}),\, v \in \mathbb{N}^n\}$ is stable under $\partial_1, \ldots, \partial_{n-1}$.*
Let $v = u + k^n$, with $k \in \mathbb{N}$. Then $Y_v^t(\mathbf{x}) = Y_{k^n}(\mathbf{x}, \mathbf{y}) \left(Y_u^t(\mathbf{x})\big|_{y_i \to y_{i+k}} \right)$.

Proof. Let v and i be such that $v_i > v_{i+1}$, and let $u = [\ldots, v_{i-1}, v_{i+1}, v_i{-}1, v_{i+1}, \ldots]$. Then $\langle u^t \rangle = \langle v^t \rangle s_i$, and the proof that $Y_u^t \mathbf{x})$ is equal to $Y_v^t \mathbf{x})\partial_i$ is the same than in the case $t = 1$.

As for the factorization property of $Y_{u+k^n}^t(\mathbf{x})$, the polynomial $Y_v^t(\mathbf{x})$ does have to satisfy more vanishing properties than $Y_u^t(\mathbf{x})$, but the factor $Y_{k^n}(\mathbf{x}, \mathbf{y})$ takes care of all the points $w \in \mathbb{N}^n$ having at least one component not greater than k. Thus, one is left with the vanishing in all points $w \subseteq k^n$, $|w| \leq |v|$, $w \neq v$, and this is insured by the image of t-Schubert polynomial $Y_u^t(\mathbf{x})$ under the uniform translation of indices $y_i \to y_{i+k}$. QED

As a consequence, t-Schubert polynomials are determined by the dominant ones. However, contrary to the case of the usual Schubert polynomials, when v is dominant, $Y_v^t(\mathbf{x})$ is not a product of linear factors in general. t-Schubert polynomials indexed by antidominant weights are symmetrical in $x_1, \ldots, x_n$, since they belong to the images of $\partial_1, \ldots, \partial_{n-1}$.

Monk's formula extends smoothly.

Lemma 13.2.2. *Let $w \in \mathbb{N}^n$, i be such that $w_i > 0$, v be the image of w under $w_i \to w_i{-}1$. Then there exists rational functions c_u in $t, y_1, y_2, \ldots$ such that one has*

$$\frac{(x_i - \langle v^t \rangle_i)}{(\langle w^t \rangle_i - \langle v^t \rangle_i)} \frac{Y_w^t(\langle w^t \rangle)}{Y_v^t \langle w^t \rangle} = Y_w^t(\mathbf{x}) + \sum_{u:\, |u|=|w|,\, u \neq w} c_u Y_u^t(\mathbf{x}) \, . \tag{13.2.2}$$

Proof. Both sides of the equation are polynomials of degree $|w|$ which vanish in all points $\langle u^t\rangle : |u| < |w|$. One can therefore determine the required coefficients c_u by specializing the LHS in $\langle u^t\rangle : |u| = |w|, u \neq w$, to insure the equality of both sides of the equation in maximal degree. QED

In the case of Schubert polynomials, the specialization $Y_v(\langle v\rangle, \mathbf{y})$ is equal to the inversion polynomial $⋒(v) = \prod_{(ji)\in\mathfrak{Inv}(\sigma)}(y_j - y_i)$, with σ of code v. Let $[\alpha_1,\ldots,\alpha_n] = [\sigma_1,\ldots,\sigma_n]\!\uparrow$, and

$$\mathbf{y}^v = [y_1,\ldots,y_{\alpha_1}, ty_{\alpha_1+1},\ldots,ty_{\alpha_2},\ldots,t^{n-1}y_{\alpha_{n-1}+1},\ldots,t^{n-1}y_{\alpha_n}, \\ t^n y_{\alpha_n+1},\ldots,t^n y_\infty]\,.$$

Then we conjecture that

$$Y_v^t\langle w^t\rangle = \prod_{(ji)\in\mathfrak{Inv}(\sigma)} (\mathbf{y}_j^v - \mathbf{y}_i^v)\,. \tag{13.2.3}$$

Instead of expanding the basis $\{Y_v^t(\mathbf{x})\}$ in the basis $\{Y_u(\mathbf{x},\mathbf{y})\}$, one rather choose the bases $\{Y_u(\mathbf{x},\mathbf{y}^v)\}$ or $\{Y_u(\mathbf{x},\mathbf{y}^u)\}$ to obtain more compact expressions.
For example

$$Y_{012}^t(\mathbf{x}) = Y_{012}(\mathbf{x},\mathbf{y}^{012}) + \frac{(t-1)\,(t^2y_4+ty_3-ty_2-y_1)\,y_2}{(t^2y_4-y_1)\,(ty_3-y_1)}\,Y_{111}(\mathbf{x},\mathbf{y}^{012})\,,$$

with $\mathbf{y}^{012} = [y_1, ty_2, ty_3, t^2y_4, t^2y_5, t^3y_6,\ldots]$, while the expansion of $Y_{012}^t(\mathbf{x})$ in the Schubert basis involves

$$Y_{012}(\mathbf{x},\mathbf{y}), Y_{111}(\mathbf{x},\mathbf{y}), Y_{002}(\mathbf{x},\mathbf{y}), Y_{011}(\mathbf{x},\mathbf{y}), Y_{001}(\mathbf{x},\mathbf{y}), Y_{000}(\mathbf{x},\mathbf{y})\,.$$

Contrary to the case of Schubert polynomials, one cannot concatanate 0 to the right of v, since in general $Y_{v\,0}^t(\mathbf{x}) \neq Y_v^t(\mathbf{x})$. In fact, concatanating sufficiently many 0's gives back the Schubert polynomials, as shows the next lemma.

Lemma 13.2.3. *Let $v \in \mathbb{N}^n$, $k \geq |v|$. Then*

$$Y_{v\,0^k}^t(\mathbf{x}) = Y_v(\mathbf{x},\mathbf{y}^v)\,.$$

Proof. One has to test vanishing in all $u \in \mathbb{N}^{n+k}$, $|u| \leq |v|$, $u \neq [v, 0^k]$. The permutation σ of code u belongs to $\mathfrak{S}_{n+k}$ and the spectral vector $\langle u^t\rangle$ is the image of $\langle u\rangle$ under $y_i \to t^{i-1}y_i$. Thus, the set of vanishing conditions is the set defining $Y_{v\,0^k}(\mathbf{x},\mathbf{y}^v)$. QED

For example, $Y_2^t(\mathbf{x}) = (x_1-y_1)(x_1-y_2)$, $Y_{20}^t(\mathbf{x}) = Y_2^t(\mathbf{x}) + \frac{(t-1)y_2}{ty_3-y_1}(x_1-y_1)(x_2-ty_3)$, $Y_{200}^t(\mathbf{x}) = (x_1-y_1)(x_1-ty_2)$.

One can reduce the number of equations in the determination of Y_v^t, v dominant, and avoid testing all $u : |u| \leq |v|$, $u \neq v$, as show the next factorization.

Lemma 13.2.4. *Let v be dominant, $v = [v_1, \dots, v_r, 0, \dots, 0]$ with $v_r > 0$. Then there exist a polynomial $P_v(\mathbf{x})$ of degree $|v| - r$ such that*

$$Y_v^t(\mathbf{x}) = (x_1 - y_1) \dots (x_r - y_r)\, P_v(\mathbf{x}) ,$$

which is determined by the conditions

$$\begin{cases} P_v\big(\langle u^t \rangle\big) = 0 \text{ for all } u : \, u \geq [1^r 0^{n-r}],\ |u| \leq |v|,\ u \neq v\,, \\ P_v(\mathbf{x})\Big|_{t=1} = Y_v(\mathbf{x},\mathbf{y})/Y_{1^r}(\mathbf{x},\mathbf{y})\,. \end{cases} \tag{13.2.4}$$

Proof. One notices that $(x_1 - y_1) \dots (x_r - y_r)$ vanishes in all points $\langle y^t \rangle$ such that $0 \in \{u_1, \dots, u_r\}$, because if $u_i = 0$ is the first occurrence of 0 in u, then $\langle u^t \rangle_i = y_1$. Therefore, it remains to satisfy the vanishing conditions for all points $u >\geq [1^r, 0^{n-r}]$, points that one can write $w + [1^r, 0^{n-r}]$, with $|w| \leq |v| - r$, $w \neq v - [1^r, 0^{n-r}]$. QED

For example, $Y_{210}^t = (x_1 - y_1)(x_2 - y_2) P_{210}(\mathbf{x})$, with

$$P_{210}(\mathbf{x}) = x^{100} + \frac{y_2(t-1)}{t^2 y_4 - y_1} x^{001} - \frac{y_2(t^3 y_4 - y_1)}{t^2 y_4 - y_1} x^{000} ,$$

determined by the conditions

$$P_{210}(\langle 110^t \rangle) = 0 = P_{210}(\langle 111^t \rangle) = P_{210}(\langle 120^t \rangle) ,$$

$$P_{210}(\mathbf{x})\Big|_{t=1} = Y_{210}(\mathbf{x},\mathbf{y})/Y_{110}(\mathbf{x},\mathbf{y}) = x_1 - y_2 .$$

Notice that $Y_{100}^t(\mathbf{x})$ is determined by similar conditions

$$Y_{100}^t(\langle 000^t \rangle) = 0 = Y_{100}^t(\langle 001^t \rangle) = Y_{100}^t(\langle 010^t \rangle) \ , \ Y_{100}^t(\mathbf{x})\Big|_{t=1} = Y_{100}(\mathbf{x},\mathbf{y})$$

but there is no change of variables which allows to pass from one system of equations to the other.

It is interesting to specialize the variables y_i to 1, but the specialized equations (13.2.4) are not sufficient to determine the specialization of $Y_v^t(\mathbf{x})$. For example, all the polynomials $Y_v^t(\mathbf{x})\big|_{y_i=1}$, $v \in \mathbb{N}^2$, $v \neq [0,0], [1,0]$ vanish in the points $[1, t]$ and $[t, 1]$.

One computes

$$\begin{aligned} Y_{002}^t(\mathbf{x})\big|_{y_i=1} &= (x_1 + x_2 + x_3 - 1 - t - t^2)^2 \\ Y_{012}^t(\mathbf{x})\big|_{y_i=1} &= (x_1 + x_2 + x_3 - 1 - t - t^2)\, S_2(x_1 + x_2 + x_3 - 1 - t) \\ Y_{022}^t(\mathbf{x})\big|_{y_i=1} &= \big(S_2(x_1 + x_2 + x_3 - 1 - t)\big)^2 . \end{aligned}$$

This leads to the conjecture that the specialization $y_i = 1$ of the symmetric polynomial $Y_v^t(\mathbf{x})$, v anti-dominant, coincides with the specialization $q = 1$ of the Macdonald polynomial $M_{v\downarrow}$ seen in (**??**).

The non-symmetric case is more subtle. Let us write that a sum of Y_v^t is equivalent to a sum of Macdonald polynomials if they become equal after specialization $y_i \to 1$, $q \to 1$ and reversal of the alphabet $x_i \to x_{n+1-i}$ in the Macdonald polynomials. Then one has $Y^t_{320} \sim M_{023}$, $Y^t_{302} \sim M_{203}$, $Y^t_{230} \sim M_{032}$, $Y^t_{023} \sim M_{320}$, but

$$Y^t_{203} \sim M_{104} \quad \& \quad Y^t_{032} \sim M_{230} + \frac{t^2-t}{t^2-1} M_{203} \,.$$

13.3 Polynomials under C-action

We have seen that $\mathfrak{Pol}(\mathbf{x}_n)$ is a free $\mathfrak{Sym}(\mathbf{x}_n)$-module. More generally, the ring of Laurent polynomials in $\mathbf{x}_n$ is a free module over the invariants of the Weyl groups of type B_n, C_n, D_n respectively. For a basis of monomials, in the more general case of a symmetrizable KacMoody group, in relation with the K-theory of the associated flag variety, see the article of Griffeth and Ram [58, Th.2.9].

Let us give in this section an explicit description of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free module under the action of the Weyl group W of type C_n.

As usual, we have to start with the smallest possible case, which is here, in disagreement with Bourbaki, $n = 1$.

We thus have polynomials in x_1, x_1^{-1}, and the operator $s_1^C : \; x_1 \to x_1^{-1}$. To understand the underlying structure, we may rather use two indeterminates x, y satisfying the relation $xy = 1$. Any polynomial $f(x, y)$ can be written

$$f(x,y) = f_1 + x f_2, \;\; f_1, f_2 \in \mathfrak{Sym}(x,y)\,.$$

Indeed, $f_2 = f\partial_{x,y}$ and $f_1 = -yf\partial_{x,y}$.

In our case, this means that $f(x_1) = f_1 + x_1 f_2$, with f_1, f_2 invariant under s_1^C. But a polynomial invariant under s_1^C is a polynomial in the variable $x_1^\bullet := x_1 + x_1^{-1}$, and one may rephrase the preceding construction as

Lemma 13.3.1. *$\mathfrak{Pol}(x_1^\pm)$ is a free-module over $\mathfrak{Pol}(x_1^\bullet) = \mathfrak{Sym}(x_1^\bullet)$ with basis $1, x_1$.*

As a corollary, one deduces

Lemma 13.3.2. *$\mathfrak{Pol}(\mathbf{x}_n) := \mathfrak{Pol}(x_1^\pm, \ldots, x_n^\pm)$ is a free module over the ring of usual polynomials $\mathfrak{Pol}(\mathbf{x}_n^\bullet) := \mathfrak{Pol}(x_1^\bullet, \ldots, x_n^\bullet)$, with basis $\{x^v : v \in \{0,1\}^n\}$.*

In other words, the ring of Laurent polynomials, say in coefficients in $\mathbb{C}$, may be identified with the tensor product of the two-dimensional spaces $\langle 1, x_i\rangle$, $i = 1, \ldots, n$, with the ring of polynomials in $x_1^\bullet, \ldots, x_n^\bullet$.

Choosing furthermore a basis of $\mathfrak{Pol}(\mathbf{x}_n^\bullet)$ as a free $\mathfrak{Sym}(\mathbf{x}_n^\bullet)$-module and using the preceding lemma, furnishes a basis of $\mathfrak{Pol}(\mathbf{x})$ as a module over its W-invariants. For example, one can take the Schubert polynomials[1] $X_\sigma(\mathbf{x}^\bullet, \mathbf{0})$, $\sigma \in \mathfrak{S}_n$, which, by definition, are all the different images of $X_{n\ldots 1}(\mathbf{x}^\bullet, \mathbf{0}) := (x_1^\bullet)^{n-1} \cdots (x_n^\bullet)^0$ under products of divided differences

$$\partial_i^\bullet := (1 - s_i)\frac{1}{x_i^\bullet - x_{i+1}^\bullet} = \partial_i \frac{1}{1 - x_i^{-1}x_{i+1}^{-1}}\,.$$

In summary, one has the following structure.

---

[1] It is more convenient to index the polynomials by permutations in $\mathfrak{S}_n$ rather than by their code.

Proposition 13.3.3. *The ring of Laurent polynomials* $\mathfrak{Pol}(\mathbf{x}_n)$ *is a free module over* $\mathfrak{Sym}(\mathbf{x}_n^\bullet)$*, with basis*

$$\{x^v\, X_{n\ldots 1}(\mathbf{x}^\bullet, \mathbf{0}):\ v \in \{0,1\}^n,\ \sigma \in \mathfrak{S}_n\}\,.$$

Since one has taken Schubert polynomials in $\mathbf{x}_n^\bullet$, it is natural to use the quadratic form

$$(f,g)^C := fg\,\partial_1^C \cdots \partial_n^C\,\partial_\omega^\bullet = fg\,x^{-\rho^C}\,\pi_{w_0}^C\,. \tag{13.3.1}$$

Since $x_i\partial_i^C = 1$, $1\partial_i^C = 0$, and $X_\sigma(\mathbf{x}^\bullet,\mathbf{0})\partial_\omega^\bullet = 0$, except $X_\omega(\mathbf{x}^\bullet,\mathbf{0})\partial_\omega^\bullet = 1$, it is immediate to evaluate the scalar product of all the elements of the basis with 1 :

Lemma 13.3.4.

$$\Big(x^v X_\sigma(\mathbf{x}^\bullet,\mathbf{0})\,,\,1\Big)^C = 0 \text{ except } \Big(x^{1\ldots 1} X_\omega(\mathbf{x}^\bullet,\mathbf{0}),\,1\Big)^C = 1\,. \tag{13.3.2}$$

Having a basis and a quadratic form, we have now to look for the adjoint basis, or equivalently, to look for a reproducing kernel.

Back to the case $n=1$, this is achieved by taking an extra indeterminate y_1. Then one instantly checks that $\{1, x_1 - y_1\}$ and $\{1, y_1 - x_1^{-1}\}$ are adjoint bases with respect to $(f,g)^C = fg\,\partial_1^C$.

Thus it is appropriate to introduce indeterminates $y_1,\ldots,y_n$, to use the polynomials $(x_1-y_1)^{v_1}\cdots(x_n-y_n)^{v_n}$, $v \in \{0,1\}^n$, instead of the monomials x^v, and to use the Schubert polynomials $X_\sigma(\mathbf{x}^\bullet,\mathbf{y}^\bullet)$ in the two alphabets $\mathbf{x}^\bullet$, $\mathbf{y}^\bullet = \{(y_1+y_1^{-1}),\ldots,(y_n+y_n^{-1})\}$. These polynomials are, by definition, all the different images of

$$X_\omega(\mathbf{x}^\bullet,\mathbf{y}^\bullet) := \prod_{i,j:\, i+j\le n} (x_i^\bullet - y_j^\bullet) = \prod_{i,j:\, i+j\le n} (x_i + x_i^{-1} - y_j - y_j^{-1})$$

under products of divided differences $\partial_i^\bullet$ (which act only on $\mathbf{x}^\bullet$).

Let us choose an indexing compatible with the tensor product structure of $\mathfrak{Pol}(\mathbf{x}_n)$. Any element w of W is identified with a *bar permutation* (or *signed permutation*). In other words, we write $w \in W$ as

$$w = [(-1)^{\epsilon_1}\sigma_1,\ldots,(-1)^{\epsilon_n}\sigma_n], \text{ with } \epsilon \in \{0,1\}^n,\ \sigma \in \mathfrak{S}_n\,.$$

Let, for any $w \in W$, $\sigma = [|w_1|,\ldots,|w_n|]$, $v \in \{0,1\}^n$ such that $v_i = 1$ whenever $-i \in w$. Define

$$\begin{aligned} X_w^C(\mathbf{x},\mathbf{y}) &= (x_1-y_1)^{v_1}\cdots(x_n-y_n)^{v_n}\, X_\sigma(\mathbf{x}^\bullet,\mathbf{y}^\bullet)\,, && (13.3.3)\\ \widetilde{X}_w^C(\mathbf{x},\mathbf{y}) &= (-1)^{\ell(\sigma)} \left(y_1 - \frac{1}{x_1}\right)^{v_1} \cdots \left(y_n - \frac{1}{x_n}\right)^{v_n} X_\sigma((\mathbf{x}^\bullet)^\omega,\mathbf{y}^\bullet)\,. && (13.3.4)\end{aligned}$$

Notice that in the last Schubert polynomials, we have reversed the alphabet $\mathbf{x}^\bullet$ and used $(\mathbf{x}^\bullet)^\omega = [x_n^\bullet,\ldots,x_1^\bullet]$.

Knowing that, in the case of the usual ring of polynomials, $\{(-1)^{\ell(\sigma)}X_\sigma(\mathbf{x}^\omega,\mathbf{y})\}$ is the basis adjoint to the basis $\{X_\sigma(\mathbf{x},\mathbf{y})\}$, with respect to the scalar product $(f,g)^\partial = fg\partial_\omega$, combining with the analysis for $n=1$, one obtains

Theorem 13.3.5. *The ring of Laurent polynomials $\mathfrak{Pol}(\mathbf{x}_n)$ is a free module over $\mathfrak{Sym}(\mathbf{x}_n^\bullet)$, with pairs of adjoint bases $\{X_w^C(\mathbf{x},\mathbf{y}) : w \in W\}$ and $\{\widetilde{X}_w^C(\mathbf{x},\mathbf{y}) : w \in W\}$. More precisely, let write $-w\omega$ for $[-w_n,\dots,-w_1]$. Then, for any $w \in W$, one has*

$$\Big(X_w^C(\mathbf{x},\mathbf{y})\,,\,\widetilde{X}_{-w\omega}^C(\mathbf{x},\mathbf{y})\Big)^C = 1$$

and the other scalar products are 0.

For example, for $n = 2$, the two adjoint bases are

$$\Big\{X^C_{[(-1)^{\epsilon_1}1,\,(-1)^{\epsilon_2}2]} = (x_1-y_1)^{\epsilon_1}\,(x_2-y_2)^{\epsilon_2}\ ,$$
$$X^C_{[(-1)^{\epsilon_2}2,\,(-1)^{\epsilon_1}1]} = (x_1-y_1)^{\epsilon_1}\,(x_2-y_2)^{\epsilon_2}\big(x_1^\bullet - y_1^\bullet\big),\ \epsilon_1,\epsilon_2\in\{0,1\}\Big\}\,,$$

$$\Big\{\widetilde{X}^C_{[(-1)^{\epsilon_1}1,\,(-1)^{\epsilon_2}2]} = (y_1-\frac{1}{x_1})^{\epsilon_1}\,(y_2-\frac{1}{x_2})^{\epsilon_2}\ ,$$
$$\widetilde{X}^C_{[(-1)^{\epsilon_2}2,\,(-1)^{\epsilon_1}1]} = (y_1-\frac{1}{x_1})^{\epsilon_1}\,(y_2-\frac{1}{x_2})^{\epsilon_2}\big(y_1^\bullet - x_2^\bullet\big),\ \epsilon_1,\epsilon_2\in\{0,1\}\Big\}\,.$$

The second alphabet $\mathbf{y}$ may be thought as a set of arbitrary parameters. One may "specialize" it to 0, i.e. specialize all y_i to 0 in the linear factors occurring in the expression of $X_w^C(\mathbf{x},\mathbf{y})$ and $\widetilde{X}_w^C(\mathbf{x},\mathbf{y})$, as well as specializing all $y_i^\bullet$ to 0 inside the Schubert polynomials.

Notice that Weyl's character formula for type C uses the operator $\pi_{w_0}^C$ which is equal to $x_1^n\cdots x_n^1\,\partial_1^C\cdots\partial_n^C\partial_\omega^\bullet$. Weyl's formula $x^\lambda\,\pi_{w_0}^C = Sp_\lambda(\mathbf{x}_n)$ may be written $(x^\lambda\,,\,x_1^n\cdots x_n^1) = Sp_\lambda(\mathbf{x}_n)$.

Key polynomials for type C may be written as linear combinations of $X_w^C(\mathbf{x}_n,\mathbf{0})$ with coefficients expressed in terms of symplectic Schur functions. For example, for $n=2$, writing X_w instead of $X_w^C(\mathbf{x}_2,\mathbf{0})$, and Sp_λ instead of $Sp_\lambda(\mathbf{x}_2)$, one has

$$K^C_{-3,1} = Sp_2\big(X_{-2,1}-X_{2,-1}-X_{-1,-2}\big) - Sp_{11}X_{1,2} + Sp_1Sp_2X_{-1,2}\,.$$

As in the case of type A, it is not difficult to reformulate the preceding construction of pairs of adjoint bases in terms of a kernel.

Theorem 13.3.6. *Let*

$$\Theta_n(\mathbf{x},\mathbf{y}) := \prod_{i=1}^n\left(x_i-\frac{1}{y_i}\right)\prod_{i<j}\left(x_i+\frac{1}{x_i}-y_j-\frac{1}{y_j}\right)\,.$$

Then $\Theta_n(\mathbf{x},\mathbf{y})$ is a reproducing kernel, modulo the identification $\mathfrak{Sym}(\mathbf{x}^\bullet)=\mathfrak{Sym}(\mathbf{y}^\bullet)$, i.e. one has

$$\forall f\in\mathfrak{Pol}(\mathbf{x})\ ,\ \Big(f(x_1,\dots,x_n)\,,\,\Theta_n(\mathbf{x},\mathbf{y})\Big)\Big|_{\mathbf{x}=\mathbf{y}} = f(y_1,\dots,y_n)\,. \qquad (13.3.5)$$

Proof. Writing the scalar product as a summation over W, thanks to (1.10.4), the LHS becomes

$$\Big(\sum \pm f^w \Theta_n(\mathbf{x}^w, \mathbf{y})\Big)\ \frac{1}{\Delta^C}\Big|_{\mathbf{x}=\mathbf{y}} .$$

However, all $\Theta_n(\mathbf{x}^w, \mathbf{y})$ vanish under the specialization $\mathbf{x} = \mathbf{y}$, i.e. $x_1 = y_1, \ldots, x_n = y_n$, but when w is the identity, in which case $\Theta_n(\mathbf{y}, \mathbf{y}) = \Delta^C(\mathbf{y})$. QED

The quadractic form takes values in $\mathfrak{Sym}(\mathbf{x}^\bullet)$ and is $\mathfrak{Sym}(\mathbf{x}^\bullet)$-bilinear, this forces the identification of $\mathfrak{Sym}(\mathbf{x}^\bullet)$ with $\mathfrak{Sym}(\mathbf{y}^\bullet)$.

The kernel diagonalizes into any pair of adjoint bases. For example, given any $\mathbf{z} = \{z_1, z_2\}$, writing X_w for $X_w^C(\mathbf{x}, \mathbf{z})$ and writing $\widetilde{X}_w$ for $\widetilde{X}_w^C(\mathbf{y}, \mathbf{z})$, one may check directly for $n = 2$ that

$$\begin{aligned}\Theta_2(\mathbf{x}, \mathbf{y}) &= \left(x_1 - \frac{1}{y_1}\right)\left(x_2 - \frac{1}{y_2}\right)\left(x_1 + \frac{1}{x_1} - y_2 - \frac{1}{y_2}\right)\\ &= X_{1,2}\widetilde{X}_{-2,-1} + X_{-1,2}\widetilde{X}_{-2,1} + X_{1,-2}\widetilde{X}_{2,-1} + X_{-1,-2}\widetilde{X}_{2,1} + X_{2,1}\widetilde{X}_{-1,-2}\\ &\qquad + X_{2,-1}\widetilde{X}_{1,-2} + X_{-2,1}\widetilde{X}_{-1,2} + X_{-2,-1}\widetilde{X}_{1,2} .\end{aligned}$$

13.4 Polynomials under D-action

There are more functions which are invariant under D-action than C-action. For example, for $n=2$, $x_1+x_2^{-1}$ is invariant under s_1, s_2^D, hence is a D-invariant, but it is not invariant under s_2^C.

To describe $\mathfrak{Pol}(\mathbf{x}_n)$ as a module over the ring $\mathfrak{Sym}^D(n)$ of D-invariant, one can start from the C-basis $\{X_w^C(\mathbf{x}_n,\mathbf{0})\}$. However, we shall see that polynomial coefficients are not sufficient. Thus, one defines $\mathfrak{Sym}^D(n)$ to be the ring of symmetric rational functions in $x_1^{\pm},\ldots,x_n^{\pm}$ which are invariant under s_n^D.

Adapting the type C-case, one takes the quadratic form

$$(f,g)^D = fg\,x^{-\rho^D}\,\pi_{w_0}^D = fg\theta_n^D\partial_\omega^\bullet\,, \tag{13.4.1}$$

using the notation of (1.10.6).

The Weyl group of type D may be considered as the subgroup of signed permutations with an even number of signs. This embedding $W_n^D \hookrightarrow W_n^C$ does not preserve the pairing:

$$(X_{-2,-1}^C(\mathbf{x}_n,\mathbf{0}),\widetilde{X}_{12}^C(\mathbf{x}_n,\mathbf{0}))^C = 1\ ,\ (X_{-2,-1}^C(\mathbf{x}_n,\mathbf{0}),\widetilde{X}_{-1,-2}^C(\mathbf{x}_n,\mathbf{0}))^C = 0,\ldots$$

but

$$(X_{-2,-1}^C(\mathbf{x}_n,\mathbf{0}),\widetilde{X}_{12}^C(\mathbf{x}_n,\mathbf{0}))^D = x_1x_2+x_1^{-1}x_2^{-1},\ ,(X_{-2,-1}^C(\mathbf{x}_n,\mathbf{0}),\widetilde{X}_{-1,-2}^C(\mathbf{x}_n,\mathbf{0}))^D = 2\,,\ldots$$

To show nevertheless that $\{X_w^C(\mathbf{x}_n,\mathbf{0}) : w\in W_n^D\}$ is a $\mathfrak{Sym}^D(n)$-basis, one uses another quadratic form

$$((f,g))^D = (f,g)^D \cap \left(K_{-1,\ldots,-1}^D - K_{-1,\ldots,-1,1}^D\right)\,,$$

the notation meaning that, after expressing $(f,g)^D$ in the K^D basis, that is, $(f,g)^D = aK_{-1,\ldots,-1}^D + bK_{-1,\ldots,-1,1}^D + \cdots$, one puts $((f,g))^D = a-b$.

The next lemma, which is immediate to verify, shows that the two special key-polynomials $K_{-1,\ldots,-1}^D, K_{-1,\ldots,-1,1}^D$ occur in the images of monomials under θ_n^D.

Lemma 13.4.1. *Let $v\in\{-1,0,1\}^n$, $m(k)$ be the number of components of v equal to k. Let $u=[|v_1|,\ldots,|v_n|]$, $\epsilon=(-1)^{n+m(-1)+1}$. Then*

$$x^v\theta_n^D = \begin{cases} K_{(-1)^{n-1},\epsilon}^D - K_{(-1)^{n-2},0^2}^D + K_{(-1)^{n-4},0^4}^D - \cdots & \text{if } m(0)=0\\ 2^{m(0)-1}(x^\bullet)^u & \text{if } m(0)>0\,. \end{cases} \tag{13.4.2}$$

Proposition 13.4.2. *For $w,w'\in W_n^C$, one has*

$$\left(\left(X_w^C(\mathbf{x}_n,\mathbf{0})\,,\,(-1)^n\widetilde{X}_{w'}^C(\mathbf{x}_n,\mathbf{0})\right)\right)^D = \delta_{w',-w\omega}\,. \tag{13.4.3}$$

Consequently, the set $\{X_w^C(\mathbf{x}_n,\mathbf{0}) : w\in W_n^D\}$ is a $\mathfrak{Sym}^D(n)$-basis.

Proof. The product of the two polynomials X^C_w, $\widetilde{X}_{w'}$ is equal to the product of two Schubert polynomials in $\mathbf{x}^\bullet$ times a monomial x^v, with $v \in \{-1, 0, 1\}^n$. If v has at least a component equal to 0, then its image under θ^D_n is a monomial in $\mathbf{x}^\bullet$. This monomial, multiplied by the two Schubert polynomials belongs to the span of monomials (in $\mathbf{x}^\bullet$) of exponent $\le [n, \ldots, n]$. It image under $\partial^\bullet_\omega$ belongs to the span of Schubert polynomials $Y_{0^{n-i},1^i}(\mathbf{x}^\bullet, \mathbf{0})$. If $i \ne n$, then $Y_{0^{n-i},1^i}(\mathbf{x}^\bullet, \mathbf{0})$ has no component in the K^D-basis of index $[(-1)^{n-1}, \pm 1]$. Moreover $Y_{1^n}(\mathbf{x}^\bullet, \mathbf{0}) = x^\bullet_1 \cdots x^\bullet_n = K^D_{(-1)^n} + K_{(-1)^{n-1},1} + \cdots$. Therefore, to avoid nullity, v must have no component equal to 0. But in that case $x^v \theta^D_n$ is D-invariant and commutes with $\partial^\bullet_\omega$. One is reduced to the case of two Schubert polynomials (2.6.6), and this allows to conclude. QED

To express $X^C_w(\mathbf{x}_n, \mathbf{0})$, when $w \in W^C_n \backslash W^D_n$, in the above basis is not immediate. For example, for $n = 2$, writing w instead of $X^C_w(\mathbf{x}_n, \mathbf{0})$, K_v instead of K^D_v, putting $\gamma = (x_1 x_2 - x_1^{-1} x_2^{-1})^2$, one has the following expansions

	$[12]$	$[\bar{1}\bar{2}]$	$[21]$	$[\bar{2}\bar{1}]$
$\gamma[\bar{1}2]$	$K_{\bar{2}\bar{1}}$	$-2K_{\bar{1}0}$	$-K_{\bar{1}\bar{1}}-K_{00}$	$K_{\bar{1}\bar{1}}+K_{00}$
$\gamma[1\bar{2}]$	$-2K_{\bar{1}0}$	$K_{\bar{2}\bar{1}}$	$K_{\bar{1}\bar{1}}+K_{00}$	$-K_{\bar{1}\bar{1}}-K_{00}$
$\gamma[\bar{2}1]$	$-K_{\bar{2}\bar{2}}-K_{\bar{2}0}-K_{\bar{1}1}+K_{00}$	$K_{\bar{2}\bar{2}}+K_{\bar{2}0}+K_{\bar{1}1}-K_{00}$	$K_{\bar{2}\bar{1}}$	$-2K_{\bar{1}0}$
$\gamma[2\bar{1}]$	$K_{\bar{2}\bar{2}}+K_{\bar{2}0}+K_{\bar{1}1}-K_{00}$	$-K_{\bar{2}\bar{2}}-K_{\bar{2}0}-K_{\bar{1}1}+K_{00}$	$-2K_{\bar{1}0}$	$K_{\bar{2}\bar{1}}$

13.5 Hecke algebras of types B, C, D

In type A, we have obtained the Hecke algebra by replacing the simple transpositions s_i by $T_i = (t_1 + t_2)\pi_i - t_2 s_i$. Having at our disposal $\pi_n^\heartsuit$ and $s_i^\heartsuit$ for $\heartsuit = B, C, D$, we have therefore candidates for $T_n^\heartsuit$. One can in fact take independent parameters for types B and C. We prefer to rename the parameters for type A, and define for $i \geq 1$,

$$T_i = (q_1{+}q_2)\pi_i - q_2 s_i \ \&\ T_i^B = (t_1{+}t_2)\pi_i^B - t_2 s_i^B \ \&\ T_i^C = (t_1{+}t_2)\pi_i^C - t_2 s_i^C\,,$$

and, for $i \geq 2$,

$$T_i^D = (q_1{+}q_2)\pi_i^D - q_2 s_i^D\,.$$

These operators satisfy the braid relations, together with

$$(T_i^B - t_1)(T_i^B - t_2) = 0,\ (T_i^C - t_1)(T_i^C - t_2) = 0,\ (T_i^D - q_1)(T_i^D - q_2) = 0\,.$$

The collection $T_1, \ldots, T_{n-1}, T_n^\heartsuit$ generates the Hecke algebra of type $\heartsuit = B, C, D$.

One can order differently the Dynkin graph. In type C, instead of using the divided difference relative to the pair x_n, x_n^{-1}, one takes the pair x_1^{-1}, x_1, and, correspondingly, $\pi_0^C = \pi_{x_1^{-1}, x_1}$, and $s_0^C = s_1^C$. In type B, one takes $\pi_0^B = \pi_{x_1^{-1/2}, x_1^{1/2}}$, and $s_0^B = s_1^B$. In type D, one puts $s_0^D = s_2^D$,

$$f(x_1, x_2, \ldots)\,\pi_0^D = \big(x_1^{-1}x_2^{-1} f - f^{s_0^D}\big)(x_1^{-1}x_2^{-1} - 1)^{-1}\,.$$

Thus

$$T_0^B = (t_1{+}t_2)\pi_0^B - t_2 s_0^B \ \&\ T_0^C = (t_1{+}t_2)\pi_0^C - t_2 s_0^C \ \&\ T_0^D = (q_1{+}q_2)\pi_0^D - q_2 s_0^D\,.$$

The operators T_0^B, T_0^C are characterized by the fact that they commute with the functions of $x_1^\bullet = x_1 + x_1^{-1}$, and by the images of $1, x_1$, which are

$$1T_0^B = t_1 = 1T_0^C, \quad x_1^{-1}T_0^B = -t_1 - t_2 - t_2 x_1^{-1}, \quad x_1^{-1}T_0^C = t_2 x_1^{-1}\,.$$

The operator T_0^D is characterized by the fact that it commutes with functions which are invariant under s_0^D, and by the images of $1, x_1, x_2, x_2x_1^{-1}$, which are

$$1T_0^D = q_1,\ x_1 T_0^D = -q_2 x_2^{-1},\ x_2 T_0^D = -q_2 x_1^{-1},\ x_2 x_1^{-1} T_0^D = q_1 x_2 x_1^{-1}\,.$$

The set $\{T_0^\heartsuit, T_1, \ldots, T_{n-1}\}$ generates another realization of the Hecke algebra of type $\heartsuit$, as an algebra of operators on polynomials. This is this realization that we shall retain in this section.

One can, of course, combine both realizations, or use simultaneously the operators for the different types. From the explicit images of $1, x_1, x_2, x_2x_1^{-1}$, one discovers that T_0^D coincides with the specialization $t_1 = 1$, $t_2 = -1$ of $T_0^C T_1 T_0^C$. Thus, the Hecke algebra of type D may be obtained as a subalgebra of the Hecke algebra of type C for $t_1 = 1$, $t_2 = -1$.

As for type A, inserting parameters in the braid relations is a powerful way of obtaining interesting elements of the Hecke algebra. We refer to the work of Cherednik [21, 22, 23] for a Yang-Baxter philosophy and its application to mathematical physics. For our part, we shall restrict to the construction of *Yang-Baxter graphs* for the Hecke algebras of the Weyl groups, wich allow to insert parameters inside reduced decompositions in a coherent way.

A *Yang-Baxter graph* is a graph with vertices labelled by pairs $\mathcal{Y}_w, v$, w in the Weyl group $W^{\heartsuit,n}$, v a vector of length n, satisfying the following conditions.

The starting point is the pair consisting of $1 = \mathcal{Y}_{12\ldots n}$ and of an arbitrary *spectral vector*. The other elements are recursively defined by the same rule than in type A, which is, for $i > 0$,

$$(\mathcal{Y}_w, v) \to \left(\mathcal{Y}_w \left(T_i + \frac{q_1+q_2}{v_{i+1}v_i^{-1}-1}\right), vs_i\right) \text{ when } \ell(ws_i) > \ell(w)\,, \qquad (13.5.1)$$

the rule for type D being

$$(\mathcal{Y}_w, v) \to \left(\mathcal{Y}_w \left(T_0^D + \frac{q_1+q_2}{v_1v_2-1}\right), vs_0^D\right) \text{ when } \ell(ws_0^D) > \ell(w)\,, \qquad (13.5.2)$$

and finally, the rule for type $\heartsuit = B, C$ being

$$(\mathcal{Y}_w, v) \to \left(\mathcal{Y}_w \left(T_0^\heartsuit + \frac{t_1+t_2}{v_1^2(-t_1t_2)^{-1}-1}\right), vs_0^\heartsuit\right) \text{ when } \ell(ws_0^\heartsuit) > \ell(w)\,, \quad (13.5.3)$$

with $vs_0^B = vs_0^C = [-t_1t_2v_1^{-1}, v_2, \ldots]$ and $vs_0^D = [v_2^{-1}, v_1^{-1}, v_3, \ldots]$.

The fact that the elements $\mathcal{Y}_w$ are well defined translates algebraically in the *Yang-Baxter equations for types B,C,D.* The relation $s_0^D s_2 s_0^D = s_2 s_0^D s_2$ corresponds to an embedding of $\mathfrak{S}_3$ into $W^{D,3}$, thus comes from type A. Thus, there is only one new relation, which is for type C (or B). On the following graphical display, this relations translates into the fact that the two paths from top to bottom give equal elements in the Hecke algebra (each path being evaluated as the product of the labeling of its edges).

Write pairs $w, T_i(\gamma)$ instead of $\mathcal{Y}_w^C, T_i + \frac{t_1+t_2}{\gamma-1}$, for the vertices of the Yang-Baxter graph.

$$
\begin{tikzcd}
& 12,[a,b] \arrow[dl, "T_1(\frac{b}{a})"'] \arrow[dr, "T_0(\frac{-a^2}{t_1t_2})"] & \\
21,[b,a] \arrow[d, "T_0(\frac{-b^2}{t_1t_2})"'] & & \bar{1}2,[\frac{-t_1t_2}{a},b] \arrow[d, "T_1(\frac{-ab}{t_1t_2})"] \\
\bar{2}1,[\frac{-t_1t_2}{b},a] \arrow[d, "T_1(\frac{-ab}{t_1t_2})"'] & & 2\bar{1},[b,\frac{-t_1t_2}{a}] \arrow[d, "T_0(\frac{-b^2}{t_1t_2})"] \\
1\bar{2},[a,\frac{-t_1t_2}{b}] \arrow[dr, "T_0(\frac{-a^2}{t_1t_2})"'] & & \bar{2}\bar{1},[\frac{-t_1t_2}{b},\frac{-t_1t_2}{a}] \arrow[dl, "T_1(\frac{b}{a})"] \\
& \bar{1}\bar{2},[\frac{-t_1t_2}{a},\frac{-t_1t_2}{b}] &
\end{tikzcd}
$$

Notice that the labelings of edges are reversed when exchanging the two paths from top to bottom. In other words, the Yang-Baxter equation equals two products obtained by reversal of each other. To simplify the picture, we could have taken $-t_1t_2 = 1$. However, keeping the two parameters t_1, t_2 instead of using $t, -t^{-1}$ is essential in some problems.

The construction of a Yang-Baxter product corresponding to the choice of a reduced decomposition of $w_0^{\heartsuit}$, which is obtained by choosing a path from top to bottom in the Yang-Baxter graph, amounts to list inversions in the order that they are created, exactly as when computing the Poincaré polynomial. This remark is clear when starting with the spectral vector $[y_1, \ldots, y_n]$. For example, for type C, the Yang-Baxter expression is obtained from the case n–1 to the case n by multiplication by the factor

$$
\left(T_{n-1} + \frac{q_1+q_2}{\frac{y_ny_{n-1}}{-t_1t_2}\text{-}1}\right) \cdots \left(T_1 + \frac{q_1+q_2}{\frac{y_ny_1}{-t_1t_2}\text{-}1}\right) \left(T_0^C + \frac{q_1+q_2}{\frac{y_n^2}{-t_1t_2}\text{-}1}\right)
\left(T_1 + \frac{q_1+q_2}{y_ny_1^{-1}\text{-}1}\right) \cdots \left(T_{n-1} + \frac{q_1+q_2}{y_ny_{n-1}^{-1}\text{-}1}\right),
$$

the negative roots of the root systems of type C being encoded as

$$
\ldots, y_ny_{n-1}, \ldots, y_ny_1, y_n^2, y_ny_1^{-1}, \ldots, y_ny_{n-1}^{-1}
$$

(we have changed the orientation of the Dynkin graph, compared to when we were computing the Poincaré polynomial).

The alternating sum $\frac{1}{|W|}\sum_w(-1)^{\ell(w)}w$ on a Weyl group is a fundamental idempotent. We have seen (1.9.10) how fruitful it is in type A to factorize the corresponding element of the Hecke algebra. Let us do the same for types B, C, D.

In type $\heartsuit = B, C$, instead of the usual length, one defines $\ell_0(w)$ (resp. $\ell_1(w)$) to be the degree in $s_0^\heartsuit$ (resp. in $s_1, s_2, \ldots$) of any reduced decomposition of w. Let

$$\nabla^{\heartsuit,n} = \sum\nolimits_{w\in W}(-t_1)^{n-\ell_0(w)}(-q_1)^{n(n-1)-\ell_1(w)}\,T_w\,,\quad \heartsuit = B, C\,, \tag{13.5.4}$$

$$\nabla^{D,n} = \sum\nolimits_{w\in W}(-q_1)^{n(n-1)-\ell(w)}\,T_w \tag{13.5.5}$$

The description of the canonical reduced decompositions of the elements of the Weyl groups of type $\heartsuit = B, C, D$ entails the following factorizations of the sums $\nabla^{\heartsuit,n}$:

$$\begin{aligned}\nabla^{\heartsuit,n} = \nabla^{\heartsuit,n-1}\Big(&T_{n-1}\cdots T_0^\heartsuit\cdots T_{n-1} - q_1 T_{n-1}\cdots T_0^\heartsuit\cdots T_{n-2} + \cdots\\ &+(-q_1)^{n-1}T_{n-1}\cdots T_0^\heartsuit - t_1(-q_1)^{n-1}T_{n-1}\cdots T_1 + \cdots\\ &- t_1(-q_1)^{2n-3}T_{n-1} - t_1(-q_1)^{2n-2}\Big)\,,\ \heartsuit = B, D\,.\end{aligned} \tag{13.5.6}$$

$$\begin{aligned}\nabla^{D,n} = \nabla^{D,n-1}\Big(&T_{n-1}\cdots T_2T_0^DT_1\cdots T_{n-1} + \cdots + (-q_1)^{n-2}T_{n-1}\cdots T_2T_0^DT_1\\ &+(-q_1)^{n-1}T_{n-1}\cdots T_2T_0^D + (-q_1)^{n-1}T_{n-1}\cdots T_2T_1\\ &+(-q_1)^nT_{n-1}\cdots T_2 + \cdots + (-q_1)^{2n-2}\Big)\end{aligned} \tag{13.5.7}$$

However, $\nabla^{\heartsuit,n}$ is a Yang-Baxter element, as shows the following proposition.

Proposition 13.5.1. *For types B, C, D, $\nabla^{\heartsuit,n}$ is the bottom element of the Yang-Baxter graph corresponding to the spectral vectors $[t_2, (-q_2q_1^{-1})t_2, \ldots, (-q_2q_1^{-1})^{n-1}t_2]$ for types B, C, and $[1, -q_2q_1^{-1}, \ldots, (-q_2q_1^{-1})^{n-1}]$ for type D.*

In particular, one has the following factorizations, for types $\heartsuit = B, C$,

$$\begin{aligned}\nabla^{\heartsuit,n} = \nabla^{\heartsuit,n-1}&\left(T_{n-1} + \frac{q_1+q_2}{\frac{-t_2}{t_1}\left(\frac{-q_2}{q_1}\right)^{2n-3}-1}\right)\cdots\left(T_1 + \frac{q_1+q_2}{\frac{-t_2}{t_1}\left(\frac{-q_2}{q_1}\right)^{n-1}-1}\right)\\ &\left(T_0^C + \frac{q_1+q_2}{\frac{-t_2}{t_1}\left(\frac{-q_2}{q_1}\right)^{2n-2}-1}\right)\\ &\left(T_1 + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{n-1}-1}\right)\cdots\left(T_{n-1} + \frac{q_1+q_2}{\frac{-q_2}{q_1}-1}\right)\,,\end{aligned} \tag{13.5.8}$$

and for type D,

$$\nabla^{D,n} = \nabla^{D,n-1} \left(T_{n-1} + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{2n-3} -1} \right) \cdots \left(T_2 + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{n} -1} \right)$$
$$\left(T_0^D + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{n-1} -1} \right) \left(T_1 + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{n-1} -1} \right)$$
$$\left(T_2 + \frac{q_1+q_2}{\left(\frac{-q_2}{q_1}\right)^{n-2} - 1} \right) \cdots \left(T_{n-1} + \frac{q_1+q_2}{\frac{-q_2}{q_1} - 1} \right) . \quad (13.5.9)$$

Proof. Using the Yang-Baxter relations, one can factor on the left of the right hand-side of the two above expressions each of the simple factors $(T_1{-}q_1), \ldots,$ $(T_{n-1}{-}q_1)$, and $T_0^D{-}q_1)$ or $(T_0^{B/C}{-}t_1)$. Moreover, the term of maximal length is $T_{w_0}^{\heartsuit}$. Therefore these factorized expressions are equal to the quasi-idempotents $\nabla^{\heartsuit,n}$. QED

For types B, C, there are other factorizations which do not correspond to the choice of a spectral vector, but present the advantage of having the parameters t_1, t_2 appear in only the factors containing T_0.

Recall the notation (1.9.1) $T_i(-k) = T_i - q_1^k/[k]$ for type A, with $[k] = q_1^{k-1} - q_2 q_1^{k-2} + \cdots + (-q_2)^{k-1}$. Let

$$\beta_k = t_1 + \frac{[k{-}1]}{[k]}(t_1 q_2 {+} t_2 q_1) , \quad T_0^{\heartsuit}(-k) = T_0^{\heartsuit}{-}\beta_k \text{ for } k \geq 1 , \ \heartsuit = B, C . \quad (13.5.10)$$

Then one proves as above the following factorization of $\nabla^{B/C}$.

Proposition 13.5.2. *For type $\heartsuit = B, C$, one has*

$$\nabla^{\heartsuit,n} = \nabla^{\heartsuit,n-1}\, T_{n-1}(-n{+}1) \cdots T_1(-1) T_0^{\heartsuit}(-n) T_1(-n{+}1) \cdots T_{n-1}(-1) . \quad (13.5.11)$$

For example,

$$\begin{aligned}\nabla^{C,2} &= (T_0^C{-}t_1)(T_1{-}q_1) \left(T_0^C - t_1 {-} \frac{t_1 q_2 + t_2 q_1}{q_1 - q_2} \right) (T_1{-}q_1) \\ &= T_0^C T_1 T_0^C T_1 - q_1 T_0^C T_1 T_0^C - t_1 T_1 T_0^C T_1 + t_1 q_1 T_0^C T_1 + t_1 q_1 T_1 T_0^C \\ &\qquad - t_1^2 q_1 T_1 - t_1 q_1^2 T_0^C + t_1^2 q_1^2 .\end{aligned}$$

All $\nabla^{\heartsuit,n}$, $\heartsuit = A, B, C, D$, are quasi-idempotents. As in the case of Weyl's character formula, they send $x^{\rho^{\heartsuit}}$ to a generalization of the Vandermonde.

Let

$$V^{A,n} = \prod_{1 \leq i < j \leq n} (q_2 x_i + q_1 x_j) \quad \& \quad V^{D,n} = \prod_{1 \leq i < j \leq n} (q_2 x_i + q_1 x_j) \left(q_1 + \frac{q_2}{x_i x_j} \right) ,$$

$$V^{B,n} = V^{D,n} \prod_{i=1}^{n} \left(\frac{t_2}{\sqrt{x_i}} + t_1\sqrt{x_i} \right) \quad \& \quad V^{C,n} = V^{D,n} \prod_{i=1}^{n} \left(\frac{t_2}{x_i} + t_1 x_i \right) .$$

The following theorem shows that $V^{\heartsuit,n}$ is a right factor of $\nabla^{\heartsuit,n}$.

Theorem 13.5.3. *With the notations of Propositions 1.10.1, 1.10.2, one has, for $\heartsuit = B, C, D$,*

$$\nabla^{\heartsuit,n} = x^{-\rho^{\heartsuit}} \pi^{\heartsuit}_{w_0} V^{\heartsuit,n} \tag{13.5.12}$$

$$= \left(\sum_{w} (-1)^{\ell(w)} w \right) V^{\heartsuit,n} \left(\Delta^{\heartsuit,n} \right)^{-1} \tag{13.5.13}$$

Moreover, for $\heartsuit = B, C$, one has

$$(-1)^{\binom{n+1}{2}} \nabla^{\heartsuit,n} = \partial_1^{\heartsuit} \cdots \partial_n^{\heartsuit} \partial_\omega^{\bullet} V^{\heartsuit,n} = \partial_\omega^{\bullet} \partial_1^{\heartsuit} \cdots \partial_n^{\heartsuit} V^{\heartsuit,n} \tag{13.5.14}$$

Proof. For each type, the different assertions are equivalent, thanks to the different factorizations of the maximal divided difference. Let us test (13.5.8), for type C, on the basis $\{P_{v,\sigma} = x^v X_\sigma(\mathbf{x}_n^{\bullet}, \mathbf{0}) : v \in \{0,1\}^n, \sigma \in \mathfrak{S}_n\}$ of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free module over the invariant under C, defined in (13.3.3).

If $v_1 = 0$, then $P_{v,\sigma}$ is sent to 0 by ∂_1^C, as well as by $T_0^C - t_1$, which is a left factor of $\nabla^{C,n}$. Since $\sigma\nabla^{C,n} = (-1)^{\ell(\sigma)}\nabla^{C,n}$ for any permutation σ, non-vanishing implies that $v = [1,\ldots,1]$. Since ∂_ω is a left factor of $\nabla^{A,n}$, hence of of $\nabla^{C,n}$, as well as of $\partial_\omega^{\bullet}$, non-vanishing requires that $\sigma = \omega$. The image of $x^{1\ldots1}X_\omega(\mathbf{x}_n^{\bullet}, \mathbf{0})$ under the right-hand side of (13.5.8) is $V^{C,n}$. The image under $\nabla^{C,n}$ is a polynomial f which belongs to the linear span of monomials of exponents u such that $[|u_1|,\ldots,|u_n|] \downarrow\leq [n,\ldots,1]$. Since moreover $\nabla^{C,n}T_i = \nabla^{C,n}q_2$ for $i = 1,\ldots,n{-}1$, and $\nabla^{C,n}T_0^C = \nabla^{C,n}t_2$, then f must be equal to $V^{C,n}$, up to a non-zero constant. One finds this constant by computing, in the image of $x^{1,2,\ldots,n}$, the coefficient of $1/x^{1,\ldots,n}$, which is $(-t_2^n)q_2^{n(n-1)}$ under $T_{w_0}^C$ (hence under $\nabla^{C,n}$), and $(-1)^{\binom{n}{2}} t_2^n q_2^{n(n-1)}$ under the right-hand side. This settles the case of type C, the case of type B being similar. The case D requires checking only the images of $P_{v,\sigma}$ for v having an even number of components equal to 1. QED

For example,

$$\text{-}\nabla^{2,C} = \partial_1^C \partial_2^C \partial_{21}^{\bullet} (q_2 x_1 + q_1 x_2) \left(q_1 + \frac{q_2}{x_1 x_2} \right) \left(\frac{t_2}{x_1} + t_1 x_1 \right) \left(\frac{t_2}{x_2} + t_1 x_2 \right) .$$

Exchanging t_1, t_2, q_1, q_2, one obtains another family of quasi-idempotents

$$\mathbb{U}^{\heartsuit,n} = \sum\nolimits_{w\in W} (-t_2)^{n-\ell_0(w)} (-q_2)^{n(n-1)-\ell_1(w)} T_w \,, \quad \heartsuit = B, C \,, \tag{13.5.15}$$

$$\mathbb{U}^{D,n} = \sum\nolimits_{w\in W} (-q_2)^{n(n-1)-\ell(w)} T_w \,. \tag{13.5.16}$$

This exchange of parameters give for $\mathbb{U}^{\heartsuit,n}$ relations corresponding to (13.5.6), (13.5.7), (13.5.8), (13.5.9), taking factors $T_i(k) = T_i - q_2^k (q_2^{k-1} + \cdots + (-q_1)^{k-1})^{-1}$ instead of $T_i(-k)$.

However, change of parameters is not sufficient for what concerns the expression of $\Bbbk^{\heartsuit,n}$ in terms of divided differences. One must also change the order of operations, as we have seen in type A. Let $\widetilde{V}^{\heartsuit,n}$ be the image of $V^{\heartsuit,n}$ under the symmetry $q_1 \leftrightarrow q_2$, $t_1 \leftrightarrow t_2$. Then one has

Theorem 13.5.4. *Given n, let $\epsilon^A = 1$, $\epsilon^B = \epsilon^C = (-1)^{\binom{n+1}{2}}$, $\epsilon^D = (-1)^{\binom{n}{2}}$. Then*

$$\Bbbk^{\heartsuit,n} = \epsilon^{\heartsuit}\widetilde{V}^{\heartsuit,n}x^{-\rho^{\heartsuit}}\,\pi^{\heartsuit}_{w_0} = \widetilde{V}^{\heartsuit,n}\nabla^{\heartsuit,n}\left(V^{\heartsuit,n}\right)^{-1} . \tag{13.5.17}$$

For example,

$$\Bbbk^{C,2} = -\,(q_1x_1{+}q_2x_2)\left(q_2{+}\frac{q_1}{x_1x_2}\right)\left(\frac{t_1}{x_1}{+}t_2x_1\right)\left(\frac{t_1}{x_2}{+}t_2x_2\right)\frac{1}{x_1^2x_2}\,\pi_0^C\pi_1\pi_0^C\pi_1\,.$$

We have used the simultaneous transposition of t_1, t_2, and q_1, q_2. In type C (or B), one has two other ways to produce a quasi idempotent from ∇^C, using a single transposition. Let $\left(\nabla^{C,n}\right)^{(t_2,t_1)}$ (resp. $\left(\nabla^{C,n}\right)^{(q_2,q_1)}$) be the image of $\nabla^{C,n}$ under the transposition (t_2, t_1) (resp. (q_2, q_1)). Write $V^{C,n}(q_1, q_2, t_1, t_2) = V^{C,n}$, $V^{C,n}(q_1, q_2, \emptyset) = V^{D,n}$, $V^{C,n}(\emptyset, t_1, t_2) = \prod_{i=1}^{n}\left(\frac{t_2}{x_i}{+}t_1x_i\right)$. We have just seen that $\Bbbk^{C,n} = \epsilon^{\heartsuit}V^{C,n}(q_2, q_1, t_2, t_1)x^{-\rho^{\heartsuit}}\,\pi^{C}_{w_0}$. The following proposition shows that the two other quasi idempotents similarly factorize in terms of divided differences.

Proposition 13.5.5. *Defining $\epsilon^C = (-1)^{\binom{n+1}{2}}$ as in (13.5.17), using the factorization (1.10.3) of $\pi^C_{w_0}$, one has*

$$\begin{aligned}\left(\nabla^{C,n}\right)^{(q_2,q_1)} &= \epsilon^C V^{C,n}(q_2, q_1, \emptyset)\,\partial_1^C\cdots\partial_n^C\partial_\omega^\bullet\,V^{C,n}(\emptyset, t_1, t_2) &(13.5.18)\\ \left(\nabla^{C,n}\right)^{(t_2,t_1)} &= \epsilon^C V^{C,n}(\emptyset, t_2, t_1)\,\partial_1^C\cdots\partial_n^C\partial_\omega^\bullet\,V^{C,n}(q_1, q_2, \emptyset)\,. &(13.5.19)\end{aligned}$$

For example,

$$\begin{aligned}\left(\nabla^{C,2}\right)^{(t_2,t_1)} &= (T_0^C{-}t_2)(T_1T_0^C{+}t_2q_1)(T_1 - q_1)\\ &= (T_0^C - t_2)(T_1 - q_1)\left(T_0^C - \frac{q_1(t_1{+}t_2)}{q_1{-}q_2}\right)(T_1 - q_1)\\ &= -\left(\frac{t_1}{x_1} + t_2x_1\right)\left(\frac{t_1}{x_2} + t_2x_2\right)\partial_1^C\partial_2^C\partial_{21}^\bullet\,(q_2x_1 + q_1x_2)\left(q_1 + \frac{q_2}{x_1x_2}\right).\end{aligned}$$

Thus

$$\begin{aligned}-x_1\left(\nabla^{C,2}\right)^{(t_2,t_1)} &= \left(x_1\left(\frac{t_1}{x_1} + t_2x_1\right)\partial_1^C\right)\left(\left(\frac{t_1}{x_2} + t_2x_2\right)\partial_2^C\right)\partial_{21}^\bullet V^{C,2}(q_1, q_2, \emptyset)\\ &= (t_2(x_1 + x_1^{-1}))(t_2 - t_1)\partial_{21}^\bullet V^{C,2}(q_1, q_2, \emptyset) = t_2(t_2 - t_1)V^{C,2}(q_1, q_2, \emptyset)\,,\end{aligned}$$

value which would be more complicated to obtain using the definition $\left(\nabla^{C,2}\right)^{(t_2,t_1)} = \sum_{w\in W}(-t_2)^{n-\ell_0(w)}(-q_1)^{n(n-1)-\ell_1(w)}\,T_w$.

The Hecke algebra $\mathcal{H}_n^D$ contains two copies of $\mathcal{H}_n^A$, one generated by $\{T_1, T_2, \ldots, T_{n-1}\}$, the other by $\{T_0^D, T_2, \ldots, T_{n-1}\}$. However, these two copies act differently on polynomials. In fact, the identity $s_0^C T_0^D s_0^C = T_1$ exchanges the two actions. Thus,

$$\nabla^{A,3} = T_1(-1)T_2(-2)T_1(-1) = \partial_{321}\,(q_2x_1{+}q_1x_2)\,(q_2x_1{+}q_1x_3)\,(q_2x_2{+}q_1x_3)$$

entails

$$T_0^D(-1)T_2(-2)T_0^D(-1) = s_0^C\partial_{321}s_0^C\left(\frac{q_2}{x_1}{+}q_1x_2\right)\left(\frac{q_2}{x_1}{+}q_1x_3\right)(q_2x_2{+}q_1x_3)\ .$$

Since the work of Young, one knows that 1-dimensional idempotents are the elementary bricks with which to build representations of the symmetric group, and, by extension, of the Hecke algebras of the classical groups.

Due to the factorization (13.5.12), one can also use polynomials of the type $V^{\heartsuit,n}$ (c.f. [111] in type A). We have seen in the preceding chapters the importance of Yang-Baxter graphs to generate families of polynomials or bases of representations. Essentially, to describe a representation of the group algebra, or of the Hecke algebra of the fundamental groups, one needs a starting element. An appropriate graph will take care of generating a basis from this element.

For example, in the case of the group algebra of the symmetric group, one can start from a product of Vandermonde determinants on blocks of consecutive variables to generate a "Specht module". Standard Young tableaux index the elements of the Specht or Young basis, the starting element being indexed by the tableau having its columns filled with consecutive integers. Thus, one has to find similar "first elements" to generate representations of the different Hecke algebras associated to the classical groups.

Young orthogonal bases are nowadays characterized as eigenvectors of Jucys-Murphy elements [69, 154, 163]. In that respect, the fundamental property of the polynomial $\Delta_\lambda^{t_1,t_2}$ encoutered in (1.9.14) is that it is an eigenvector of the Jucys-Murphy elements $\xi_1^A = 1$, $\xi_2^A = T_1T_1/(-q_1q_2)$, $\xi_3^A = T_2T_1T_1T_2/(q_1q_2)^2, \ldots$

Jucys-Murphy elements are recursively defined [172] by

$$\xi_1^A = 1,\ \xi_1^B = T_0^B,\ \xi_1^C = T_0^C,\ \xi_2^D = T_0^DT_1,\ \ \xi_i^\heartsuit = T_{i-1}\xi_{i-1}^\heartsuit T_{i-1}(-q_1q_2)^{-1}\,.$$

Irreducible representations of the Hecke algebra of type C are indexed by pairs of partitions, and, correspondingly, bases are indexed by bitableaux (pairs of standard tableaux). Representations may be realized as subspaces of the Hecke algebra, the elements corresponding to bitableaux being eigenvectors of the Jucys-Murphy elements with special eigenvalues [172]. According to what we have said, to obtain irreducible polynomial representations of the Hecke algebra $\mathcal{H}_n^C$, we need only to exhibit, for each pair of partitions $(\lambda,\mu) :\ |\lambda|{+}|\mu| = n$, a polynomial which is an eigenvector of the Jucys-Murphy elements, with the same eigenvalues as the bitableau of shape (λ,μ) filled with consecutive numbers in columns.

We do not have a solution in general, but only in the case where $[\lambda, \mu]$ is a partition. We give without proof these polynomials, to prompt a reader to describe the general case.

Given three positive integers a, b, k, with $a < b$, let

$$\phi(a,b,k) \;=\; \prod_{a\le i<j\le b} (q_2x_i + q_1x_j)\left(q_1^k - \frac{(-q_2)^k}{x_ix_j}\right) \tag{13.5.20}$$

$$\phi^C(a,b,k) \;=\; \phi(a,b,k)\prod_{i=a}^{b}\left(t_2x_i^{-1} + t_1x_i\right) \tag{13.5.21}$$

Thus $\phi^C(1,n,1) = V^{C,n}$.

To a pair of partitions $\lambda = [\lambda_1, \ldots, \lambda_r]$, $\mu = [\mu_1, \ldots, \mu_\ell]$, one associates a *content-vector* of length $\lambda|+|\mu|$

$$c(\lambda,\mu) = \big[[0,-1,\ldots,-\lambda_1+1],[1,0,\ldots,-\lambda_2+2],\ldots,[r-1,\ldots,-\lambda_r+r],\\ [0,-1,\ldots,-\mu_1+1],[1,0,\ldots,-\mu_2+2],\ldots,[\ell-1,\ldots,-\lambda_\ell+\ell]\big]$$

(this vector is made of blocks that we have figured, one should erase the inside "[" and "]").

Claim. *Let $\lambda = [\lambda_1, \ldots, \lambda_r]$, $\mu = [\mu_1, \ldots, \mu_\ell]$ be two partitions, with $\lambda|| = m$ $\lambda|+|\mu| = n$, such that $\lambda_r \ge \mu_1$. Let $c = c(\lambda, \mu)$ be the content-vector, and*

$$v = [\lambda_1, \lambda_1+\lambda_2, \ldots, \lambda_1+\cdots+\lambda_r, \lambda_1+\cdots+\lambda_r+\mu_1, \ldots, \lambda_1+\cdots+\mu_\ell]\,.$$

Then the polynomial

$$\phi^C(1,v_1,1)\phi^C(v_1+1,v_2,2)\cdots\phi^C(v_{r-1}+1,v_r,r)\phi(v_r+1,v_{r+1},r+1)\cdots\phi(v_{r+\ell-1},v_{r+\ell},r+\ell)$$

is an eigenvector for the Jucys-Murphy elements $\xi_1^C, \ldots, \xi_n^C$ with eigenvalues

$$t_2, t_2(-q_1/q_2)^{c_2}, \ldots, t_2(-q_1/q_2)^{c_m}, t_1, t_1(-q_1/q_2)^{c_{m+2}}, \ldots, t_1(-q_1/q_2)^{c_n}\,.$$

For example, for $\lambda = [3, 2], \mu = [2]$, the content-vector is $c = [0, -1, -2, 1, 0, 0, -1]$ and the polynomial

$$(q_2x_1+q_1x_2)\left(q_1+\frac{q_2}{x_1x_2}\right)(q_2x_1+q_1x_3)\left(q_1+\frac{q_2}{x_1x_3}\right)(q_2x_2+q_1x_3)\\ \left(q_1+\frac{q_2}{x_2x_3}\right)\left(\frac{t_2}{x_1}+t_1x_1\right)\left(\frac{t_2}{x_2}+t_1x_2\right)\left(\frac{t_2}{x_3}+t_1x_3\right)(q_2x_4+q_1x_5)\\ \left({q_1}^2-\frac{{q_2}^2}{x_4x_5}\right)\left(\frac{t_2}{x_4}+t_1x_4\right)\left(\frac{t_2}{x_5}+t_1x_5\right)(q_2x_6+q_1x_7)\left({q_1}^3+\frac{{q_2}^3}{x_6x_7}\right)\,.$$

is an eigenvector of the Jucys-Murphy elements, with eigenvalues

$$t_2,\ -t_2q_2/q_1,\ t_2q_2^2/q_1^2,\ -t_2q_1/q_2,\ t_2,\ t_1,\ -t_1q_2/q_1\,.$$

13.6 Noncommutative symmetric functions

We have already used noncommutative methods in the theory of symmetric functions, by embedding the ring $\mathfrak{Sym}$ into $\mathfrak{Plac}$. Since $\mathfrak{Sym}(\mathbf{x}_\infty)$ is a ring of polynomials in $S_1, S_2, \ldots$, one can use another approach to noncommutativity by deciding that $S_1, S_2, \ldots$ do not commute any more, and look for the bases analogous to the bases of $\mathfrak{Sym}$ other than products of complete functions.

Instead of using quasi-determinants as in [52], let us adopt a more down-to-earth point of view, and use a combinatorics of compositions[2].

Given a composition v of n, let $\mathcal{D}(v) = [v_1, v_1{+}v_2, \ldots, v_1{+}v_2{+}\ldots{+}v_{r-1}] = [d_1, \ldots, d_{r-1}]$ be the list of *descents* of v, and $\langle v\rangle \in \mathbb{N}^{n-1}$ be the exponent of the monomial $x_{d_1} \ldots x_{d_{r-1}}$.

Let $\mathfrak{Pol}^1$ be the vector space with basis $\{x^{\langle v\rangle}\}$ indexed by all compositions, and $\mathfrak{Pol}^1_n$ the subspace corresponding to compositions of n. Define the product

$$\mathfrak{Pol}^1_n \times \mathfrak{Pol}^1_m \ni f \times g \\ \to f(x_1, \ldots, x_{n-1})\,(1 + x_n)\,g(x_{1+n}, \ldots x_{n+m-1}) \in \mathfrak{Pol}^1_{m+n}\,.$$

Let $\mathbf{Sym}$ be the free associative algebra generated by $S[1], S[2], \ldots$. Given any composition $v = [v_1, \ldots, v_r]$, denote $S[v]$ the product $S[v_1] \cdots S[v_r]$, and let $\mathbf{Sym}_n$ be the linear span of $\{S[v] : |v| = n\}$.

Other bases of $\mathbf{Sym}_n$ have been defined in [52] through generating functions. Let $\sigma(t) = \sum_{k\ge 0} t^k S[k]$. Then one defines $L[k], \Psi[k], \Phi[k]$ by

$$\sum_{k\ge 0} t^k L[k] \;=\; \sigma(-t)^{-1} \tag{13.6.1}$$

$$\sum_{k\ge 1} t^{k-1}\Phi[k] \;=\; \frac{d}{dt}\log\bigl(\sigma(t)\bigr) \tag{13.6.2}$$

$$\sum_{k\ge 1} t^{k-1}\Psi[k] \;=\; \sigma(t)^{-1}\frac{d}{dt}\sigma(t)\,. \tag{13.6.3}$$

By product, one obtains three linear bases of $\mathbf{Sym}$:

$$L[v] = L[v_1]L[v_2]\ldots\;,\;\; \Phi[v] = \Phi[v_1]\Phi[v_2]\ldots\;,\;\; \Psi[v] = \Psi[v_1]\Psi[v_2]\ldots\,.$$

Another important basis, the basis of *ribbon functions* $R[v]$, is recursively defined by $R[k] = S[k]$,

$$R[v,a]\,R[b,w] = R[v,a,b,w] + R[v,a{+}b,w]\;,\;\; a,b \in \mathbb{N}^+\,.$$

One can identify $\mathbf{Sym}$ and $\mathfrak{Pol}^1$ by sending $S[k]$ to 1, and requiring the compatibility with the product. In more details, let $\theta : \mathbf{Sym} \to \mathfrak{Pol}^1$ be defined by $\theta(S[k]) = 1$ for all $k \in \mathbb{N}$, and, for any two compositions v, w,

$$\theta(S[v]S[w]) = \theta(S[v])\,(1 + x_{|v|})\,\theta(S[w])\,.$$

[2] In [52], one mostly uses quasi-determinants of almost triangular matrices (i.e. null under the subdiagonal), in which case the theory is simpler than the general theory.

Thus, given $v = [v_1, \ldots, v_r]$, let $\mathcal{D}(v) = [d_1, \ldots, d_{r-1}]$ be the descents of v. Then

$$\theta\big(S[v_1, \ldots, v_r]\big) = (1 + x_{d_1})(1 + x_{d_2}) \ldots (1 + x_{d_{r-1}}) \tag{13.6.4}$$
$$\theta\big(R[v_1, \ldots, v_r]\big) = x_{d_1} x_{d_2} \ldots x_{d_{r-1}} = x^{\langle v \rangle} \,. \tag{13.6.5}$$

It is easy to check that

$$\theta(L[k]) = x_1 \ldots x_{k-1} \tag{13.6.6}$$
$$\theta(\Psi[k]) = 1 - x_1 + x_1 x_2 - \cdots + (\text{-}1)^{k-1} x_1 \ldots x_{k-1} \;. \tag{13.6.7}$$

A little more effort is required to show that

$$\theta(\Phi[k]) = 1 - \binom{k\text{-}1}{1}^{-1} e_1 + \binom{k\text{-}1}{2}^{-1} e_2 - \cdots + (\text{-}1)^{k-1} e_{k-1} \,, \tag{13.6.8}$$

where $e_1, \ldots, e_{k-1}$ are the elementary symmetric functions in $x_1, \ldots x_{k-1}$.

The above values induce $\theta(L[v])$, $\theta(\Psi[v])$, $\theta(\Phi[v])$. For example, for $n = 3$, the polynomial images of the different bases are given by the following table :

basis	3	21	12	111
$\theta(R)$	1	x_2	x_1	x_1x_2
$\theta(S)$	1	$1{+}x_2$	$1{+}x_1$	$(1{+}x_1)\,(1{+}x_2)$
$\theta(L)$	x_1x_2	$x_1\,(1{+}x_2)$	$(1{+}x_1)\,x_2$	$(1{+}x_1)\,(1{+}x_2)$
$\theta(\Psi)$	$1{-}x_1{+}x_1x_2$	$(1{+}x_2)\,(1{-}x_1)$	$(1{-}x_2)\,(1{+}x_1)$	$(1{+}x_1)\,(1{+}x_2)$
$\theta(\Phi)$	$1{-}1/2\,x_2{-}1/2\,x_1{+}x_1x_2$	$(1{+}x_2)\,(1{-}x_1)$	$(1{-}x_2)\,(1{+}x_1)$	$(1{+}x_1)\,(1{+}x_2)$

Notice that the expression of any element f of **Sym** in the basis $R[v]$ can be obtained by expanding $\theta(f)$ in terms of monomials, and that the expression in the basis $S[v]$ as the same coefficients as the expansion of the image of $\theta(f)$ under the translation $x_i \to x_i\text{-}1$. For example $\theta(Ps[4]) = x^{000} - x^{100} + x^{110} - x^{111}$. This polynomial becomes under the translation $4x^{000} - 2x^{010} - 3x^{100} + 2x^{110} + x^{101} + x^{011} - x^{111} - x^{001}$, and therefore

$$\Psi[4] = 4S[4] - S[31] - 2S[22] + S[211] - 3S[13] + S[121] + 2S[112] - S[1111] \,.$$

Some other linear bases of $\mathfrak{Sym}$ have been introduced, for example, the multiplicative basis $K[v]$ of [52, p. 279] which is such that

$$\theta(K[k]) = (1 + qx_1)(1 + q^2 x_2) \ldots (1 + q^{k-1} x_{k-1}) \,.$$

Florent Hivert [63] defined a deformation of the ribbon functions, the *noncommutative Hall-Littlewood functions* $H[v]$, which are such that, using descents as above, one has

$$\theta(H[v]) = (x_{d_1} + q)(x_{d_2} + q^2) \ldots (x_{d_{r-1}} + q^{r-1}) \,.$$

The space $\mathbf{Sym_n}$ is dual to the space $\mathbf{Qsym_n}$ of *noncommutative quasi-symmetric functions*, the basis $\{M[v]\}$ dual of $\{S[v]\}$ being the basis of *quasi-monomial functions*. The basis $\{E[v]\}$ is dual to the basis $\{L[v]\}$. Another basis, the *quasi-ribbon functions* $\{F[v]\}$, has been defined by Gessel [53].

The space $\mathbf{Qsym_n}$ can also be identified with $\mathfrak{Pol}_n^1$. Under this identification that we still denote θ, one has, in terms of the descents $\mathcal{D}(v) = \{v_1, v_1{+}v_2, v_1{+}v_2{+}v_3 \ldots\}$ of v,

$$\theta(F[v]) = \prod\nolimits_{i\in\mathcal{D}(v)} x_i = x^{\langle v\rangle} \tag{13.6.9}$$

$$\theta(E[v]) = \prod\nolimits_{i\notin\mathcal{D}(v)} (x_i{-}1) \tag{13.6.10}$$

$$\theta(M[v]) = x^{\langle v\rangle} \prod\nolimits_{i\notin\mathcal{D}(v)} (1{-}x_i)\,. \tag{13.6.11}$$

basis	3	21	12	111
$\theta(F)$	1	x_2	x_1	x_1x_2
$\theta(E)$	$(x_1{-}1)\,(x_2{-}1)$	$(x_1{-}1)$	$(x_2{-}1)$	1
$\theta(M)$	$(1{-}x_1)\,(1{-}x_2)$	$(1{-}x_1)\,x_2$	$x_1\,(1{-}x_2)$	x_1x_2

The pairing between $\mathbf{Sym}_n$ and $\mathbf{Qsym_n}$ induces a quadratic form on $\mathfrak{Pol}_n^1$. In fact, the space $\mathfrak{Pol}_n^1$ being the tensor product of 2-dimensional spaces with bases $1, x_i$, there is not much choice to define a pairing compatible with this structure. Given $f, g \in \mathfrak{Pol}_n^1$, one puts

$$(f,g) = CT\big(f(x_1,\ldots,x_{n-1})g(x_1^{-1},\ldots,x_{n-1}^{-1})\big)\,.$$

Because $(1, 1{-}x_i) = 1 = (1{+}x_i, x_i)$, and $(1{+}x_i, 1{-}x_i) = 0 = (1, x_i)$, one has indeed that $\big(\theta(S[v]), \theta(M[u])\big) = \delta_{u,v}$ as required by the definition of the pairing. The monomials $\theta(R[v])$ and $\theta(F[v])$ being both equal to $x^{\langle v\rangle}$, the basis $F[v]$ is dual to the basis $R[v]$.

As usual, it is convenient to use a Cauchy kernel, having two alphabets $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_{n-1}$, and two morphisms $\theta = \theta^x$ and θ^y. Let

$$\Omega_n = (1 + x_1y_1)\ldots(1 + x_{n-1}y_{n-1})\,.$$

Then the duality between the bases $R[v], F[v]$ (resp. $S[v], M[v]$) reads now as

$$\Omega_n = \sum_v \theta^x\big(R[v]\big)\,\theta^y\big(F[v]\big) = \sum_v \theta^x\big(S[v]\big)\,\theta^y\big(M[v]\big)\,, \tag{13.6.12}$$

sum over all compositions of n.

One can determine $\big(\theta(\Phi[v]),\theta(M[u])\big)$ without writing $\Phi[v]$ in the S-basis, by applying recursively the property

$$CT_{x_i}\Big(\theta(\Phi[k{+}1])\,(1-x_i^{-1})\Big)$$
$$= \frac{k{+}1}{k}\left(1-\binom{k{-}1}{1}^{-1} e_1+\binom{k{-}1}{2}^{-1} e_2-\cdots+(-1)^{k-1}e_{k-1}\right). \quad (13.6.13)$$

where $e_1,\ldots,e_{k-1}$ are the elementary symmetric functions in $\{x_1,\ldots,x_k\}\setminus\{x_i\}$.

Computations in $\mathfrak{Pol}_n^1$ are usually simpler than in $\mathbf{Sym}_n$ and $\mathbf{Qsym_n}$, and allow to recover all the transition matrices given in [52]. Let us illustrate the advantage of the polynomial point of view by determining the basis adjoint to $\{\Phi[v]\}$, denoted $\{\Phi^\star[v]\}$ (compare to [52, Prop. 4.29]).

Lemma 13.6.1. *For $k=1,2,3,\ldots,$ let $\hbar_k(x_1,\ldots,x_k)$ be equal to $\theta(\Phi[1^{k+1}])$. Then for any composition v of n, with descents $\mathcal{D}(v)$, one has*

$$\theta(\Phi^\star[v]) = \frac{1}{\prod_{i=1}^r v_i^2}\prod_{j\notin\mathcal{D}(v)}(1-x_j)\,\hbar_{r-1}(x_{d_1},\ldots,x_{d_{r-1}})\,. \quad (13.6.14)$$

Proof. Let $\{f[v]\}$ be a multiplicative basis of $\mathbf{Sym}_n$, and $g_v\in\mathfrak{Pol}_n^1$ be such that $g_v = g'\prod_{i\notin\mathcal{D}(v)}(1-x_i)$. Then $\big(\theta(f[u]),g_v\big)=0$ if $\mathcal{D}(u)\not\subseteq\mathcal{D}(v)$. Thus, $\big(\theta(\Phi[u]),g_v\big)=0$ if $\mathcal{D}(u)\not\subseteq\mathcal{D}(v)$. Let us now impose that $\big(\theta(\Phi[u]),g_v\big)=0$ for the other compositions u different from v. One can use (13.6.13) to eliminate the factors $(1-x_j)$ in g. Renaming $x_1,\ldots,x_{r-1}$ the remaining variables, one is left with the equations $\big(\theta(\Phi[u']),g'\big)=0$ for all compositions u' of r, $u'\neq 1^r$. Therefore $g'(x_1,\ldots,x_{r-1})$ is equal, up to a scalar, to $\theta(\Phi^\star[1^r])$. QED

For example, $2\hbar_1(x_1)=1{+}x_1$, $6\hbar_2(x_1,x_2)=1+2x_1{+}2x_2+x_1x_2$, $4!\hbar_3(x_1,x_2,x_3) = 1+3x_1{+}5x_2{+}3x_3+3x_1x_2{+}5x_1x_3{+}3x_2x_3+x_1x_2x_3$. Hence, for $v=[2,4,3]$, one has $\mathcal{D}(v)=[2,6]$ and

$$\theta(\Phi^\star[243]) = \frac{1}{24}(1{-}x_1)(1{-}x_3)(1{-}x_4)(1{-}x_5)(1{-}x_7)(1{-}x_8)\left(\frac{1}{6}{+}\frac{1}{3}x_2{+}\frac{1}{3}x_6{+}\frac{1}{6}x_2x_6\right).$$

We hope that the reader will be willing to show that the polynomials $r!\hbar_{r-1}$ are the *descent polynomials* filtering permutations according to their descents:

$$r!\hbar_{r-1}(x_1,\ldots,x_{r-1}) = \sum\nolimits_{\sigma\in\mathfrak{S}_r}\prod\nolimits_{i\in Desc(\sigma)} x_i\,.$$

One method is to use Cauchy formula

$$\Omega_n = \sum\nolimits_v \theta(\Phi[v])\theta(\Phi^\star[v])\,,$$

supposing by induction that all $\Phi^\star[v]$ are known, except $\Phi^\star[1^n]$.

The ring **Sym** is a Hopf algebra. As such, it possesses an *antipode* $\mathcal{A}$ which may be characterized, for $v = [v_1, \ldots, v_\ell]$ and $v\omega = [v_\ell, \ldots, v_1]$, by

$$\mathcal{A}\big(S[v]\big) = (\text{-}1)^{|v|} L[v\omega]\,.$$

Therefore, $\mathcal{A}$ induces on $\mathfrak{Pol}_n^1$ the transformation

$$x^u \to (\text{-}1)^n x^{1^{n-1}-u\omega} = (\text{-}1)^n x^{1-u_{n-1},\ldots,1-u_1}\,.$$

For example, one has

$$\begin{array}{ccc} S[2,4,3] & \xrightarrow{\mathcal{A}} & -L[3,4,2] \\ \theta\downarrow & & \downarrow\theta \\ (1{+}x_2)(1{+}x_7) & \xrightarrow{\mathcal{A}} & \text{-}(x_1\ldots x_{10})(1{+}\frac{1}{x_9})(1{+}\frac{1}{x_4}) \end{array}$$

Using θ, one sees instantly that

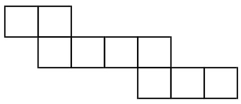
$$\mathcal{A}\big(\Psi[v]\big) = (\text{-}1)^{\ell(v)}\Psi[v\omega] \quad \& \quad \mathcal{A}\big(\Phi[v]\big) = (\text{-}1)^{\ell(v)}\Phi[v\omega]\,.$$

Ribbon functions are exchanged, up to sign, by the antipode :

$$\mathcal{A}\big(R[v]\big) = (\text{-}1)^{|v|} R[v^\sim]\,,$$

denoting by $v^\sim$ the *conjugate* composition (obtained by reading the number of boxes of the diagram of v by columns, from the right).

$$\mathcal{A}\big(R[243]\big) = -R[1121121] \quad \Leftarrow$$



1 2 1 1 2 1 1

Requiring the compatibility with θ, one has no choice for extending the antipode to **QSym** :

$$\mathcal{A}\big(F[v]\big) = (\text{-}1)^{|v|} F[v^\sim]\,.$$

The formula for $\theta(\Phi^\star[v])$ shows that

$$\mathcal{A}\big(\Phi^\star[v]\big) = (\text{-}1)^{\ell(v)}\Phi^\star[v\omega]\,.$$

This extends the property of the commutative power sums that

$$p_\lambda(-X) = (\text{-}1)^{\ell(\lambda)} p_\lambda(X)\,.$$

There are other ways to use tableaux in relation with $\mathbf{Sym_n}$ and $\mathbf{QSym_n}$. Indeed, the sum of all the tableaux in $1,\dots,N$ which standardize to a given tableau t has a commutative image which is clearly quasi-symmetric. More precisely, if t has (maximal) subwords in consecutive letters $[1,\dots,d_1]$, $[d_1{+}1,\dots,d_2]$, $\dots$, $[d_r{+}1,\dots,n]$, let $v(t)$ be the composition of n with descents $d_1,\dots,d_r$. Then

$$ev\left(\sum\nolimits_{st(T)=t} T\right) = F[v(t)]\,.$$

For example, the tableaux which standardize to 34125 are all the tableaux $b_1b_2a_1a_2b_3$ with $a_1 \le a_2 < b_1 \le b_2 \le b_3$, and their sum evaluates to $F[2,3] = \sum_{u,v\colon |u|=2,\,|v|=3} x^{uv}$.

Let $v \in \mathbb{N}_+^r$, $n = |v|$, $N \ge r$. Following Haglund, Luoto, Mason, Willigenburg [60], define the *quasisymmetric Schur function* $Q\widehat{K}_v$ to be the sum

$$Q\widehat{K}_v = \sum\nolimits_{w\colon w\backslash 0=v} \widehat{K}_w\,. \tag{13.6.15}$$

If two tableaux T_1, T_2 have the same standardization, with $T_1 \in \widehat{K}^{\mathcal{F}}_\alpha$ and $T_2 \in \widehat{K}^{\mathcal{F}}_\beta$, then $\alpha\backslash 0 = \beta\backslash 0$, as is seen from the construction of the right key. Therefore, one has that [60, Th.6.2]

$$Q\widehat{K}_v = \sum\nolimits_t F[v(t)]\,, \tag{13.6.16}$$

sum over all standard tableaux belonging to some $\widehat{K}^{\mathcal{F}}_w$ with $w\backslash 0 = v$. The image of this identity under θ is

$$\theta\left(Q\widehat{K}_v\right) = \sum\nolimits_t \prod\nolimits_{i\colon [i+1,i]\in t} x_i\,, \tag{13.6.17}$$

over the same set of standard tableaux[3].

Since the Schur function $s_\lambda(\mathbf{x}_n)$ of index $\lambda \in \mathbb{N}^n$, with $n = |\lambda|$, is equal to the sum $\sum_{v\colon v\downarrow=\lambda} \widehat{K}_v$, the preceding formula gives

$$\theta(s_\lambda) = \sum\nolimits_t \prod\nolimits_{i\colon [i+1,i]\in t} x_i\,, \tag{13.6.18}$$

sum over all standard tableaux of shape λ, as stated in [113].

It is clear that the transition matrix between $\{Q\widehat{K}_v\}$ and $\{F[v]\}$ is uni-triangular (for the lexicographic order from the right), the terms on the diagonal corresponding to the tableaux congruent to the words $st(\dots 2^{v_2}1^{v_1})$.

Instead of giving the transition matrix, one can as well write the generating function $\sum \theta(Q\widehat{K}_v)Q\widehat{K}_v$.

For $n = 3, 4, 5$, these generating functions are

$Q\widehat{K}_3 + x_2Q\widehat{K}_{2,1} + x_1Q\widehat{K}_{1,2} + x_1x_2Q\widehat{K}_{1,1,1}$

[3] the integers i such t contains the subword $[i{+}1, i]$ are called the *recoils* of the tableau.

$Q\widehat{K}_4 + x_3 Q\widehat{K}_{3,1} + (x_1x_3{+}x_2)\, Q\widehat{K}_{2,2} + (x_1{+}x_2)\, Q\widehat{K}_{1,3} + x_3x_2 Q\widehat{K}_{2,1,1} + x_1x_3 Q\widehat{K}_{1,2,1} + x_1x_2 Q\widehat{K}_{1,1,2} + x_3x_1x_2 Q\widehat{K}_{1,1,1,1}$

$Q\widehat{K}_5 + x_4 Q\widehat{K}_{4,1} + (x_3{+}x_1x_4 + x_2x_4)\, Q\widehat{K}_{3,2} + (x_1x_3{+}x_2)\, Q\widehat{K}_{2,3} + (x_1{+}x_2{+}x_3)\, Q\widehat{K}_{1,4} + x_3x_4 Q\widehat{K}_{3,1,1} + (x_2x_4{+}x_1x_3x_4)\, Q\widehat{K}_{2,2,1} + x_3x_2 Q\widehat{K}_{2,1,2} + (x_1x_4{+}x_2x_4)\, Q\widehat{K}_{1,3,1} + (x_1x_3{+}x_1x_2x_4)\, Q\widehat{K}_{1,2,2} + (x_1x_2{+}x_1x_3{+}x_3x_2)\, Q\widehat{K}_{1,1,3} + x_2x_3x_4 Q\widehat{K}_{2,1,1,1} + x_1x_3x_4 Q\widehat{K}_{1,2,1,1} + x_1x_2x_4 Q\widehat{K}_{1,1,2,1} + x_3x_1x_2 Q\widehat{K}_{1,1,1,2} + x_1x_2x_3x_4 Q\widehat{K}_{1,1,1,1,1}\,.$

For example, the coefficient of $Q\widehat{K}_{32}$ is $x_3 + x_1x_4 + x_2x_4$, due to the three tableaux

$$\begin{array}{ccc}\boxed{4}&\boxed{5}&\\ \boxed{1}&\boxed{2}&\boxed{3}\end{array} \equiv \begin{array}{ccc}\boxed{1}&\boxed{4}&\boxed{5}\\ &\boxed{2}&\boxed{3}\end{array} \in \widehat{K}^{\mathcal{F}}_{00302},\quad \begin{array}{ccc}\boxed{2}&\boxed{5}&\\ \boxed{1}&\boxed{3}&\boxed{4}\end{array} \equiv \begin{array}{ccc}\boxed{2}&\boxed{3}&\boxed{5}\\ &\boxed{1}&\boxed{4}\end{array} \in \widehat{K}^{\mathcal{F}}_{00032},$$
$$\begin{array}{ccc}\boxed{3}&\boxed{5}&\\ \boxed{1}&\boxed{2}&\boxed{4}\end{array} \equiv \begin{array}{ccc}\boxed{1}&\boxed{3}&\boxed{5}\\ &\boxed{2}&\boxed{4}\end{array} \in \widehat{K}^{\mathcal{F}}_{00032}\,.$$

Permutations occur in many different ways in the theory of **Sym** and **Qsym**. In fact, Solomon [183] has shown that the subspace of $\mathbb{C}[\mathfrak{S}_n]$ generated by $Sol_v := \sum\{\sigma : Desc(\sigma) = \mathcal{D}(v)\}$ is a sub-algebra[4] of $\mathbb{C}[\mathfrak{S}_n]$. The space $\mathbf{Sym_n}$ is isomorphic, as a vector space, to Solomon's sub-algebra of $\mathbb{C}[\mathfrak{S}_n]$, and thus inherits a product which is called the *internal product*.

Under this correspondence, $R[v]$ is sent onto Sol_v, that is, on the sum of permutations which can be displayed as a ribbon tableau of shape v. For example, for $v[2,1,2]$, one has

$$Sol_{212} = \begin{array}{ccc}\boxed{4}&\boxed{5}&\\ &\boxed{3}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{3}&\boxed{4}&\\ &\boxed{2}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{4}&\boxed{5}&\\ &\boxed{2}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{3}&\boxed{5}&\\ &\boxed{2}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{2}&\boxed{5}&\\ &\boxed{3}&\\ &\boxed{1}&\boxed{5}\end{array}$$
$$+ \begin{array}{ccc}\boxed{2}&\boxed{4}&\\ &\boxed{3}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{2}&\boxed{5}&\\ &\boxed{4}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{3}&\boxed{5}&\\ &\boxed{4}&\\ &\boxed{1}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{1}&\boxed{5}&\\ &\boxed{4}&\\ &\boxed{2}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{1}&\boxed{4}&\\ &\boxed{3}&\\ &\boxed{2}&\boxed{5}\end{array} + \begin{array}{ccc}\boxed{1}&\boxed{5}&\\ &\boxed{3}&\\ &\boxed{2}&\boxed{5}\end{array}\,.$$

Plactic considerations can also be used. Twist the previous morphism between $\mathbf{Sym_n}$ and $\mathbb{C}[\mathfrak{S}_n]$ by inversion of permutations, that is use the morphism

$$\mathbf{Sym_n} \ni R[v] \to \sum\nolimits_{\sigma:\, Desc(\sigma)=\mathcal{D}(v)} \sigma^{-1}\,.$$

Then $S[v]$ is sent over all permutations having subwords $[i{+}1, i]$ for all $i \in \mathcal{D}(v)$, and the image of Ω_n is

$$\widetilde{\Omega}_n = \sum\nolimits_{\sigma\in\mathfrak{S}_n} \left(\prod\nolimits_{[i+1,i]\in\sigma} y_i\right) \sigma\,.$$

Therefore, $\widetilde{\Omega}_n$ is a sum of plactic classes of all standard tableaux of n boxes.

$$\widetilde{\Omega}_3 = \mathcal{C}l\big(\boxed{1}\boxed{2}\boxed{3}\big) + y_1\mathcal{C}l\left(\begin{array}{cc}\boxed{2}&\\ \boxed{1}&\boxed{3}\end{array}\right) + y_2\mathcal{C}l\left(\begin{array}{cc}\boxed{3}&\\ \boxed{1}&\boxed{2}\end{array}\right) + y_2y_1\mathcal{C}l\left(\begin{array}{c}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{array}\right)\,.$$

[4]His construction is valid for any Coxeter group.

Projecting the plactic class of a tableau to the Schur function of index the shape of this tableau, one finally obtains a symmetric function

$$\sum_T \left(\prod\nolimits_{[i+1,i]\in T} y_i\right) s_{\mathfrak{sh}(T)} \,,$$

sum over all standard tableaux.

Let us mention that one can define non-commutative Macdonald polynomials indexed by compositions [6, 135].

Quasisymmetric functions may be used to study problems in the classical theory of symmetric functions. For example, the *plethysm* (i.e. the composition) of symmetric functions is a fundamental issue (it is the third axiom in the definition of a λ-anneau).

In [115], the plethysm of power sums and products of complete functions is studied using ribbon tableaux, this allowing to introduce an extra parameter q pointing the connection of plethysm with representations of $U_q(\widehat{\mathfrak{sl}}_n)$. Let us restrict to the $q = 1$ case and consider the plethysm of a power sum p_k with a Schur function s_λ. In plain words,

$$p_k\big(s_\lambda(\mathbf{x}_n)\big) = s_\lambda(x_1^k, \ldots, x_n^k)\,.$$

The observation that for a quasimonomial function, one has

$$p_k(M[v_1, \ldots, v_r]) = M[kv_1, \ldots, kv_r]$$

shows that the image under θ of the plethysm with p_k is the morphism

$$\mathfrak{Pol}_n^1 \ni f(x_1, \ldots, x_{n-1}) \to \left(\prod_{\substack{i=1\ldots kn-1\\ i\neq 0 \mod k}} (1-x_i)\right) f(x_k, x_{2k}, \ldots, x_{kn-k})\,.$$

As a consequence, one has the following description of the plethysm of a power sum with a Schur function.

Proposition 13.6.2. *Let λ be a partition of n. Then*

$$\theta\big(p_k(s_\lambda)\big) = \left(\prod_{\substack{j=1\ldots kn-1\\ j\neq 0 \mod k}} (1-x_j)\right) \sum\nolimits_t \prod\nolimits_{i:\,[i+1,i]\in t} x_{ki}\,, \tag{13.6.19}$$

sum over all standard tableaux of shape λ.

For example, one reads from the two tableaux $\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$ that $\theta(s_{21}) = x_1 + x_2$ and that

$$\theta(p_2(s_{21}) = (1-x_1)(1-x_3)(1-x_5)(x_2+x_4)\,.$$

This last polynomial determines the explicit expansion

$$p_2(s_{21}) = -s_{411} + s_{222} - s_{2211} + s_{42} - s_{33} + s_{3111}\,.$$

The plethysm of p_2 and any Schur function is described by Carré and Leclerc in terms of domino tableaux in [17].

The space $\mathfrak{Pol}_n^1$ has dimension 2^{n-1}, number bigger than the number of partitions of n. One can use a projection of $\mathfrak{Pol}_n^1$ onto a space of monomials in bijection with partitions. Indeed, for each partition λ, the tableau $st\left(\ldots 2^{\lambda_2}1^{\lambda_1}\right)$ has tallest shape among the tableaux having recoils $\lambda_1, \lambda_1{+}\lambda_2, \lambda_1{+}\lambda_2{+}\lambda_3 \ldots$: $\begin{array}{|c|cc} \hline 6 & & \\ \hline 4 & \multicolumn{1}{c|}{5} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$ is taller than $\begin{array}{|c|ccc} \hline 6 & & & \\ \hline 4 & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$ and $\begin{array}{|c|c|cc} \hline 4 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$. Therefore the morphism

$$s_\lambda \to x^{\langle\lambda\rangle} = x_{\lambda_1} x_{\lambda_1+\lambda_2} x_{\lambda_1+\lambda_2+\lambda_3} \cdots$$

is unitriangular, and any symmetric function f in $\mathfrak{Sym}_n$ is determined by the restriction $\widetilde{\theta}(f)$ of $\theta(f)$ to the linear span $\langle x^{\langle\lambda\rangle},\, \lambda \in \mathfrak{Part}_n\rangle$.

For example, for $n = 6$ one has the following correspondence between monomials and Schur functions.

	6	51	42	411	33	321	3111	222	2211	21111	111111
00000	1	·	·	·	·	·	·	·	·	·	·
00001	·	1	−1	·	·	1	·	−1	−1	·	·
00010	·	·	1	·	−1	−1	·	2	1	·	·
00011	·	·	·	1	·	−1	·	1	1	·	·
00100	·	·	·	·	1	·	·	−1	·	·	·
00101	·	·	·	·	·	1	·	−2	−1	·	·
00111	·	·	·	·	·	·	1	·	−1	·	·
01010	·	·	·	·	·	·	·	1	·	·	·
01011	·	·	·	·	·	·	·	·	1	·	·
01111	·	·	·	·	·	·	·	·	·	1	·
11111	·	·	·	·	·	·	·	·	·	·	1

The row of index 00100 $\leftrightarrow x^{00100} = x_3$ must be interpreted as $x_3 = \widetilde{\theta}(s_{33} - s_{222})$.

Going back to our example, instead of using $\theta\big(p_2(s_{21})\big)$, one takes $\widetilde{\theta}\big(p_2(s_{21})\big) = x^{00010} - x^{00011} + x^{00111} = x_4 - x_4x_5 + x_3x_4x_5$, to determine $p_2(s_{21})$.

Bibliography

[1] A. Arabia. *Cohomologie T-équivariante de G/B pour un groupe G de Kac-Moody*, Inv. Math.**85** (1986) 39–52. 94

[2] J-C. Aval. *Keys and Alternating Sign Matrices*, Séminaire Lotharingien de Combinatoire, SLC **B59f** (2007). 218

[3] J. Bandlow, J. Morse. Combinatorial expansions in K-theoretic bases, arXiv:1106.1594. 178

[4] W. Baratta. *Pieri-type formulas for nonsymmetric Macdonald polynomials* Int. Math. Res. Not., **15** (2009) 2829–2854. 262

[5] R. Baxter. *Exactly sovable models in statistical mechanics*, Academic Press (1982) 26

[6] N. Bergeron and M. Zabrocki. *q and q,t-analogs of non-commutative symmetric functions*, Discrete Math. **298** (2005), no. 1-3, 79–103. 331

[7] N. Berline, M. Vergne. *Zéros d'un champ de vecteurs et classes caractéristiques équivariantes*, Duke Math. **50** (1983) 539–549. 94

[8] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand. *Schubert cells and the cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), 1-26.

[9] S. Billey, W. Jockush, R.P. Stanley. *Some combinatorial properties of Schubert polynomials* J. Alg. Comb. **2** (1993) 115–207. 214

[10] Billey, M. Haiman *Schubert polynomials for the classical groups* J. of the Amer. Math. Soc. **8** (1995) 443–482. 4

[11] A. Björner, F. Brenti. *Combinatorics of Coxeter groups* 298, 299, 303, 304

[12] N. Bourbaki. *Algèbre*, Paris.

[13] M. Bousquet-Mélou, L. Habsieger, *Sur les matrices à signes alternants*, Discrete Math. **139** (1995) 55–72. 220

[14] A.S.Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians, Acta. Ma th. 189 (2002) 37–78. 178

[15] A. Buch, A. Kresch, H. Tamvakis, Yong. *Alexander Grothendieck polynomials and quiver formulas* Amer. J. Math. 127 (2005) 551–567.

[16] W. Cai, N. Jing. *Applications of Laplace-Beltrami operator for Jack functions*, `arXiv:1101.5544` (2011) 49

[17] C. Carré and B. Leclerc. *Splitting the square of a Schur function into its symmetric and antisymmetric parts*, J. Alg. Comb **4** (1995), 201–231. 332

[18] J. Carrell. *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Proc. Symp. Pure M. **56** (1994)53–61. 286

[19] W.Y.C Chen, B. Li, J.D. Louck. *The flagged double Schur function*, J. Alg. Comb. **15** (2002) 7-26.

[20] W.Y.C Chen, J.D. Louck. *Interpolation for symmetric functions*, Adv. in Math. **117** (1996) 147–156. 100

[21] I. Cherednik. *Factorizing particles on a half-line and root systems*, english translation in: Theor. Math. Phys. **61** (1984) 977–983. 316

[22] I. Cherednik. *On special bases of irreducible finite-dimensional representations of the degenerate affine Hecke algebra*, Funct. Analysis Appl. **20** (1986), 87–89. 316

[23] I. Cherednik. *Notes on affine Hecke algebras*, Max-Planck Intitut preprint MPI/90-04 (1990). 316

[24] S.S. Chern. *Classes of Hermitian manifolds*, Annals of Math. **47** (1946) 85-121.

[25] S.S. Chern. *On the multiplication in the characteristic ring of a sphere*, Annals of Math. **49** (1948) 362-372.

[26] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math **21** (1973), 287-301.

[27] M. Demazure. *Une formule des caractères*, Bull. Sc. Math., **98** (1974) 163–172. 61

[28] M. Demazure. *A Moebius-like formula in the Schubert calculus*, Invent. Math. **35**(1976) 317–319.

[29] V. Deodhar. *One some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials*, J. Algebra **111** (1987) 483-506. 292

[30] F. Descouens, A. Lascoux. *Non-Symmetric Hall-Littlewood Polynomials* Séminaire Lotharingien de Combinatoire, SLC **54A** B54Ar (2006). 268, 270

[31] G. Duchamp, F. Hivert and J.-Y. Thibon. *Noncommutative symmetric functions VI*, (2001)

[32] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras*, Discrete Math. and Theor. Comp. Sci. **1** (1997), 159–216.

[33] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, J.Y. Thibon. Euler-Poincaré characteristics and polynomial representations of Iwahori-Hecke algebras *Pub. RIMS (Kyoto)*, **31** (1995) 179–201. 38

[34] P. Edelman, C. Greene. *Balanced tableaux*, Advances in Maths **63** (1987) 42–99. 193, 195, 211, 301

[35] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Annals of Math.(2) **35** (1934) 396–443. 192, 301

[36] S. Fomin and C. Greene. *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), 179–200.

[37] S. Fomin, C. Greene, V. Reiner, M.Shimozono. *Balanced labellings and Schubert polynomials*, Europ. J. Combin. **18** (1997) 373–389. 301

[38] S. Fomin and A.N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proceedings of FPSAC'94, DIMACS, Rutgers University, 183–189. 225, 226

[39] S. Fomin and A. Kirillov. *The Yang-Baxter equation, symmetric functions and Schubert polynomials*, Discrete Math., **153** (1996) 123–143. 29, 123, 225

[40] S. Fomin and A. Kirillov. *Quadratic algebras, Dunkl elements and Schubert calculus*, Advances in Geometry, Progress in Math. **172** Birkauser (1998) 147–182. 4, 121

[41] S. Fomin, A.N. Kirillov. *Combinatorial B_n-analogs of Schubert polynomials*, Trans. A.M.S. **348** (1996) 3591–3620. 4

[42] S. Fomin and R. Stanley. *Schubert polynomials and the nilCoxeter algebra*, Advances in Math. **103** (1994) 196–207.

[43] P. Di Francesco, P. Zinn-Justin. *Quantum Knizhnik Zamolodchikov equation, totally symmetric self- complementary plane partitions and alternating sign matrices*, Theor. Math. Phys. **154** (2008) 331-348. 294

[44] Amy Mei Fu, A. Lascoux. *Non symmetric Cauchy kernels for the classical Groups* Journal of Combinatorial Theory, Series A **116** (2009) 903–917. 74, 75, 89, 236

[45] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1991), 381-420.

[46] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle* Duke Math. J. **76** (1994) , 711–729. 215

[47] W. Fulton. *Determinantal formulas for orthogonal and symplectic degeneracy loci*, J. Diff. Geom. **43** (1996) 276–290. 4

[48] W. Fulton. *Universal Schubert polynomials*, Duke Math. **96** (1999) 575–594. 203

[49] W. Fulton, P. Pragacz. *Schubert varieties and degeneracy loci*, Springer Lecture Notes **1689** (1998). 4

[50] A. Garsia. *The saga of reduced factorizations of elements of the symmetric group*, Publications du LACIM, **29**, Université du Quebec à Montréal (2002). 125

[51] R. Geck, S. Kim, *Bases for the Bruhat-Chevalley order on all finite Coxeter groups*, J. of Algebra **197** (1997) 278–310. 302

[52] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh et J.Y. Thibon. *Noncommutative symmetric functions*, Advances in Math, **112** (1995)218–348. 324, 325, 327

[53] I. Gessel *Multipartite P-partitions and inner products of skew Schur functions*, Contemp. Math. **34** (1984), 289–301. 326

[54] I. Gessel, X. Viennot. *Binomial determinants, paths and hook length formulae*, Advances in M. **58** (1985) 300–321. 187

[55] Z. Giambelli. *Alcune proprieta delle funzioni simmetriche carateristiche*, Atti Torino **38** (1902-1903) 823-844. 128

[56] J. de Gier, P. Pyatov. *Factorised solutions of Temperley-Lieb qKZ equations on a segment*, `arXiv:0710.5362` (2007). 294

[57] J. de Gier, A. Lascoux, M. Sorrell. *Deformed Kazhdan-Lusztig elements and Macdonald polynomials*, J. Combin. Theory Ser. A **119** (2012), no. 1, 183–211. 294

[58] S. Griffeth, A. Ram. *Affine Hecke algebras and the Schubert calculus*, European J. of Combinatorics **25** (2004) 1263-1283. 309

[59] J. Haglund, M. Haiman, N. Loehr. *A combinatorial formula for non-symmetric Macdonald polynomials*, math.CO/0601693 (2006). 264

[60] J. Haglund, K. Luoto, S. Mason, S. van Willigenburg. *Quasisymmetric Schur functions*, J. Comb. Th. A (2011) 463-–490. 329

[61] A. Hamel, R.C. King. *Bijective proof of a symplectic dual pair identity*, Siam J. Discrete Math. **25** (2011) 539-560. 250

[62] K. Hasegawa. *Spin module versions of Weyl's reciprocity theorem for classical Kac-Moody Lie algebras*, Publ. Res. Inst. Math. Sci., 25 (1989) 741–828. 250

[63] F. Hivert, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. Math. **155** (2000), 181–238. 325

[64] Qing-Hu Hou, Yan-Ping Mu. *Recurrent sequences and Schur functions*, Adv. in Appl. M. **31** (2003) 150–162.

[65] J.E. Humphreys. *Reflection groups and Coxeter groups*, Cambridge Studies in advanced mathematics **29** (1990). 281

[66] B. Ion. *Nonsymmetric Macdonald polynomials and Demazure characters* Duke Math. J. **116** (2003), 299–318. 265, 267

[67] S. Iso. *Anyon Basis of c=1 Conformal Field Theory*, `arXiv:hep-th/9411051` Nucl.Phys.B443 (1995),581–595. 49

[68] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevally groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965) 5–48. 57

[69] A.A. Jucys. *On the Young operators on the symmetric groups*, Lietuvos fizikos rinkinys **VI:2** (1966) 163-180; *Factorization of Young projection operators for the symmetric group*, ibidem, **X1:1** (1971) 5–10. 322

[70] M. Kasatani, V. Pasquier. On polynomials interpolating between the stationary state of a O(n) model and a Q.H.E. ground state, cond-mat/0608160 (2006). 264, 294

[71] M. Kashiwara. On crystal bases, *Canadian M. Soc. Conf. Proc.*, **16**(1995) 155–197.

[72] D. Kazhdan and G. Lusztig. *Representations of Coxeter groups and Hecke algebras*, Inv. Math. (1970) 165–184. 279, 282

[73] D. Kazhdan, G. Lusztig. *Schubert varieties and Poincaré duality* Proc. Symp. Pure M. **34**, A.M.S. (1980) 185–203. 285, 286

[74] C. Kassel, A. Lascoux, C. Reutenauer, *Factorizations in Schubert cells*, Adv. in Maths **150** (2000) 1–35. 301

[75] R.C. King. *S-functions and characters of Lie algebras and superalgebras*, Invariant Theory and Tableaux, IMA volumes in Math. and its Applications, **19**, Springer (1988) 226–261.

[76] A. Kirillov, A. Lascoux. *Factorization of Kazhdan-Lusztig elements for Grassmannians*, "Combinatorial Methods in Representation Theory", Advanced Studies in Pure Mathematics **28** (2000) 143–154. 295

[77] F. Knop. Integrality of two variable Kostka functions. *J. Reine Ang. Math.*, **482** (1997) 177–189. 257

[78] F. Knop. Symmetric and non-symmetric quantum Capelli polynomials. *Comment. Math. Helvet.*, **72** (1997) 84–100. 262

[79] D. Knuth. Permutations, matrices, and generalized Young tableaux, *Pacific J.M.* **34** (1970) 709–727. 185

[80] F. Knop and S. Sahi. Difference equations and symmetric polynomials defined by their zeros. *Int. Math. Res. Not.*, **10** (1996) 473–486.

[81] A. Kohnert, S. Veigneau. *Using Schubert basis to compute with multivariate polynomials*, Adv. Appl. Math. **19** (1997) 45–60. 109

[82] K. Koike, I. Terada. *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, J. Algebra **107** (1987) 466–511.

[83] B. Kostant, S. Kumar. *The nil Hecke ring and cohomology of G/P for a Kac-Moody group*, Adv. in Math. **62** (1986) 187–237. 82, 94, 100, 102

[84] B. Kostant, S. Kumar. *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. (1990) 549–603. 82, 94

[85] C. Krattenthaler. *Identities for classical group characters of nearly rectangular shape*, J. Algebra **209** (1998) 1–64.

[86] D. Krob, B. Leclerc and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets*, Internat. J. Alg. Comp. **7** (1997), 181–264.

[87] D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$*, J. Algebraic Combinatorics **6** (1997), 339–376.

[88] D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions V: A degenerate version of $U_q(gl_N)$*, Internat. J. Alg. Comp. **9** (1999), 405–430.

[89] S. Kumar. *The nil Hecke ring and singularity of Schubert varieties*, Invent. math. **123** (1996) 471–506.

94

[90] V. Lakshmibai, B. Sandhya. *Criterion for smoothness of Schubert varieties*, Proc.Indian Acad. Sc. M. **100** (1990) 45–52. 285

[91] V. Lakshmibai, C.S. Seshadri. Geometry of G/P V, *J. Algebra*, **100**(1986) 462–557.

[92] D. Laksov, A. Lascoux, A. Thorup. *On Giambelli's theorem on complete correlations*, Acta.Math. **162** (1989) 163–199. 253

[93] T. Lam, P. Pylyavskyy. *Combinatorial Hopf algebras and K-homology of Grassmanians*, Int. Math. Res. Not. **24** (2007) 178, 179

[94] A. Lascoux. *Puissances extérieures, déterminants et cycles de Schubert*, Bull. Soc. Math. Fr. **102**(1974) 161–179. 65, 133

[95] A. Lascoux. *Foncteurs de Schur et Grassmanniennes*, Thèse, Université Paris 7 (1977).

[96] A. Lascoux. Syzygies des variétés déterminantales, Adv in Maths **30** (1978) 203–237.

[97] A. Lascoux. *Classes de Chern des variétés de drapeaux*, Comptes Rendus **295**(1982) 393–398. 77

[98] A. Lascoux. *Cyclic permutations on words, tableaux and harmonic polynomials*, Proc of Hyderabad Conference on Algebraic Groups, Manoj Prakashan, Madras (1991) 323–347. 277

[99] A. Lascoux. *Anneau de Grothendieck de la variété de drapeaux*, in "The Grothendieck Festchrift", vol III, Birkhäuser(1990) 1–34. 45, 144, 148

[100] A. Lascoux. *Ordonner le groupe symétrique: pourquoi utiliser l'algèbre de Iwahori-Hecke ?*, Congrès International Berlin 1998, Documenta Mathematica, extra volume ICM 1998, III (1998) 355–364. 285, 286, 294, 296

[101] A. Lascoux, *Square-ice enumeration*, Séminaire Lotharingien de Combinatoire, The Andrews festschrift, Springer (2001) **42** B42 (1999) 335-348. 47

[102] A. Lascoux. Yang-Baxter graphs, Jack and Macdonald polynomials, Tianjin 1999, *Ann. Comb.* **5** (2001) 397–424 255, 258

[103] A. Lascoux. *Chern and Yang through Ice*, preprint (2002). 218

[104] A. Lascoux. *Transition on Grothendieck polynomials*, Proc. Nagoya Workshop on Physics and Combinatorics (2000), ed. A Kirillov, N. Liskova, World Scientific (2001) 164–179. 147, 171

[105] A. Lascoux. *Littlewood's formulas for characters of orthogonal and symplectic groups*, in Algebraic Combinatorics and Quantum Groups, ed. by Naihuan Jing, World Scientific (2003) 125–133. 23

[106] A. Lascoux. *Double Crystal graphs*, Studies in Memory of Issai Schur, Progress In Math. **210**, Birkhäuser (2003) 95–114.

[107] A. Lascoux. *Schubert et Grothendieck: Un bilan bidécennal*, Séminaire Lotharingien de Combinatoire SLC50 (2003). 230

[108] A. Lascoux. *Symmetric Functions and Combinatorial Operators on Polynomials*, CBMS/AMS Lecture Notes **99** (2003). 39, 50, 66, 101, 103, 128

[109] A. Lascoux. *Gaudin functions, and Euler-Poincaré characteristics*, arXiv:0709.1635 (2007). 47, 264

[110] A. Lascoux. *Pfaffians and Representations of the Symmetric Group*. Acta Mathematica Sinica **25** (2009) 1929–1950. 252

[111] A. Lascoux. *Polynomial representations of the Hecke algebra of the symmetric group*, International J. of Algebra and Computations (2013). 322

[112] A. Lascoux. *How many alphabets can a Schur function accomodate ?* Journal of Combinatorics **3** (2013).

[113] A. Lascoux, B. Leclerc and J.Y. Thibon. *Crystal graphs and q-analogue of weight multiplicities for the root system A_n*, Letters in Math. Phys., **35** (1995) 359–374. 329

[114] A. Lascoux, B. Leclerc and J.Y. Thibon. *Flag Varieties and the Yang-Baxter Equation* , Letters in Math. Phys., **40** (1997) 75–90. 34

[115] A. Lascoux, B. Leclerc and J.Y. Thibon. *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties*, J. of Math. Physics **38** (1997) 1041–1068. 331

[116] A. Lascoux, B. Leclerc and J.Y. Thibon. *The plactic monoid*, Combinatorics on words vol 2, Lothaire ed., Cambridge Univ. Press (2002). 277

[117] A. Lascoux, H. Naruse. *Finite sum Cauchy identity for dual Grothendieck polynomials*, preprint (2012). 179

[118] A. Lascoux & M.P. Schützenberger. *Le monoïde plaxique*, in Non-commutative structures in algebra and geometric combinatorics, Napoli 1978, Quaderni de “La Ricerca Scientifica”, **109**, C.N.R., Roma (1981) 129–156. 184, 185, 277

[119] A. Lascoux & M.P. Schützenberger. *Sur une conjecture de H.O. Foulkes*, Comptes Rendus **286** (1978) 323–324. 276

[120] A. Lascoux & M.P. Schützenberger. *Polynômes de Kazhdan-Lusztig pour les grassmanniennes*, in Tableaux de Young et Foncteurs de Schur, Torun 1980, Astérique **87-88** (1981) 294, 296

[121] A. Lascoux & M.P. Schützenberger. *Polynômes de Schubert*, Comptes Rendus **294** (1982) 447–450.

[122] A. Lascoux & M.P. Schützenberger. *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, Comptes Rendus **295** 629–633. 193, 195, 211, 225, 226, 227, 228

[123] A. Lascoux & M.P. Schützenberger. *Symmetry and Flag manifolds*, Invariant Theory, Springer L.N., **996** (1983) 118–144. 62

[124] A. Lascoux & M.P. Schützenberger. *Schubert polynomials and the Littlewood-Richardson rule*, Letters in Math.Phys, **10** (1985) 111–124. 154

[125] A. Lascoux & M.P. Schützenberger. *Symmetrization operators on polynomial rings*, Funk. Anal. **21** (1987) 77–78. 18

[126] A. Lascoux & M.P. Schützenberger. Keys and standard bases, *Invariant Theory and Tableaux, IMA volumes in Math. and its Applications*, **19**, Springer (1988) 125–144. 184, 189, 192

[127] A. Lascoux & M.P. Schützenberger. *Fonctorialité des polynômes de Schubert* in Invariant Theory, Contemporary Maths, **88** (1989) 585–598. 203

[128] A. Lascoux & M.P. Schützenberger. Tableaux and non commutative Schubert Polynomials, *Funk.Anal* **23**(1989) 63–64. 212

[129] A. Lascoux & M.P. Schützenberger. *Algèbre des différences divisées*, Discrete Maths, **99** (1992) 165–179.

[130] A. Lascoux & M.P. Schützenberger. *Treillis et bases des groupes de Coxeter*, Electronic Journal of Combinatorics **3** R27 (1996).

[131] A. Lascoux and P. Pragacz. *Ribbon Schur functions*, Europ. J. Combinatorics, **9**, (1988), 561-574. 128, 129

[132] Lascoux, P. Pragacz. *Operator calculus for $\mathbb{Q}$-polynomials and Schubert polynomials*, Adv. in Math. **140** (1998) 1–43. 4

[133] A. Lascoux, P. Pragacz. *Orthogonal divided differences and Schubert polynomials, P-functions, and vertex operators* Michigan Math. J. 48 (2000), 417–441. 4

[134] A. Lascoux, E.M. Rains, S.O. Warnaar. Nonsymmetric interpolation Macdonald polynomials and gl_n basic hypergeometric series, Transfo Groups **14** (2009) 613–647. 260, 262

[135] A. Lascoux, J-C. Novelli, J-Y. Thibon, *Noncommutative symmetric functions with matrix parameters*, `arXiv:1110.3209` (2011). 331

[136] C. Lenart. *Noncommutative Schubert calculus and Grothendieck polynomials*, Adv. in Math.**143** (1999) 159–183.

[137] C. Lenart. *Combinatorial aspects of the K-theory of Grassmannians*, Ann. Combin. **4** (2000) 67–82. 175, 179, 230

[138] C. Lenart. *A K-theory version of Monk's formula*, J. Pure Appl. Algebra 179 (2003), no. 1-2, 137–158 167

[139] C. Lenart, S. Robinson, F. Sottile. *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, Am. J. Maths (2005)

[140] C. Lenart, F. Sottile. *A Pieri-type formula for the K-theory of a flag manifold*. Trans. Amer. Math. Soc. **359** (2007) 2317–2342. 167

[141] C. Lenart, A. Postnikov. *Affine Weyl Groups in K-theory and representation theory*. Int. Math. Res. Not. (2007) 1–65 144

[142] P. Littelmann. *A plactic algebra for semisimple Lie algebras*, Adv. in Math. **124** (1996) 312–331.

[143] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford University Press (1950). 22

[144] D.E. Littlewood. *On certain symmetric functions*, Proc. London. M.S. **43** (1961) 485–498. 272

[145] G. Lusztig. *Equivariant K-theory and representations of Hecke Algebras*, Proc. Amer. Math. Soc. **94** (1985), 337-342.

[146] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, Oxford, 2nd edition (1995). 22, 128, 129, 253, 254

[147] I. G. Macdonald. *The Poincaré series of a Coxeter group*, Math. Annal **199** (1972) 161–174. 54

[148] I. G. Macdonald. *Notes on Schubert polynomials* LACIM, Publi. Université Montréal, (1991).

[149] I. G. Macdonald. *Schur functions: theme and variations*, Séminaire Lotharingien de Combinatoire, SLC **28** 1992 5–39. 128

[150] I. G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki, 47ème année, **797** (1995).

[151] P. Magyar. *Schubert polynomials and Bott-Samelson varieties*, Commentarii Math. Helvet. **73** (1998), 603–636. 210

[152] Molev

[153] Monk, D. *the Geometry of flag manifolds*, Proc.London M.S 3 bf9 (1959) 253–286.

[154] G.E. Murphy. *On the representation theory of the symmetric groups and associated Hecke algebras*, J. Algebra **152** (1992) 492–513. 322

[155] I. Newton. *Philosophiae Naturalis Principiae Mathematica, Liber III*, London (1687) 481-83. 61

[156] S.Okada. Applications of minor summation formulas to rectangular-shaped representations of classical groups ????? 254

[157] S. Okada. *Enumeration of symmetry classes of alternating sign matrices and characters of classical groups*, J. Alg. Comb. **23** (2006) 43-69.

[158] A. Okounkov. *Quantum immanants and higher Capelli identities*, Transformation Groups **1** (1996) 99–126. 103

[159] A. Okounkov. A characterization of interpolation Macdonald polynomials *Adv. in Applied Math.* **20** (1998) 395–428. 264

[160] A. Okounkov. Shifted Macdonald polynomials: q-integral representation and combinatorial formula, *Compositio Math.* **112** (1998) 147–182. 264

[161] A. Okounkov. Combinatorial formula for Macdonald polynomials, Bethe Ansatz and generic Macdonald polynomials, math.QA/0008094. 264

[162] A. Okounkov and G. Olshanski. *Shifted Schur functions*, St. Petersburg Math. J. **9** (1998), 239-300. 100

[163] A. Okounkov, A. Vershik. A new approach to representation theory of symmetric groups, Selecta Math., **2** (1996) 581–605. 32, 322

[164] G. Olshanski, A. Regev, A. Vershik. *Frobenius-Schur functions* (1999) 128

[165] E.M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.*, **175** (1995) 75–121.

[166] P. Polo. *Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups.* Represent. Theory **3** (1999), 90–104. 291

[167] V. Pons. *Interval structure of the Pieri formula for Grothendieck polynomials* 144

[168] A. Postnikov. *Permutohedra, Associahedra and Beyond*, preprint (2004). 276

[169] R. Proctor. *Interconnections between symplectic and orthogonal characters*, Contemp. Math. **88** (1989) 145–162.

[170] E. M. Rains, *BC_n-symmetric polynomials*, Transform. Groups **10** (2005), 63–132. 259

[171] E. M. Rains, *BC_n-symmetric Abelian functions*, Duke Math. J. **135** (2006), 99–180.

[172] A. Ram. *Seminormal representations of Weyl groups and Iwahori-Hecke algebras*, Proceedings of the London Math. Soc. (3) 75 (1997), 99-133. 322

[173] V. Reiner. *The distribution of descents and length in a Coxeter group* Electron. J. Combin. **2** (1995), Research Paper 25. 57

[174] V. Reiner, M. Shimozono. *Key polynomials and a flagged Littlewood-Richardson rule*,J. Comb. Th. A **70** (1995) 107–143. 214

[175] V. Reiner, M. Shimozono. *Plactification*, J. of Algebraic Combinatorics **4** (1995) 331–351. 194

[176] V. Reiner, M. Shimozono. *Percentage-avoiding, northwest shapes and peelable tableaux*, J. Combin. Theory A **82** (1998) 1–73. 213

[177] H.A.Rothe, *Ueber permutationen, in Beziehung auf die Stellen ihrer Elemente*, Sammlung combinatorisch -analytischer Abhandlungen, Leipzig, (cited by Muir, History of determinants, vol1) (1800). 300

[178] S. Sahi. Interpolation, integrality and a generalization of Macdonald's polynomials. *Int. Math. Res. Not.*, **10** (1996) 457–471.

[179] S. Sahi. The binomial formula for nonsymmetric Macdonald polynomials. q-alg/9703024.

[180] Y. B. Sanderson. *On the connection between Macdonald polynomials and Demazure characters*, J. Algebraic Combin. **11** (2000), 269–275. 267

[181] C. Schensted. Longest increasing and decreasing subsequences, *Canad. J. Math.*, **13** (1961) 179–191.

[182] M. Shimozono and M. Zabrocki. Stable grothendieck polynomials and Ω-calculus, preprint 2003. 178

[183] L. Solomon. *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **21** (1976) 255–268. 330

[184] R. Stanley. *On the number of reduced decompositions of elements of Coxeter groups*, Journal Europ. de Comb. **5** (1984) 359–372. 195

[185] J. R. Stembridge. *Non-intersecting paths, Pfaffians and plane partitions*, Adv. Math. **83** (1990), 96–131. 253, 254

[186] J. R. Stembridge, D. J. Waugh. *A Weyl group generating function that ought to be better known*, Indag.M. **9** (1998)451–457) 57

[187] J. Stembridge A Short Derivation of the Möbius Function for the Bruhat Order, *J. Algebraic Combinatorics* **25** (2007), 141–148. 45, 304

[188] S. Sundram. *Tableaux in the representation theory of the classical Lie groups*, Invariant Theory and Tableaux, IMA volumes in Math. and its Applications, **19**, Springer (1988) 191–225.

[189] T. Sundquist. *Two variable Pfaffian identities and symmetric functions*, J. Algebraic Combin. 5 (1996), 135–148. 252

[190] D.N. Verma. *Möbius inversion for the Bruhat ordering on a Weyl group*, Annales E.N.S. **4** (1971) 393–398. 192

[191] M. Wachs. *Flagged Schur functions, Schubert polynomials*, J.of Comb. theory **A 40** (1985) 276–289.

[192] S. O. Warnaar. *Bisymmetric functions, Macdonald polynomials and* sl_3 *basic hypergeometric series*, math.CO/0511333 (2005). 264

[193] H. Weyl. *The classical groups, their invariants and their representations*, Princeton University Press, Princeton, (1946). 239

[194] C.N. Yang. *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312–1315. 26

[195] A. Young. *The Collected Papers of Alfred Young*, University of Toronto Press (1977). 26

[196] A. Zelevinsky. *Small resolutions of singularities of Schubert varieties* Funct. Anal. Pry. **17** (1983) 142–144.