Lecture 8.

Hochschild cohomology of an associative algebra and its Morita-invariance. Hochschild cohomology complex. Multiplication and the Eckman-Hilton argument. Derivations of the tensor algebra and the Gerstenhaber bracket on Hochschild cohomology. Hochschild cohomology and deformations. Quantizations. Kontsevich formality (statements).

8.1 Generalities on Hochschild cohomology.

Up to now, we were studying Hochschild homology of associative algebras and related concepts — cyclic homology, regulator maps, and so on. We will now turn to the other half of the story: Hochschild cohomology.

We recall (see Definition 1.1) that the *Hochschild cohomology* $HH^{\bullet}(A, M)$ of an associative unital algebra A over a field k with coefficients in an A-bimodule $M \in A\text{-}\mathsf{bimod}$ is given by

$$HH^{\bullet}(A, M) = \operatorname{Ext}^{\bullet}_{A\text{-}\mathsf{bimod}}(A, M),$$

where A in the right-hand side is the diagonal bimodule $A \in A\text{-}\mathsf{bimod}$. *Hochschild cohomology of an algebra* A is its cohomology with coefficients in the diagonal bimodule, $HH^{\bullet}(A) = HH^{\bullet}(A, A)$.

We note right away that the Hochschild cohomology groups $HH^{\bullet}(A)$ are Morita-invariant — that is, they only depend on the category $A\text{-}\mathsf{mod}$ of left A-modules. Indeed, all we need to compute $HH^{\bullet}(A)$ is the tensor abelian category $A\text{-}\mathsf{bimod}$ with its unit object $A \in A\text{-}\mathsf{bimod}$; as we have seen already in Lecture 6, these only depend on $A\text{-}\mathsf{mod}$.

When A is commutative and $X = \operatorname{Spec} A$ is smooth, the Hochschild-Kostant-Rosenberg Theorem (Theorem 1.2) provides a canonical identification

$$HH^{\bullet}(A) \cong H^0(X, \Lambda^{\bullet}\mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent bundle to X. Roughly speaking, Hochschild cohomology is in the same relation to Hochschild homology as vector fields are to differential forms. We note, however, that to describe $HH^{\bullet}(A)$, we need not only the tangent bundle $\mathcal{T}_X$, but all its exterior powers $\Lambda^{\bullet}\mathcal{T}_X$, so that Hochschild cohomology contains not only vector fields, but all the polyvector fields, too. In the non-commutative setting, there is no reasonable way to work only with vector fields, we have to treat all the polyvector fields as a single package.

Just as in the case of Hochschild homology, we can compute Hochschild cohomology $HH^{\bullet}(A)$ of an algebra A by using the canonical bar resolution $C_{\bullet}(A)$ of the diagonal bimodule A. This gives the *Hochschild cohomology complex* with terms

$$\operatorname{Hom}(A^{\otimes n}, A), \quad n \geq 0,$$

where Hom means the space of all k-linear maps. Maps $f \in \operatorname{Hom}(A^{\otimes n}, A)$ are called *Hochschild cochains*; we can treat an n-cochain as an n-linear A-valued form on A. The differential δ in the Hochschild cohomology complex is given by

(8.1)
$$\delta(f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) - \sum_{0 \leq j < n} (-1)^j f(a_0, \dots, a_j a_{j+1}, \dots, a_n) + (-1)^{n+1} f(a_0, \dots, a_{n-1}) a_n.$$

For example, if $f = a \in A$ is a 0-cochain, then $\delta(f)$ is given by $\delta(f)(b) = ab - ba$; if $f : A \to A$ is a 1-cochain, then we have

$$\delta(f)(a, b) = af(b) + f(a)b - f(ab).$$

We conclude that the space $HH^0(A) \subset A$ of Hochschild 0-cocycles is the center of the algebra A; the space of Hochschild 1-cocycles is the space of all *derivations* $f : A \to A$ (that is, maps that satisfy the Leibnitz rule $f(ab) = af(b) + f(a)b$). The Hochschild cohomology group $HH^1(A)$ is the space of all derivations $A \to A$ considered modulo the *inner derivations* given by $b \mapsto ab - ba$.

8.2 Multiplication and the Eckman-Hilton argument.

By definition, Hochschild cohomology $HH^{\bullet}(A) = \operatorname{Ext}^{\bullet}(A, A)$ of an associative unital algebra A is equipped with an additional structure: an associative multiplication, given by the Yoneda product on Ext-groups.

However, the abelian category A-bimod is a tensor category, and the diagonal bimodule $A \in A$-bimod is its unit object. This defines a second multiplication operation on $HH^{\bullet}(A)$: given two elements $\alpha, \beta \in \operatorname{Ext}^{\bullet}(A, A)$, we can consider their tensor product $\alpha \otimes_A \beta \in \operatorname{Ext}^{\bullet}(A, A)$.

Both multiplications are obviously associative, and it seems that this is all we can claim. However, a moment's reflection shows that more is true.

Lemma 8.1. *The two multiplications on Hochschild cohomology $HH^{\bullet}(A)$ are the same, and moreover, this canonical multiplication is (graded)commutative.*

Proof. It is easy to see that the two multiplications we have defined obey the following distribution law:

$$(\alpha_1 \otimes_A \alpha_2) \cdot (\beta_1 \otimes_A \beta_2) = (-1)^{\deg \alpha_2 \deg \beta_1} \alpha_1\beta_1 \otimes_A \alpha_2\beta_2, \tag{8.2}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in HH^{\bullet}(A)$. This formally implies the claim:

$$\alpha\beta = (\alpha \otimes_A 1) \cdot (1 \otimes_A \beta) = (\alpha \cdot 1) \otimes_A (1 \cdot \beta) = \alpha \otimes_A \beta,$$

and similar for the commutativity, which we leave to the reader. □

This observation is known as the *Eckman-Hilton argument*: two associative multiplications which commute according to (8.2) are commutative and equal. It first appeared in algebraic topology — essentially the same argument shows that the homotopy groups $\pi_i(X)$ of a topological space X are abelian when $i \geq 2$. Although the Eckman-Hilton argument is very elementary, it captures an essential feature of the whole story: in fact, all the results about Hochschild cohomology can be deduced from an elaboration of this semi-trivial observation. A good reference for this is a paper by M. Batanin, `arXiv:math/0207281`. In these lectures, we will not attempt such an extreme treatment and follow a more conventional path, only refering to the Eckman-Hilton argument when it simplifies the exposition.

One example of this is an explicit description of the product in $HH^{\bullet}(A)$ in terms of Hochschild cochains. Writing down the Yoneda product in terms of Ext's computed by an explicit resolution is usually rather cumbersome, and the resulting formulas are not nice. However, the tensor product $f \otimes_A g$ of two Hochschild cochains $f : A^{\otimes n} \to A$, $g : A^{\otimes m} \to A$ is very easy to write down: it is given by

$$(f \otimes_A g)(a_1, \dots, a_{n+m}) = f(a_1, \dots, a_n)g(a_{n+1}, \dots, a_{n+m}). \tag{8.3}$$

By Lemma 8.1, the Yoneda product is given by the same formula.

8.3 The Gerstenhaber bracket.

Recall now that the space of vector fields on a smooth algebraic variety has an additional structure: the Lie bracket. It turns out that such a bracket, known as the *Gerstenhaber bracket*, also exists for an arbitrary associative unital algebra A. To define it, we need to introduce a completely different construction of the Hochschild cohomology complex.

Assume given a k-vector space V, and consider the free graded associative coalgebra $T_{\bullet}(V)$ generated by V placed in degree 1 — explicitly, we have

$$T_nV = V^{\otimes n}, \quad n \geq 0.$$

Consider the graded Lie algebra $DT^\bullet(V)$ of all *coderivations* of the coalgebra $T_\bullet(V)$ — the notion of a coderivation of a coalgebra is dual to that of a derivation of an algebra, and we leave it to the reader to write down a formal definition. Then since the coalgebra $T_\bullet(V)$ is freely generated by V, every $\delta \in DT^\bullet(V)$ is uniquely determined by its composition with the projection $T_\bullet(V) \to V$, so that we have

$$DT^{n+1}(V) \cong \operatorname{Hom}(V^{\otimes n}, V), \quad n \geq 0. \tag{8.4}$$

Lemma 8.2. *Assume that* $\operatorname{char} k \neq 2$. *A coderivation* $\mu \in DT^1(V) = \operatorname{Hom}(V^{\otimes 2}, V)$ *satisfies* $\mu^2 = 0$ *if and only if the corresponding binary operation* $V^{\otimes 2} \to V$ *is associative.*

Proof. Since μ is an odd derivation, $\mu^2 = \frac{1}{2}\{\mu,\mu\} : T_{\bullet+2}(V) \to T_\bullet(V)$ is also a derivation; thus it suffices to prove that the map $\mu^2 : V^{\otimes 3} \to V$ is equal to 0 if and only if the map $\mu : V^{\otimes 2} \to V$ is associative. This is obvious: by the Leibnitz rule, we have

$$\mu^2(v_1, v_2, v_3) = \mu(\mu(v_1, v_2), v_3) - \mu(v_1, \mu(v_2, v_3))$$

for any $v_1, v_2, v_3 \in V$. ∎

Thus if we are given an associative algebra A, the product in A defines an element $\mu \in DT^1(A) = \operatorname{Hom}(A^{\otimes 2}, A)$ such that $\{m, m\} = 0$. Then setting $\delta(a) = \{\mu, a\}$ for any $a \in DT^\bullet(A)$ defines a differential $\delta : DT^\bullet(A) \to DT^{\bullet+1}(A)$ and turns $DT^\bullet(A)$ into a graded Lie algebra. But as we can see from (8.4), the space $DT^n(A)$ is exactly the space of Hochschild $(n+1)$-cochains of the algebra A.

Exercise 8.1. *Check that under the identification* (8.4)*, the differential* δ *in* $DT^\bullet(A)$ *becomes equal to the differential in the Hochschild cohomology complex.*

Thus the Hochschild complex for the algebra A becomes a graded Lie algebra, with a Lie bracket of degree -1, and we get an induced graded Lie bracket on Hochschild cohomology $HH^\bullet(A)$. This is known as the *Gerstenhaber bracket.* Explicitly, the Gerstenhaber bracket $\{f, g\}$ of two cochains $f : A^{\otimes n} \to A$, $g : A^{\otimes m} \to A$ is given by

$$\begin{aligned} \{f,g\}(a_1,\dots,a_{n+m-1}) = &\sum_{1\leq i<n} (-1)^i f(a_1,\dots,g(a_i,\dots,a_{i+m-1}),\dots,a_{n+m-1}) \\ &- \sum_{1\leq i<m} (-1)^i g(a_1,\dots,f(a_i,\dots,a_{i+n-1}),\dots,a_{n+m-1}). \end{aligned} \tag{8.5}$$

Exercise 8.2. *Prove this. Hint: use the Leinitz rule.*

We note that if we take $g = \mu$, (8.5) recovers the formula (8.1) for the differential δ in the Hochschild cohomology complex. On the other hand, if both f and g are 1-cochains — that is, k-linear maps from A to itself — then $\{f, g\} : A \to A$ is their commutator, $\{f, g\} = fg - gf$. If f and g are also 1-cocycles, that is, derivations of the algebra A, then so is their commutator $\{f, g\}$: the Gerstenhaber bracket on $HH^1(A)$ is given by the commutator of derivations.

Thus we have two completely different interpretation of the Hochschild complex, and two natural structures on it: the multiplication and the Lie bracket. These days, the corresponding structure on $HH^\bullet(A)$ is usually axiomatized under the name of a *Gerstenhaber algebra.*

Definition 8.3. A *Gerstenhaber algebra* is a graded-commutative algebra $B^\bullet$ equipped with a graded Lie bracket $\{-,-\}$ of degree -1 such that

$$\{a, bc\} = \{a, b\}c + (-1)^{\deg b \deg c}\{a, c\}b \tag{8.6}$$

for any $a, b, c \in B^\bullet$.

Exercise 8.3. *Check that the Hochschild cohomology algebra $HH^{\bullet}(A)$ equipped with its Gerstenhaber bracket satisfies* (8.6)*, so that $HH^{\bullet}(A)$ is a Gerstenhaber algebra in the sense of Definition* 8.3.

We note that the definition of a Gerstenhaber algebra is very close to that of a *Poisson algebra* — the difference is that the bracket has degree -1, and (8.6) acquires a sign. We will discuss this analogy in more detail at a later time.

8.4 Hochschild cohomology and deformations.

By far the most common application of Hochschild cohomology is its relation to deformations of associative algebras. We will explain this in the form of the so-called *Maurer-Cartan* formalism popularized by M. Kontsevich.

Assume given an Artin local algebra S with maximal ideal $\mathfrak{m} \in S$ and residue field $k = A/\mathfrak{m}$. By an *S-deformation* $\widetilde{A}$ of an associative unital k-algebra A we will understand a flat S-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/\mathfrak{m} \cong A$.

Assume given such a deformation $\widetilde{A}$, choose a k-linear splitting $A \to \widetilde{A}$ of the projection $\widetilde{A} \to \widetilde{A}/\mathfrak{m} \cong A$, and extend it to an S-module map $\widetilde{A} \cong A \otimes_k S$ — since $\widetilde{A}$ is flat, this map is an isomorphism. We leave it to the reader to check that Lemma 8.2 extends to flat S-modules, with the same statement and proof. Then the multiplication map $\mu : \widetilde{A} \otimes_S \widetilde{A} \to \widetilde{A}$ can be rewritten as

$$\mu = \mu_0 + \gamma \in \operatorname{Hom}(A^{\otimes 2}, A) \otimes S, \tag{8.7}$$

where μ_0 is the multiplication map in A. If the splitting map $A \to \widetilde{A}$ is compatible with the multiplication, then $\gamma = 0$; but in general, it is a non-trivial correction term with values in $\operatorname{Hom}(A^{\otimes 2}, A) \otimes \mathfrak{m} \subset \operatorname{Hom}(A^{\otimes 2}, A) \otimes S$. All we can say is that, since both μ_0 and μ are associative, by Lemma 8.2 we have $\{\mu, \mu\} = 0$ and $\{\mu_0, \mu_0\} = 0$. This can be rewritten as the *Maurer-Cartan equation*

$$\delta(\gamma) + \frac{1}{2}\{\gamma, \gamma\} = 0, \tag{8.8}$$

where δ is the Hochschild differential of the algebra A. Conversely, every solution γ of the Maurer-Cartan equation defines by (8.7) an associative product structure on the S-module $A \otimes_k S$.

This establishes the correspondence between S-deformations of the algebra A and $\mathfrak{m}$-valued degree-1 solutions of the Maurer-Cartan equation in the differential graded Lie algebra $DT^{\bullet}(A)$. We denote the set of these solutions by $MC(DT^{\bullet}(A), \mathfrak{m})$; by definition, it only depends on the differential graded Lie algebra $DT^{\bullet}(A)$ and the local Artin algebra S with its maximal ideal $\mathfrak{m} \subset S$.

How canonical is this correspondence? There is one choice: that of an S-module identification $\widetilde{A} \cong A \otimes S$. The set of all such identifications is a torsor over the algebraic group $GL_{S,\mathfrak{m}}(A)$ of all S-linear invertible maps $A \otimes S \to A \otimes S$ which are equal to identity modulo $\mathfrak{m}$. Assume now that $\operatorname{char} k = 0$. Then we note that since S is local and Artin, this algebraic group is unipotent, and therefore it is completely determined by its Lie algebra $\operatorname{Hom}(A, A) \otimes \mathfrak{m} \cong DT^0(A) \otimes \mathfrak{m}$. Changing an identification $\widetilde{A} \cong A \otimes S$ changes the solution $\gamma \in MC(DT^{\bullet}(A), \mathfrak{m})$, so that we have an action of the group $GL_{S,\mathfrak{m}}(A)$ on $MC(DT^{\bullet}(A), \mathfrak{m})$. The corresponding action of its Lie algebra $DT^0(A) \otimes \mathfrak{m}$ is easy to describe: an element $l \in DT^0(A) \otimes \mathfrak{m}$ sends μ to $\{\mu, l\}$, which in terms of γ is given by

$$\gamma \mapsto \{\mu_0, l\} + \{\gamma, l\} = \delta(l) + \{\gamma, l\},$$

where $\delta : DT^0(A) \to DT^1(A)$ is the differential in $DT^{\bullet}(A)$.

This is the general pattern of deformation theory in the Maurer-Cartan formalism. To a deformation problem, one associates a differential graded Lie algebra $L^{\bullet}$, which "controls" the problem in the following sense: isomorphism classes of deformations over a local Artin base $\langle S, \mathfrak{m} \rangle$ are in one-to-one correspondence with solutions of the Maurer-Cartan equation in $L^1 \otimes \mathfrak{m}$, considered modulo the natural action of the unipotent algebraic group corresponding to the nilpotent Lie algebra $L^0 \otimes \mathfrak{m}$ (because of this passage from a Lie algebra to a unipotent group, the formalism only works well in characteristic 0). In the case of deformations of an associative algebra A, we have just shown that the controlling differential graded Lie algebra is the Hochschild cohomology complex $DT^\bullet(A)$.

As an interesting special case, one can consider the so-called first-order deformations — that is, one takes $S = k[h]/h^2$, the algebra of dual numbers. Then $\mathfrak{m} = k$ and $\mathfrak{m}^2 = 0$, so that the Lie algebra $L^0 \otimes \mathfrak{m} \cong L^0$ is abelian, the corresponding unipotent group is simply the vector space $L^0 \otimes \mathfrak{m}$, and its action is given by $\gamma \mapsto \gamma + dl$, $l \in L^0$. On the other hand, the term $\{\gamma, \gamma\}$ in the Maurer-Cartan equation vanishes. Thus the set of isomorphism classes of deformations is naturally identified with the degree-1 cohomology classes of the complex $L^\bullet$. We note that this special case does not require the assumption $\operatorname{char} k = 0$ — indeed, integrating an *abelian* Lie algebra to a unipotent group does not require exponentiation, so that no denomitators occurs.

In particular, the first-order deformations of an associative algebra A are classified, up to an isomorphism, by elements in the second Hochschild cohomology group $HH^2(A)$.

We also note that while we have introduce the Maurer-Cartan formalism in the case of a local Artin base S, it immediately extends to complete deformations over a complete local Noetherian base: the only difference is that the Lie algebra $L^\bullet \otimes \mathfrak{m}$ should be replaced with its $\mathfrak{m}$-adic completion, and its degree-0 term $L^\bullet \otimes \mathfrak{m}$ becomes not nilpotent but pro-nilpotent.

8.5 Example: quantizations.

A useful particular case of the deformation formalism described above is that of a commutative algebra A: assume given a commutative algebra A, and assume that $X = \operatorname{Spec} A$ is a smooth algebraic variety. Under the Hochschild-Kostant-Rosenberg isomorphism

$$HH^\bullet(A) = H^0(X, \Lambda^\bullet \mathcal{T}_X),$$

the group $HH^1(A)$ corresponds to the space of vector fields on X, and the Gerstenhaber bracket is the usual Lie bracket of vector fields. The bracket between $HH^1(A) = H^0(X, \mathcal{T}_X)$ and $HH^0(A) = H^0(X, \mathcal{O}_X)$ is given by the action of a vector field on the space of functions. The bracket on $HH^i(A)$, $i \geq 2$ is uniquely defined by (8.6); it is known as the *Schouten bracket* of polyvector fields.

Deformations of the algebra A are classified by $HH^2(A) = H^0(X, \Lambda^2 \mathcal{T}_X)$, the space of bivector fields on X. Such a field $\Theta \in H^0(X, \Lambda^2 \mathcal{T}_X)$ defines a bracket operation $\{-,-\}$ on $\mathcal{O}_X$ by the rule

$$\{f, g\} = \langle df \wedge dg, \Theta \rangle.$$

This bracket is obviously a derivation with respect to either of the arguments: we have $\{f_1 f_2, g\} = f_2\{f_1, g\} + f_1\{f_2, g\}$. Moreover, it satisfies the Jacobi identity if and only if $[\Theta, \Theta] = 0$ with respect to the Schouten bracket. In this case, Θ is called a *Poisson bivector*, and A acquires a structure of a *Poisson algebra*.

Definition 8.4. A *Poisson algebra* is a commutative algebra A equipped with a Lie bracket $\{-,-\}$ such that $\{f_1 f_2, g\} = f_2\{f_1, g\} + f_1\{f_2, g\}$ for any $f_1, f_2, g \in A$.

A natural source of Poisson algebra structures on A is given by its *quantizations*.

Definition 8.5. A *quantization* $\widetilde{A}$ of the algebra A is a flat complete associative unital $k[[h]]$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/h \cong A$.

For any quantization $\widetilde{A}$, there obviously exists a unique bracket $\{-,-\}$ on A such that

$$\widetilde{f}\widetilde{g} - \widetilde{g}\widetilde{f} = h\{f, g\} \mod h^2 \tag{8.9}$$

for any $f, g \in A$ and arbitrary $\widetilde{f}, \widetilde{g} \in \widetilde{A}$ such that $\widetilde{f} = f \mod h$, $\widetilde{g} = g \mod h$. It is easy to check that this bracket defines a Poisson algebra structure on A. On the other hand, $\widetilde{A}$ can be treated as a $k[[h]]$-deformation of A, so that we have a solution $\gamma \in \operatorname{Hom}(A^{\otimes 2}, A)[[h]]$ of the Maurer-Cartan equation. Its leading term $\Theta \in \operatorname{Hom}(A^{\otimes 2}, A)$ is a Hochschild cocycle, thus gives a bivector on X.

Exercise 8.4. *Check that the bracket on A defined by the bivector Θ is equal to the bracket given by* (8.9)*.*

The equation $[\Theta, \Theta] = 0$ also immediately follows from the Maurer-Cartain equation.

8.6 Kontsevich formality: the statement.

For some time, an important open question was whether the above construction can be reversed: given a Poisson algebra structure on a commutative smooth algebra A, can we extend it to a quantization $\widetilde{A}$? Or, equivalently: given an element $\Theta \in HH^2(A)$ such that $\{\Theta, \Theta\} = 0$, can we extend it to a solution of the Maurer-Cartan equation in $DT^{\bullet}(A)[[h]]$? A positive answer to this was first conjectured and then proved by M. Kontsevich. In fact, he proved the following stronger fact.

Theorem 8.6 (Kontsevich Formality Theorem). *Let $A = k[x_1, \dots, x_n]$ be a polynomial algebra over a field k of characteristic* 0*. Then the DG Lie algebra $DT^{\bullet}(A)$ is* formal *— that is $DT^{\bullet}(A)$ is quasiisomorphic to its cohomology $HH^{\bullet+1}(A)$ (the DG Lie algebra formed by the Hochschild cohomology groups of A, with trivial differential).*

Here the precise meaning of "quasiisomorphic" is the following: there exists a chain of DG Lie algebras $L_i^{\bullet}$ and DG Lie algebra maps $DT^{\bullet}(A) \leftarrow L_1^{\bullet} \to L_2^{\bullet} \leftarrow \cdots \to HH^{\bullet+1}(A)$ such that all the maps induces isomorphisms on cohomology of the complexes. Unfortunately, in general there does not exist a single DG Lie algebra quasiisomorphism $HH^{\bullet+1}(A) \to DT^{\bullet}(A)$ (in particular, the canonical Hochschild-Kostant-Rosenberg map is not compatible with the bracket). However, this is not important for the deformation theory.

Exercise 8.5. *Check that for any local Artin $\langle S, \mathfrak{m}\rangle$, a DG Lie algebra quasiisorphism $L_1^{\bullet} \to L_2^{\bullet}$ between two DG Lie algebras $L_1^{\bullet}$, $L_2^{\bullet}$ induces a map between the solution sets $MC(L_1^{\bullet}, \mathfrak{m})$ and $MC(L_2^{\bullet}, \mathfrak{m})$ of the Maurer-Cartan equation which identified the sets of equivalence classes of the solutions.*

This together with the Formality Theorem implies that quantizations of the algebra A are in one-two one correspondence with equivalence classes of the solutions of the Maurer-Cartan equations in the DG Lie algebra $HH^{\bullet+1}(A)$. However, since the differential in this algebra is trivial, the Maurer-Cartan equation simply reads $\{\Theta, \Theta\} = 0$. In particular, any Poisson bivector on A canonically gives such a solution.

There are two proofs of the Kontsevich Formality Theorem: the original proof of Kontsevich, which is largely combinatorial, and a second proof by D. Tamarkin — this is more conceptual, but it requires a much more detailed study of the the Hochschild cohomology complex $DT^{\bullet}(A)$. Roughly speaking, one proves an even stronger theorem: $DT^{\bullet}(A)$ and $HH^{\bullet+1}(A)$ are quasiisomorphic not only as DG Lie algebras, but as Gerstenhaber algebras. This stronger statement is actually easier; in fact, Tamarkin shows without much difficulty that any Gerstenhaber algebra which has cohomology algebra $HH^{\bullet}(A)$ must be formal. The real difficulty in the proof is the following: *a priori*, the

Hochschild cohomology complex $DT^{\bullet}(A)$ *is not a Gerstenhaber algebra* — indeed, while it does have a Lie bracket and a multiplication, the multiplication (8.3) is commutative only on the level of cohomology, not on the nose. What precise structure does exist on $DT^{\bullet}(A)$ is a subject of the so-called *Deligne Conjecture*. We will return to this later, after introducing some appropriate machinery.