Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus

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Analytical Table of Contents

1. Introduction

1.0. The subject of this study is the differential, the fundamental concept of the infinitesimal calculus, as it was understood and used by LEIBNIZ and those mathematicians who, in the late seventeenth century and the eighteenth, developed the differential and integral calculus along the lines on which LEIBNIZ had introduced it. More precisely, this study is concerned with the influence of certain conceptual and technical aspects of first-order and higher-order differentials on the development of the infinitesimal calculus from LEIBNIZ' time until EULER's.

This part of the history of the calculus belongs to the wider history of analysis. This makes it necessary to discuss in this first chapter certain key processes in the history of analysis, which form the context of the development of the concepts of differential, higher-order differential and derivative; and my study of this development may provide some new insights into these processes.

The first chapter will also serve as an indication of the relation which the subjects treated in the subsequent chapters have to general questions in the history of analysis.

1.1. There are three processes in the history of analysis in the seventeenth and eighteenth centuries which are of crucial importance for the history of the concept of the differential. The first is the introduction, in the 1680's and 1690's, of the LEIBNIZIAN infinitesimal analysis within the body of the CARTESIAN analysis, which at that time may be characterised as the study of curves by means of algebraic techniques.[1]

The second process, occurring roughly in the first half of the eighteenth century, was the separation of analysis from geometry. From being a tool for the study of curves, analysis developed into a separate branch of mathematics, whose subject matter was no longer the relations between geometrical quantities connected with a curve, but relations between quantities in general as expressed by formulas involving letters and numbers.

This change of interest from the curve to the formula induced a change in fundamental concepts of analysis. While in the geometrical phase the fundamental concept in the analytical study of curves was the *variable geometrical quantity*, the separation of analysis from geometry made possible the emergence of the concept of *function of one variable*, which eventually replaced the variable geometrical quantity as the fundamental concept of analysis.

In this process of separation from geometry the differential underwent a corresponding change; it was stripped of its geometric connotations and came to be treated as a mere symbol, like the other symbols occurring in formulas. However, throughout the first half of the eighteenth century the differential kept its position as the fundamental concept of the LEIBNIZIAN infinitesimal calculus.

The third process in which we are interested is the replacement of the differential by the derivative as fundamental concept of infinitesimal analysis. Usually this process is connected with the works of LAGRANGE and CAUCHY, but I shall argue that an important aspect of it is to be found in the works of EULER.

1.2. From consideration of the chronological order of the three processes mentioned above, it is clear that the early LEIBNIZIAN infinitesimal calculus, as it was practised by LEIBNIZ and by his followers in the 1680's and 1690's, was part of an analysis primarily concerned with curves or with the relations between variable geometrical quantities as embodied in the curve. Thus the LEIBNIZIAN calculus cannot be understood without reference to its geometric interpretation. I devote the second chapter of the present study to a detailed description of the concepts of this calculus, and I indicate there how far these concepts were influenced by their geometric context and how they consequently were changed when analysis was separated from geometry. Thus it will become clear how far the early LEIBNIZIAN calculus differed from the mathematical theory and practice which we now indicate by the term "calculus".

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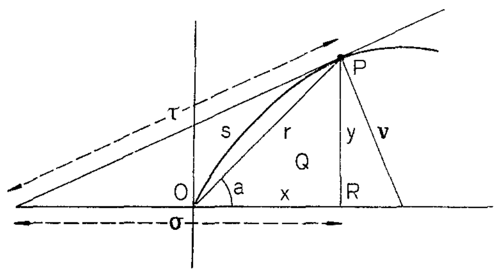
[1] Compare the opening sentence of the *Préface* of L'HÔPITAL *1696*: "L'Analyse qu'on explique dans cet Ouvrage, suppose la commune, mais elle en est fort différente. L'Analyse ordinaire ne traitte que des grandeurs finies: celle-ci penetre jusques dans l'infini même." The "common" or "ordinary" analysis is the CARTESIAN analysis; compare the "communis calculus" in the title of LEIBNIZ' *Elementa*.

Moreover, in Chapter 3 I discuss examples of the influence of the concepts discussed in Chapter 2 both on the choice of problems and on the technique of the calculus in its early stage.

1.3. As a preliminary to these chapters, I insert here some general remarks on the geometric character of the seventeenth century analysis. This analysis was a corpus of analytical tools (algebraic equations and operations, later the differential and the rules of the calculus) for the study of geometric objects, namely curved lines. The first textbook of the infinitesimal calculus had the most significant title *Analyse des infiniment petits pour l'intelligence des lignes courbes.*[2]

The fundamental object of inquiry, therefore, was the curve. A curve embodies relations between several variable geometrical quantities[3] defined with respect to a variable point on the curve. Such variable geometrical quantities—or vari-



x: abscissa, y: ordinate, s: arclength, r: radius, a: polar arc, σ: subtangent, τ: tangent, v: normal, $Q = \widehat{OPR}$: area between curve and X-axis, xy: circumscribed rectangle

ables as I shall call them for short—are for instance (see the figure): ordinate, abscissa, arclength, radius, polar arc, subtangent, normal, tangent, areas between curve and axes, circumscribed rectangle, solids of revolution with respect to the axes, distance to the X-axis (or the Y-axis) of the centre of gravity of the arc, or of the centres of gravity of the areas between curve and axes.

The relations between these variables were expressed, if they could be, by means of equations. This was not always possible; until just before the end of the seventeenth century there were no formulas for transcendental relations, and these were expressed by means of certain circumlocutions in prose, which basically expressed a method of geometric construction for the curve representing the transcendental relation in question.

[2] L'Hôpital *1696*.

[3] These variable geometrical quantities are, in terms of Menger's classification of the concepts designated by the term "variable" (*cf. 1955*, pp. xi–xii), of the type which he calls "consistent classes of quantities" or "fluents"—with one important restriction, however. Menger's "fluents" presuppose the choice of a unit. They are pairs, consisting of a "thing" and a corresponding number, the number indicating the value or the measure of the thing with respect to a unit (*1955*, p. 167). However, the variable geometric quantities of seventeenth century mathematics (and also of physics in that period) were not, or not necessarily, related to a unit and expressed as numbers; compare § 1.5.

1.4. CARTESIAN analysis introduced the use of equations to represent and analyse the relations between the variables connected with the curve; usually the relation between ordinate and abscissa was taken as fundamental.

It is important to notice the absence of the concept of *function* in this context of algebraic relations between variables. Neither the equations nor the variables are functions in the sense of a mapping $x \to y(x)$, that is, a unidirectional relation between an "independent" variable x and a "dependent" variable y. A relation between x and y was considered as one entity, not a combination of two mutually inverse mappings $x \to y(x)$ and $y \to x(y)$. Thus the curve was not seen as a graph of a function $x \to y(x)$, but as a figure embodying the relation between x and y.

Variables are not functions, because the concept of variable does not imply dependence on another, specially indicated "independent" variable.

I shall use the absence of the concept of function to explain several aspects of the early differential calculus, such as, for instance, the lack of the concept of *derivative*. A derivative [function] presupposes the prior concept of function and hence could not play a fundamental role in the early calculus.

1.5. The variables of geometric analysis referred to geometric quantities, which were not real numbers[4]. For geometric quantity, or quantity in general, as conceived by mathematicians up to the seventeenth century, lacks a multiplicative structure and a unit element. Quantities were conceived as having a dimension. Geometric quantities could have the dimension of a line (*e.g.* ordinate, arc length, subtangent), of an area (*e.g.* the area between curve and axis) or of a solid (*e.g.* the solid of revolution). Outside geometry there are the quantities of different dimensions such as velocity, corporeity (or mass), force, *etc.* Furthermore, the algebraic manipulation, especially with geometric quantities, led to dimensions higher than that of the solid. Although these quantities of higher dimension, like for instance powers like a^4 and b^5 of line segments a and b were felt to be not directly interpretable in space; they were accepted in analysis and their dimension was determined by the number of factors with the dimension of a line.

Only quantities of the same dimension could be added. In certain cases the multiplication of quantities was interpretable, as for instance in the case of two line segments, the product of which would be an area. But multiplication was never a closed operation; that is, the product of two quantities of equal dimension could not have the same dimension. Hence within the set of quantities of the same dimension there was no multiplicative structure and no unit element. A choice of a privileged element in the set of quantities of the same dimension (as a base for measuring, for instance, or as fundamental constant for certain curves or actually as unit element) was therefore always arbitrary; the structure of quantity itself did not offer such a privileged element.

1.6. These possibilities of multiplication and addition made possible the algebraic treatment of quantities, although with certain restrictions. The special nature of multiplication induced a law of dimensional homogeneity for the equations occurring in this algebraic treatment: all the terms of an equation had to be of the same dimension.

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[4] On the concept of quantity, compare ITARD *1953*.

It is well known that as early as 1637 DESCARTES had indicated how the requirements of dimensional homogeneity could be circumvented and how a multiplication of line segments—as the prototype of quantity in general—could be defined so as to render the product also a line segment[5]. DESCARTES chose an arbitrary line segment as unit segment 1 and defined the product of two line segments a and b as the line segment c satisfying the proportion

$$1:a=b:c.$$

In particular he interpreted powers this way: if x is a line segment, x^2 is the line segment such that

$$1:x=x:x^2.$$

This solution of the problem of dimension was useful in the theory of equations in one unknown. These could now be interpreted as relations between line segments, and the roots would also be line segments, by which both irrational solutions of equations and dimensions higher than the solid became interpretable.

But in the analytical study of curves, dimensional homogeneity of equations continued to be a major requirement of neatness until well into the eighteenth century[6]. This is not too surprising[7] because in that part of mathematics dispensing with dimensional homogeneity had no direct advantages, apart from rendering higher powers interpretable. The introduction of a unit requires an arbitrary choice which infringes on the generality of the treatment, and also dimensional homogeneity assures natural geometric interpretation of every step in the algebraic analysis and thus it provides a useful check on complicated calculations.

In a geometric analysis which keeps to dimensional homogeneity it is not necessary to introduce a unit length, and therefore the geometric quantities such as length, area, *etc.* are not scaled; they are not real numbers, representing a ratio

[5] DESCARTES *1637*, opening sections.

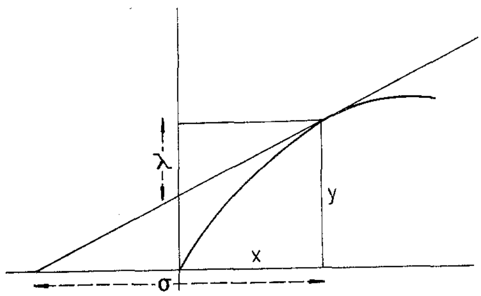
[6] As an illustration of the persistence of the dimensional interpretation of formulas I quote JOHANN BERNOULLI's definition of a homogeneous differential equation: a differential equation in which "nullae occurrunt quantitates constantes, quae dimensionum numerum adimplent." (BERNOULLI to LEIBNIZ, 19-V-1694; *Math. Schr.* III, pp. 138–139.) The definition presupposes homogeneity; absence of constant quantities as factors to adjust the homogeneity means that all terms are, apart from numerical factors, products of an equal number of variable factors. Even in the 1720's BERNOULLI objected to a mathematician who overlooked dimensional homogeneity: "Pardon, Monsieur, c'est là encore une façon de parler contre l'usage des Géométres; car vous savez que chez eux *multiplier un rectangle par une ligne, c'est faire un parallelépipede*, et non pas un autre rectangle ..." (*Opera* IV, p. 164.)

One of the reasons why the requirement of dimensional homogeneity was eventually left behind was the emergence of transcendental relations, especially the exponential functions, Indeed, a^x does not have a well defined dimension. Compare L'HÔPITAL's reaction to BERNOULLI's treatment of exponential functions: "... car que peut signifier m^n si m et n marquent des lignes? une ligne elevée à la puissance designée par une autre ligne?" (L'HÔPITAL to JOHANN BERNOULLI, 16-V-1693; BERNOULLI *Briefwechsel*, p. 172.)

[7] BOYER, in *1956* (especially, 84–85, pp. 140, 162), emphasizes that dimensional homogeneity was abandoned only almost a century after DESCARTES, but he seems to consider this as an unexplained delay in the development towards modern analytic geometry.

to a standard unit. Real numbers appeared in analysis only as integer or fractional factors in the terms of equations, or as ratios of two quantities of the same dimension.

1.7. In Chapter 2 I shall explore the implications of the fact that the early Leibnizian infinitesimal calculus was a geometric calculus. Here I shall conclude the general remarks on its geometric nature by indicating how the geometric background of the early Leibnizian calculus explains why a concept of *derivative* was absent in that calculus. First of all, the concept of derivative presupposes the concept of *function* (because the derivative dy/dx is the derivative of a function $y(x)$), and since the latter was virtually absent in the analysis of geometric problems (see 1.4 above), so the former could not be there either. In the configuration of the curve, the tangent and the connected variables (see the figure)



the derivative dy/dx, occurs only as the ratio of the ordinate y to the subtangent σ. This ratio has no obvious central position in the configuration and its choice as fundamental concept would therefore be very arbitrary. Indeed it is not clear why y/σ rather than x/λ should be chosen. Put in other words, the choice of y/σ implies the arbitrary choice of considering y as a function of x, rather than x as a function of y, or both x and y as functions of some other variable.

But there is still another reason why the derivative could not occur naturally in the geometric context, and this reason is connected with the dimensional interpretation of geometric quantities. If y/σ is considered as the derivative of the variable y, then derivation would correlate a ratio (the derivative) to a variable that has the dimension of length. This implies that the operation cannot be repeated in a natural way because it is not clear what sort of quantity it would correlate with a ratio. The only way to introduce repeated derivatives would be to interpret the ratio y/σ in some way as a line segment, and then to plot a new curve along the X-axis with ordinate y/σ. The ratio of ordinate and subtangent of this new curve would then be the derivative of the derivative. But the ratio y/σ is a real number, and therefore its interpretation as a line segment involves the choice of a unit length. Since the unit is not given at the outset, this implies an arbitrary choice; in a purely geometric context, higher-order derivatives are not uniquely defined.

Thus the derivative could not occur in the geometric phase of the infinitesimal calculus, and this may help us to understand why the early infinitesimal calculus was built upon the concept of the differential with all its concomitant problems concerning the infinitely small. Also in differentiation, interpreted as correlating a differential to a variable, the repetition of the operation involves an arbitrary choice, namely the choice of the progression of the variables (*cf.* § 2.16 *sqq.*). This aspect of the concept of the differential forms one of the main themes of my study; it is especially important in Chapter 5.

1.8. Two separate causes for the absence of the derivative in the early period of the calculus have been mentioned above: the absence of the concept of function and the requirements of dimensional interpretation. Both features were changed as analysis was separated from geometry. In the first half of the eighteenth century, a shift of interest occurred from the curve and the geometric quantities themselves to the formulas which expressed the relations among these quantities. The analytical expressions involving numbers and letters, rather than the geometric objects for which they stood, became the focus of interest. The concern about the dimensional homogeneity of formulas faded. Homogeneity in this sense survived only as a technical term for a special property of formulas. This meant that tacitly it was supposed that a unit quantity was chosen, for otherwise homogeneity would be an essential requirement for all formulas. Hence the letters in the formulas represented scaled quantities, so that we may say that the practitioners of analysis in this phase worked with real numbers based on a number-line model; but there was little interest in what the letters in formulas signified.

1.9. This change of interest towards the formula made possible the emergence of the concept of *function of one variable*. The term "function" has its origin in the geometrical phase of analysis. LEIBNIZ introduced it into mathematics and used it for variable geometric quantities such as coordinates, tangents, radii of curvature, *etc.* These were the "functiones" of a curve; they were not considered as dependent on one specified independent variable[8]. Later JOHANN BERNOULLI wrote about the powers of a variable "or any function in general" of a variable[9]. LEIBNIZ agreed[10] to this use of the term, which thus lost its initial geometric connotations and became a concept connected with formulas rather than with figures.

Indeed it is only natural that as analysis was separated from geometry, the basic components of formulas should become fundamental concepts. The function, as defined by JOHANN BERNOULLI and EULER, was such a basic component part of formulas, namely an expression involving constant quantities (letters and numbers) and only one variable quantity (letter).

[8] As a mathematical term, the word function occurs for the first time in print in LEIBNIZ *1692a*, but LEIBNIZ had used it in much earlier manuscripts. In *1694a* he wrote: "*Functionem* voco portionem rectae, quae ductis ope sola puncti fixi et puncti curvae cum curvedine sua dati rectis abscinditur." (*Math. Schr.* V, p. 306.) As examples, he gave abscissa, ordinate, tangent, perpendicular, subtangent, subperpendicular, parts of the axes cut off by the tangent and the perpendicular, radius of curvature.

[9] "... (curva) cujus applicatae FP ad datam potestatem elevatae seu generaliter earum quaecunque functiones ..." (Appendix to a letter of JOHANN BERNOULLI to LEIBNIZ, 5-VII-1698; LEIBNIZ *Math. Schr.* III, pp. 506–507.)

[10] "Placet etiam, quod appellatione Functionum uteris more meo." (LEIBNIZ to JOHANN BERNOULLI, 19-VII-1698; LEIBNIZ *Math. Schr.* III, p. 525.)

Thus we have BERNOULLI's definition:

> Here we call *function* of a variable quantity, a quantity composed in whatever way of that variable quantity and of constants[11].

and EULER's:

> A function of a variable quantity is an analytical expression composed in whatever way of that variable quantity and of numbers or constant quantities[12].

EULER, in fact, moved slightly away from analytical representability; he allowed implicit relations as functions[13] and in his *1755* he gave a very general formulation of the concept of function:

> If quantities depend on others in such a way that if the latter are changed, the former undergo a change as well, then the former are called functions of the latter. This terminology is a very general one and covers all ways in which one quantity can be determined by others[14].

Also, EULER extended the concept of function to expressions involving more than one variable[15]. The emergence of functions of more than one variable marks another decisive move away from the geometric paradigm of the curve with connected geometric quantities, namely a move from problems (as about curves) involving only one degree of freedom, to those with, in principle, any number of degrees of freedom.

1.10. Thus the separation of analysis from geometry introduced the concept of function and removed the dimensional interpretation of the objects of study; the way was open for the introduction of the derivative. Still the differential kept its position as fundamental concept of the infinitesimal calculus until long after analysis had ceased to be geometric. And even when, through the works of LAGRANGE, BOLZANO and CAUCHY[16], the derivative had replaced the differential as fundamental concept of the calculus, the differential withstood all attempts

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[11] "On appelle ici *Fonction* d'une grandeur variable, une quantité composée de quelque manière que ce soit de cette grandeur variable et de constantes." (JOHANN BERNOULLI *1718*; *Opera* II, p. 241.)

[12] "Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus." (EULER *1748*, § 4.)

[13] "Quin etiam functiones algebraicae saepe numero ne quidem explicite exhiberi possunt, cuiusmodi functio ipsius z et Z, si definiatur per huiusmodi aequationem

$$Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1.$$

Quanquam enim haec aequatio resolvi nequit, tamen constat Z aequari expressioni cuipiam ex variabili z et constantibus compositae ac propterea fore Z functionem quandam ipsius z." (EULER *1748*, § 7.)

[14] "Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipsae mutationes subeant, eae harum functiones appellari solent; quae denominatio latissime patet atque omnes modos, quibus una quantitas per alias determinari potest, in se complectitur." (EULER *1755*; *Opera* (I) X, p. 4.)

[15] As for instance in EULER *1755*, Chapter VII.

[16] Compare BOYER *1949* (pp. 251, 268, 275). Unlike LAGRANGE, BOLZANO and CAUCHY saw that, for a sufficiently rigorous formulation of the calculus, the derivative itself had to be defined in terms of the limit concept.

to eliminate it completely from analysis. It still appears in mathematics, either as the unrigorously introduced, but didactically helpful, infinitesimal in introductions to the calculus[17], or redefined as element of the dual of a tangent space, or, again, but now rigorously introduced, as infinitesimal in non-standard analysis[18].

The question of why the derivative replaced the differential as the fundamental concept of the infinitesimal calculus, needs further scrutiny. This replacement is usually thought to have been caused by an embarrassment, increasingly felt throughout the eighteenth century, over the logical inconsistencies of the infinitely small, and hence the inadequacy of the differential as fundamental concept of the calculus. The reasons why such a concern may bring the derivative to the fore are evident even in certain studies of LEIBNIZ himself on the foundations of the calculus. These studies, which were not published and therefore remained without influence upon the development of the infinitesimal calculus, are discussed in Chapter 4.

However, there were more reasons for the emergence of the derivative. One of them is the study of functions of more than one variable. The usual conceptions and techniques of differentials break down when applied to such functions, and the ensuing difficulties have to be solved by the systematic use of derivatives and partial derivatives[19].

Another reason for the emergence of the derivative is connected with the higher-order differentials. I shall discuss this reason in Chapter 5; suffice it here to remark that, unlike the hardy first-order differentials, the higher-order differentials were banished quite early. It is reasonable to suppose that the technical and conceptual difficulties associated with higher-order differentials

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[17] APOSTOL has collected in his chapter on the differential (*1969*, pp. 167–189) six articles from the *Amer. Math. Monthly*, published between 1942 and 1952, on how to introduce and use the differential in teaching practice. In the last article the editors of the *Monthly* come to the conclusion that "there is no commonly accepted definition of the differential which fits all uses to which the notation is applied." (p. 186.)

[18] ROBINSON *1966*; compare Appendix 2.

[19] The usual concept of the differential was connected with the concept of the variable as ranging over an ordered sequence of values; the differential was the infinitesimal difference between two successive values of the variable (see § 2.4 and § 2.6). Variables which are functions of two independent variables cannot be conceived as ranging over an ordered sequence in this sense, and hence the concept of the differential as the infinitesimal difference between *successive* values of the variable breaks down. The differential dV of a function $V(x, y)$ is therefore directly introduced in terms of its relation with the ordinary differentials of x and y:

$$dV = P\,dx + Q\,dy$$

(*cf.* EULER *1755*, § 213 *sqq*). Here P and Q are the partial derivatives, which EULER (*ibid.*, § 231) indicated by brackets:

$$P = \left(\frac{dV}{dx}\right), \quad Q = \left(\frac{dV}{dy}\right).$$

For such expressions the usual technique for dealing with dx and dy (for insta ce the cancelling of differentials in a quotient) cannot be applied; the dx's in $\left(\frac{dV}{dx}\right)$ and in $P\,dx$ are not the same; $\left(\frac{dV}{dx}\right)dx \neq dV$.

were so severe that these differentials had to be eliminated. I shall argue in the fifth Chapter that this was indeed the case, and that the attempts, especially those of EULER, to eliminate higher-order differentials formed one of the main causes of the emergence of the derivative.

2. The LEIBNIZIAN Infinitesimal Calculus

2.0. This chapter provides an outline of the theory, the techniques and the underlying concepts of the infinitesimal calculus practised by LEIBNIZ and his early followers such as JAKOB I and JOHANN I BERNOULLI and L'HÔPITAL.

The presentation of such an outline presents methodological problems connected with the idea of underlying concepts, for the concepts are not always made explicit in the original writings (as for instance in the case of the *progression of the variables*, discussed below). Still, even if not formulated explicitly, particular concepts may strongly influence and direct the development of a branch of science, and the historian cannot understand such a development unless he makes these concepts explicit for himself. An outline of the LEIBNIZIAN calculus presents therefore a twofold task: first, to write as if it were a modern textbook version of the LEIBNIZIAN calculus as close as possible to what LEIBNIZ and his followers thought and practised; secondly, to indicate how far the elements of such a unified and explicit theory are abstracted from the actual practice in which they appeared.

In the following I make a typographical distinction between these two aspects of the outline. The paragraphs in italics contain abstracts of the underlying theory; each of these paragraphs is followed by a discussion of the texts on which the abstract is based and an assessment of the deviation between my presentation of the theory and actual practice.

Two further preliminary remarks are necessary. The outline of the LEIBNIZIAN calculus does not cover the genesis of this calculus in the 1670's, which is described most fully in HOFMANN *1949*. Rather, it describes the calculus after a certain consolidation, in which inconsistencies, induced by influences of the calculus of finite number sequences[20] and by the theory of indivisibles, were removed. Appendix 1 contains some remarks on the relations between the LEIBNIZIAN calculus and indivisible techniques; the outline covers the consolidated LEIBNIZIAN calculus from about the year 1680.

The outline accepts infinitely small and infinitely large quantities as genuine mathematical entities. To do otherwise would depart too far from the LEIBNIZIAN calculus. By accepting these quantities, the outline accepts all the inconsistencies which during the 18^{th} century were increasingly felt as embarrassment and which were removed in the 19^{th} century by eliminating altogether the infinitesimal quantities from the calculus. These inconsistencies and the resulting deficiency of the foundations of the calculus have attracted more attention from historians of mathematics than the question of how, on such insecure foundations, the

[20] The calculus of number sequences had as effect that LEIBNIZ's earliest studies on the calculus (discussed by HOFMANN in his *1949*) were less strictly geometrical than his later work. For instance, in these earliest studies formulas often occur which violate the requirement of dimensional homogeneity.

calculus could develop in so prolific a manner as it did from LEIBNIZ's time to CAUCHY'S. I shall therefore accept the inconsistencies in the outline and discuss them later only as far as they caused actual technical difficulties or induced certain directions of development.

A preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities is provided by the recently developed *non-standard analysis*[21], which shows that it is possible to remove the inconsistencies without removing the infinitesimals themselves. I discuss how *non-standard analysis* relates to the LEIBNIZIAN calculus in Appendix 2.

2.1. *The* LEIBNIZIAN *calculus has its origins in the theory of number sequences and the difference sequences and sum sequences of such sequences.* LEIBNIZ *explored this theory in the 1670's*[22]. *He applied it to the study of curves by considering sequences of ordinates, abscissas* etc., *and supposing the differences between the terms of these sequences infinitely small (that is, negligible with respect to finite quantities, but unequal to zero). Therefore, the fundamental concepts of the* LEIBNIZIAN *infinitesimal calculus can best be understood as extrapolations to the actually infinite of concepts of the calculus of finite sequences. I use the term "extrapolation" here to preclude any idea of taking a limit. The differences of the terms of the sequences were not considered each to approach zero*[23]. *They were supposed fixed, but infinitely small.*

Compare LEIBNIZ's assertion:

> The consideration of differences and sums in number sequences had given me my first insight, when I realized that differences correspond to tangents and sums to quadratures[24].

Also:

> For instance $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ etc. or $\int\frac{dx}{xx-1}$, with x equal to 2, 3, 4, *etc.* is a sequence which taken entirely to infinity, can be summed, and dx is here 1. For in the case of numbers the differences are assignable. (...) But if x or y were not discrete terms, but continual terms, that is, not numbers whose differences are assignable intervals, but straight line abscissas increasing con-

[21] ROBINSON *1966*.

[22] See HOFMANN & WIELEITNER *1931* and HOFMANN *1949*, pp. 6–13.

[23] Thus the following assertion of BOURBAKI (*1960*, p. 208) is misleading: "(LEIBNIZ) se tient très près du calcul des différences, dont son calcul différentiel se déduit par un passage à la limite que bien entendu il serait fort en peine de justifier rigoureusement." For the same reason the following remark by HOFMANN on LEIBNIZ's invention (1675) of the calculus must be modified: "Schliesslich erkannte er (*i.e.* LEIBNIZ) als gemeinsame Grundlage der zahlreichen und bis dahin nur umständlich durch individuellen Ansätze gewonnenen Einzelergebnisse, den *Grenzprozess*." (*1966*, p. 210.)

[24] "Mihi consideratio Differentiarum et Summarum in seriebus Numerorum qrimam lucem affuderat, cum animadverterem differentias tangentibus, et summas puadraturis respondere." (LEIBNIZ to WALLIS, 28-V-1697; *Math. Schr.* IV, p. 25.)

tinually or by elements, that is, by inassignable intervals, so that the sequence of terms constitutes the figure, ...[25]

The following quotation reveals LEIBNIZ's opinion about infinitely small quantities:

And such an increment (*namely the addition of an incomparably smaller line to a finite line*) cannot be exhibited by any construction. For I agree with Euclid Book V Definition 5 that only those homogeneous quantities are comparable, of which the one can become larger than the other if multiplied by a number, that is, a finite number. I assert that entities, whose difference is not such a quantity, are equal. (...) This is precisely what is meant by saying that the difference is smaller than any given quantity[26].

For LEIBNIZ's further arguments about the nature of the infinitely small see Chapter 4.

2.2. *The importance of theories of finite sequences for the problems about curves, to which the* LEIBNIZIAN *calculus was primarily applied, lies in the fact that it is often useful to approximate the curve by a polygon. The ordinates and abscissas corresponding to the vertices of the polygon form finite sequences*[27]. *In accord with the conception of the differential calculus as being an extrapolation of the calculus of finite sequences to the actually infinite, the practitioners of the* LEIBNIZIAN *calculus emphasized that the key to the calculus was to conceive the curve as an infinitangular polygon.*

The concept of the curve as an infinitangular polygon played an important role in the new infinitesimal methods developed in the 17^{th} century. LEIBNIZ stressed its importance for his calculus for instance as follows:

I feel that this method and others in use up till now can all be deduced from a general principle which I use in measuring curvilinear figures, *that a curvilinear figure must be considered to be the same as a polygon with infinitely many sides.*[28]

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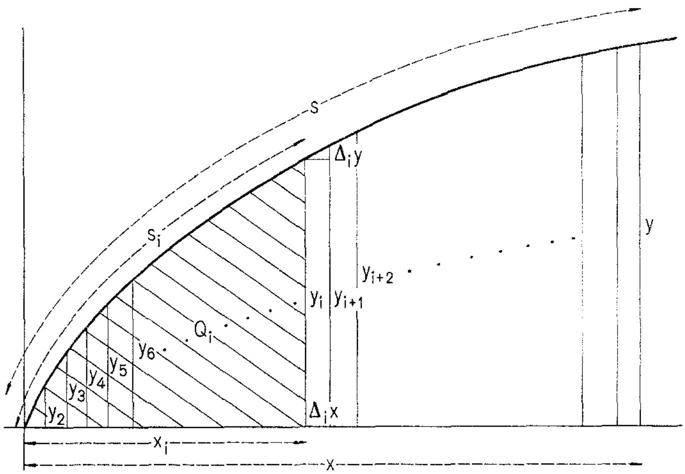
[25] "Exempli gratia $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ etc. seu $\int \frac{dx}{xx-1}$, posito x esse 2 vel 3 vel 4 etc. est series quae tota in infinitum sumta summari potest, et dx quidem hoc loco est 1. In numericis enim differentiae sunt assignabiles. (...) Quodsi x vel y essent non termini discreti, sed continui, id est non numeri intervallo assignabili differentes, sed lineae rectae abscissae, continue sive elementariter hoc est per inassignabilia intervalla crescentes, ita ut series terminorum figuram constituat; ..." (LEIBNIZ *1702b*; *Math. Schr.* V, pp. 356–357.)

[26] "Nec ulla constructione tale augmentum exhiberi potest. Scilicet eas tantum homogeneas quantitates comparabiles esse, cum Euclide lib. 5 defin. 5 censeo, quarum una numero, sed finito multiplicata, alteram superare potest. Et quae tali quantitate non differunt, aequalia esse statuo (...). Et hoc ipsum est, quod dicitur differentiam esse data quavis minorem." (LEIBNIZ *1695a*; *Math. Schr.* V, p. 322.)

[27] Such sequences occur especially in ARCHIMEDEAN style studies of geometrical problems, in which the method to prove the results was the so-called method of exhaustion, of which WHITESIDE (*1961*, pp. 331–348) gives an authorative account.

[28] "Sentio autem et hanc [methodum] et alias hactenus adhibitas omnes deduci posse ex generali quodam meo dimentiendorum curvilineorum principio, *quod figura curvilinea censenda sit aequipollere Polygono infinitorum laterum.*" (LEIBNIZ *1684b*; *Math. Schr.* V, p. 126.) The method refered to is an infinitesimal method which J. CHR. STURM had exposed in an article in the *Acta Erud.* of March 1684.

2.3. *It will prove rewarding to study in detail how theories of sequences, as applied to curves and approximating polygons, can be extrapolated to the actually infinite. In the case of the approximation of a curve by a polygon of a finite number of sides (see the figure), the polygon induces sequences of ordinates $\{y_i\}$, of abscissas $\{x_i\}$, of*



arc-lengths $\{s_i\}$, of quadratures[29] *$\{Q_i\}$, and in general of all variables which may be considered in the problem at hand. These sequences consist of a finite number of finite terms. (If one branch of the curve extends to infinity, the number of terms may be infinite, but this does not affect my argument.)*

The operators of forming sequences of differences or sums of a given sequence, operators which I indicate by Δ and Σ, respectively, yield new finite sequences of finite terms:

$$\Delta\{x_i\} = \{\Delta_i x\}$$

with

$$\Delta_i x \underset{D}{=} x_{i+1} - x_i,$$

and

$$\Sigma\{y_i\} = \{\Sigma_{j=1}^{i} y_j\},$$

etc.

In his early studies on difference schemes and sequences in general[30], LEIBNIZ dealt with the relations indicated here and in the following paragraphs.

2.4. *In the extrapolation from the finite array to the actually infinite the polygon becomes a polygon whose sides are infinitely small and whose angles are infinitely many. This infinitangular polygon is considered to coincide with the curve; its infinitely small sides, if prolonged form tangent lines to the curve.*

[29] The term "quadrature" is here used for the area between curve, ordinate and axis, not for the process of calculating (or squaring) this area. Both meanings of the term occur in seventeenth century mathematical texts.

[30] See HOFMANN & WIELEITNER *1931* and HOFMANN *1949*, pp. 6–13.

The sequences of ordinates abscissas etc. *now consist of infinitely many terms. Successive terms of these sequences have infinitely small differences; anachronistically speaking, one might say that the terms lie dense in the range of the corresponding variable. In the practice of the* LEIBNIZIAN *calculus, the variable is conceived as taking only the values of the terms of the sequence. Thus the conception of a variable and the conception of a sequence of infinitely close values of that variable, come to coincide.*

The operators Δ and Σ of the finite array act on sequences. Thus, in the extrapolation to the actually infinite, Δ and Σ are transformed into operators d and $\int$ (see the next section), which act on the sequences of infinitely close values of variables. But as these sequences are indiscernible from the variables themselves, d and $\int$ are operators which act on variables.

The conception of the variable as ranging over an ordered sequence of values—LEIBNIZ uses the terms "series" and "progressio"—is clearly expressed in the quotation given above in § 2.1. Another example is LEIBNIZ's discussion of the rule $d(xy) = xdy + ydx$; it shows that also the area xy of the circumscribed rectangle was considered as a variable ranging over a sequence of values:

> $d(xy)$ is the same as the difference between two adjacent xy, of which let one be xy, the other $(x+dx)(y+dy)$. Then $d(xy) = (x+dx)(y+dy) - xy$ or $xdy + ydx + dxdy$, and this will be equal to $xdy + ydx$ if the quantity $dxdy$ is omitted, which is infinitely small with respect to the remaining quantities, because dx and dy are supposed infinitely small (namely if the term of the sequence represents lines, increasing or decreasing continually by minima).[31]

See also the quotations given below in § 2.8 and § 2.9.

LEIBNIZ used the adjective "continuus" for a variable ranging over an infinite sequence of values. He also used terminology of growth and motion, speaking for instance about "increasing by minima" ("per minima crescentes"), "continually increasing by inassignables" ("continue crescentes per inassignabilia"), "momentaneously increasing" ("momentanee crescentes"), in which "minima" and "inassignables" stand for the differentials as differences between successive terms of the sequence. If these differences are all equal, LEIBNIZ sometimes used the term "uniformly increasing" ("aequabiliter crescere").

2.5. *Considering now how the finite difference sequences and sum sequences are affected by the extrapolation to the actually infinite, we see that a difference sequence is transformed into a sequence of an infinite number of infinitely small terms; these terms are called the* differentials. *A finite sum sequence is transformed into a sequence of an infinite number of infinitely large terms; these terms are called the* sums.

---

[31] "$d\,\overline{xy}$ idem est quod differentia duorum xy sibi propinquorum quorum unum esto xy, alterum $x+dx$ in $y+dy$ (that is: $(x+dx)(y+dy)$) fiet: $d\,\overline{xy}$ aequ. $\overline{x+dx}$ in $\overline{y+dy} - xy$ seu $+xdy+ydx+dxdy$ et omissa quantitate $dxdy$, quae infinite parva est respectu reliquorum, posito dx et dy esse infinite parvas (cum scilicet per seriei terminum lineae continue per minima crescentes vel decrescentes intelliguntur) prodibit $xdy+ydx$." (LEIBNIZ *Elementa*, p. 154.)

Differentials and sums form sequences and are therefore variables of the same sort as the sequences of the ordinary variables discussed in the preceding paragraph. The differential is an infinitely small variable; the sum is an infinitely large variable. Thus the operator Δ, by the extrapolation, transforms into an operator differentiation, *indicated by the symbol d, which assigns an infinitely small variable to a finite variable, for instance dy to y. Similarly, the operator Σ transforms, by the extrapolation, into the operator* summation, *indicated by the symbol $\int$, which assigns an infinitely large variable to a finite variable, for instance $\int y$ to y.*

The Latin terms are *differentia* or *differentiale*, and *summa*; the latter was little used and was soon replaced by the term *integrale*; for the operator $\int$ accordingly the terms *summatio* and *integratio* occur; see § 2.10 and § 2.11. The operator d is called *differentiatio*.

It is important to stress the concept of the differential as a variable, and of differentiation as an operator assigning variables to variables. On the concept of variable, see § 1.4. As I explained there, the concept of variable differs from the concept of function in that it is not necessary to specify on which "independent" variable a given variable depends. Differentials and sums have different values according to where in the geometrical figure they occur; although infinitely small, or infinitely large respectively, they have thus the same characteristics which make ordinate, abscissa *etc.* variables; they are therefore rightly considered as variables. The fact that a differential is sometimes supposed constant, is not at variance with its status as variable. Constant variables occur in many situations, as for instance the constant ordinate of a horizontal straight line, the constant radius of curvature of the circle and the constant subtangent of the logarithmic curve.

The common concern of historians with the difficulties connected with the infinite smallness of differentials[32] has distracted attention from the fact that in the practice of the LEIBNIZIAN calculus differentials as single entities hardly ever occur. The differentials are ranged in sequences along the axes, the curve and the domains of the other variables; they are variables[33], themselves depending on the other variables involved in the problem, and this dependence is studied in terms of differential equations.

Moreover, to introduce higher-order differentials (see § 2.8), first-order differentials have to be conceived as variables ranging over an ordered sequence; if only a single dx is considered, ddx does not make sense. The following quotation from LEIBNIZ illustrates this:

> Further, ddx is the element of the element or the *difference of the differences*, for the quantity dx itself is not always constant, but usually increases or decreases continually.[34]

[32] The attitude is evident, for instance, in BOYER *1949*.

[33] The only reference I have found in works on the history of mathematics to the fact that differentials are variables and that the way in which they vary can be chosen arbitrarily by choosing the progression of the variables, is in COHEN *1883* (especially, p. 75). However, as COHEN's prime objective is to ascertain the reality of differentials in the sense of an *Erkenntniskritik*, the historical sections of his book are of little further interest for present-day historians of mathematics.

[34] "Porro ddx est elementum elementi seu *differentia differentiarum*, nam ipsa quantitas dx non semper constans est, sed plerumque rursus (continue) crescit aut decrescit." (LEIBNIZ *1710a*; *Math. Schr.* VII, p. 322–323.)

2.6. *The infinitely small differential and the infinitely large summa are considered actually as a difference or a sum; the differential dy of a finite variable y is conceived as the difference between y^{I} and y, if y^{I} is the ordinate next to y in the infinite sequence of ordinates. The sum $\int y$ is conceived as the sum of all the terms in the sequence of the ordinates, from the ordinate at the origin (or another fixed ordinate) to the ordinate y.*

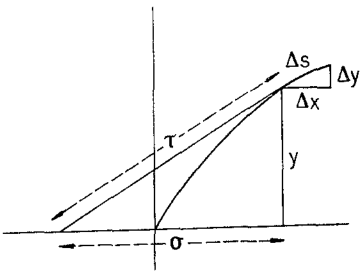
Compare Leibniz's explanation:

> Here dx means the element, that is, the (instantaneous) increment or decrement, of the (continually) increasing quantity x. It is also called difference, namely the difference between two proximate x's which differ by an element (or by an inassignable), the one originating from the other, as the other increases or decreases (momentaneously).[35]

On the concept of sums, see the quotation in § 2.9. On the relatively scarce occurrence of infinitely large sums in the calculus, see Appendix 1. As one example of its occurrence I quote some lines of Johann Bernoulli, in which he evaluates sums as quotients with infinitely small denominators:

> Now because (if dz is supposed constant) $\int z, \int^2 z, \int^3 z, \int^4 z$, *etc.*, are equal to $\frac{zz}{1\cdot 2\cdot dz}, \frac{z^3}{1\cdot 2\cdot 3\cdot dz^2}, \frac{z^4}{1\cdot 2\cdot 3\cdot 4\cdot dz^3}, \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot dz^4}$, *etc.* ...[36]

2.7. *In the finite array, the ratios $\Delta x:\Delta y:\Delta s$ are approximately equal to the ratios $\sigma:y:\tau$ of subtangent, ordinate and tangent (see the figure). In the extrapolation*



to the actually infinite the triangle becomes the differential triangle with sides dx, dy and ds. The hypotenuse of the differential triangle is a side of the infinitangular polygon, and therefore, if prolonged, it forms a tangent line to the curve. Hence

$$dx:dy:ds=\sigma:y:\tau;$$

this relation is fundamental for the application of differentials to problems about tangents.

[35] "Hic dx significat elementum, id est incrementum vel decrementum (momentaneum) ipsius quantitatis x (continue) crescentis. Vocatur et *differentia*, nempe inter duas proximas x elementariter (seu inassignabiliter) differentes, dum una fit ex altera (momentanee) crescente vel decrescente." (Leibniz *1710a*; *Math. Schr.* VII, pp. 222–223.)

[36] "quoniam nunc (posita dz constante) $\int z, \int^2 z, \int^3 z, \int^4 z$ etc. aequantur ipsis $\frac{zz}{1\cdot 2\cdot dz}, \frac{z^3}{1\cdot 2\cdot 3\cdot dz^2}, \frac{z^4}{1\cdot 2\cdot 3\cdot 4\cdot dz^3}, \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot dz^4}$ etc. ..." (Johann Bernoulli to Leibniz, 27-VII-1695; *Math. Schr.* III, p. 199.)

LEIBNIZ became aware of the importance of the differential triangle while studying work of PASCAL[37]. In his first publication on the calculus (*1684a*), LEIBNIZ used the relation $dx:dy=\sigma:y$ to introduce the differential as a finite line. I discuss this definition, which is rather anomalous in LEIBNIZ's work on the calculus, in Chapter 4, where I also investigate the reasons why he adopted it for his first publication.

Compare further LEIBNIZ's explanation:

> ... to find a *tangent* is to draw a straight line which joins two points of the curve which have an infinitely small distance, that is, the prolonged side of the infinitangular polygon which for us is the same as the *curve*.[38]

2.8. *The operators Δ and Σ of the finite array can be applied repeatedly:*

$$\Delta\Delta\{y_i\}=\{\vartriangle_i^2 y\}$$

with

$$\vartriangle_i^2 y=\vartriangle_{i+1} y-\vartriangle_i y=y_{i+2}-2y_{i+1}+y_i,$$

and

$$\Sigma\Sigma\{y_i\}=\{\Sigma_{j=1}^{i}\Sigma_{k=1}^{j} y_k\}$$

etc. *Accordingly, d and $\int$ can be applied repeatedly, which application yields the differentio-differentials, or higher-order differentials, and the higher-order sums. In the case of the variable y, for instance, d applied to the variable dy yields the second-order differential ddy, a variable infinitely small with respect to dy. ddy can be conceived as the difference between dy^{I} and dy, if dy^{I} is the differential adjacent to dy in the infinite sequence of differentials. Further application of d yields the higher-order differentials $dddy$ (or d^3y), d^4y, d^5y,* etc. *$\int$, applied to the variable $\int y$, yields $\int\int y$, a variable infinitely large with respect to $\int y$, which can be conceived as the sum of the terms in the sequence $\int y$. Repeated application yields $\int\int\int y$ (or $\int^3 y$), $\int^4 y$,* etc.

Compare LEIBNIZ's explanation, already quoted in part in § 2.5:

> Further, ddx is the element of the element, or the *difference of the differences*, for the quantity dx itself is not always constant, but usually increases or decreases continually. And in the same way one may proceed to $dddx$ or d^3x and so forth.[39]

On the repeated sums see the quotation in § 2.6.

2.9. *The operators Δ and Σ in the finite array are, in a sense, reciprocal:*

$$\Delta\Sigma\{y_i\}=\{y_{i+1}\};\qquad \Sigma\Delta\{y_i\}=\{y_{i+1}-y_1\}.$$

These properties are reflected in a reciprocity of d and $\int$:

$$d\int y=y;\qquad \int dy=y.$$

[37] See HOFMANN *1949*, pp. 28–29.

[38] "... *tangentem* invenire esse rectam ducere, quae duo curvae puncta distantiam infinite parvam habentia jungat, seu latus productum polygoni infinitanguli, quod nobis *curvae* aequivalet." (LEIBNIZ *1684a*; *Math. Schr.* V, p. 223.)

[39] "Porro ddx est elementum elementi seu *differentia differentiarum*, nam ipsa quantitas dx non semper constans est, sed plerumque rursus (continue) crescit aut decrescit. Et similiter procedi potest ad $dddx$ seu d^3x, et ita porro; ..." (LEIBNIZ *1710a*; *Math. Schr.* VII, pp. 222–223.)

In the latter formula a constant should be added, but it is usually left out; $\int dy = y$ *is easily visualised as stating that the sum of the differentials in a segment equals the length of the segment.* $d\int y = y$ *lacks an obvious geometrical interpretation, because* $\int y$ *is a sequence of infinitely large terms. However, if instead of the finite variable* y *an infinitely small variable, say* $y\,dx$*, is considered, then* $d\int y\,dx = y\,dx$ *can be understood as stating that the differences between the terms of the sequence of areas* $\int y\,dx$ *are* $y\,dx$*.*

Compare LEIBNIZ's assertion:

> *Foundation of the calculus:* Differences and sums are reciprocal to each other, that is, the sum of the differences of a sequence is the term of the sequences, and the difference of the sums of a sequence is also the term of the sequence. The former I denote thus: $\int dx = x$; the latter thus: $d\int x = x$.[40]

Elsewhere, LEIBNIZ explained:

> Reciprocal to the Element or differential is the *sum*, because if a quantity decreases (continually) till it vanishes, then that quantity is the sum of all the successive differences, so that $d\int y\,dx$ is the same as $y\,dx$. But $\int y\,dx$ means the area which is the aggregate of all rectangles, any of which has an (assignable) length y and (elementary) width dx corresponding in the sequence to y. There are also *sums of sums* and so forth, for instance $\int dx \int y\,dx$, which is the solid built up of all areas such as $\int y\,dx$ multiplied by the elements dz which correspond in the sequence.[41]

2.10. *The reciprocity of the operators* d *and* $\int$ *suggests the possibility of introducing* $\int$ *as the inverse of* d *per definitionem. In fact, such a definition underlies the calculus as developed in the early studies of the Bernoullis.*

In the terminology introduced by the Bernoullis, integration, *symbol* $\int$*, is the operator which assigns to an infinitely small variable its* integral, *defined by the property that the differential of the integral equals the original quantity. So defined, the integral, like the sum, is a variable.*

The contrast between integration *and* summation *may be illustrated by the case of the quadrature*

$$\int y\,dx = Q. \tag{1}$$

In terms of summation, (1) *asserts that the sum of the infinitely small rectangles* $y\,dx$ *equals* Q*. In terms of* integration (1) *asserts that* Q *is a quantity whose differential is* $y\,dx$*.*

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[40] "*Fundamentum calculi:* Differentiae et summae sibi reciprocae sunt, hoc est summa differentiarum seriei est seriei terminus, et differentia summarum seriei est ipse seriei terminus, quorum illud ita enuntio: $\int dx$ aequ. x; *hoc* ita: $d\int x$ aequ. x." (LEIBNIZ *Elementa*, p. 153.)

[41] "Contrarium ipsius Elementi vel differentiae est *summa*, quoniam quantitate (continue) decrescente donec evanescat, quantitas ipsa semper est summa omnium differentiarum sequentium, ut adeo $d\int y\,dx$ idem sit quod $y\,dx$. At $\int y\,dx$ significat aream quae est aggregatum ex omnibus rectangulis, quorum cujuslibet longitudo (assignabilis) est y aliqua, et latitudo (elementaris) est dx ipsi y ordinatim respondens. Dantur et *summae summarum*, et ita porro, ut si sit $\int dz \int y\,dx$, significatur solidum quod conflatur ex omnibus areis, qualis est $\int y\,dx$, ordinatim ductis in respondens cuique elementum dz." (LEIBNIZ *1710a*; *Math. Schr.* VII, pp. 222–223.)

JAKOB and JOHANN BERNOULLI acquainted themselves with the LEIBNIZIAN calculus between 1687 and 1690[42]. Until 1690 the only articles by LEIBNIZ on which they could base their studies were *1684a*, which concerns differentiation only, and *1686*. The latter article mentioned summation, used the symbol $\int$, and indicated the reciprocity of sums and differentials; the sums mentioned are sums of differentials. It is not surprising, therefore, that the BERNOULLIS developed a concept of integration as the reciprocal of differentiation. For example, in JOHANN BERNOULLI's *Integral Calculus*, the integrals are introduced as follows:

> We have seen above how the *Differentials* of quantities are to be found; we shall now show how, conversely, the *Integrals* of differentials can be found, that is those quantities of which they are the differentials.[43]

LEIBNIZ, who saw use of the term "integral" for the first time in JAKOB BERNOULLI *1690*, tried later to persuade JOHANN BERNOULLI to adopt the terminology of "sums":

> I leave it to your deliberation if it would not be better in the future, for the sake of uniformity and harmony, not only between ourselves but in the whole field of study, to adopt the terminology of summation instead of your integrals. Then for instance $\int y\,dx$ would signify the sum of all y multiplied by the corresponding dx, or the sum of all such rectangles. I ask this primarily because in that way the geometrical summations, or quadratures, correspond best with the arithmetical sums or sums of sequences. (...) I do confess that I found this whole method by considering the reciprocity of sums and differences, and that my considerations proceeded from sequences of numbers to sequences of lines or ordinates.[44]

This request served as occasion for JOHANN BERNOULLI to explain the origin of the term integral:

> Further, as regards the terminology of the sum of differentials I shall gladly use in the future your terminology of summations instead of our integrals. I would have done so already much earlier if the term integral were not so much appreciated by certain geometers [a reference to French mathematicians, especially l'HÔPITAL, who had studied BERNOULLI's *Integral Calculus*] who acknowledge me as the inventor of the term. It would therefore be thought that I rather obscured matters, if I indicated the same thing now with one term and now with another. I confess that indeed the terminology does not aptly agree with the thing itself

[42] Apparently, no manuscript record of these early BERNOULLIAN studies has survived. JAKOB BERNOULLI's diary, the *Meditationes*, does not contain material on this crucial period; see HOFMANN *1956*, p. 16.

[43] "Vidimus in praecedentibus quomodo quantitatum *Differentiales* inveniendae sunt: nunc vice versa quomodo differentialium *Integrales*, id est, eae quantitates quarum sunt differentiales, inveniantur, monstrabimus." (JOHANN BERNOULLI *Integral Calculus*, p. 387.)

[44] "Unde Tibi deliberandum relinquo, annon, pro Integralibus vestris, praestet in posterum uniformitatis et harmoniae gratia non inter nos tantum, sed in ipsa doctrina adhiberi Summatorias expressiones, ita ut, exempli gratia, $\int y\,dx$ significet summam omnium y in dx respondentes ductorum, seu summam omnium hujusmodi rectangulorum: praesertim cum tali ratione summationes geometricae seu quadraturae optime cum arithmeticis seu serierum summis conferantur. (...) Ego certe in totam hanc methodum me fateor, ex hac consideratione reciprocationis inter summas differentiasque, incidisse, et a Seriebus numerorum ad linearum seu ordinatarum considerationes processisse." (LEIBNIZ to BERNOULLI 28-II-1695; *Math. Schr.* III, p. 168.)

(the term suggested itself to me as I considered the differential as the infinitesimal part of a whole or *integral*; I did not think further about it).[45]

The matter was left there, and gradually the terminology of integrals replaced LEIBNIZ's original terminology of sums.

2.11. *The calculus built on the concept of* integration *and that built on the concept of* summation *differ also in that* summation *leads naturally to infinitely large quantities (see Appendix 1), whereas in a calculus based on the concept of* integration, *such quantities are less likely to appear, since* integration *is applied only to quantities which are themselves differentials.*

2.12. *The differentials and sums, introduced by the operators d and $\int$, are quantities, and therefore they have a dimension. If these infinitesimal quantities are of the same dimension, they can be added; also products of such quantities can be formed and the dimension of the product will be related to the dimensions of the factors in the same way as in the case of finite quantities (see § 1.5).*

In the finite array, the terms of the difference and sum sequences have the same dimension as the terms of the original sequence (if y_i are line segments, then so are $\Delta_i y$ and $\Sigma_{j=1}^{i} y_j$). Consequently, d and $\int$ preserve the dimension. If y is a variable line segment, then dy is an infinitely small variable line segment and $\int y$ is an infinitely large variable line segment. If Q is a quadrature, dQ is an infinitely small area, etc.[46]

Compare JOHANN BERNOULLI's explanation of the conservation of dimension by differentiation:

The parts of a solid, although infinitely small, are always solids; those of a surface are always surfaces, and the parts of a line are always lines, for it is not possible that a kind of quantity can be changed by division into another kind of quantity.[47]

2.13. *Differentials and sums form classes having distinct orders of infinity. Thus for instance dy is infinitely small with respect to y; ddy is infinitely small with respect to dy, and in general $d^{k+1}y$ is infinitely small with respect to $d^k y$. Similarly $\int^{k+1} y$ is infinitely large with respect to $\int^k y$,* etc.

All first-order differentials of finite variables have the same order of infinity (that is, any two of them have a finite ratio, except at singularities). Consequently, for every k, all k^{th}-order differentials have the same order of infinity. This rule, by

[45] "Caeterum, quod nomenclationem differentialium summae attinet, lubentissime pro integralibus nostris Tuas in posterum adhibeo summatorias expressiones; quod diu ante fecissem, si nomen integralium non adeo invaluisset apud quosdam Geometras, qui me hujus nominis authorem agnoscunt, ut satis obscurus visus fuissem, unam eandemque rem, nunc hoc, nunc alio nomine designans. Fateor enim nomenclationem istam (quae, considerando differentialem tanquam partem infinitesimam totius vel *integri*, mihi non ulterius cogitanti, venit in mentem) rei ipsi non apte convenire." (JOHANN BERNOULLI to LEIBNIZ, 30-IV-1695; *Math. Schr.* III, p. 172.)

[46] The conservation of the dimension by the operator d marks the fundamental difference between infinitely small elements and indivisibles; compare WALLNER *1903*.

[47] "Les parties d'un corps, quoique infiniment petites, sont toujours corps; celles d'une surface, sont toujours surfaces; et les parties d'une ligne sont toujours lignes: n'étant pas possible qu'un genre de quantité puisse être changé par la division en un autre genre de quantité." (JOHANN BERNOULLI *Opera* IV, p. 162.)

no means obvious, relates to assumptions about the regularity of the infinitangular polygon which I shall discuss in § 2.18. Moreover, the order of infinity of k^{th}-order differentials is the same as that of k^{th} powers of first-order differentials (that is, $d^k y$ bears a finite ratio, except at singularities, to $(dy)^k$). This rule (see § 2.18) results from assumptions about the regularity of the infinitangular polygon.

Similarly, the sums and the repeated sums form classes having distinct orders of infinity. Because of the above mentioned relations between the elements of classes of different orders of infinity, the number of orders of infinity is infinite, but denumerable; every infinitely small quantity has a finite ratio to $(dx)^k$ for some natural number k and every infinitely large quantity has a finite ratio to $[\int y]^k$ for some natural number k (see § 2.15 below, and Appendix 2).

As an example of the terminology with which these orders of infinity were indicated I quote some lines by JOHANN BERNOULLI:

> Let a be a finite line, adx an infinitely small of the first sort, $dddy$ an infinitely small of the third sort. It is to be proved that $\frac{adx}{dddy}$ is an infinitely large of the second sort. For that purpose, let $\frac{adx}{dddy}$ be called z; hence $adx = zdddy$; hence $dx:dddy = z:a$. Now dx is infinite-infinitely larger than $dddy$; hence also z, which is the quotient resulting from the division, will be infinite-infinitely larger than a, which is a finite line; it follows that z will be an infinitely large of the second sort.[48]

It is instructive to cite in this context a proof by LEIBNIZ that ddx is a quantity infinitely small with respect to dx. The proof occurs as a refutation of NIEUWENTIJT'S opinion[49] that second-order differentials do not exist:

> For whenever the terms do not increase uniformly, the increments necessarily have differences themselves, and obviously these are the differences of the differences. Further, the renowned author [that is, NIEUWENTIJT] concedes that dx is a quantity. Now the third proportional of two quantities is again a quantity, and the quantity ddx is of this kind with respect to the quantities x and dx, which I prove thus: Let x be in geometrical progression and y in arithmetical progression, then dx will be to the constant dy as x to a constant a, or $dx = xdy:a$. Hence $ddx = dxdy:a$. Removing $dy:a$ from this by the former equation, one has $xddx = dxdx$, whence it is clear that x is to dx as dx to ddx.[50]

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[48] "Soit a une ligne finie, adx un infiniment petit du premier genre, $dddy$ un infiniment petit du troisiéme genre, il faut prouver que $\frac{adx}{dddy}$ est un infiniment grand du second genre. Pour cette fin, soit $\frac{adx}{dddy}$ nommé z; donc $adx = zdddy$; donc $dx:dddy = z:a$. Or dx est infini-infiniment plus grand que $dddy$; donc aussi z, qui est le quotient de la division, sera infini-infiniment plus grand que a, qui est une ligne finie; et partant z sera un infiniment grand du second genre." (JOHANN BERNOULLI *Opera* IV, p. 166.)

[49] Expressed in NIEUWENTIJT *1694*.

[50] "Nam quotiens termini non crescunt uniformiter, necesse est incrementa eorum rursus differentias habere, quae sunt utique differentiae differentiarum. Deinde concedit Cl. Autor, dx esse quantitatem; jam duabus quantitatibus tertia proportionalis utique est etiam quantitas; talis autem, respectu quantitatum x et dx, est quantitas ddx, quod sic ostendo. Sint x progressionis Geometricae, et y arithmeticae, erit dx ad constantem dy, ut y ad constantem a, seu $dx = xdy:a$; ergo $ddx = dxdy:a$. Unde tollendo $dy:a$ per aequationem priorem fit $xddx = dxdx$, unde patet esse x ad dx, ut dx ad ddx." (LEIBNIZ *1695a*; *Math. Schr.* V, p. 325; compare *ibid.* II, p. 288.)

This passage has repeatedly bewildered historians of mathematics.[51] It is, however, a perfectly acceptable argument, if one bears in mind that LEIBNIZ does not claim that ddx is always the third proportional of x and dx but rather gives an example in which such is the case. The example then proves the existence of quantities infinitely small with respect to dx. The curve in question is, of course, the logarithmic curve ($x = b e^{y/a}$), which was usually defined as the curve in which a geometric sequence of ordinates (or abscissas) corresponds to an arithmetic sequence of abscissas (or ordinates). Hence LEIBNIZ takes dy constant and knows that the dx form a geometrical sequence.

2.14. *To avoid ambiguities, there are certain rules of notation. If no brackets are used, the operators d, dd, d^3, etc. have to be interpreted as acting on the one letter variable following it. If the operator is meant to act on a composite variable, brackets must be added. Thus dx^2 means $(dx)^2$, as d acts only on x; the differential of x^2 is indicated as $d(x^2)$. Similarly d^2x^3 means $(d^2x)^3$. Differential quotients like $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$,* etc. *have to be interpreted as $\frac{d^2y}{(dx)^2}$, $\frac{d^3y}{(dx)^3}$,* etc. *The operator $\int$ is interpreted as acting on all letters which follow it. Thus $\int y\,dx$ means $\int(y\,dx)$.*

LEIBNIZ used overbars rather than brackets, *e.g.* $d\overline{xy}$ for $d(xy)$. He also used the comma as separating symbol; thus $d, xy + a^2$ for $d(xy + a^2)$. EULER gives these rules of notation explicitly in *1755* (§ 144).

2.15. *I turn now to a difficulty which necessarily arises in any attempt to set up an infinitesimal calculus which takes the differential as fundamental concept, namely the indeterminacy of differentials.*

The first differential dx of the variable x is infinitely small with respect to x, and it has the same dimension as x. These are the only conditions it has to satisfy, and they do not determine a unique dx, for if dx satisfies the conditions then clearly so do $2dx$ and $\frac{1}{2}dx$ and in general all $a\,dx$ for finite numbers a. That is, all quantities that have the same dimension and the same order of infinity as dx might serve as dx.

Moreover, there are elements not from this class which satisfy the conditions for dx; for instance dx^2/a and $\sqrt{a\,dx}$, for finite positive a of the same dimension as x. dx^2/a is infinitely small with respect to dx and $\sqrt{a\,dx}$ is infinitely large with respect to dx, so that there is even not a privileged class of infinite smallness from which dx has to be chosen; there is no "first" class of infinite smallness adjacent to finiteness. Thus first-order differentials involve a fundamental indeterminacy.

The early practitioners of the LEIBNIZIAN calculus seem not to have noticed this indeterminacy of first-order differentials. Compare Appendix 2 (especially § 7.8), where I discuss a study of EULER's which shows that he was aware of this problem.

It is difficult to give reasons for, or to draw conclusions from the fact that this problem was recognized late. One important aspect doubtless is that it does not influence the computational techniques or the interpretation of first-order differential equations. Geometric intuition convinces us that the finite ratios $dx:dy:ds$ are independent of the choice of dx in any class of infinitely small

[51] Compare WEISSENBORN *1856*, p. 99 and BOYER *1949*, p. 211.

quantities, so that, although the first-order differentials themselves are indeterminate, the relations between them are determined. Also the summation of differentials is not affected by this indeterminacy; $\int dx = x$ applies for every choice of the dx's. Thus in the treatment of the most common problems of the infinitesimal calculus, quadratures, tangent problems, inverse tangent problems, rectifications, cubatures, *etc.*, the indeterminacy of the fundamental concept did not influence the technique of the analysis.

However, there is another kind of indeterminacy, which affects higher-order differentials and which did profoundly influence the concepts and the techniques of the early differential calculus. I discuss this indeterminacy in the following paragraphs.

2.16. *There are many ways to approximate a curve by a polygon. To fix ideas, I mention three possibilities:*

a) polygons with equal sides,

b) polygons, the projection of whose sides on the X-axis are all equal,

c) polygons, the projection of whose sides on the Y-axis are all equal.

In these three cases the operators Δ and Σ can be applied to the appropriate sequences, but the results of this application may differ. In Case a, $\Delta_i s$ is constant; consequently $\Delta_i^k s = 0$ if $k \geqq 2$; but in general $\Delta_i^k x$ and $\Delta_i^k y$ will not be equal to zero. In Case b, $\Delta_i x$ is constant (say equal to Δx); hence $\Delta_i^k x = 0$ if $k \geqq 2$, but $\Delta_i^k y$ and $\Delta_i^k s$ will in general not be equal to zero.

Moreover, in Case b, $\Delta x \Sigma \{y_i\}$ is an approximation of the quadrature; in other words, the sequence $\{\Sigma_{j=1}^{i} y_j\}$ is approximately proportional to the sequence of quadratures $\{Q_i\}$. In Cases a and c this approximation does not apply. Therefore, the justice of such an approximation depends on the choice of the polygon.

The form of the polygon defines the sequences of abscissas, ordinates, arc lengths, etc. *Conversely, if the sequence of values of one variable is given (and if it is agreed that the vertices of the polygon are on the curve), then the polygon is determined and hence also the sequences of values of the other variables. Cases b and c, discussed above, may thus be described as polygons induced by arithmetic sequences of abscissas and ordinates, respectively.*

The indeterminacy of the approximating polygon in the finite array, or the freedom to impose an additional requirement (such as to form an arithmetic progression) on the sequence of values of one variable, is preserved in the extrapolation to the actually infinite. Thus the concept of infinitangular polygon implies an indeterminacy; it allows the free choice of an additional supposition about the sequence over which the values of one variable range. The most obvious way of making such an additional supposition is to extend the concept of arithmetic sequence to infinitesimals. Thus the supposition that the sequence of values of x is arithmetic becomes, for infinitesimals, the supposition that dx is constant.

Corresponding to the three cases discussed above there are the following possibilities for additional suppositions about the infinitangular polygon:

a′) ds constant,

b′) dx constant,

c′) dy constant.

Taking over LEIBNIZIAN *terminology, I shall refer to the imposing of an additional supposition about the infinitangular polygon as the* choice *or the* specification of the progression of the variables; *for one may conceive this choice or specification as concerning the way the variables proceed along their domains.*

The freedom to choose the progression of the variables is described in the following quotations from LEIBNIZ:

To take sums it is quite unnecessary that the dx or the dy be constant and the $ddx = 0$, but one assumes the progression of the x or y (whichever one wishes to take as abscissa) as one likes it.[52]

... namely that the progression of the x can be assumed ad libitum ...[53]

That many different progressions of the variables could be studied appears from a letter of VARIGNON to LEIBNIZ, where he writes about a problem involving variables x, y, s, and z:

Apart from these 18 formulas (...) of which the last 12 are deduced from the first six by supposing successively dx, dy, ds, dz constant, one can still deduce an infinity of other formulas from these first six by supposing in the same way anything else constant (...) for instance by successively supposing also $\frac{dy}{y}$, $\frac{ds^2}{y}$, $y^m dx$, $y^m ds$ *etc.* constant.[54]

As appears from this quotation, the progression of the variables is specified by indicating which first-order differential is supposed constant. Sometimes this is described fully in prose: "the arc length growing uniformly" for ds constant, and "the x growing uniformly"[55] for dx constant.

2.17. *The rules for the operators d and* $\int$ *discussed so far do not depend on the choice of the progression of the variables, but as long as the progression is not specified, the variables introduced by the operators d and* $\int$ *are affected by the same indeterminacy as the infinitangular polygon. For instance, in Case a',* $dds = 0$ *(because* ds *is constant), but in Case b',* dds *is not equal to zero. The differentials and the relations between them depend on the progression of the variables. Also the sums depend on the progression of the variables. The relation of* $\Sigma\{y_i\}$ *to the quadrature, discussed in connection with Case b, transforms, by the extrapolation, into the assertion that, under the supposition of a constant* dx, $\int y$ *is proportional to the quadrature* Q,

[52] "Es ist gantz nicht nöthig ad summandum, dass die dx oder dy constantes und die $ddx = 0$ seyen, sondern man assumiret die progression der x oder y (welches man pro abscissa halten wil) wie man es gut findet." (LEIBNIZ to VON BODENHAUSEN, *Math. Schr.* VII, p. 387.)

[53] "... ut scilicet progressio ipsarum x assumi posset qualiscunque ..." (LEIBNIZ *1684a*; *Math. Schr.* V, p. 233.)

[54] "Outre ces 18 formules (...) dont les 12 dernieres sont déduites des six premieres en y supposant dx, dy, ds, dz successivement constantes, l'on peut encore en deduire une infinité d'autres de ces six premieres en y supposant de même toutte autre chose de constante, (...) par example en y supposant aussi $\frac{dy}{y}$, $\frac{ds^2}{y}$, $y^m dx$, $y^m ds$ etc. successivement constantes, ..." (VARIGNON to LEIBNIZ, 4-XII-1710; LEIBNIZ *Math. Schr.* IV, p. 173.)

[55] "arcu aequabiliter crescente"; "x uniformiter crescentes." (LEIBNIZ *Math. Schr.* V, pp. 285 and 233.)

with dx as infinitely small proportionality factor: $dx\int y = Q$. This relation does not *apply under any other supposition about the progression of the variables.*

This point will be discussed further in relation with CAVALIERIAN theories in Appendix 1. Suffice here the following quotation, in which LEIBNIZ explains that if dx is taken constant, one may treat the quadrature as $\int y$ ("sum of all y"), as is done in the theory of indivisibles, but if one wishes to consider different progressions of the variables, the quadrature has to be evaluated as $\int y\,dx$:

> And this indeed is also one of the advantages of my differential calculus, that one does not say, as was formerly customary, the sum of all y, but the sum of all $y\,dx$, or $\overline{\int y\,dx}$, for in this way I can make dx explicit and I can transform the given quadrature into others in an infinity of ways, and thus find the one by means of the other.[56]

2.18. *The properties of the differentials and the sums as outlined above imply certain conditions of regularity of the infinitangular polygon. The requirement that the second-order differentials be infinitely small with respect to the first-order differentials implies that the first-order differentials must vary smoothly; two adjacent differentials must be approximately equal. This requirement does not follow immediately from the extrapolation from the finite array. Indeed, in the finite array one can imagine a polygon with sides of alternating lengths h and $2h$, in which the difference sequence $\Delta_i s$ of the arc lengths would be $\{h, 2h, h, 2h, h, \ldots\}$ and the second-difference sequence $\{h, -h, h, -h, h, -h, \ldots\}$. Extrapolating this case to the actually infinite makes the second-order differential dds of the same order of infinity as the first-order differential ds.*

Such anomalous progressions of the variables have to be excluded; they can be so effectively by considering only progressions in which the first differential of one of the variables is constant. This can be understood in hindsight from the fact that the curves which were studied implied, except at singularities, sufficiently often differentiable relations between the variables. Hence if u is the variable with constant first differential, the corresponding sequence of, say, $y\,(y = f(u))$, is formed by extrapolation from a finite sequence like $f(a), f(a+h), f(a+2h), f(a+3h), \ldots$. The property that dy, ddy, d^3y etc. are of successive different orders of infinity then relates to the different orders of h of

$$\begin{aligned}
\Delta y &= f(a+h) - f(a) = O(h),\\
\Delta^2 y &= f(a+2h) - 2f(a+h) + f(a) = O(h^2),\\
\Delta^3 y &= f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a) = O(h^3).
\end{aligned}$$

From these relations it can also be seen that, if the first differential of one of the variables is supposed constant, the k^{th}-order differentials are of the same order of infinity as the k^{th}-powers of the first-order differentials.

The argument above suggests that the variable with constant first differential acquires the role of independent variable. This aspect if discussed further in § 2.20.

[56] "Und das ist eben auch eines der avantagen meines calculi differentialis, dass man nicht sagt die summa aller y, wie sonst geschehen, sondern die summa aller $y\,dx$ oder $\overline{\int y\,dx}$, denn so kan ich das dx expliciren und die gegebene quadratur in andere infinitis modis transformiren und also eine vermittelst der andern finden." (LEIBNIZ to VON BODENHAUSEN; *Math. Schr.* VII, p. 387.)

I have found very few traces of an awareness that the usual suppositions about the progression of the variables imply regularity conditions not implicit in the concept of infinitangular polygon. Most likely this lack of awareness was caused by the fact that if the rules of the calculus are followed and if one specifies the progression of the variables by specifying a constant differential, one hardly ever encounters problems which throw up this question. Still, the question did occur, namely in connection with the fact that zero has no fixed order of infinity. As an example I quote JAKOB BERNOULLI's discussion of the differential of x^2.[57] He wrote

$$d(x^2) = (x + dx)^2 - x^2 = 2x\,dx + (dx)^2,$$

and concluded from this that, for $x \neq 0$, $d(x^2) = 2x\,dx$, but that, for $x = 0$, $d(x^2) = (dx)^2$. The last formula violates the regularity condition that first-order differentials must all be of the same order of infinity; with respect to first-order differentials, $(dx)^2$ has to be discarded and $d(x^2)$ has to be evaluated as equal to zero for $x = 0$.

2.19. *The curve embodies relations between the relevant variables. Like the finite variables, the differentials bear relations to each other induced by the curve. The equations which express these relations are the differential equations.*

The terms of the equations which express the relations between the finite variables are analytic combinations (products, sums etc.*) of these variables. Therefore these terms are themselves variables and the operator d can be applied to them. The* rules of the calculus *teach how the differentials of such analytic combinations relate to the differentials of their component terms and factors. These rules are:*

$$\begin{aligned}
d(x+y) &= dx + dy\\
d(xy) &= x\,dy + y\,dx\\
d\frac{x}{y} &= \frac{x\,dy - y\,dx}{y^2}\\
dx^a &= a x^{a-1} dx \quad \textit{(also for fractional a)}\\
d\log x &= \frac{a\,dx}{x} \quad \textit{(a depending on the kind of logarithm involved)}\\
db^x &= a b^x dx \quad \textit{(with } a = \ln b\textit{)}\\
d\sin x &= \cos x\,dx\\
d\arcsin x &= \frac{dx}{\sqrt{1-x^2}} \quad \text{etc.}
\end{aligned}$$

Because these rules are independent of the choice of the progression of the variables, one can apply them without making any supposition about this progression.

In *1684a* LEIBNIZ published the rules of differentiation for sums, products, quotients, powers and roots.[58] It may be noticed that the applicability of the LEIBNIZIAN algorithm to roots and complicated irrationalities constituted one of its great advantages over the already known rules for tangents and extreme values (FERMAT, SLUSE), which applied only to polynomial equations for algebraic curves. The computation of such equations for given curves (for instance

[57] JAKOB BERNOULLI *Opera* II, p. 1088; see for further examples BOYER *1949*, p. 251.

[58] LEIBNIZ *1684a*; *Math. Schr.* V, p. 225.

LEIBNIZ's example: the locus of points whose distances to six given points add up to a given constant) often required long and tedious calculations because the roots had to be eliminated. Hence the title of *1684a*: *A new method for maxima and minima, and also for tangents, which is not impeded by fractions or irrational quantities, and a singular kind of calculus for these.*[59]

The rules for differentiating non-algebraic compositions of variables (exponentials, logarithms, trigonometric relations) were not given in this article of LEIBNIZ's. They involve certain difficulties connected with the concept of dimension; see note 6.

2.20. *By applying the operator d to both sides of the equation of the curve and then working out the results using the rules, the differential equation of the curve is derived. Repeated application of d yields the higher-order differential equations of the curve. As the rules of the calculus are independent of the choice of the progression of the variables, the resulting differential equations are valid with respect to every such progression. However, the choice of a progression of the variables may transform the second and higher-order differential equations into simpler ones, which then, of course, are valid only for the progression chosen.*

This aspect of higher-order differential equations, which is related to the indeterminacy of the infinitangular polygon discussed above in § 2.16, may best be illustrated by an example, for which I take the parabola $a y = x^2$. Repeated application of d on both sides of the equation yields the first-order and higher-order differential equations, valid for every progression of the variables:

$$\begin{aligned} a\,d y &= 2\,x\,d x, \\ a\,d\,d y &= 2(d x)^2 + 2\,x\,d\,d x, \\ a\,d^3 y &= 6\,d x\,d\,d x + 2\,x\,d^3 x, \\ a\,d^4 y &= 6(d\,d x)^2 + 8\,d x\,d^3 x + 2\,x\,d^4 x \end{aligned} \tag{2}$$

etc.

If the progression of the variables is specified by $d y$ constant ($d\,d y = 0$), these equations are transformed into

$$\begin{aligned} a\,d y &= 2\,x\,d x, \\ 0 &= 2(d x)^2 + 2\,x\,d\,d x, \\ 0 &= 6\,d x\,d\,d x + 2\,x\,d^3 x, \\ 0 &= 6(d\,d x)^2 + 8\,d x\,d^3 x + 2\,x\,d^4 x, \\ &\text{etc.,} \end{aligned} \tag{3}$$

and if $d x$ is supposed constant, ($d\,d x = 0$), the equations are transformed into

$$\begin{aligned} a\,d y &= 2\,x\,d x, \\ a\,d\,d y &= 2(d x)^2, \\ a\,d^3 y &= 0, \\ a\,d^4 y &= 0, \end{aligned} \tag{4}$$

etc.

[59] "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus." (LEIBNIZ *1684a*; *Math. Schr.* V, p. 220.)

The example shows that the general higher-order differential equations of a curve may be considerably simplified by the choice of an appropriate progression of the variables. Hence there are two kinds of differential equations in the calculus: those which apply regardless of the progression of the variables, and those which apply only for a specified progression.[60] *In treating a differential equation, it must always be clear to which kind it belongs, and if it belongs to the second kind, the progression has to be specified. This is done by specifying which first-order differential is supposed constant.*

Higher-order differential equations of the same curve, but applying with respect to different progressions of the variables will differ considerably. Conversely, the same higher-order differential equation, if understood with respect to different progressions of the variables, will define different curves. I shall treat this dependence of higher-order differential equations on the progression of the variables in more detail in Chapters 3 and 5.

In the techniques for the derivation of higher-order differential equations from the data in a physical or geometric problem and in the techniques for the solution of such equations the choice of appropriate progressions of the variables plays a most important role. I shall discuss examples of this technical aspect of the LEIBNIZIAN calculus in Chapter 3.

The choice of the progression of the variables is related to what would be the choice of an independent variable if one wanted to consider the variables as functions. This is illustrated by equations (3) and (4). Equations (3) in which dy is supposed constant, correspond to

$$\begin{aligned} a &= 2xx', \\ 0 &= 2(x')^2 + 2xx'', \\ 0 &= 6x'x'' + 2xx''', \\ 0 &= 6(x'')^2 + 8x'x''' + 2xx'''', \\ &\text{etc.} \end{aligned}$$

in which x', x'' *etc.* are the derivatives of x as function of y $(x = \sqrt{ay})$. Similarly, equations (4), which presuppose dx constant, correspond to

$$\begin{aligned} ay' &= 2x, \\ ay'' &= 2, \\ ay''' &= 0, \\ ay'''' &= 0, \\ &\text{etc.} \end{aligned}$$

where y', y'' *etc.* are the derivatives of y as function of x $(y = x^2/a)$,

The correspondence between the variable with constant first-order differential and the "independent" variable occurring in functions may also be clarified by

[60] BERNOULLI used the terms "complete" and "incomplete" for the two kinds of differential equations; see note 71.

considering the formula which is at present still in use for the second derivative:

$$\frac{d^2 y}{d x^2}.$$

For $y = f(x)$, the derivative is defined by

$$\frac{d y}{d x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The second derivative is usually introduced as the derivative of the derivative. However, one can also introduce it as

$$\frac{d^2 y}{d x^2} = \lim_{h \to 0} \frac{[f(x+2h) - f(x+h)] - [f(x+h) - f(x)]}{h^2}$$

which is analogous to

$$\frac{d^2 y}{d x^2} = \frac{d y^{\mathrm{I}} - d y}{d x^2}.$$

For this definition of the second derivative it is essential that one take two equal segments h along the X-axis. This becomes clear if we consider how the second derivative could be defined directly as a limit of a quotient of finite differences with respect to unequal segments h_1 and h_2 along the X-axis. The numerator of such a quotient would be

$$[f(x+h_1+h_2) - f(x+h_1)] - [f(x+h_1) - f(x)].$$

But there is a problem of choice for the denominator, for which h_1^2 or h_2^2 or, as a comprise, $h_1 h_2$ might be chosen. But for no choice of the denominator will the double limit as $h_1 \to 0$, $h_2 \to 0$ exist, as can be checked easily in the example $f(x) = x$. So we have to suppose $h_1 = h_2$, which is equivalent to what in LEIBNIZIAN terminology is rendered as supposing $d x$ constant. Hence *only* if $d x$ is taken constant does $d^2 y$ have a relation to the second derivative of y as function of x.[61] Thus the variable whose first-order differential is supposed constant takes a role equivalent of that of the independent variable.

[61] This, incidentally, is the reason why the suggestive cancelling of the differentials in the chain rule for derivatives,

$$\frac{d y}{d t} = \frac{d y}{d x} \cdot \frac{d x}{d t},$$

does not occur in the chain rule for higher order derivatives. A similar cancelling of $d x^2$ in the case of second derivatives would lead to

$$\frac{d^2 y}{d t^2} = \frac{d^2 y}{d x^2} \cdot \frac{d x^2}{d t^2} = \frac{d^2 y}{d x^2} \cdot \left(\frac{d x}{d t}\right)^2;$$

but in order that this equation be interpretable as a relation between second derivatives $\frac{d^2 y}{d t^2}$ and $\frac{d^2 y}{d x^2}$, both $d t$ and $d x$ must be supposed constant, which can only apply in the case that $x = a t + b$. In general, the relation between the second derivatives of $y(t)$, $y(x)$ and $x(t)$ is given by

$$\frac{d^2 y}{d t^2} = \frac{d^2 y}{d x^2} \cdot \left(\frac{d x}{d t}\right)^2 + \frac{d y}{d x} \cdot \frac{d^2 x}{d t^2},$$

in which indeed the last term vanishes if $x = a t + b$.

2.21. *In equations (2) (3) and (4) it appears that the first-order differential equations are not affected by the change of the progression of the variables. This is a general rule, and its effect is that in the treatment of first-order differential equations the progression of the variables need not be specified and can be left undetermined. Hence in that case no variable need be singled out and given a constant first-order differential, and so all variables have equal status in the calculus. Also the solution of first-order differential equations is not affected by specification or change of the progression of the variables.*

*The rule applies to first-order differential equations of any degree (*i.e. *the equation may involve powers and products of first-order differentials). It may be proved as follows: Differential equations are homogeneous with respect to order of infinity (see § 2.22). In the case of equations involving only first-order differentials this means that they are homogeneous in these differentials. Hence multiplication of all differentials by the same factor does not affect the equation. Now at every point of the curve, the relation*

$$dx:dy:ds=\sigma:y:\tau$$

applies independently of the progression of the variables. Hence if dx, dy, ds and dx^*, dy^*, ds^* are induced by two different progressions of the variables,

$$dx:dx^*=dy:dy^*=ds:ds^*,$$

that is, in changing from one progression of the variables to another, the differentials are all multiplied by the same factor, so that the relation between them, expressed by the differential equation, remains the same. (The argument can be extended to cover cases involving variables other than x, y and s.)

The rule plays an important role in arguments of Johann Bernoulli *and* Euler *about the transformation of higher-order differential equations by different choices of the progression of the variables, a matter I discuss in Chapters 3 and 5.*

In general, the authors conscientiously specify the progression of the variables in those cases where that is necessary. I have found few examples where the specification is omitted. One such case shows how crucial the specification is for understanding the calculations. It occurs in Johann Bernoulli's *Integral Calculus:*

> Because $s=adx:dy$ [this is the differential equation which Bernoulli discusses] we have $ds=\sqrt{dx^2+dy^2}=addx:dy$, and hence $dy=addx:\sqrt{dx^2+dy^2}$. In order that the integrals can be taken on both sides, both sides are multiplied by dx, which results in $dxdy=adxddx:\sqrt{dx^2+dy^2}$. Taking integrals, we arrive at $xdy=a\sqrt{dx^2+dy^2}$, and after reducing the equation, we find $dy=adx:\sqrt{x^2-a^2}$ as before. [Bernoulli had previously discussed this differential equation.][62]

These calculations are incomprehensible because Bernoulli fails to indicate that he takes dy constant.

[62] "Quia $s=adx:dy$, erit $ds=\sqrt{(dx^2+dy^2)}=addx:dy$ ideoque $dy=addx:\sqrt{(dx^2+dy^2)}$. Ut utrobique possit sumi integrale, multiplicetur utrumque per dx, habebitur $dxdy=adxddx:\sqrt{(dx^2+dy^2)}$. Sumptis integralibus, erit $xdy=a\sqrt{(dx^2+dy^2)}$, reductaque aequatione, erit $dy=adx:\sqrt{(xx-aa)}$, ut ante." (Johann Bernoulli *Integral Calculus*, p. 426.)

2.22. *The geometric interpretation of the quantities entering the analysis requires the equations to be homogeneous in dimension. In addition, there is a second kind of homogeneity, which requires that all the terms of an equation should be of the same order of infinity. A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,*

$$a + dx = a$$

$$dx + ddy = dx$$

etc. *The resulting equations satisfy this second requirement of homogeneity.*

LEIBNIZ valued the two laws of homogeneity highly, as appears from his *1710b*, where he introduced a new notation for powers and extended the notation for differentials in order to display the analogy between powers and differentials, and, correspondingly, the analogy between the laws of dimensional homogeneity and homogeneity of orders of infinity. He wrote $p^k x$ for x^k (thus stressing the fact that taking powers is, like taking differentials, an *operator*), and he extended $d^n x$ to the case in which $n = 0$ by defining $d^0 x = x$. He then exhibited the analogy between powers of sums and differentials of products, which is, in fact, "LEIBNIZ's rule":

$$p^e(x+y) = 1\, p^e x p^0 y + \frac{e}{1} p^{e-1} x p^1 y + \frac{e(e-1)}{1.2} p^{e-2} x p^2 y + \frac{e(e-1)(e-2)}{1.2.3} p^{e-3} x p^3 y + \mathit{etc.}$$

$$d^e(xy) = 1\, d^e x d^0 y + \frac{e}{1} d^{e-1} x d^1 y + \frac{e(e-1)}{1.2} d^{e-2} x d^2 y + \frac{e(e-1)(e-2)}{1.2.3} d^{e-3} x d^3 y + \mathit{etc.}$$[63].

He extended the analogy to sums of three terms and products of three factors. After this he remarked:

> And this analogy even goes so far that, in this way of notation (which may surprise you), also $p^0(x+y+z)$ actually corresponds to $d^0(xyz)$, for
>
> $$p^0(x+y+z) = 1 = p^0 x p^0 y p^0 z$$
>
> and
>
> $$d^0(xyz) = xyz = d^0 x d^0 y d^0 z.$$
>
> At the same time a transcendental law of homogeneity appears, which is not equally obvious in the usual way of notation for differentials. For instance, if we use this new kind of *Characteristica*, it appears that $addx$ and $dxdx$ are not only algebraically homogeneous (as in both cases two quantities are multiplied), but that they are also transcendentally homogeneous and comparable. For the former can be written as $d^0 a d^2 x$, and the latter as $d^1 x d^1 x$, and in both cases the differential exponents have the same sum, for $0+2 = 1+1$. The transcendental law of homogeneity presupposes the algebraical law.[64]

[63] LEIBNIZ *Math. Schr.* V, pp. 379–380.

[64] "Eaque analogia eousque porrigitur, ut tali scribendi more (quod mireris) etiam $p^0(x+y+z)$ et $d^0(xyz)$ sibi respondeant et veritati, nam

$$p^0(x+y+z) = 1 = p^0 x p^0 y p^0 z$$

2.23. *Dimension and order of infinity of finite and infinitesimal quantities are affected by multiplication and by the application of the operators d and $\int$ as follows:*

Multiplication changes the order of infinity unless the factor is finite; it changes the dimension unless the factor is a number or a ratio.

The operator d preserves the dimension and changes the order of infinity; for any variable quantity A, dA is infinitely small with respect to A.

The operator $\int$ preserves the dimension and changes the order of infinity; $\int A$ is infinitely large with respect to A.

Some examples may suffice for further clarification:

$a\dfrac{ddy}{dx^2}$	*is a finite ratio,*
$\dfrac{dy}{ds}\int\int x$	*is a line segment, infinitely large of second order,*
abx	*is an infinitely small area,*
$\dfrac{ds}{a}ddx$	*is a line, infinitely small of third order* etc.

2.24. *It is appropriate to end this outline of the Leibnizian calculus by indicating how its key concepts,* differentiation *and* summation, *contrast with the concepts of* derivation *and* integration *as used in present-day infinitesimal calculus of real functions. To be explicit:* Derivation *is the operator which assigns to a function f its derivative f′, which is again a function, defined by* $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$; *and* Integration *(which term I now use in a sense different from that in § 2.10 above, where I discussed Bernoulli's concept of integration) is the operator which assigns to a function f an integral* $\int f(t)\,dt$ *of f, which is again a function, determined (modulo a constant term) by the requirement that its derivative equals f, or, alternatively, defined as* $\int_a^x f(t)\,dt$, *using a direct definition of the definite integral by means of limits of sums over refining partitions (Riemann integral).*

Comparison of these two pairs of concepts reveals three important contrasts:

(I) Differentiation *and* summation *apply to* variables, *irrespective of the dependency of these on other "independent" variables;* derivation *and* integration *apply to* functions *of one specified variable.*

(II) Differentiation *and* summation *depend on the progression of the variables in the sense that the first-order and higher-order differentials and sums remain undetermined until the progression of the variables is specified, although in*

---

et

$$d^0(xyz) = xyz = d^0x\,d^0y\,d^0z.$$

Eadem etiam opera apparet, quaenam sit *Lex homogeneorum transcendentalis,* quam vulgari modo scribendi differentias non aeque agnoscas. Exempli gratia, novo hoc *Characteristicae* genere adhibito, apparebit $addx$ et $dxdx$ non tantum Algebraic (dum utro-bique binae quantitates in se invicem ducuntur) sed etiam transcendentaliter homogeneas esse et comparabiles inter se, quoniam illud scribi potest $d^0\,a\,d^{\,2}x$, hoc $d^1x\,d^1x$, et utrobique exponentes differentiales conficiunt eandem summam, nam $0+2=1+1$. Caeterum lex homogeneorum transcendentalis vulgarem seu Algebraicam praesupponit." (Leibniz *1710b*; *Math. Schr.* V, pp. 381–382; compare also *ibid.* IV, p. 55.) The transcendental law of homogeneity is also mentioned in Leibniz *1684a*; *Math. Schr.* V, p. 224.

some cases the relations between the differentials and sums are independent of the progression of the variables and are therefore not affected by this indeterminacy.

(III) Differentiation *and* summation *change the order of infinity and leave the dimension unchanged;* derivation *and* integration *change the dimension and leave the order of infinity (in this case, the finiteness) unchanged.*

The third point needs some clarification as here the anachronism, implicit in any comparison of concepts which were used in different periods, becomes evident: derivation and integration do not occur in a specifically geometric context. Nevertheless, to consider the obvious geometric interpretation of these operators is illuminating. Let, therefore, x *and* $y = f(x)$ *have the dimension of a line; then* $y' = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, *the limit of a ratio of lines, is dimensionless (a ratio or a number), and* $\int_a^x f(t)\,dt$ *is an area. Hence derivation and integration change the dimension. On the other and both* $f'(x)$ *hand* $\int_a^x f(t)\,dt$ *are finite, so that the operators conserve the order of infinity.*

The three contrasts illustrate the fundamental change which the infinitesimal calculus underwent from the time of Leibniz till roughly the end of the nineteenth century. The change has been a gradual and most complex process which cannot be understood unless the conceptual foundations of the calculus in its beginning stage are made explicit—which may justify this outline and indeed the whole present study.

3. Aspects of Technique and Choice of Problems in the Leibnizian Calculus

3.0. In this chapter I discuss certain passages from the writings of the early practitioners of the LEIBNIZIAN calculus, which show how the conceptual foundations of the calculus, discussed in the previous Chapter, influenced choice of problems and techniques of solution. I concentrate on examples relevant to the indeterminacy of the progression of the variables and the laws of homogeneity, because these are features which the calculus lost in its later development. Thus discussion of these will contribute most to our understanding of the early stage of the calculus. There are three groups of examples; the first two deal with techniques connected with the choice of the progression of the variables, and the third deals with the laws of homogeneity.

3.1.0. As I discussed in Chapter 2, higher-order differential equations, and in general expressions involving higher-order differentials, depend on the progression of the variables. The appropriate choice of the progression can considerably simplify such expressions, and different choices lead to different formulas for the same geometrical relations or entities. Most higher-order differential expressions are interpretable only if the progression of the variables with respect to which they are meant to apply is specified. As we shall see, the choice of the progression can be made in different stages of the argument; sometimes it can even be avoided entirely.

In this section I illustrate this aspect of the techniques of higher-order differentials by various deductions of formulas for the radius of curvature at a point of a given curve. These deductions and the resulting formulas differ greatly among each other, and it will become clear that these differences are related to the different ways in which the choice of the progression of the variables is introduced in the deductions.

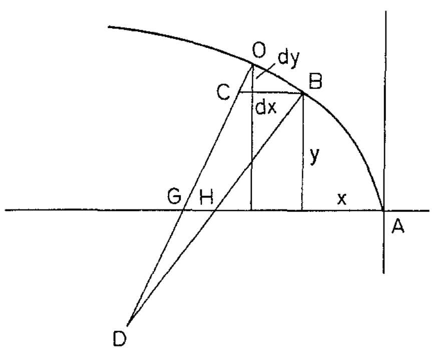
3.1.1. As I shall restrict myself to the technical aspects of the several deductions of formulas for the radius of curvature, I give here only a concise indication of the relation between the relevant texts.

When Johann Bernoulli arrived in Paris in 1691, he possessed a formula for the radius of curvature, the use of which impressed l'Hôpital so much that Bernoulli was asked to become the Marquis's private teacher (*cf.* Johann Bernoulli *Briefwechsel* p. 136). Probably the formula involved was the one which appears in Bernoulli's *Integral Calculus*, the deduction of which I shall discuss. Jakob Bernoulli, independently of his brother, also possessed formulas for the radius of curvature. He used these in deriving the results on diacaustic curves that were published, without proofs, in Jakob Bernoulli *1693*. In his *1693* (published in May 1694) l'Hôpital provided the proofs of Jakob Bernoulli's results as well as deductions of formulas for the radius of curvature, one in a kind of polar coordinates and one in rectangular coordinates, the latter derived in a way slightly different from Johann Bernoulli's in the *Integral Calculus*. (This derivation of l'Hôpital together with other formulas for the radius of curvature, is found also in l'Hôpital *1696*, §§ 77–79.)

Meanwhile Jakob Bernoulli published, in his *1694*, formulas for the radius of curvature, in rectangular and a kind of polar coordinates, with an infinitesimal geometric deduction of the former. I shall discuss these, as well as the proof for the formulas in polar coordinates provided by the editor of Jakob Bernoulli's *Opera*, G. Cramer. Leibniz discussed Jakob Bernoulli's formulas in Leibniz *1694b* and gave other formulas, which I discuss, deduced by a method related to his theory of envelopes.

The discussions on the radius of curvature in the above mentioned writings were partly related to a controversy between Jakob Bernoulli and Leibniz about the number of coinciding intersections of the curve and the osculating circle. Also, they reveal a growing tension between the brothers Bernoulli. However, this is not the place to discuss these aspects. Finally it may be remarked that the authors did not use the term radius of curvature, but rather radius of the osculating circle.

3.1.2. The first example is Johann Bernoulli's deduction of a formula for the radius of curvature in his *Integral Calculus* (*Opera* III 437), dating from 1691. The radii OD and BD (see the figure) are perpendicular to the curve AB; they meet



in the centre of curvature D. OB is the arc length differential, corresponding to the differentials dx and dy. $DB = r$ is the radius of the curvature. Because HB is normal to the curve,

$$AH = x + y\,\frac{dy}{dx}\,.$$

GH is the differential of AH, and BERNOULLI evaluates this after choosing the progression of the variables by taking dx constant ("posito $ddx = 0$"):

$$\begin{aligned} HG = d(AH) &= d\left(x + y\,\frac{dy}{dx}\right) \\ &= dx + \frac{dy^2 + yddy}{dx}\,. \end{aligned}$$

HG occurs in the proportion

$$BC : HG = BD : HD,$$

in which $BC = \dfrac{dx^2 + dy^2}{dx}$, $BD = r$ and $HD = r - BH = r - \dfrac{y\sqrt{dx^2 + dy^2}}{dx}$ so that r can be calculated, which yields

$$r = \frac{-(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{dxddy}\,,$$

a formula which is valid only under the supposition that dx is constant.

By substituting $ds = \sqrt{dx^2 + dy^2}$, which JOHANN BERNOULLI does not do in the passage discussed although he certainly has seen the possibility, one gets

$$r = \frac{ds^3}{dxddy} \qquad \text{for constant } dx,$$

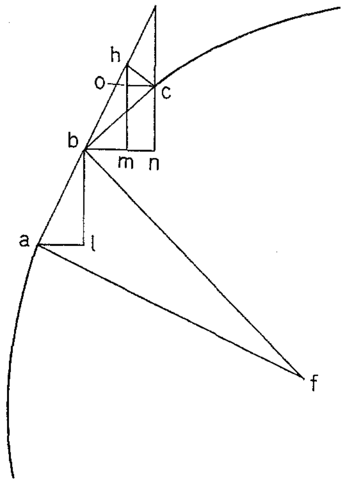
which is one of the formulas given by JAKOB BERNOULLI; see below. As I have pointed out in § 2.20, the choice of the progression of the variables by taking a constant dx corresponds to the choice of x as independent variable in a treatment of the problem in terms of functions. The formula, therefore, corresponds to the well-known formula

$$r = \left[\frac{ds}{dx}\right]^3 \Big/ \left[\frac{d^2y}{dx^2}\right] = \frac{\left[1 + \dfrac{dy^2}{dx}\right]^{\frac{3}{2}}}{\left[\dfrac{d^2y}{dx^2}\right]}\,.$$

3.1.3. In the example above, the choice of the progression of the variables is made in the analytical part of the deduction, after certain relations between first-order infinitesimals (GH, CB) are deduced from an inspection of the figure. The next example shows that relations between higher-order differentials can be directly deduced from a figure, in which case the choice of the progression of the variables can be made in drawing the figure. The example is JAKOB BERNOULLI's deduction of a formula for the radius of curvature as it occurs in his *1694*. In the figure, it is supposed that ds is constant, that is $ab = bc$. af is perpendicular to ab, bf is perpendicular to bc, so that f is the centre of curvature and $bf = r$ the radius of curvature. Furthermore, ab is prolonged to h, $bh = bc$, whence $al = bm$, and the following similarities hold (approximatively):

$$\Delta bmh \sim \Delta hoc,$$

$$\Delta hcb \sim \Delta abf.$$



Hence

$$\frac{ho}{bc}=\frac{ho}{hc}\cdot\frac{hc}{bc}$$
$$=\frac{bm}{bh}\cdot\frac{ab}{bf}$$
$$=\frac{al}{ab}\cdot\frac{ab}{bf}$$

(here the constancy of ds is used), so that

$$\frac{ho}{bc}=\frac{al}{bf}.$$

Now

$$bf=r,$$
$$al=dx,$$
$$bc=ds,$$
$$ho=hm-nc=bl-nc=ddy$$

(note that no signs are taken into consideration). Hence

$$\frac{dx}{r}=\frac{ddy}{ds}$$

so that $r=\dfrac{dxds}{ddy}$, for constant ds.

As the supposition of a constant ds corresponds to taking s as independent variable (see above), the related formula in terms of functions is

$$r=\left[\frac{dx}{ds}\right]\Big/\left[\frac{d^2y}{ds^2}\right].$$

JAKOB BERNOULLI considers in this article also other progressions of the variables; he deduces, by a similar infinitesimal geometric argument in which al is supposed equal to bn (*i.e.* dx constant), the formula

$$r=\frac{ds^3}{dxddy} \quad \text{for } dx \text{ constant}$$

and analogously

$$r = \frac{ds^3}{dy\,ddx} \quad \text{for } dy \text{ constant.}$$

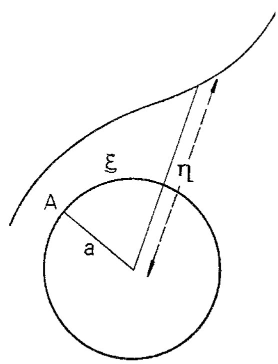
In terms of functions, these correspond to

$$r = \left[\frac{ds}{dx}\right]^3 \Big/ \left[\frac{d^2y}{dx^2}\right]$$

and

$$r = \left[\frac{ds}{dy}\right]^3 \Big/ \left[\frac{d^2x}{dy^2}\right].$$

In the same article JAKOB BERNOULLI gives, without deduction, formulas for the radius of curvature in a kind of polar coordinates ξ and η (differing from the modern polar coordinates in that both have the dimension of a line; ξ is the arc length of a fixed base circle from a fixed point A to the intersection of the radius η with the circle; the base circle has radius a. (See the figure.)



These formulas are:

$$r = \frac{a\,d\eta\,ds}{2d\xi\,d\eta + \eta\,dd\xi} \qquad \text{for } ds \text{ constant,}$$

$$r = \frac{a\eta\,d\xi\,ds}{\eta\,d\xi^2 - aa\,dd\eta} \qquad \text{for } ds \text{ constant,}$$

$$r = \frac{a\,ds^3}{d\xi\,ds^2 + d\xi\,d\eta^2 - \eta\,d\xi\,dd\eta} \qquad \text{for } d\xi \text{ constant,}$$

$$r = \frac{a\,ds^3}{d\xi\,ds^2 + d\xi\,d\eta^2 + \eta\,d\eta\,dd\xi} \qquad \text{for } d\eta \text{ constant,}$$

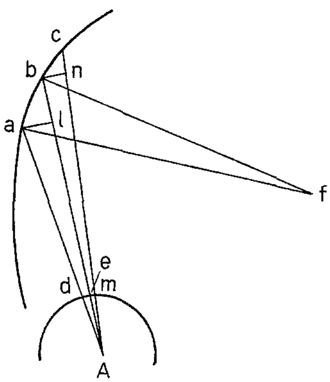
in which formulas, as BERNOULLI points out, the differential of arc length ds has to be evaluated as

$$ds = \frac{\sqrt{\eta^2 d\xi^2 + a^2 d\eta^2}}{a}.$$

3.1.4. The editor of JAKOB BERNOULLI'S *Opera* (1744), G. CRAMER has added a note to the reprint of JAKOB BERNOULLI *1694* in the *Opera*, in which he provided an infinitesimal geometrical proof for these formulas in polar coordinates (*Opera* 579). The proof is remarkable because it does not make suppositions about the progression of the variables in the figure, and thus CRAMER arrived at a formula for the radius of curvature which applies to all progressions, namely

$$r = \frac{a\,ds^3}{d\xi\,ds^2 + d\xi\,d\eta^2 + \eta\,d\eta\,dd\xi - \eta\,d\xi\,dd\eta},$$

from which he derived the four formulas above by taking $d\,ds = 0$, $dd\xi = 0$ and $d\,d\eta = 0$ respectively. I shall not give here the rather complicated infinitesimal geometrical deduction, but only its starting point, the indications of the various differentials in the figure:



$$Ad = a,\ Aa = \eta,\ de = d\xi,$$

$$lb = d\eta, \quad em = d\xi + d\,d\xi,$$

$$nc = d\eta + d\,d\eta,$$

$$ab = ds = \sqrt{a^2 d\eta^2 + \eta^2 d\xi^2}/a,$$

$$bc = ds + d\,ds, \quad af = r,$$

$$al = \frac{\eta\,d\xi}{a},$$

$$bn = \frac{\eta\,d\xi + d\eta\,d\xi + \eta\,dd\xi}{a}.$$

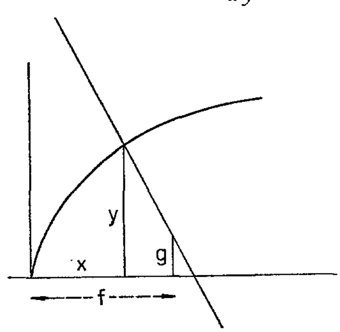
3.1.5. My last example is from Leibniz's article *1694b*, in which he commented on the formulas for the radius of curvature in Jakob Bernoulli *1694*. Leibniz remarked that these formulas are implicit in his own treatment of the evolute (the locus of the centres of curvature of a curve) as envelope of the family of the normals to the curve. In his *1692a* and *1694a* Leibniz had discussed the calculus of envelopes, or *calculus differentialis reciprocus* as he called it, which shows how to find the envelope of a family

$$F(x, y, c) = 0 \tag{1}$$

of straight lines by differentiating (1) with respect to the parameter c, and subsequently eliminating c from the resulting equation and (1).

This procedure can be applied to find the evolute of a curve as the envelope of the normals to the curve. The equation of the normal in the point (x, y) of the curve is (see the figure)

$$y - g = (f - x)\,\frac{dx}{dy} \tag{2}$$



and this equation describes the family of normals if x and $y(x)$ are considered as parameters (analogous to c in (1)). Thus one has to differentiate (2) supposing g and f constant and x and y variable, which yields

$$dy=(f-x)\,d\frac{dx}{dy}-dx\frac{dx}{dy}. \tag{3}$$

Now from the equation of the curve, in combination with (2) and (3), the parameters x and y can be eliminated to yield the equation of the evolute in f and g.

This procedure involves differentio-differentials, but LEIBNIZ indicated that these can be removed by calculating the differential equation of the curve, which yields an expression of $\frac{dx}{dy}$ in terms of x and y; if this expression is inserted in (3), no higher-order differentials will occur. The formulas for the radius of curvature which result from this procedure of removing differentio-differentials are independent of the progression of the variables; this property of the formulas constitutes in LEIBNIZ's opinion an advantage over JAKOB BERNOULLI's formulas.[65]

In the actual deduction of the formulas LEIBNIZ did not explicitly use the *calculus differentialis reciprocus*, so that I can illustrate the procedure directly by his deduction of two formulas, namely

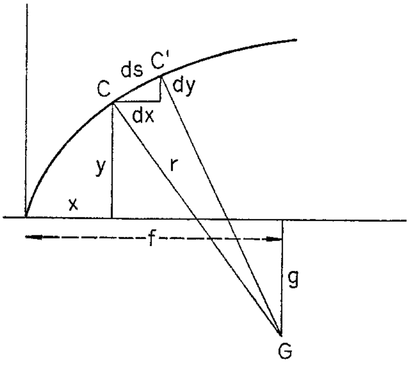
$$r=dy\Big/d\left[\frac{dx}{ds}\right] \quad\text{and}\quad r=(-)\,dx\Big/d\left[\frac{dy}{ds}\right],$$

or, as LEIBNIZ gives them in prose:

> The radius of the osculating circle is to unity as the element of one of the coordinates is to the element of the ratio of the elements of the other coordinate and of the curve.[66]

The radius of curvature CG (see figure) is perpendicular to the curve ACC', whence

$$r:(f-x)=ds:dy, \quad\text{or}\quad r\frac{dy}{ds}=f-x.$$



[65] "Sed et pro centris non minus ac radiis circulorum osculantium theoremata generaliora formari possunt, quae certorum elementorum aequalitate non indigent." (LEIBNIZ *1694b*; *Math. Schr.* V, p. 309.)

[66] "... radius osculi est ad unitatem, ut elementum unius coordinatae est ad elementum rationis elementorum alterius coordinatae et curvae." (LEIBNIZ *1694b*; *Math. Schr.* V, p. 309.)

Leibniz differentiated this equation, considering r and f as constants, which gives

$$r\,d\left[\frac{dy}{ds}\right] = -dx.$$

This procedure is the analogue of differentiating the equation of the family of normals with respect to x and y, keeping g and f constant. It follows that

$$r = -dx \Big/ d\left[\frac{dy}{ds}\right],$$

and, by a similar argument,

$$r = dy \Big/ d\left[\frac{dx}{ds}\right]$$

is derived.

This example is important for three reasons. First the formulas involve only first order differentials of the finite variable quantities x, y, s, $\frac{dy}{ds}$, $\frac{dx}{ds}$, and are therefore independent of the progression of the variables, an aspect which, as we have seen, Leibniz valued highly. Secondly, this independence of the progression of the variables is achieved by introducing the differential quotients $\frac{dy}{ds}$ and $\frac{dx}{ds}$ as new variables. These two features, the endeavour to find formulas independent of the progression of the variables and the resulting introduction of differential quotients, will be discussed further in Chapter 5, where I shall show that they underlay a program of Euler to eliminate all higher-order differentials from the calculus.

Thirdly, the example shows how different the Leibnizian calculus is from the calculus involving functions; indeed the formulas which Leibniz deduced, in contrast to the formulas of the Bernoullis, cannot be translated directly in terms of functions and derivatives, just because the progression of the variables is not, and need not, be specified.

3.2.0. In the following sections I discuss Johann Bernoulli's deduction of rules for transforming from one progression of the variables to another. Bernoulli's deduction shows that such transformation rules involve the introduction of differential quotients or differential coefficients; they are therefore important in connection with the emergence of the concept of derivative.

I use the term differential quotient to denote a quotient of differentials, say, dy/dx; and the term differential coefficient to denote a coefficient in an equality between differentials, such as p in $dy = p\,dx$. Obviously, differential quotients and differential coefficients only differ in the way they are introduced in calculations. Their role in analysis is akin to the role of derivatives, but there is an important difference: differential quotients or coefficients are not defined by means of limits, and they need not be conceived as functions.

3.2.1. The formulas for the radius of curvature are expressions involving higher-order differentials. Such expressions in general depend on the progression of the variables. That is, given a variable V, whose definition involves higher-order differentiation (such as the radius of curvature), then analytical expressions A_i for this variable, calculated with respect to different progressions P_i of the variables, will in general differ among each other; and there will also be an analytical expression A which represents the variable V with respect to every progression of the variables[67]. The question which suggests itself in this situation is how A_i and A are related, and whether there are transformation rules by which A_i and A can be calculated from given A_1, P_1 and P_i.

The same situation occurs in the case of higher-order differential equations. In Chapter 5 I shall deal in somewhat greater detail with the problems connected with the dependence of higher-order differential equations on the progression of the variables. Suffice it here to remark that a higher-order differential equation $E_1 = 0$, valid with respect to a specified progression P_1 of the variables, defines a curve or a relationship between certain finite variables (or, if no boundary conditions are imposed, a set of curves or relationships). With respect to other progressions P_i of the variables, the same curve or relationship will be defined by differential equations $E_i = 0$, and there will also be a differential equation $E = 0$ which defines the curve or relationship with respect to every progression of the variables (I shall use the term "general differential equation" for $E = 0$).[68] Again, the obvious question to ask in this situation is how the E_i and E are related, and whether there are transformation rules by which E_i and E can be derived from given E_1, P_1 and P_i.

3.2.2. About the middle of the eighteenth century this problem had been recognised and its solution had become one of the standard techniques of the calculus.[69] I shall discuss the solution as given by JOHANN BERNOULLI in an

[67] To take the radius of curvature as example:

$$V = r,$$

$$A_1:\ r = \frac{ds^3}{dx\,ddy} \qquad \text{for } P_1\text{: } dx \text{ constant,}$$

$$A_2:\ r = \frac{dx\,ds}{ddy} \qquad \text{for } P_2\text{: } ds \text{ constant,}$$

$$A_3:\ r = \frac{ds^3}{dy\,ddx} \qquad \text{for } P_3\text{: } dy \text{ constant,}$$

$$A:\ r = \frac{dy}{d\left(\frac{dx}{ds}\right)} \qquad \text{for any progression of the variables.}$$

It should be stressed that the A_i and A are not uniquely determined, as is illustrated by the two formulas which LEIBNIZ gave for the radius of curvature independent of the progression of the variables.

[68] To take the third-order differential equation of the parabola $ay = x^2$ as example (*cf.* § 2.20):

$$E_1:\ ad^3y = 0 \qquad \text{for } P_1\text{: } dx \text{ constant,}$$

$$E_2:\ 0 = 6dx\,ddx + 2x\,d^3x \qquad \text{for } P_2\text{: } dy \text{ constant,}$$

$$E:\ ad^3y = 6dx\,ddx + 2dx\,d^3x \qquad \text{for any progression of the variables.}$$

[69] EULER dealt with the technique in great detail in his *1755*; §§ 252–262 and 272–278 of Chapter 8, concern the case of formulas or expressions in general, and Chapter 9, §§ 298–306 (*cf.* § 5.11–§ 5.12) the case of differential equations. D'ALEMBERT, in his article *Différentiel* in the *Encyclopédie*, gave rules to transform a second order differential equation in which dx is supposed constant into the pertaining general differential equation, and he noted: "Cette regle est expliquée dans plusieurs ouvrages, et surtout dans la *seconde partie du calcul intégral* de M. de Bougainville, qui ne tardera pas à paroitre. En attendant on peut avoir recours aux *oeuvres* de Jean Bernoulli, *tom IV, pag. 77*; ..." (References are to BOUGAINVILLE *1754* and JOHANN BERNOULLI *Opera*.)

"Anecdoton" dating probably from shortly after 1715 but published only in 1742.[70] The title of the short note is

> Problem. To render incomplete differential equations of arbitrary degree complete, that is, to transform them into others, in which no differential has to be supposed constant.[71]

Underlying BERNOULLI's solution is a fact I explained in § 2.21: differential equations that involve only first-order differentials of finite variables are independent of the progression of the variables. Thus if one can transform the given differential equation into a differential equation involving first differentials only, then one can drop the restriction to the specified progression of the variables. In his note BERNOULLI worked this out for the case of differential equations valid under the supposition of a constant dx.

First he introduced differential coefficients (or differential quotients, but BERNOULLI did not use a separate term for them) z, t, v, *etc.* These are finite variables, and their definition involves only first-order differentials, so that they are independent of the progression of the variables. z is defined by

$$dy = z\,dx \tag{4}$$

or

$$z = \frac{dy}{dx}.$$

Differentiation of (4) yields (because dx is constant)

$$ddy = dz\,dx,$$

and BERNOULLI introduced t by

$$ddy = dz\,dx = t\,dx^2, \tag{5}$$

whence

$$t = \frac{dz}{dx}.$$

Again, differentiation of (5) yields

$$d^3y = dt\,dx^2,$$

and v is introduced by

$$d^3y = dt\,dx^2 = v\,dx^3,$$

[70] JOHANN BERNOULLI *Opera* IV, pp. 77–79. The note opened with a reference to TAYLOR *1715*. TAYLOR discussed there the following problem: "Aequationem fluxionalem, in quâ sunt fluentes tantum duae z et x, quarum z fluit uniformiter, ita transmutare ut fluat x uniformiter." This, of course, is the formulation in the terminology of fluxions of the problem of transforming a differential equation applying for constant dz into the corresponding differential equation applying for constant dx.

[71] "*Problema.* Aequationes differentiales incompletas cujuscunque gradus reddere completas, *hoc est*, eas transmutare in alias, in quibus nulla differentialis supponatur constans." (JOHANN BERNOULLI *Opera* IV, p. 77.) Thus the problem is, if expressed by means of the notation introduced above, to derive E from E_1 and P_1. BERNOULLI used the adjective "complete" for the general differential equation and conceived the differential equations for specified progressions of the variables as "incomplete", presumably because they are derived from the "complete" differential equation by discarding those terms which, in the case of the specified progression of the variables, are equal to zero.

that is

$$v = \frac{dt}{dx}.$$

Obviously, this process can be repeated till the highest-order differential involved is reached.

If now, in the original differential equation, the following substitutions are made:

$$dy \to dy, \quad ddy \to dz\,dx, \quad d^3y \to dt\,dx^2, \quad d^4y \to dv\,dx^3$$

etc., then the resulting differential equation will involve only first-order differentials of finite variables (namely of x, y, z, t, v, *etc.*), and will therefore be independent of the progression of the variables. From this resulting differential equation, the differential coefficients have now to be eliminated, but this without losing the independence of the progression of the variables. To do this BERNOULLI applied the rules of the calculus without making a supposition about the progression of the variables

$$dz = d\left(\frac{dy}{dx}\right) = \frac{dx\,ddy - dy\,ddx}{dx^2},$$

$$dt = d\left(\frac{dz}{dx}\right) = d\left(\frac{dx\,ddy - dy\,ddx}{dx^2}\right)$$

$$= \frac{dx^2 d^3y - 3\,dx\,ddx\,ddy + 3\,ddx^2 dy - dx\,dy\,d^3x}{dx^4},$$

$$dv = d\left(\frac{dt}{dx}\right) = d\left(\frac{dx^2 d^3y - 3\,dx\,ddx\,ddy + 3\,ddx^2 dy - dx\,dy\,d^3x}{dx^4}\right)$$

$$= \frac{1}{dx^6}\,(dx^3 d^4y - 6\,dx^2\,ddx\,d^3y + 15\,dx\,ddx^2\,ddy - 15\,ddx^3\,dy - 4\,dx^2\,d^3x\,ddy + 10\,dx\,ddx\,d^3x\,dy - dx^2\,d^4x\,dy).$$

Substitution of these results yields a differential equation which is independent of the progression of the variables (or, in BERNOULLI's terminology, "complete") and which involves only the original variables x and y and their differentials.

The introduction of the differential coefficients z, t, v, *etc.* was necessary to prove the transformation rules, which now can be stated directly: In order to derive the general differential equation from the original differential equation applying for constant dx, one has to perform the following substitutions:

$$dy \to dy,$$

$$ddy \to \frac{dx\,ddy - dy\,ddx}{dx},$$

$$d^3y \to \frac{dx^2 d^3y - 3\,dx\,ddx\,ddy + 3\,ddx^2 dy - dx\,dy\,d^3x}{dx^2},$$

$$d^4y \to (dx^3\,d^4y - 6\,dx^2\,ddx\,d^3y + 15\,dx\,ddx^2\,ddy - 15\,ddx^3\,dy - 4\,dx^2\,d^3x\,d\,dy + 10\,dx\,ddx\,d^3x\,dy - dx^2\,d^4x\,dy)/dx^3.$$

3.2.3. In a Scholium which follows these transformation rules BERNOULLI turned to the problem of deriving the differential equation for any specified progression of the variables from the differential equation applying for the progression with constant dx, or, as he put it in not too rigorous terminology:

> This rule is of use in transforming constant differentials into other constant differentials.[72]

To do this, Bernoulli indicated, one first derives the general differential equation by the transformation rules and then one applies the property of the differentials implied in the specification of the new progression of the variables to transform the general differential equation into the required differential equation. The procedure is explained by examples: If the new progression of the variables requires dy constant, all terms in the general differential equation involving ddy, d^3y *etc.* are to be discarded. If the element of arc ds is supposed constant, it follows that $d\sqrt{dx^2+dy^2}=0$, whence $dx\,ddx+dy\,ddy=0$, so that $ddy=-\frac{dx\,ddx}{dy}$. From this, by repeated differentiation, formulas for d^3y, d^4y, *etc.* can be found, which, if substituted in the general differential equation, yield the differential equation applying for constant ds. Similarly Bernoulli discussed the case in which $y\,dx$ is supposed constant.

3.2.4. Two remarks on Bernoulli's treatment of the transformation rules are appropriate. First, as in the case of Leibniz's formula for the radius of curvature, independence of the progression of the variables is gained by introducing the differential coefficients, or differential quotients $z=\frac{dy}{dx}$, $t=\frac{dz}{dx}$, *etc.*, so that we see here an example of the fact that consideration of problems relevant to the indeterminacy of higher-order differentials induces differential coefficients or differential quotients to emerge.[73] In Chapters 4 and 5 I shall discuss examples from studies of Leibniz and Euler in which this process is also evident.

Secondly, as I indicated in § 2.21, the choice of progression of the variables corresponds to the choice of an independent variable in a treatment of the problem in terms of functions. However, in Bernoulli's study, as indeed in most of the writings on these transformation rules, the terminology of constant differentials is used, that is, a concept of function of one specified variable is not involved, the problem is conceived and treated entirely in terms of variables and their progressions. How strong this conception was, is shown by the fact that when Cauchy, in 1823, presented the transformation rules discussed above as rules describing the change of independent variable, he still used the terminology of the constant differential:

> It is by substitutions of this kind that one can operate a *change of independent variable* (...) To return to the case in which x is the independent variable, it would suffice to suppose the differential dx constant, and hence $d^2x=0$, $d^3x=0$, ...[74]

[72] "Hujus Regulae est usus in transformandis differentialibus constantibus in alias constantes." (Johann Bernoulli *Opera* IV, p. 78.)

[73] The fact is even more evident in Euler *1755*, which I discuss in Chapter 5.

[74] "C'est par des substitutions de cette nature qu'on peut opérer un *changement de variable indépendente* (...). Pour revenir au cas où x est variable indépendente, il suffirait de supposer la differentielle dx constante, et par suite $d^2x=0$, $d^3x=0$, ..." (Cauchy *1823*; *Oeuvres* (II) IV, p. 74.) Later, the assumption that the differential of the independent variable is constant caused confusion. Compare for instance Hadamard *1935*: "J'ai lu, comme tout le monde, l'histoire de la différentielle de la variable independente qui doit être constante (et qui est d'ailleurs forcément variable puisque infiniment petite)." (p. 341.)

3.3.0. In seventeenth-century analysis relations between variable quantities were usually represented by equations, but this was by no means the only way. In fact, as I mentioned in § 1.3, there were types of relations which could not be represented by equations, such as the relation between the coordinates of transcendental curves. Another way of representing relations between variable quantities which was very common in the seventeenth century, was proportionality. It was used especially in those cases in which representation by an equation would involve dimensional difficulties.

For the representation of relations between infinitesimal variable quantities both equations and proportionalities were used. The former, of course, were the differential equations, and I shall refer to the latter as differential proportions. In this section I shall discuss the role of the progression of the variables with respect to differential proportions.

3.3.1. Differential proportions occur especially in the treatment of physical, more precisely mechanical problems. Therefore I have to make some preliminary remarks about the mathematical treatment of physical problems in the seventeenth and early eighteenth centuries. This subject deserves more space and attention than I can devote to it here; indeed the unfortunate habit of historians of science of transferring the mathematical treatment of physical problems directly into modern mathematical symbolism has obscured many important aspects of seventeenth century physics. I am sure that an extended study of the influence of the mathematical methods and styles on the development of physics will show important new insights.

Mathematics is used in the treatment of physical problems to represent and analyse the relations between physical quantities such as length, weight, time, mass, velocity, force, momentum, *etc.* Representation of these relations by equations involved, for the seventeenth-century mathematician, considerable conceptual difficulties connected with the requirement of dimensional homogeneity. As I have indicated in § 1.5, quantities of different dimension could not be added, and multiplication of quantities always involved a change of dimension. These conceptual difficulties were solved later in the eighteenth and nineteenth centuries by accepting in the formulas any combination of a restricted number of basic dimensions (mass, length, time and a few others), and by allowing dimensioned factors in equations to make dimensions on both side of the equality sign equal. But in the seventeenth century such dimensioned factors were not acceptable, and thus direct comparison of quantities of different dimension by means of equations was virtually impossible.

In view of these conceptual difficulties related to dimensional homogeneity it is not surprising that two other ways of representing relations between physical quantities were prominent in seventeenth century mathematical physics, namely proportions and proportional representation by line segments. Proportions apply to linear dependence between variable quantities, a relation which is perhaps the oldest and certainly the most important relation between physical quantities for which a special technical terminology was developed. Two interdependent variable quantities, say X and Y, are said to be proportional, or to vary proportionally, if for any two pairs of corresponding values X, Y and

X', Y', always

$$X:X' = Y:Y'.$$

The terminology (X is "as" Y) as well as the interpretation avoids all dimensional difficulties because it considers only ratios between quantities of the same dimension. All physical laws which seventeenth-century natural philosophy discovered and which concerned linear relations between different physical quantities were represented in the terminology of proportions.

3.3.2. To represent non-linear relations between physical quantities the seventeenth-century mathematician could use a method which can be called proportional representation by line segments. This procedure involved the introduction of variable line segments proportional to the original physical quantities. Thus if a relation between the physical variable quantities ξ and η was studied, one introduced variable line segments x and y, x proportional to ξ, y proportional to η, and the induced relation between x and y could be represented by a curve drawn with respect to an X- and an Y-axis. This introduction of line segments proportional to physical quantities is very clearly expressed in the following passage from an article by LEIBNIZ, in which he discussed a certain case of retarded motion where a relation between velocity (v), time (t) and space traversed (s) applied which we should express by an equation

$$\alpha t - s = \beta v$$

(α and β constants), but which LEIBNIZ indicated as follows:

> There are straight lines proportional to the times elapsed, and if from each of these the straight line is subtracted which is equal to the corresponding space traversed by the moving point, then the remaining straight line will be proportional to the acquired velocity.[75]

It is important to stress that both for proportions and for proportional representation no unit lengths or unit quantities were introduced. Hence the relations are not reduced to relations between real numbers (as in modern mathematical physics), but essentially as relations between unscaled line segments. The mathematical physics of the seventeenth century was a truly geometric physics.

Moreover, proportional representation, in the absence of fixed units, implied a freedom of choice which the seventeenth-century mathematicians often aptly used: if two physical quantities are proportional, one can take one variable line segment to represent both. Thus, for instance, in the case of free fall, where velocity is proportional to time, both velocity and time can be represented by the same geometrical quantity. This is indeed what LEIBNIZ and HUYGENS did in their discussion on motion in resisting media (see § 3.3.4). Thus, in their geometrical analysis, the law of fall was taken as $v = t$; of course the final results were formulated again in terms of proportionalities.

3.3.3. The branch of physics in which these geometric methods were applied with most spectacular success was mechanics, especially the study of forces and

[75] "Dantur rectae proportionales temporibus insumtis, a quarum unaquaque si detrahatur recta aequalis respondenti spatio percurso a puncto mobili, residua recta erit proportionalis velocitati acquisitae." (LEIBNIZ *1689a*; *Math. Schr.* VI, p. 138.)

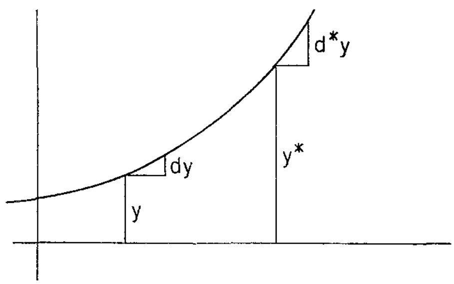
of the resulting changes of motion. This study of change of motion involved infinitesimals, and thus we find differential proportions in dynamics. Like differential equations, differential proportions in general depend on the progression of the variables, that is, the same differential proportion may represent different relations between the variables involved according to the different progressions of the variables with respect to which the proportion is supposed to apply.

In contrast to the case of first-order differential equations, which are independent of the progression of the variables, there are differential proportions involving only first-order differentials which do depend on the progression of the variables. An example is

$$dy \sim y$$

which means (see the figure) that for every corresponding y, dy and y^*, d^*y:

$$dy : d^*y = y : y^*.$$



Obviously this interpretation is inconclusive unless the relation between dy and d^*y is indicated; choosing different progressions of the variables affects the left-hand side but not the right-hand side. For instance if dx is supposed constant, $dy \sim y$ implies $y = ce^x$; if $y\,dx$ is supposed constant, $dy \sim y$ implies $y = \frac{1}{x}$; and if dy is supposed constant, the interpretation is not clear, because $dy \sim y$ would imply $y = c$, and $dy = 0$, so that y does not take part in a progression of the variables.

The cases in which differential proportions do not depend on the progression of the variables are those in which the proportions are directly reducible to differential equations which are independent of the progression. That is, the differential proportion

$$A \sim B$$

is independent of the progression of the variables if A and B are of the same order of infinity and both involve only first-order differentials. In that case the proportion is equivalent to

$$A = cB$$

which is a differential equation of the type described in § 2.21.

3.3.4. I turn now to a discussion between LEIBNIZ and HUYGENS which illustrates the difficulties connected with the requirement that the progression of the variables in the case of differential proportions be specified. In his *1689a*, LEIBNIZ published some results on motion in resisting media. He distinguished between two kinds of resistance, absolute and relative, the distinction being concerned with the dependence of the resistance on the velocity. LEIBNIZ considered resistance to be the action of the medium which diminishes the "force" of the body. He took the diminution of the body's velocity to be proportional to the diminution of its "force".

His definitions of the two kinds of resistance were:

> *Absolute resistance* is the resistance which absorbs equal amounts of the forces of the moving body, whether it moves with a small or with a large velocity, if only it moves, and this resistance depends on the glutinosity of the medium (...).
> *Relative resistance* is caused by the density of the medium, and it is greater in proportion as the velocity of the moving body is greater (...).[76]

Later on in the article he made it explicit that in the case of relative resistance, the motion is retarded in proportion to the velocity. Diminution of force, or of velocity, is a differential, so these definitions imply differential proportions, namely

$$\begin{aligned} &\text{absolute resistance: } dv \text{ constant,} \\ &\text{relative resistance: } dv \sim v. \end{aligned} \tag{7}$$

Both proportions (and therefore both LEIBNIZ's definitions) are meaningless, unless the progression of the variables be specified. In this case, that means unless it be stated whether the diminutions are taken over equal intervals of time (dt constant) or over equal intervals of some other variable. As appears from LEIBNIZ's article he considered the diminution over equal intervals of space (ds constant), which is understandable because he considered the resistance as a property of the medium. Indeed he specified that in the case of absolute resistance:

> The elements of the velocity which the body loses are as the elements of the space traversed.[77], (8)

and in the case of relative resistance:

> The diminutions of the velocity are in the composite ratio of the actual velocity and the increments of the space traversed.[78] (8)

(8) corresponds to

$$\text{absolute resistance: } dv \sim ds \tag{9}$$

and

$$\text{relative resistance: } dv \sim v\,ds. \tag{9}$$

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[76] "*Absoluta resistentia* est, quae tantundem virium mobilis absorbet, sive id parva sive magna velocitate moveatur, dummodo moveatur, et pendet a medii glutinositate (...)
Resistentia respectiva oritur ex medii densitate, et major est pro majori mobilis velocitate (...)." (LEIBNIZ *1689a*; *Math. Schr.* VI, p. 136.)

[77] "... elementa velocitatum amissarum sunt ut elementa spatiorum percursorum, ..." (LEIBNIZ *1689a*; *Math. Schr.* VI, p. 137.)

[78] "Diminutiones velocitatum sunt in ratione composita velocitatum praesentium et incrementorum spatii." (LEIBNIZ *1689a*; *Math. Schr.* VI, p. 140.)

The formulas (9) are differential proportions between terms of the same order of infinity and involving only first-order differentials; they are therefore independent of the progression of the variables. But it is clear that the translation of (6) into (9) applies only if ds is taken constant, so that the specification of the progression of the variables plays a crucial role in translating the prose description of this kind of retarded motion into effective mathematical symbolism: only if ds is considered constant can the absolute resistance be called independent of the velocity and the relative resistance proportional to the velocity.

3.3.5. However, in his article LEIBNIZ was not explicit about the need to specify the progression of the variables, and he was forced to elaborate on this point in a very revealing correspondence with HUYGENS on this matter. Writing to HUYGENS on 6-II-1691, LEIBNIZ compared his own results with HUYGENS' and NEWTON'S studies on motion in resisting media, and he found that the results on what he called relative resistance, or resistance proportional to the velocity, coincided with the results which HUYGENS and NEWTON had derived for resistance proportional to the square of the velocity. He concluded that this discrepancy in the formulation of the starting points was caused by the fact that HUYGENS and NEWTON had considered change of velocity in equal intervals of time, whereas he himself had considered change of velocity in equal intervals of space; and indeed, if we consider the formula for relative resistance (9) which is independent of the progression of the variables

$$dv \sim v\,ds,$$

and if we suppose dt constant, then (because $ds \sim v\,dt$)

$$dv \sim v\,ds \sim v^2\,dt.$$

Thus if dt is considered constant, one can say that the relative resistance is proportional to the square of the velocity.

LEIBNIZ objected to HUYGENS that he and NEWTON should have made this clear:

> To put it exactly, one is only allowed to say that the resistances are proportional to the velocity, or to the square of the velocity, if one also indicates the time or the medium, as I have done.[79]

He came back to this question in his addition *1691* to his article on motion in resisting media, where he wrote:

> About relative resistance I find that our arguments are based on the same foundation, although at first sight this may not seem to be the case. For they [*i.e.* HUYGENS and NEWTON] suppose the resistances in the duplicate proportion of the velocities, while I, speaking in absolute terms, have stated that the resistances (which I measure by the decrements of the velocity caused by the density of the medium) are in the composite ratio of the velocities and the elements of the space which are to be traversed with the corresponding velocities. But if then the elements of the time are taken equal (in which case the elements of the space to be traversed are proportional to the velocities) the resistances are indeed in the duplicate ratio of the velocities.[80]

[79] "A parler exactement on ne doit pas dire que les resistences sont en raison de velocité ny en raison des quarrés des vélocités, si ce n'est qu'on adjoute le temps ou le milieu, comme j'ay fait." (HUYGENS *Oeuvres* X, p. 12.)

[80] "Circa respectivam (that is, resistentiam) video nos iisdem fundamentis inaedificasse, etsi prima fronte aliud videri possit. Ipsi (that is, HUYGENS and NEWTON)

HUYGENS eventually agreed that LEIBNIZ's results corresponded to his own and NEWTON's, but he still objected to calling the resistance in that case proportional to the velocity; he maintained that the constancy of the intervals had nothing to do with the question, resistance was a force in the same way as gravity is a force, and considering the diminutions of velocity in certain elements of time or space as the resistance was taking the effect for the cause (letter of HUYGENS to LEIBNIZ 23-II-1691, HUYGENS' *Oeuvres*, X 19).

The discussion is important because it centered on the special role of the variable *time* in the study of force in terms of acceleration, which was made prominent by NEWTON in the *Principia*. The algorithm of differentials made this role explicit: acceleration is the derivative of velocity with respect to time; hence if one wants to introduce differentials, one has to assume the progression of the variables with dt constant. In other words, if one applies this NEWTONIAN concept of force, one can only compare forces by comparing the changes of motion they produce in equal (infinitesimal) intervals of time.

3.3.6. Not only is the constant differential crucial in the interpretation of differential proportions, it also plays an important role in the technique of treating and eventually solving these proportions. In the tansformation of the proportions (7), above, into (9), the constant ds is used to make the order of infinity on both sides of the proportion equal. In order to transform (9) further into differential equations the introduction of dimensioned factors would have been necessary, which, as I indicated above, would involve conceptual difficulties for the mathematician of the seventeenth century. However, in the case of differential proportions between geometric quantities these difficulties were not felt; the factor of proportionality would have an acceptably interpretable geometric dimension. Indeed, if the proportionality factor has to be of dimension m and order of infinity n, and if dt is the constant differential of a variable line segment t, the required factor will be $a^{m-n}(dt)^n$, in which a is a line segment.

An example of the use of the constant differential and of dimensioned factors to reduce geometric differential proportions to differential equations is provided by a series of problems which LEIBNIZ proposed in his *1692b* in connection with the catenary. As LEIBNIZ and others had noted, the catenary satisfies the differential proportion

$$d\,d\,x \sim (d\,y)^3 \quad (d\,s \text{ constant}).$$

This property provided LEIBNIZ the occasion to put the question which curves have the properties

$$d\,d\,x \sim (d\,y)^2 \quad (d\,s \text{ constant})$$

enim statuunt resistentias in duplicata ratione velocitatum, ego vero absolute loquendo resistentias (quas decrementis velocitatis a medii densitate ortis existimo) esse dixi in ratione composita velocitatum et elementorum spatii, quae scilicet velocitatibus respondentibus decurri inchoantur; unde jam elementis temporis sumtis aequalibus (quo casu elementa spatii decurrenda velocitatibus proportionalia sunt) utique resistentiae erunt in duplicata ratione velocitatum, ..." (LEIBNIZ *1691*; *Math. Schr.* VI, p. 144.)

and

$$ddx \sim dy \qquad (ds \text{ constant}).$$

LEIBNIZ, in fact, described these differential proportions entirely in prose, and the passage is a good example of this style:

> Also I can solve without difficulty the following problem: to find the line with the property that if its arc increases uniformly, the elements of the elements of the abscissas are proportional to the cubes of the increments or elements of the ordinates; it is very true that this occurs in the case of the catenary or funicular. But because this is already noted by the Bernoullis I shall add here that if, instead of the cubes of the elements of the ordinates, the squares are taken, the required line will be logarithmic. And I find that if the elements themselves of the ordinates are proportional to the elements of the elements, or the second differentials of the abscissas, the required line is the circle.[81]

Now JAKOB BERNOULLI, commenting on these differential proportions in his *1693*, transformed them into differential equations by adjusting, in the way I indicated above, appropriate powers of an arbitrary line segment a and of the constant ds. The result was

$$\begin{aligned} a\,ds\,ddx &= (dy)^3 && (ds \text{ constant}),\\ a\,ddx &= (dy)^2 && (ds \text{ constant}),\\ a\,ddx &= ds\,dy && (ds \text{ constant}). \end{aligned}$$

It is of interest to note that if these differential equations are transformed into the corresponding derivative equations, the constant ds is used in a similar way: both sides of the equation are divided by the appropriate power of ds in order to make them finite. Thus the corresponding derivative equations are

$$\begin{aligned} a\frac{d^2x}{ds^2} &= \left(\frac{dy}{ds}\right)^3 && (\text{division by } ds^3),\\ a\frac{d^2x}{ds^2} &= \left(\frac{dy}{ds}\right)^2 && (\text{division by } ds^2),\\ a\frac{d^2x}{ds^2} &= \frac{dy}{ds} && (\text{division by } ds^2). \end{aligned}$$

4. LEIBNIZ'S Studies on the Foundations of the Infinitesimal Calculus

4.0. The present chapter is devoted to certain aspects of LEIBNIZ's studies on the foundations of the infinitesimal calculus. The importance of these studies lies primarily in the fact that they show how deeply LEIBNIZ understood the questions about the nature and the existence of differentials and higher-order

[81] "Caeterum a me quoque non difficulter solvitur illud problema: Invenire lineam cujus arcu aequabiliter crescente elementa elementorum, quae habent abscissae, sint proportionalia cubis incrementorum vel elementorum, quae habent ordinatae, quod in catenaria seu funiculari succedere verissimum est. Sed quoniam id jam a *Bernoulliis* est notatum, adjiciam, si pro cubis elementorum ordinatarum adhibeantur quadrata, quaesitam lineam fore logarithmicam; si vero ipsa simplicia ordinatarum elementa sint proportionalia elementis elementorum seu differentiis secundis abscissarum, inveni lineam quaesitam esse circulum ipsum." (LEIBNIZ *1692b*; *Math. Schr.* V, p. 285.)

differentials and how successful he was in his attempts to solve the problem of the foundations of the calculus. Moreover, in examining these studies, we can achieve an explanation of the occurrence of an alternative definition of the differential in some of LEIBNIZ'S earlier articles on the calculus. Also, the studies show how an interest in fundamental questions concerning the differential leads naturally to the introduction of the concept of function and the differential quotient, and thus to a concept which comes close to that of the derivative.

One preliminary remark has to be made, however; these studies of LEIBNIZ did not exert any influence on the actual development of the calculus in the eighteenth century. The prime source I discuss is a manuscript first published in 1846. LEIBNIZ'S studies share this lack of direct influence with the other more publicly conducted discussions on the foundations of the calculus, such as NIEUWENTIJT'S critique[82], the controversy in the French Royal Academy[83] and the most famous of the debates on foundation of infinitesimal mathematics, those started by BERKELEY[84]. It seems that none of these had significant influence on the actual practice and the results of infinitesimal analysis in the first half of the eighteenth century.

4.1. Most of the early practitioners of the LEIBNIZIAN calculus (although not LEIBNIZ himself) accepted the existence of infinitesimal quantities and justified the rules of the calculus by appealing to this existence. The usual criticism of the calculus denied, or at any rate questioned the existence of infinitesimal quantities. LEIBNIZ himself had a much deeper understanding of the nature of the problem. He was aware that in fact there are two separate questions: one, whether infinitesimal quantities actually exist; the other, whether analysis by means of differentials, following the rules of the calculus, leads to correct solutions of problems.[85]

On the first, metaphysical, question LEIBNIZ did not commit himself definitively; indeed he doubted the possibility of proving the existence of infinitesimal quantities. His answer to the second question, the justification of the calculus, had therefore to be independent of the first; he could not invoke the existence of infinitesimals in answer to objections to the validity of the calculus. Instead, he had to treat the infinitesimals as "fictions" which need not correspond to actually

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[82] *Cf.* NIEUWENTIJT *1694* and *1696*, LEIBNIZ *1695a* and *1695b*, and HERMANN *1700*.

[83] Compare note 89.

[84] See BOYER *1949*, pp. 224–229.

[85] "Interim an status ille transitionis momentaneae, ab inaequalitate ad aequalitatem, a motu ad quietem, a convergentia ad parallelismum, vel similis in sensu rigoroso ac metaphysico sustineri queat, seu an extensiones infinitae aliae aliis majores aut infinite parvae aliae aliis minores, sint reales; fateor posse in dubium vocari: et qui haec discutere velit, delabi in controversias Metaphysicas de compositione continui, a quibus res Geometricas dependere non est necesse. (...) Si omnino ultimum aliquod vel saltem rigorose infinitum quis intelligat, potest hoc facere, etsi controversiam de realitate extensorum aut generatim continuorum infinitorum aut infinite parvorum non decidat, imo etsi talia impossibilia putet; suffecerit enim in calculo utiliter adhiberi, uti imaginarias radices magno fructu adhibent Algebristae." (LEIBNIZ *Cum prodiisset*, p. 43.)

existing quantities, but which nevertheless can be used in the analysis of problems.[86]

LEIBNIZ attempted, with considerable success, to justify the calculus. However, in the writings that were published in his lifetime, he always wrote rather elusively about the question, so that his remarks caused more confusion than clarification; and even after the publication, in the nineteenth and twentieth centuries, of manuscripts which contain fuller accounts of these attempts, much of the confusion about LEIBNIZ's opinion on these questions has remained.[87]

4.2. LEIBNIZ considered two different approaches to the foundations of the calculus; one connected with the classical methods of proof by "exhaustion", the other in connection with a law of continuity. In the first approach he conceived the calculus as an abbreviated language for proofs by exhaustion. Considered in that way, equality between two expressions involving differentials meant that, if instead of the differentials the corresponding finite differences were substituted, the difference between the values of these expressions could be made arbitrarily small (with respect to the values themselves) by choosing the differences small enough. Thus the discarding of higher-order differentials with respect to first-order differentials could be justified.[88]

This approach forms the background of LEIBNIZ's remark (in a letter to PINSON[89], which was published in 1701), that the differential may be supposed to stand to the variable in the proportion of a grain of sand to the earth:

[86] "Ego philosophice loquendo non magis statuo magnitudines infinite parvas quam infinite magnas, seu non magis infinitesimas quam infinituplas. Utrasque enim per modum loquendi compendiosum pro mentis fictionibus habeo, ad calculum aptis, quales etiam sunt radices imaginariae in Algebra. Interim demonstravi, magnum has expressiones usum habere ad compendium cogitandi adeoque ad inventionem, ..." (LEIBNIZ to DES BOSSES, 17-III-1706; *Phil. Schr.* II, p. 305.)

[87] The most important manuscript in this respect is LEIBNIZ *Cum prodiisset* (1701 or somewhat later) which was published by GERHARDT in 1846; SCHOLTZ (*1932*) for the first time stressed its significance for LEIBNIZ's ideas on the foundations of the calculus; she also showed that LEIBNIZ *Quad. Arith. Circ.* (1676) contains valuable information on this matter. It seems that SCHOLTZ *1932* has not aroused the interest which it deserves. BOYER (*1959*, pp. 210–213) has not recognised any consistency in LEIBNIZ's ideas on the foundations of the calculus; he has therefore presented the many quotations of LEIBNIZ on this subject in a random way—which of course strongly suggests the absence of any inner structure in LEIBNIZ's thought.

[88] Compare also the following lines on the rule $dxy = xdy + ydx$: "... restat $xdy + ydx + dxdy$. Sed hic $dxdy$ rejiciendum, ut ipsis $xdy + ydx$ incomparabiliter minus, et fit $d, xy = xdy + ydx$, ita ut semper manifestum sit, re in ipsis assignabilibus peracta, errorem, qui inde metui queat, esse dato minorem, si quis calculum ad Archimedis stylum traducere velit." (LEIBNIZ to WALLIS, 30-III-1699; *Math. Schr.* IV, p. 63.)

[89] The letter (LEIBNIZ to PINSON, 29-VIII-1701; *Math. Schr.* IV, pp. 95/96—part of it was published as LEIBNIZ *1701*; *Math. Schr.* V, p. 350) was an important piece of evidence in the controversy on the infinitesimal calculus which raged in the Académie des Sciences about 1701 and in which the main contestants were VARIGNON and ROLLE. The letter was a reaction to certain remarks of le père GOUYE (*1701*) on the differential calculus. VARIGNON opened a correspondence with LEIBNIZ on this matter (VARIGNON to LEIBNIZ, 28-XI-1701; *Math. Schr.* VI, pp. 89/90), and received a fuller account of LEIBNIZ's views on infinitesimals (LEIBNIZ to VARIGNON, 2-II-1702; *Math. Schr.* IV, pp. 91–95) which was published in the *Journal des Savans* (LEIBNIZ *1702a*). See further RAVIER *1937*, p. 77 (nr. 161).

For instead of the infinite or the infinitely small, one takes quantities as large, or as small, as necessary in order that the error be smaller than the given error, so that one differs from Archimedes' style only in the expressions, which are more direct in our method and conform more to the art of invention.[90]

Understandably, this remark caused great confusion in the French mathematical circle, in which L'Hôpital and Varignon had always defended the Leibnizian calculus by an appeal to the actual existence of infinitesimals. Now the opponents of the calculus used the letter to Pinson to attack Varignon with Leibniz's own words: the differentials were finite. Varignon asked for clarification, which resulted in Leibniz *1702a*, in which Leibniz wrote:

And to this effect I have given once some lemmas on incomparables in the Leipzig *Acta*, which one may understand as one wishes, either as rigorous infinites, or as quantities only, of which the one does not count with respect to the other. But at the same time one has to consider that these ordinary incomparables themselves are by no means fixed or determined; they can be taken as small as one wishes in our geometrical arguments. Thus they are effectively the same as rigorous, infinitely small quantities, for if an opponent would deny our assertion, it follows from our calculus that the error will be less than any error which he will be able to assign, for it is in our power to take the incomparably small small enough for that, as one can always take a quantity as small as one wishes.[91]

4.3. The chief source for Leibniz's second approach to the justification of the use of "fictitious" infinitesimals in the calculus is a manuscript[92], dating from after 1701 and published by C. I. Gerhardt in 1846. It is a draft for an article in which the rules of the calculus, as published in Leibniz *1684a*, were to be proven. Leibniz based his proofs on a law of continuity, which he formulated as:

If any continuous transition is proposed terminating in a certain limit, then it is possible to form a general reasoning, which covers also the final limit.[93]

[90] Car au lieu de l'infini ou de l'infiniment petit, on prend des quantités aussi grandes et aussi petites qu'il faut pour que l'erreur soit moindre que l'erreur donnée, de sorte qu'on ne diffère du stile d'Archimède que dans les expressions, qui sont plus directes dans nôtre méthode et plus conformes à l'art d'inventer." (Leibniz *1701*; *Math. Schr.* V, p. 350.)

[91] "Et c'est pour cet effect que j'ay donné un jour des lemmes des incomparables dans les Actes de Leipzic, qu'on peut entendre comme on veut, soit des infinis à la rigueur, soit des grandeurs seulement, qui n'entrent point en ligne de compte les unes au prix des autres. Mais il faut considerer en même temps, que ces incomparables communs mêmes n'estant nullement fixes ou detrminés, et pouvant estre pris aussi petits qu'on veut dans nos raisonnemens Geometriques, font l'effect des infiniment petits rigoureux, puis qu'un adversair voulant contredire à nostre enontiation, il s'ensuit par nostre calcul que l'erreur sera moindre qu'aucune erreur qu'il pourra assigner, estant en nostre pouvoir de prendre cet incomparablement petit, assez petit pour cela, d'autant qu'on peut tousjours prendre une grandeur aussi petite qu'on veut." (Leibniz *1702a*; *Math. Schr.* IV, p. 92.)

[92] Leibniz *Cum prodiisset.* The manuscript contains an allusion to Gouye *1701*, whence it must be dated after or in 1701. As it deals with the problems which were discussed in 1701–1702, it is probable that it originated in or not much later than 1701. I discuss here the part of the manuscript which, in the edition of 1846, begins at page 40.

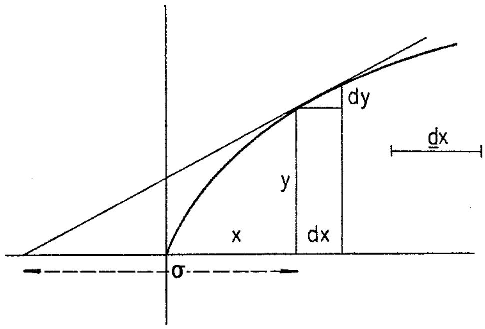
[93] "Proposito quocunque transitu continuo in aliquem terminum desinente, liceat ratiocinationem communem instituere, qua ultimus terminus comprehendatur." (Leibniz *Cum prodiisset*, p. 40.)

The law, not too clear in its formulation[94], was explained by some examples: in the case of intersecting lines, for instance, arguments involving the intersection could be extended (by introducing an "imaginary" point of intersection and considering the angle between the lines "infinitely small") to the case of parallelism; also arguments about ellipses could be extended to parabolas by introducing a focus infinitely distant from the other, fixed, focus.

Thus such extensions of "ratiocinationes" to limiting cases ("terminus") involve the use of terms or symbols which become meaningless in the limiting case, while the argument they describe remains applicable, and in such cases the terms and symbols can be kept as "fictions". According to LEIBNIZ, the use of infinitesimals belongs to this kind of argument.[95]

4.4. LEIBNIZ's proofs of the rules of the calculus based on this law of continuity, as given in the manuscript, can be summarised as follows[96]:

Let (see the figure) dx and dy denote *finite* corresponding differences, and



let $\underline{d}x$ be a fixed finite line segment. For fixed x and y, define $\underline{d}y$ by the proportionality

$$\underline{d}y : \underline{d}x = dy : dx. \tag{1}$$

$\underline{d}y$ is finite, dependent on dx and defined by (1) for $dx \neq 0$. LEIBNIZ argued that $\underline{d}y$ can also be given an interpretation in the case $dx = 0$, namely as defined by

$$\underline{d}y : \underline{d}x = y : \sigma,$$

in which σ is the subtangent; that is, he took the tangent as the limiting position of the secant. It is important to stress that for this he did not invoke the law of continuity; as will be seen, he used the law later, *presupposing* that the limiting position of the secant is the tangent.

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[94] For other formulations of LEIBNIZ's law of continuity see *Math. Schr.* IV, p. 93 and *Phil. Schr.* III, p. 52.

[95] LEIBNIZ thought that ARCHIMEDES must have used infinitesimal arguments of this kind in finding his theorems; he mentioned that such arguments were occasionally practised by DESCARTES, who considered the cycloid as an infinitangular polygon, and also "Hugenius ipse in opere de Pendulo, cum soleret sua confirmare rigorosis demonstrationibus, nonnunquam tamen vitandae nimiae prolixitatis causa infinite parve adhibuit, ..." (LEIBNIZ *Cum prodiisset*, pp. 42–43.)

[96] I have slightly changed LEIBNIZ's notation; for LEIBNIZ's (d) I use $\underline{d}$, so that $(d)x$, $(d)dx$, $(dd)x$ become $\underline{d}x$, $\underline{d}dx$, $\underline{d}\underline{d}x$, respectively. For LEIBNIZ's ${}_2(d)x$ I write $\underline{d}^{+}x$. Instead of LEIBNIZ's separating commas I use brackets.

Now if $dx \neq 0$, the ratio $\underline{d}y:\underline{d}x$ can be substituted for $dy:dx$ in the formula expressing the relation between the finite differences dx and dy. Once this supposition is made, the argument implicit in the formulas can be extended, as indeed the law of continuity asserts, to the limiting case $dx=0$, because in that case $\underline{d}y:\underline{d}x$ is still interpretable and meaningful as a ratio of finite quantities. But then one may resubstitute $dy:dx$ for $\underline{d}y:\underline{d}x$ both in the cases $dx \neq 0$ and $dx=0$, interpreting, in the latter case, the dx and dy as "fictions". To prove the rules of the calculus, it has now to be shown that these rules of manipulating the fictitious dy and dx in the case $dx=0$, are indeed interpretable as corresponding to correct manipulations with the finite $\underline{d}x$ and $\underline{d}y$.

Such proofs LEIBNIZ gave in his manuscript for the rules covering addition, subtraction, division and powers in general. The procedure appears most clearly in his proof for the differentiation rule of a product, $d(xv)=x\,dv+v\,dx$, which I quote here in full:

> *Multiplication* Let $ay=xv$, then $a\underline{d}y=x\underline{d}v+v\underline{d}x$.
>
> Proof: $ay+ady=(x+dx)(v+dv)$
> $=xv+xdv+vdx+dxdv,$
>
> and by discarding ay and xv, which are equal, this becomes
>
> $$ady=xdv+vdx+dxdv$$
>
> or
>
> $$\frac{ady}{dx}=\frac{xdv}{dx}+v+dv,$$
>
> and by transferring the matter, so far as possible, to lines which never vanish, this becomes
>
> $$\frac{a\underline{d}y}{\underline{d}x}=\frac{x\underline{d}v}{\underline{d}x}+v+dv,$$
>
> so dv is left as the only term which can vanish, and in the case of vanishing differences, because then $dv=0$, this becomes
>
> $$a\underline{d}y=x\underline{d}v+v\underline{d}x$$
>
> as was asserted.
>
> (...) Whence also, because $\underline{d}y:\underline{d}x$ is always $=dy:dx$, one may assume this in the case of vanishing dy, dx, and put
>
> $$ady=xdv+vdx.$$[97]

[97] "*Multiplicatio*. Sit $ay=xv$, fiet $a\underline{d}y=x\underline{d}v+v\underline{d}x$. Demonstratio: $ay+ady=(x+dx)(v+dv)=xv+xdv+vdx+dxdv$, et abjiciendo utrinque aequalia ay et xv fiet

$$ady=xdv+vdx+dxdv,$$

seu

$$\frac{ady}{dx}=\frac{xdv}{dx}+v+dv$$

et transferendo rem ad rectas nunquam evanescentes qua licet, fiet

$$\frac{a\underline{d}y}{\underline{d}x}=\frac{x\underline{d}v}{\underline{d}x}+v+dv$$

ut sola quae evanescere possit, supersit dv, et in casu differentiarum evanescentium, quia $dv=0$, fiet

$$a\underline{d}y=x\underline{d}v+v\underline{d}x$$

ut asserebatur, (...). Unde etiam quia $\underline{d}y:\underline{d}x$ semper $=dy:dx$, licebit hoc fingere in casu dy, dx evanescentium, et facere (...)

$$ady=xdv+vdx."$$

(LEIBNIZ *Cum prodiisset*, pp. 46–47; the few words omitted contain an obvious error in calculation and are not important for the argument.)

4.5. I wish to draw attention to two aspects of this approach to the justification of the calculus which are relevant to the general theme of my study. First, the $\underline{d}y$, introduced by LEIBNIZ, is equal, in the case in which $dx=0$, to the differential as defined by CAUCHY: if we call $y=f(x)$, then (1) asserts

$$\underline{d}y=\frac{\Delta y}{\Delta x}\underline{d}x=\frac{f(x+\Delta x)-f(x)}{\Delta x}\cdot\underline{d}x, \tag{2}$$

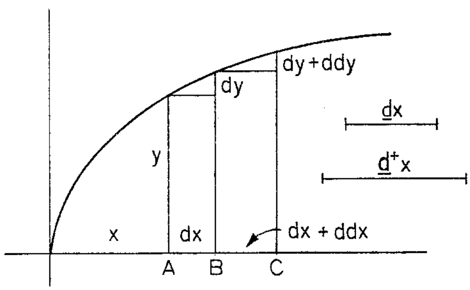
and LEIBNIZ's argument that for $\Delta x=0$ the secant becomes a tangent corresponds to taking the limit in (2):

$$\underline{d}y|_{\Delta x=0}=f'(x)\cdot\underline{d}x$$

Second, LEIBNIZ's attempts show that an endeavor to secure the foundations of the calculus naturally leads to the introduction of the concept of function. The choice of a constant $\underline{d}x$, and the introduction of the ratios $dy:dx$, $dv:dx$ to be replaced by $\underline{d}y:\underline{d}x$, $\underline{d}v:\underline{d}x$, is equivalent to the choice of x as independent variable, as functions of which the other variables are considered. As will appear later in this chapter, this choice is also equivalent to what in the context of infinitesimal differentials is the choice of dx as constant differential. This introduction of the concept of function in a primarily geometric situation of a curve with respect to axes involves, as I have stated before (§ 1.4 and § 1.7), a certain arbitrariness; indeed LEIBNIZ might as well have started by choosing a constant $\underline{d}y$ and by considering the ratios $dx:dy$, $dv:dy$ etc. Also, in order to substitute the $\underline{d}x$ and $\underline{d}y$ for the differences dx and dy, one has to consider the quotients $\frac{dy}{dx}$, and, in the limit case, the expression $\left.\frac{\underline{d}y}{\underline{d}x}\right|_{dx=0}$. This shows that the endeavor to justify the calculus leads naturally to the concepts of differential quotients and hence to derivatives.

4.6. Turning now to the last part of LEIBNIZ's study which contains an attempt to prove in the same way that the second-order differential of xv is $x\,d\,dv+v\,d\,dx+2\,dx\,dv$, I shall show how important it is that this approach implies an introduction of the function concept. Indeed this part of the study is a failure precisely because LEIBNIZ did not realise that he had to choose one of the variables as the independent variable, that is, that he had to introduce the concept of function. Although the text is often rather confused, I think that the essence of it can be rendered as follows.

LEIBNIZ considered a figure of which the essential parts are indicated in the figure[98] in which x and y are fixed and dx, $d\,dx$, dy and $d\,dy$ are *finite*. B and C



[98] The figure is adapted to my rendering of the argument.

are supposed to move simultaneously toward A until they coincide with A at the same moment. Leibniz did not assume that $AB = BC$ throughout this movement, that is, he did not suppose the sequence of the x-values to be arithmetical. He also did not stipulate the requirement of smoothness for the infinitangular polygon, which I discussed in § 2.18 and which requires that $\frac{BC - AB}{AB}$ tends to zero, so that $d\,d\,x$ becomes infinitely small with respect to $d\,x$.

Leibniz introduced two basic finite constant lines $\underline{\mathrm{d}}\,x$ and $\underline{\mathrm{d}}^{+}x$, which he allowed to be unequal, as can be inferred from the figure he gives. He then introduced $\underline{\mathrm{d}}\,y$ and $\underline{\mathrm{d}}\,v$ as defined by

$$\begin{aligned} \underline{\mathrm{d}}\,y : \underline{\mathrm{d}}\,x &= d\,y : d\,x, \\ \underline{\mathrm{d}}\,v : \underline{\mathrm{d}}\,x &= d\,v : d\,x, \end{aligned} \tag{3}$$

and furthermore a $\underline{\mathrm{d}}^{+}y$ defined by

$$\underline{\mathrm{d}}^{+}y : \underline{\mathrm{d}}^{+}x = (d\,y + d\,d\,y) : (d\,x + d\,d\,x).$$

Although eventually he did not use this $\underline{\mathrm{d}}^{+}y$ in his arguments, he seemed to assume that in the limit $\underline{\mathrm{d}}\,y$ and $\underline{\mathrm{d}}^{+}y$ are equal, which, however, is the case only if $\underline{\mathrm{d}}\,x = \underline{\mathrm{d}}^{+}x$.

Next Leibniz calculated from

$$a\,y = x\,v,$$

$$a(y + d\,y) = (x + d\,x)(v + d\,v)$$

and

$$a(y + 2\,d\,y + d\,d\,y) = (x + 2\,d\,x + d\,d\,x)(v + 2\,d\,v + d\,d\,v),$$

the difference equation

$$a\,d\,d\,y = x\,d\,d\,v + v\,d\,d\,x + 2\,d\,x\,d\,v + 2\,d\,v\,d\,d\,x + 2\,d\,x\,d\,d\,v + d\,d\,x\,d\,d\,v \tag{4}$$

in which he divided each term by $a\,d\,d\,x$ in order to introduce quotients of differences:

$$\frac{ddy}{ddx} = \frac{xddv}{addx} + \frac{v}{a} + \frac{2dxdv}{addx} + \frac{2dv}{a} + \frac{2dxddv}{addx} + \frac{ddv}{a}. \tag{5}$$

To proceed similarly to the case of the first-order differential equation, Leibniz now had to introduce finite variables, interpretable in the case $d\,x = 0$, and quotients of which could replace the quotients of differences in (5). To do this he introduced $\underline{\mathrm{d}}\,d\,x$ defined by

$$\underline{\mathrm{d}}\,d\,x : \underline{\mathrm{d}}\,x = d\,x : \underline{\mathrm{d}}^{+}x, \tag{6}$$

and similarly $\underline{\mathrm{d}}\,d\,y$ and $\underline{\mathrm{d}}\,d\,v$. He assumed

$$\frac{ddy}{ddx} = \frac{\underline{\mathrm{d}}dy}{\underline{\mathrm{d}}dx}$$

and

$$\frac{ddv}{ddx} = \frac{\underline{\mathrm{d}}dv}{\underline{\mathrm{d}}dx}. \tag{7}$$

This step remained entirely unjustified[99], and even if LEIBNIZ could argue it, it appears that he was not aware that the substitutions (7) would not solve the problem, because the $\underline{d}\,dx$, $\underline{d}\,dy$ and $\underline{d}\,dv$ as defined by (6) (which involves inhomogeneous ratios) are *not* finite variables but infinitely small variables, so that $\frac{\underline{d}\,dy}{\underline{d}\,dx}$ and $\frac{\underline{d}\,dv}{\underline{d}\,dx}$ are still uninterpretable in the case $dx=0$.

To deal with $\frac{2dxdv}{addx}$ LEIBNIZ defined a finite variable $\underline{d}\underline{d}x$ by

$$\underline{d}\underline{d}x : \underline{d}x = a\,d\,dx : (dx)^2. \tag{8}$$

$\underline{d}\underline{d}x$ is indeed finite, but the assumption is interpretable in the case in which $dx=0$ implies the condition of neatness for the infinitangular polygon that I mentioned above, namely that ddx becomes infinitely small with respect to dx. (Note the role of a in (8) to ensure homogeneity of dimension and order of infinity.)

Now

$$\begin{aligned} \frac{dxdv}{addx} &= \frac{(dx)^2 dv}{addxdx} \\ &= \frac{\underline{d}x\underline{d}v}{\underline{d}\underline{d}x\underline{d}x} \\ &= \frac{\underline{d}v}{\underline{d}\underline{d}x}. \end{aligned} \tag{9}$$

Substitution of (7) and (9) in (5) yielded

$$\frac{\underline{d}\,dy}{\underline{d}\,dx} = \frac{x\underline{d}\,dv}{a\underline{d}\,dx} + \frac{v}{a} + \frac{2\underline{d}v}{\underline{d}\underline{d}x} + \frac{2dv}{a} + \frac{2\underline{d}\,dvdx}{a\underline{d}\,dx} + \frac{ddv}{a}$$

which, as LEIBNIZ assumed wrongly, was still interpretable in the case in which $dx=0$, in which case therefore

$$\frac{\underline{d}\,dy}{\underline{d}\,dx} = \frac{x\underline{d}\,dv}{a\underline{d}\,dx} + \frac{v}{a} + \frac{2\underline{d}v}{\underline{d}\underline{d}x},$$

whence, by the same argument as used with respect to the first order differential equation, the differentials could be kept, in the case in which $dx=0$, as "fictions", so that

$$\frac{ddy}{ddx} = \frac{xddv}{addx} + \frac{v}{a} + \frac{2dxdv}{addx},$$

with which result the manuscript ends.

4.7. I have summarised this failing attempt to prove a rule for higher-order differentials, because the reason why it failed is most illuminating. As I have indicated, the approach that LEIBNIZ followed implies the concept of the variables as functions of one specified variable, in this case x. Taking $\underline{d}\,x$ constant corresponds to taking the sequence of x-values as arithmetic. But apparently LEIBNIZ wanted to conserve the freedom of choice of the progression of the

[99] Here CHILD (*1920*, p. 157), in his translation of the manuscript, inserts a note stating that, because of this error, "there is not much benefit in considering the remainder of this passage"—a judgement with which I disagree.

variables and therefore allowed $d\,d x \neq 0$ and introduced both a $\underline{d}\,x$ and a $\underline{d}^{+}x$. Thus the failure of his attempt is caused by an implied contradiction between considering the variables as functions of one specified variable and still trying to leave the progression of the variables unspecified.

4.8. Once it is assumed that the differential of one variable is constant, LEIBNIZ's approach can be followed successfully. To show this I shall prove in LEIBNIZ's way that, for $a y = x v$, the second-order differential equation is $a\,d\,d y = x\,d\,d v + 2\,d x\,d v$, under the supposition that $d\,d x = 0$. To prove this, $\underline{d}\,y$ and $\underline{d}\,v$ can be introduced as above, and I define $\underline{d}\,\underline{d}\,y$ and $\underline{d}\,\underline{d}\,v$ by

$$\begin{aligned} \underline{d}\,\underline{d}\,y : \underline{d}\,x &= \underline{d}\,x\,d\,d y : (d x)^2, \\ \underline{d}\,\underline{d}\,v : \underline{d}\,x &= \underline{d}\,x\,d\,d v : (d x)^2. \end{aligned} \tag{10}$$

Note the use of $\underline{d}\,x$ to conserve homogeneity of dimension and order of infinity. $\underline{d}\,x$ is chosen for that purpose rather than an arbitrary constant a, because in that way (10) is in agreement with (3):

$$\begin{aligned} \underline{d}\,\underline{d}\,y = \underline{d}(\underline{d}\,y) &= \frac{d(\underline{d}\,y)}{d x}\,\underline{d}\,x \\ &= \frac{d\left(\dfrac{d y}{d x}\,\underline{d}\,x\right)}{d x}\,\underline{d}\,x \\ &= \frac{d^2 y}{d x^2}\,\underline{d}\,x. \end{aligned}$$

Now I may divide by $(d x)^2$ each term of the difference equation (4) (from which the terms with $d\,d x$ are now left out):

$$\frac{a d d y}{d x^2} = \frac{x d d v}{d x^2} + \frac{2 d x d v}{d x^2} + \frac{2 d x d d v}{d x^2},$$

and I may substitute the corresponding ratios of $\underline{d}\,y$, $\underline{d}\,v$, $\underline{d}\,x$, $\underline{d}\,\underline{d}\,y$, and $\underline{d}\,\underline{d}\,v$:

$$\frac{a \underline{d}\,\underline{d}\,y}{(\underline{d}\,x)^2} = \frac{x \underline{d}\,\underline{d}\,v}{(\underline{d}\,x)^2} + \frac{2 \underline{d}\,v}{\underline{d}\,x} + \frac{2 d x \underline{d}\,\underline{d}\,v}{(\underline{d}\,x)^2}.$$

This formula remains interpretable in the case in which $d x = 0$ (the last term then vanishes), so that, following LEIBNIZ's argument, I may use the differentials as "fictions" also in the case in which $d x = 0$:

$$\frac{a d d y}{d x^2} = \frac{x d d v}{d x^2} + \frac{2 d v}{d x}$$

or

$$a\,d\,d y = x\,d\,d v + 2\,d x\,d v,$$

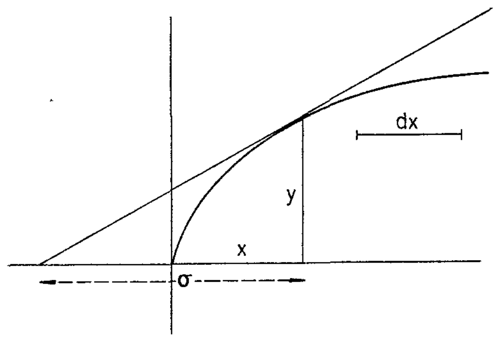
which is indeed the second-order differential equation of $a y = x v$ under the supposition that $d x$ is constant.

4.9. LEIBNIZ's fundamental idea, to choose a finite fixed $\underline{d}\,x$ and to define a finite $\underline{d}\,y$ by means of this $\underline{d}\,x$, must have occurred to him much earlier than

1701. Indeed it appears in his very first publication on the calculus, LEIBNIZ *1684a*, and in his discussion with NIEUWENTIJT on the nature of differentials in 1695.

In his *1684a* LEIBNIZ introduced differentials and stated (without proofs) the rules of differentiation. The definition of differential which he gave did not allude to infinitesimals; he assumed a fixed finite line segment called dx[100], and he defined dy as the fourth proportional to subtangent, ordinate and dx (see the figure):

$$dy:dx=y:\sigma. \tag{11}$$



The finite line segment dy, so defined, he called a *differentia*. Obviously, this dy is the same as

$$\underline{d}y|_{dx=0}$$

(see (5)).

LEIBNIZ did not give reasons for choosing this definition for the differential, but it seems most likely that he chose it to avoid controversies on infinitesimals. That it was a conscious choice may be inferred from a manuscript which GERHARDT identified as an alternative draft for the first publication of the rules of the calculus, in which the differentials are introduced as infinitesimals[101].

In LEIBNIZ *1684a* the relations of the *differentiae* as defined by (11) with infinitesimals is mentioned, almost casually, after the enunciation of the rules of the calculus:

> The proof of all these things is easy for someone who is well acquainted with these matters, if he keeps in mind one point which has not yet been sufficiently exposed, namely that the dx, dy, dv, dw, dz can be considered as proportional to the differences, or the momentaneous decreases or increases, of the corresponding x, y, v, w, z. (...)
> ... *to find a tangent* is to draw a straight line joining two points of the curve which have an infinitely small distance to each other; or the produced side of the infinitangular polygon which for us is equivalent to the *curve*. This infinitely

[100] LEIBNIZ here used the notation dx, dy; as in his later studies which I discussed, he used above $(d)x$, $(d)y$ (*cf.* note 96).

[101] LEIBNIZ *Elementa*; on the dating compare GERHARDT *1855*, p. 72.

small distance, however, can always be expressed by a given differential, such as dv, or by a relation to it, that is, by a given tangent.[102]

In fact, in later articles (with one exception in his answer to Nieuwentijt's objections) Leibniz did not use definition (11) but treated the differentials directly as infinitesimals. Thus the choice of (11) as definition in Leibniz *1684a* was an anomalous and rather unfortunate one (indeed, the term *differentia* in relation with this definition is a misnomer). It must have further obstructed the understanding of the article, which for other reasons was already very obscure[103].

4.10. Leibniz returned to definition (11) in his answer to the critique of Nieuwentijt on the calculus. Nieuwentijt (*1694*) could accept the existence of first-order differentials (he thought this was a consequence of the infinite divisibility of quantity) but he denied the existence of higher-order differentials. In his answer (*1685a*) Leibniz avoided the ontological argument in Nieuwentijt's objection; differentials, he said, were infinitely small, and true quantities in their own sense:

> Therefore I accept not only infinitely small lines, such as dx, dy, as true quantities in their own sort, but also their squares or rectangles, such as $dxdx$, $dydy$, $dxdy$. And I accept cubes and other higher powers and products as well, primarily because I have found these useful for reasoning and invention.[104]

But, feeling that this would not satisfy his opponent, Leibniz returned to the question in a later addition (*1695b*) to the article, in which he showed that, although the first-order and higher-order differentials are infinitely small, one can indicate finite variables which vary proportionally to them.

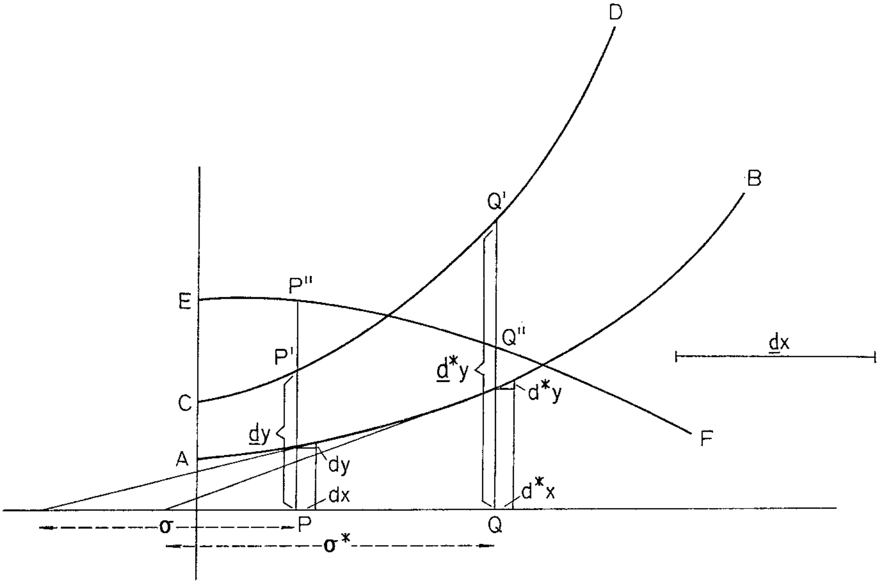
Here he used definition (11), and his argument is important because again it shows how this definition implies the function concept and the supposition that the differential of x is constant. In order to represent his argument, I indicate the constant dx and the dy defined by (11) as $\underline{d}x$ and $\underline{d}y$, respectively, now using the dx and dy exclusively to indicate the infinitesimal differentials. Leibniz explained that, given a curve AB (see the figure[105]), one can plot the $\underline{d}y$ (he

[102] "Demonstratio omnium facilis erit in his rebus versato et hoc unum hactenus non satis expensum consideranti, ipsas dx, dy, dv, dw, dz, ut ipsarum x, y, v, w, z (cujusque in sua serie) differentiis sive incrementis vel decrementis momentaneis proportionales haberi posse. (...)
.. *tangentem invenire* esse rectam ducere, quae duo curvae puncta distantiam infinite parvam habentia, jungat, seu latus productum polygoni infinitanguli, quod nobis *curvae* aequivalet. Distantia autem illa infinite parva semper per aliquam differentialem notam, ut dv, vel per relationem ad ipsam exprimi potest, hoc est per notam quandam tangentem." (Leibniz *1684a*; *Math. Schr.* V, p. 223.)

[103] Precisely in the definition of the differential, the text in Leibniz *1684a* was affected by severe typographical errors. It may be noticed that in the version published in *Math. Schr.* (V, p. 220) Gerhardt has, without indication, corrected these errors. It is important to recall here that Leibniz *1684a* and *1686* formed the source from which the Bernoullis learned the calculus in the years 1687–1690; *cf.* § 2.10 and Eneström *1908*.

[104] "Itaque non tantum lineas infinite parvas, ut dx, dy, pro quantitatibus veris in suo genere assumo, sed et earum quadrata vel rectangula $dxdx, dydy, dxdy$, idemque de cubis aliisque altioribus sentio, praesertim cum eas ad ratiocinandum inveniendumque utiles reperiam." (Leibniz *1695a*; *Math. Schr.* V, p. 322.)

[105] The figure, as well as the explanation by means of (12), is mine; Leibniz's explanation in *1695b* is entirely in prose and not accompanied by a figure.



referred here to his *1684a*) as ordinates along the X-axis, thus obtaining a new curve CD whose ordinates vary proportionally with the differentials dy. That is, if dx, dy and d^*x, d^*y are the infinitesimal differentials corresponding to P and Q, respectively, then

$$PP':QQ' = \underline{d}y:\underline{d}^*y = dy:d^*y. \tag{12}$$

LEIBNIZ's remark in *1684a*, quoted above, that the *differentia* as defined by (11) can be considered as proportional to the momentary increments, or infinitesimal differentials, obviously also concerned the proportionality (12).

Applying the same procedure to the curve CD yields a curve EF, whose ordinates are proportional to the differentials of CD, and therefore to the second-order differentials of AB:

$$PP'':QQ'' = d\,dy:d^*\,d^*y.$$

Obviously, the procedure can be repeated again, by which LEIBNIZ has shown that finite line-variables can be given proportional to differentials of any order. However, what LEIBNIZ did not indicate is that this argument is valid only if one supposes $dx = d^*x$, that is, if one supposes the progression of the variables such that dx remains constant.[106]

Indeed

$$\begin{aligned}\underline{d}y:\underline{d}^*y &= \frac{y}{\sigma}\,\underline{d}x:\frac{y^*}{\sigma^*}\,\underline{d}x\\ &= \frac{dy}{dx}:\frac{d^*y}{d^*x},\end{aligned}$$

so that

$$\underline{d}y:\underline{d}^*y = dy:d^*y$$

[106] JAKOB HERMANN, who in *1700* repeated LEIBNIZ's arguments contra NIEUWENTIJT, also failed to mention this condition.

only if

$$dx = d^* x.$$

4.11. Thus the answer to NIEUWENTIJT shows clearly the implications of the definition of differentials by (11): such a definition implies the arbitrary choice of one variable as independent variable whose differential must then implicitly be supposed constant. This needlessly restricts the generality of the differential calculus, as it imposes the choice of a special progression of the variables. For instance, the deduction of differential equations or expressions from the inspection of figures, as in the case of the radius of curvature, which I discussed as an example of this approach in § 3.1.3, would have been severely hindered if this definition had made a significant impact on the early calculus.

On the other hand, it is also evident from the LEIBNIZIAN studies discussed in this chapter, that a concern about the foundations of the calculus does lead to an introduction of differential quotients or even derivatives, and hence to a predominance of the concept of function. And indeed, as the subsequent history of the foundations of the calculus shows, it was in this direction that the solution lay.

Thus the early stage of the calculus was not favorable to studies of the foundations, such studies would have hindered, rather than invigorated, the practice of the calculus in that period. This may explain why LEIBNIZ hardly published anything about his studies in this direction, and also, in general, why such studies could become influential only much later, when the concept of function had established itself firmly in analysis.

5. Euler's Program to Eliminate Higher-Order Differentials from Analysis

5.0. In this chapter I discuss EULER'S treatment of differentials and higher-order differentials. After penetrating studies of the questions relating to the indeterminacy of higher-order differentials, EULER came to the conclusion that, precisely because of their indeterminacy, such differentials should be banished from analysis. He also indicated methods by which this could be achieved, and I shall show that in these methods the differential coefficient (see § 3.2.0) and the concept of function of one variable play crucial roles. Thus the indeterminacy of higher-order differentials was one of the main causes of the emergence of the derivative as fundamental concept of the calculus.

5.1. EULER was well aware of the problems about the inconsistencies of the infinitely small, and in the *Institutiones Calculi Differentialis* (*1755*) he devoted large parts of the preface and of Chapter II to a discussion of these problems. The aim of his arguments is to establish that, although the concept of the infinitely small cannot be rigorously upheld, still the computational practice with differentials leads to correct results. His arguments have been amply discussed by historians of mathematics[107], so that I can confine myself to a very concise summary. EULER claimed that infinitely small quantities are equal to zero, but that two quantities, both equal to zero, can have a determined ratio. This ratio of zeros was the real subject-matter of the differential calculus, which was

[107] *E.g.* BOYER *1949*, pp. 243–245.

a method of determining the ratio of evanescent increments, which any functions take when an evanescent increment is given to the variable quantity of which they are functions.[108]

EULER also considered this ratio of zeros as a limit; discussing the ratio $\Delta(x)^2 : \Delta x$, for $\Delta x = \omega$, he said:

But it is clear that the smaller the increment ω is taken, the nearer one approaches to this ratio $(2x:1)$. Hence it is correct and even very appropriate to consider these increments first as finite and also to represent them in figures, if these are necessary to illustrate the matter, as finites; next one has to imagine these increments to become smaller and smaller, and so their ratio will be found to approach more and more to a certain limit, which it can reach only when the increments vanish fully into nothing. This limit, which is as it were the ultimate ratio of the increments, is the true object of the differential calculus.[109]

The practice of calculations with differentials had to be interpreted as dealing in fact with these ratios:

Although the rules, as they are usually presented, seem to concern evanescent increments, which have to be defined; still conclusions are never drawn from a consideration of the increments separately, but always of their ratio. (...) But in order to comprise and represent these reasonings in calculations more easily, the evanescent increments are denoted by certain symbols, although they are nothing; and since these symbols are used, there is no reason why certain names should not be given to them.[110]

Thus the argument justified the use of differentials, and EULER proceeded to introduce the differential calculus on that basis. After having treated, in the first two chapters, the theory of finite difference sequences, he defined the differential calculus as the calculus of infinitesimal differences:

The analysis of infinites, with which I am dealing now, will be nothing else than a special case of the method of differences expounded in the first chapter, which occurs, when the differences, which previously were supposed finite, are taken infinitely small.[111]

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[108] "... methodus determinandi rationem incrementorum evanescentium, quae functiones quaecunque accipiunt, dum quantitati variabili, cuius sunt functiones, incrementum evanescens tribuitur." (EULER *1755* praef.; *Opera* (I) X, p. 5.)

[109] "Interim tamen perspicitur, quo minus illud incrementum ω accipiatur, eo propius ad hanc rationem accedi; unde non solum licet, sed etiam naturae rei convenit haec incrementa primum ut finita considerare atque etiam in figuris, si quibus opus est ad rem illustrandam, finite repraesentare; deinde vero haec incrementa cogitatione continuo minora fieri concipiantur sicque eorum ratio continuo magis ad certum quendam limitem appropinquare reperietur, quem autem tum demum attingant, cum plane in nihilum abierint. Hic autem limes, qui quasi rationem ultimum incrementorum illorum constituit, verum est obiectum Calculi differentialis." (EULER *1755*, praef.; *Opera* (I) X, p. 7.)

[110] "Quamvis enim praecepta, uti vulgo tradi solent, ad ista incrementa evanescentia definienda videantur accommodata, nunquam tamen ex iis absolute spectatis, sed potius semper ex eorum ratione conclusiones deducuntur. (...) Quo autem facilius hae rationes colligi atque in calculo repraesentari possint, haec ipsa incrementa evanescentia, etiamsi sint nulla, tamen certis signis denotari solent; quibus adhibitis nihil obstat, quominus iis certa nomina imponantur." (EULER *1755*, praef.; *Opera* (I) X, p. 5.)

[111] "Erit ergo analysis infinitorum, quam hic tractare coepimus, nil aliud nisi casus particularis methodi differentiarum in capite primo expositae, qui oritur, dum differentiae, quae ante finitae erant assumtae, statuantur infinite parvae." (EULER *1755*, § 114.)

which is rather at variance with his remarks quoted above, a contradiction which shows that his arguments about the infinitely small did not really influence his presentation of the calculus.

5.2. This introduction of the calculus as concerning infinitesimal difference sequences is very much akin to LEIBNIZ'S conception of the calculus as discussed in Chapter 2. However, one significant difference, reflecting the transition from a geometric analysis to an analysis of functions and formulas, should be indicated here: no longer are the infinitesimal sequences induced by an infinitangular polygon standing for a curve, but by a function which, if the dependent variable ranges through an infinitesimal sequence $x, x+dx, x+2\,dx, x+3\,dx, \ldots$, yields the sequence $f(x), f(x+dx), f(x+2\,dx), f(x+3\,dx), \ldots$.

Differentiation is, for EULER, an operator which correlates to a function, or in general to a quantity, its differential:

> In the differential calculus the rules are taught by which the first differential of any given quantity can be found. The second differentials are found by differentiation of the first, the third differentials by the same operation from the second and in the same way the successive differentials from the preceding; thus the differential calculus comprises the method for finding all differentials of whatever order. (...) *Differentiation* indicates the operation by which differentials are found.[112]

Integration is the inverse operation, but EULER also indicated the relation of integration with summation.

Differentiation raises the order of infinite smallness; integration does the converse, by which the reigns of the infinitely large are opened up. On the orders of infinity, EULER expressed views like those which I discussed in § 2.13, but also he pointed toward extensions of these ideas; on this see Appendix 2.

5.3. I now turn to EULER'S treatment of higher-order differentiation and to the role of the differential coefficient in it. In *1755* Chapter IV (§ 124), EULER introduced higher-order differentiation under the supposition of a constant dx, or $ddx=0$. This is in keeping with his view of the differential calculus as an extrapolation of the calculus of finite differences, for in the latter he had studied sequences $f(a), f(a+\omega), f(a+2\omega), \ldots$. Setting now $\omega=dx$ infinitely small, he arrived at the case where dx is constant. Consequently in Chapters V and VI of *1755* the differentiation of algebraic and transcendental functions is treated under the supposition of a constant dx.

However, already in Chapter IV EULER commented on the restriction implied in this supposition. He discussed the dependence of higher-order differentials on the progression of the variables in three most important sections. I quote these sections here because they contain a very clear exposition of the problems concerning the indeterminacy of higher-order differentials. In particular, the following points may be noticed: the progression of the variables is arbitrary; first-order differentials do not depend on the progression but higher-order differentials do:

[112] "In calculo differentiali praecepta traduntur, quorum ope cuiusvis quantitatis propositae differentiale primum inveniri potest; et quoniam differentialia secunda ex differentiatione primorum, tertia per eandem operationem ex secundis et ita porro sequentia ex praecedentibus reperiuntur, calculus differentialis continet methodum omnia cuiusque ordinis differentialia inveniendi. (...) *Differentiatio* autem denotat operationem, qua differentialia inveniuntur." (EULER *1755*, § 138.)

higher-order differentials of functions can be expressed in terms of differential coefficients and the first-order differential of the independent variable; the progression of the variables can be specified by specifying the variable with constant first-order differential.

128. We noted already in the first chapter that second and successive differentials cannot be constituted unless the successive values of x are assumed to proceed according to a certain law. As this law is arbitrary, we suppose these values in an arithmetical progression, for such a progression is the easiest and also the most suitable. For the same reason nothing can be stated with certainty about the second differentials, unless the first differentials, with which the variable quantity x is supposed to increase continually, proceed according to a given law. We therefore suppose that the first differentials of x, namely dx, dx^{I}, dx^{II}, *etc.*, are all equal to each other, whence the second differentials are

$$ddx = dx^{\mathrm{I}} - dx = 0, \qquad ddx^{\mathrm{I}} = dx^{\mathrm{II}} - dx^{\mathrm{I}} = 0 \text{ etc.}$$

Thus the second and higher order differentials depend on the order which the differentials of the variable quantity x have among each other, and this order is arbitrary. As this circumstance does not affect first order differentials, there is an immense difference, with respect to the way they are found, between first and higher differentials.

129. But if the successive values of x, x^{I}, x^{II}, x^{III}, x^{IV}, are supposed not to proceed as an arithmetical progression, but following any other law, then their first differentials dx, dx^{I}, dx^{II} etc. will not be equal to each other and hence the ddx will not be $=0$. For this reason the second differentials of any functions of x acquire another form, for if the first differential of such a function y is $=pdx$, then, to find its second differential, it will not be sufficient to multiply the differential of p with dx, but also one has to consider the differential of dx, which is ddx. Now the second differential arises if pdx is subtracted from its succeeding value, which arises if $x+dx$ is substituted for x, and $dx+ddx$ for dx. Suppose therefore that the succeeding value of p is $p+qdx$; then the succeeding value of pdx will be

$$= (p+qdx)(dx+ddx) = pdx + pddx + qdx^2 + qdxddx;$$

from which pdx is subtracted, so that the second differential is

$$ddy = pddx + qdx^2 + qdxddx = pddx + qdx^2,$$

because $qdxddx$ vanishes with respect to $pddx$.

130. Although equality is the simplest and the most useful relation which can be supposed between all the increments of x, still it happens often that not the increments of the variable quantity x, of which y is a function, are supposed equal, but those of some other quantity of which x itself is a function. Often also the first differentials of such another quantity are supposed equal although the relation of this quantity to x is unknown. In the former case the second and higher differentials of x depend on the relation of x to the quantity which is supposed to increase uniformly, and from this quantity they should be defined in the same way as we have indicated to define the second differential of y from the differentials of x. In the latter case the second and higher differentials of x have to be considered as unknowns and they have to be denoted by the symbols ddx, d^3x, d^4x, *etc.*[113]

[113] "128. In capite primo iam notavimus differentias secundas atque sequentes constitui non posse, nisi valores successivi ipsius x certa quadam lege progredi assumantur; quae lex cum sit arbitraria, his valoribus progressionem arithmeticam tanquam facillimam simulque aptissimam tribuimus. Ob eandem ergo rationem de differentialibus secundis nihil certi statui poterit, nisi differentialia prima, quibus quantitas variabilis x continuo crescere concipitur, secundum datam legem progredian-

5.4. The meaning of higher-order differentials depends on the progression of the variables with respect to which they are considered. Hence the meaning of formulas in which higher-order differentials occur depends in the same way on the progression of the variables, and to the implications of this fact Euler devoted a large part of the eighth and ninth chapters of *1755*.

In §§ 251–261 of Chapter VIII Euler introduced the indeterminacy of formulas involving higher-order differentials with the examples $d\,d\,x$ and $\frac{x^3 d^3 x}{d\,x\,d\,d\,x}$. If $d\,x$ is considered constant, $d\,d\,x = 0$ and $\frac{x^3 d^3 x}{d\,x\,d\,d\,x} = \frac{0}{0}$. But if $d(x^2)$ is supposed constant, $d\,d\,x = -\frac{d\,x^2}{x}$ and $\frac{x^3 d^3 x}{d\,x\,d\,d\,x} = -3\,x^2$. And in general, if $d(x^n)$ is supposed constant, $d\,d\,x = -\frac{n-1}{x}\,d\,x^2$ and $\frac{x^3 d^3 x}{d\,x\,d\,d\,x} = -(2n-1)\,x^2$.

For the case of formulas involving two interdependent variables x and y, Euler considered the formula $\frac{y\,d\,d\,x + x\,d\,d\,y}{d\,x\,d\,y}$, which he showed to be dependent on the progression of the variables by considering the special case of the relation

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tur; ponimus itaque differentialia prima ipsius x, nempe $d\,x$, $d\,x^{\mathrm{I}}$, $d\,x^{\mathrm{II}}$ etc., omnia inter se aequalia, unde fiunt differentialia secunda

$$d\,d\,x = d\,x^{\mathrm{I}} - d\,x = 0, \quad d\,d\,x^{\mathrm{I}} = d\,x^{\mathrm{II}} - d\,x^{\mathrm{I}} = 0 \quad \text{etc.}$$

Quoniam ergo differentialia secunda et ulteriora ab ordine, quem differentialia quantitatis variabilis x inter se tenent, pendent hicque ordo sit arbitrarius, quae conditio differentialia prima non afficit, hinc ingens discrimen inter differentialia prima ac sequentia ratione inventionis intercedit.

129. Quodsi autem successivi ipsius x valores x, x^{I}, x^{II}, x^{III}, x^{IV} etc. non secundum arithmeticam progressionem statuantur, sed alia quacunque lege progredi ponantur, tum eorum quoque differentialia prima $d\,x$, $d\,x^{\mathrm{I}}$, $d\,x^{\mathrm{II}}$ etc. non erunt inter se aequalia neque propterea erit $d\,d\,x = 0$. Hanc ob rem differentialia secunda quarumvis functionum ipsius x aliam formam induent; si enim huiusmodi functionis y differentiale primum fuerit $= p\,d\,x$, ad eius differentiale secundum inveniendum non sufficit differentiale ipsius p per $d\,x$ multiplicasse, sed insuper ratio differentialis ipsius $d\,x$, quod est $d\,d\,x$, haberi debet. Quoniam enim differentiale secundum oritur, si $p\,d\,x$ a valore eius sequente, qui oritur, dum $x + d\,x$ loco x et $d\,x + d\,d\,x$ loco $d\,x$ ponitur, subtrahatur, ponamus valorem ipsius p sequentem esse $= p + q\,d\,x$ eritque ipsius $p\,d\,x$ valor sequens

$$= (p + q\,d\,x)(d\,x + d\,d\,x) = p\,d\,x + p\,d\,d\,x + q\,d\,x^2 + q\,d\,x\,d\,d\,x;$$

a quo subtrahatur $p\,d\,x$ eritque differentiale secundum

$$d\,d\,y = p\,d\,d\,x + q\,d\,x^2 + q\,d\,x\,d\,d\,x = p\,d\,d\,x + q\,d\,x^2,$$

quia $q\,d\,x\,d\,d\,x$ prae $p\,d\,d\,x$ evanescit.

130. Quanquam autem ratio aequalitatis est simplicissima atque aptissima, quae continuo ipsius x incrementis tribuatur, tamen frequenter evenire solet, ut non eius quantitatis variabilis x, cuius y est functio, incrementa aequalia assumantur, sed alius cuiuspiam quantitatis, cuius ipsa x sit functio quaedam. Quin etiam saepe eiusmodi alius quantitatis differentialia prima statuuntur aequalia, cuius nequidem relatio ad x constet. Priori casu pendebunt differentialia secunda et sequentia ipsius x a ratione, quam x tenet ad illam quantitatem, quae aequabiliter crescere ponitur, ex eaque pari modo definiri debent, quo hic differentialia secunda ipsius y ex differentialibus ipsius x definire docuimus. Posteriori autem casu differentialia secunda et sequentia ipsius x tanquam incognita spectari eorumque loco signa $d\,d\,x$, $d^3\,x$, $d^4\,x$, etc. usurpari debebunt." (Euler *1755*, §§ 128–130.)

$y = x^2$ between x and y. In that case, if dx is constant,

$$\frac{yddx + xddy}{dxdy} = \frac{xddy}{dxdy} = \frac{x \cdot 2dx^2}{dx \cdot 2xdx} = 1;$$

but if dy is constant,

$$\frac{yddx + xddy}{dxdy} = \frac{yddx}{dxdy} = \frac{x^2 \cdot \frac{-2}{2x} dx^2}{dx \cdot 2xdx} = -\frac{1}{2}.$$

EULER concluded from this that an expression involving higher-order differentials of interdependent variables will in general be dependent on the progression of the variables. Only if the higher-order differentials cancel each other, when the relation between the variables is substituted, is the formula independent of the progression of the variables. As an example he presented $\frac{dyddx - dxddy}{dx^3}$, in which he substituted $y = x^2$, $y = x^n$, and $y = -\sqrt{1 - x^2}$ respectively, showing that in each of these cases the result is a finite expression in x only and therefore independent of the progression of the variables. To prove that $\frac{dyddx - dxddy}{dx^3}$ is independent of the progression of the variables for any relation between x and y, EULER introduced the differential coefficients p and q, defined by $dy = p\,dx$ and $dp = q\,dx$. As these definitions involve only first-order differentials, the differential coefficients p and q are independent of the progression of the variables. Now

$$ddy = p\,ddx + q\,dx^2$$

whence

$$\frac{dyddx - dxddy}{dx^3} = \frac{pdxddx - dx(pddx + qdx^2)}{dx^3} = -q,$$

so that $\frac{dyddx - dxddy}{dx^3}$ does not depend on the progression of the variables.[114]

5.5. After these examples of the consequences of the indeterminacy of higher-order differentials, EULER introduced a most important argument, the conclusion of which is that higher-order differentials should be banished from analysis, because, in every case, either they can be eliminated from the expression in which they occur, or they are inherently vague. If a particular first-order differential is assumed constant, higher-order differentials can be eliminated by expressing them in terms of first order differentials. In expressions which are independent of the progression of the variables the higher-order differentials can be eliminated because they cancel each other. In the remaining case, namely if no progression of the variables is specified and formulas are considered which do depend on the progression, the higher-order differentials are meaningless and vague and therefore not acceptable in analysis. Therefore

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[114] SPEISER (*1945* XXXVIII) has remarked that EULER's studies on dependence and independence of the progression of the variables may be considered as containing a beginning of a theory of differential invariants. Indeed, the choice of a progression of the variables is equivalent to a choice of an independent variable, and hence independence of the progression of the variables corresponds to invariance with respect to parametric representation. However, EULER's studies show no concern about invariance with respect to systems of transformations of the mathematical object (for instance the curve) itself.

> It follows from this that second and higher order differentials in reality never occur in the calculus and that, because of the vagueness of their meaning, they have no further use in Analysis. (...)
>
> It was necessary, however, that we expounded the method of treating them, because they are used often, but only fictitiously, in the calculus. But we shall soon indicate a method by which second and higher differentials can always be eliminated.[115]

5.6. EULER then went on to show how higher-order differentials can actually be eliminated from formulas.

The methods which he used for this elimination, and which I shall summarise below, are very important in the history of the fundamental concepts of analysis, because they involve the systematic use of differential coefficients. By the introduction of differential coefficients, EULER reduced higher-order differentials to first-order differentials, thus gaining independence of the progression of the variables.

Now the use of the differential coefficients p, q, r, *etc.*, of a relation between x and y, defined by $dy = p\,dx$, $dp = q\,dx$, $dq = r\,dx$, *etc.*, implies the choice of an independent variable (in this case x) of which y, p, q, r, *etc.* are considered to be functions. Thus differential coefficients are computationally and conceptually very close to derivatives—only the use of limits in their definition is lacking.

The emergence and the systematic use of differential coefficients must therefore be considered as a most important stage in the process of the emergence of the derivative as fundamental concept of the calculus.

EULER'S use of differential coefficients was directly connected with his conviction that the indeterminacy of higher-order differentials is so undesirable a feature that higher-order differentials have to be banished entirely from analysis. Thus we may say that one of the main causes for the emergence of the derivative was the indeterminacy of higher-order differentials.

5.7. The methods of eliminating higher-order differentials which EULER presented in *1755* (§§ 264–270) may be summarised as follows: If an expression involves only the variable x and its differentials, and if t is the variable whose differential dt is constant, differential coefficients p, q, r *etc.* can be introduced as follows:

$$dx = p\,dt \qquad dp = q\,dt \qquad dq = r\,dt \qquad \text{etc.}$$

The differentials can then be expressed as

$$dx = p\,dt \qquad ddx = q\,dt^2 \qquad d^3x = r\,dt^3 \qquad \text{etc.}$$

substitution of which yields a formula in which the only infinitesimal is a power of dt. Furthermore, as

$$dt = \frac{dx}{p},$$

[115] "Ex his igitur sequitur differentialia secunda et altiorum ordinum revera nunquam in calculum ingredi atque ob vagam significationem prorsus ad Analysin esse inepta. (...) Quoniam tamen saepissime apparenter tantum in calculo usurpantur, necesse fuit, ut methodus eas tractandi exponeretur. Modum autem mox ostendemus, cuius ope differentialia secunda et altiora semper exterminari queant." (EULER *1755*, § 263.)

and p, q, r, *etc.* can be considered as functions of x, one has

$$d^2x = \frac{q}{p^2}\,dx^2 \qquad d^3x = \frac{r}{p^3}\,dx^3 \quad \text{etc.},$$

so that the expression can be reduced to a form in which the only infinitesimal is a power of dx and in which t does not occur explicitly.

For expressions involving two interdependent variables x and y, the case of a constant dx is treated by introducing the differential coefficients as

$$dy = p\,dx, \qquad dp = q\,dx, \qquad dq = r\,dx, \qquad \text{etc.},$$

by which the first and higher-order differentials of y can be eliminated:

$$dy = p\,dx, \qquad ddy = q\,dx^2, \qquad d^3x = r\,dx^3, \qquad \text{etc.}$$

The case dy constant is treated analogously. If in general dt is constant and x and y depend on t one may proceed by

$$\begin{array}{llll} dx = p\,dt, & dp = q\,dt, & dq = r\,dt, & \text{etc.} \\ & ddx = q\,dt^2, & d^3x = r\,dt^3, & \text{etc.} \\ dy = P\,dt, & dP = Q\,dt, & dQ = R\,dt, & \text{etc.} \\ & ddy = Q\,dt^2, & d^3y = R\,dt^3 & \text{etc.} \end{array}$$

In the cases where the constant differential is expressed in x, y, dx and dy, the elimination of the higher-order differentials may be performed using the differential coefficients of the relation between x and y:

$$dy = p\,dx, \qquad dp = q\,dx, \qquad dq = r\,dx.$$

EULER presented this procedure in the cases of the progressions of the variables with $y\,dx$ constant and with $\sqrt{dx^2 + dy^2}$ constant. As an example I indicate his treatment of the case $y\,dx$ constant. One has then

$$y\,ddx + dx\,dy = 0,$$

whence

$$ddx = -\frac{dx\,dy}{y} = -\frac{p}{y}\,dx^2,$$

from which formulas for d^3x, d^4x, *etc.* can be obtained by further differentiation. Further

$$ddy = d(p\,dx) = q\,dx^2 + p\,ddx = \left(q - \frac{p^2}{y}\right)dx^2,$$

from which formulas for d^3y, d^4y, *etc.* can be derived. By means of these relations, any proposed expression involving higher-order differentials, under the supposition $y\,dx$ constant, can be reduced to an expression that involves a power of dx as the only infinitesimal, and hence is independent of the progression of the variables. EULER closed his exposition of the techniques of elimination of higher-order differentials with a series of examples.

5.8. Obviously, elimination of higher-order differentials profoundly affects the treatment of higher-order differential equations. In fact, such equations are transformed into equations between differential coefficients and thus acquire the form in which differential equations are treated today (despite their name), namely equations between derivatives.

It is of interest, therefore, to summarise in this place EULER'S arguments on the transformation of differential equations into equations between differential coefficients, which he inserted in the beginning of the second volume, on the integration of higher-order differential equations, of his *Institutiones Calculi Integralis* (*1768*).

EULER introduced differential coefficients in his definition of a second-order differential equation:

> Given two variables x and y, if $dy = p\,dx$ and $dp = q\,dx$, any equation defining a relation between x, y, p and q is called a second order differential equation of the two variables x and y.[116]

As advantages of this use of differential coefficients, EULER mentioned that the progression of the variables need not be indicated and that only finite quantities (for also the first-order differentials are absent in the definition) occur in the differential equation.

After having shown how an equation between differentials, for a given progression of the variables, can be reduced to an equation between differential coefficients, and *vice versa*, EULER stated as further advantage that in this way the occurrence of a multitude of differential equations for one and the same relation between x and y is avoided. For in the customary way of treating differential equations the same relation between x and y gives rise to many different forms of the relevant differential equation, according to the choice of the progression of the variables.

In addition, the differential equations valid with respect to the various progressions of the variables are usually much more complicated than the corresponding equation between differential coefficients, a feature which EULER illustrated by several examples.

5.9. The occurrence of many differential equations (according to the choice of the progression of the variables), for one and the same relation between the variables x and y, suggests the reverse question, namely whether one equation between higher-order differentials may imply different relations between x and y (different solutions) if it is considered as valid with respect to different progressions of the variables. This question of the dependence of the solution of a differential equation on the progression of the variables is treated by EULER in the ninth chapter of *1755*. Indeed, although EULER had indicated the way that higher-order differentials could be eliminated from analysis he still treated two further aspects of these differentials, namely, transformation rules for formulas with respect to different progressions of the variables and criteria that differential equations be independent of the progression of the variables.

5.10. On the transformation rules I shall be brief, because EULER'S treatment of these differs from BERNOULLI'S (discussed in §§ 3.2.2–3.2.4) only in being more

[116] "Positis binis variabilibus x et y si vocetur $dy = p\,dx$ et $dp = q\,dx$, aequatio quaecunque relationem inter quantitates x, y, p et q definiens vocatur aequatio differentialis secundi gradus inter binas variabiles x et y." (EULER *1768* (vol. II,) § 706.)

elaborate. The differential coefficients p, q, r *etc.* can be expressed in terms of higher-order differentials, independently of the progression of the variables, as follows:

$$\begin{aligned} p &= \frac{dy}{dx}, \\ q &= \frac{dx\,ddy - dy\,ddx}{dx^3}, \qquad (1) \\ r &= \frac{dx^2 d^3y - 3dx\,ddx\,ddy + 3dy\,ddx^2 - dx\,dy\,d^3x}{dx^5}, \end{aligned}$$

etc.

Transformation of a formula applying with respect to a progression P_1 of the variables into a formula representing the same mathematical entity with respect to a progression P_2, can be performed as follows. First the higher-order differentials are eliminated by introducing the differential coefficients in the way discussed above. Then substitution of (1) is effected, resulting in a formula involving higher-order differentials but independent of the progression of the variables. From this formula, by substituting the relation between the differentials which characterises the progression P_2, the required formula is derived.

Euler explained this process by means of examples at great length, arriving finally at a list of transformation rules for the most common progressions of the variables, namely dx constant, dy constant, $y\,dx$ constant and $\sqrt{dx^2+dy^2}$ constant. Formulas applying for any of these four progressions can be transformed directly by means of these rules into a form independent of the choice of progression.

5.11. Euler used these transformation rules in the ninth chapter of *1755* to explore further the dependence of the solutions of higher-order differential equations on the progression of the variables. He explained the technique of reducing higher-order differential equations with specified progression of the variables to equations between the finite variables and the differential coefficients. After that he put the question: what can be said about the solution of a higher-order differential equation if the progression of the variables is not specified? In answer to this question he showed how the transformation rules can be used to ascertain whether a given higher-order differential equation, without indication of the progression of the variables, implies a determined relation between x and y; that is, whether there is a relation between x and y which satisfies the differential equation for all possible progressions of the variables. One way to ascertain this is to suppose different progressions of the variables and to see if the corresponding equations between differential coefficients imply the same relations between x and y (§ 301).

Another method, safer and easy, is to choose a progression of the variables, for instance dx constant, and to apply the transformation rules to deduce from the given differential equation with dx constant, the corresponding general (*i.e.* progression-independent) differential equation. The comparison of the two forms of the equation can reveal a condition for $y(x)$ under which the two forms coincide; a $y(x)$ satisfying these conditions may then be a progression-independent solution of the differential equation (§ 302).

5.12. This EULER illustrated in the subsequent sections. He first considered the general second-order differential equation

$$P\,d^2x + Q\,d^2y + R\,dx^2 + S\,dx\,dy + T\,dy^2 = 0. \tag{2}$$

Under the supposition dx constant, (2) becomes

$$Q\,d^2y + R\,dx^2 + S\,dx\,dy + T\,dy^2 = 0,$$

and, applying the transformation rule

$$d^2y \rightsquigarrow d^2y - \frac{dy}{dx}\,d^2x$$

for transformation to the progression-independent case (see § 3.2.2), EULER found

$$-Q\frac{dy}{dx}\,d^2x + Q\,d^2y + R\,dx^2 + S\,dx\,dy + T\,dy^2 = 0. \tag{3}$$

Comparison of (2) and (3) shows that the function $y(x)$ satisfies (2) independently of the progression of the variables only if

$$P = -Q\frac{dy}{dx}$$

or

$$P\,dx + Q\,dy = 0$$

(§ 303). But if $P\,dx + Q\,dy = 0$ (and P and Q are not equal to zero, a condition which EULER did not mention), then, by differentiation,

$$P\,d^2x + Q\,d^2y + dP\,dx + dQ\,dy = 0,$$

which, compared with (2), yields

$$R\,dx^2 + S\,dx\,dy + T\,dy^2 = dP\,dx + dQ\,dy,$$

from which, using $dy = -\dfrac{P}{Q}\,dx$, the differentials can be eliminated, resulting in a finite equation, giving the condition for $y(x)$ in terms of a relation between x and y. It needs then still to be checked whether a $y(x)$ that satisfies this condition also satisfies the differential equation (2), but if so, this is a method for calculating the progression-independent solution of (2) without integration (§ 304).

EULER gave two examples of this procedure, one in which it leads to a solution and one in which it does not. The first example was

$$x^3\,d^2x + x^2y\,d^2y - y^2\,dx^2 + x^2\,dy^2 + a^2\,dx^2 = 0. \tag{4}$$

In this case, $P\,dx + Q\,dy = 0$ means

$$x^3\,dx + xy^2\,dy = 0.$$

Differentiating this relation, one gets

$$x^3\,d^2x + xy^2\,d^2y + 3x^2\,dx^2 + 2xy\,dx\,dy + x^2\,dy^2 = 0.$$

Comparison with (4) yields

$$a\,d^2x - y^2\,dx - 3x^2\,dx - 2xy\,dy = 0.$$

Use of $dy = -\frac{x}{y} dx$, transforms this into

$$a^2 dx - y^2 dx - x^2 dx = 0$$

or

$$y^2 + x^2 = a^2,$$

which EULER indicated as a solution of (4) applying regardless of the progression of the variables (§ 305).

The other example was

$$y^2 d^2 x - x^2 d^2 y + y dx^2 - x dy^2 + a dx dy = 0.$$

The criterion is now

$$y^2 dx - x^2 dy = 0,$$

and the finite relation between x and y derived as in § 305 is

$$x^3 - y^3 + axy = 2xy^2 + 2x^2 y,$$

which, however, appears not to be compatible with $y^2 dx - x^2 dy = 0$, unless, EULER said, dx and dy are both zero (that is, x constant and y constant), but that solution applies to every differential equation.

5.13. These researches of EULER imply as it were the counterpart of his remark quoted above, namely that one of the disadvantages of higher-order differential equations is that one and the same relation between x and y gives rise to many different differential equations, according to the progression of the variables chosen. Here, conversely, EULER showed that one and the same equation among differentials may imply many different solutions, and that only in special cases there occur solutions valid for all progressions.

The more reason, then, EULER must have had after these explorations to pursue his program of eliminating higher-order differentials, and the concomitant indeterminacy, by introducing differential coefficients.

Appendix 1. Leibniz's Opinion of Cavalierian Indivisibles, Infinitely Large Quantities

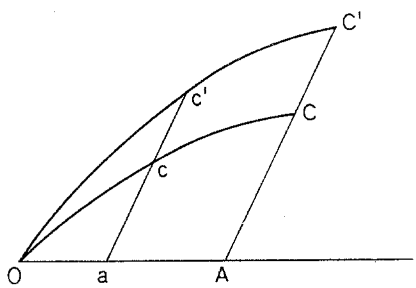
6.0 This Appendix deals with certain statements of LEIBNIZ concerning CAVALIERI's method of indivisibles and the difference between this method and his own differential calculus.

The relation of the LEIBNIZIAN calculus to the theories of CAVALIERI is of importance especially for the formative years of the LEIBNIZIAN calculus. This episode is described in detail in HOFMANN *1949*, and my present study is devoted to the LEIBNIZIAN calculus in a later stage (see § 2.0). I shall therefore confine myself to a few remarks concerning the relevant quotations of LEIBNIZ.

The importance of the quotations lies in the fact that LEIBNIZ expressed his opinions in terms of progressions of the variables and the free or restricted choice of these progressions. My study of this concept may therefore provide some new insight in the question of the relation of the LEIBNIZIAN calculus to the methods of CAVALIERI.

Moreover, the quotations are relevant to the question of the role of the infinitely large in the LEIBNIZIAN calculus. Compared with the infinitely small, the infinitely large hardly ever occurs in the calculus. This feature might at first sight seem at variance with LEIBNIZ's concept of the operators of differentiation and summation as being reciprocal (*cf.* §2.9); for just as differentiation introduces infinitely small differentials, so summation could introduce infinitely large sums. The reason why the infinitely large occurs but rarely is that LEIBNIZ consistently evaluated quadratures as (finite) sums of area-differentials, and not as (infinitely large) sums of ordinates. He consciously chose for the former approach, having become aware that the disadvantages of the latter are apparent in the CAVALIERIAN method of indivisibles.

6.1. The evaluation of quadratures as aggregates or sums of finite line-variables is implied in CAVALIERI's method of indivisibles (*cf.* WALLNER *1903* and BOYER *1941*). The area between the curve OC and the axis OA was conceived as the aggregate of all ordinates ac extending from the axis OA under a fixed angle towards the curve. CAVALIERI used the term "omnes lineae" ("all lines") for this aggregate.



This concept of the quadrature offers the possibility of finding relations between the quadratures of curves from relations between their ordinates. For instance, if, throughout AC, the ordinates of OC and OC' are in a fixed proportion, $ac:ac' = p:q$, then the quadratures are in the same proportion, $\overset{\frown}{OCA} : \overset{\frown}{OC'A} = p:q$. The concept that a figure is built up from its indivisibles can also be applied to space-figures, in which case the indivisible "ordinates" are parallel plane sections of the figure.

CAVALIERI's method admits a far-reaching translation into mathematical symbols. The aggregate of the ordinates y of a curve can be denoted by omn $\cdot\, y$, and with help of this symbolism various relations between quadratures can be represented analytically, and a calculus of these quadratures can be elaborated.

6.2. LEIBNIZ, following CAVALIERI and FABRI, used such a symbolism in his studies of October and November 1675 (LEIBNIZ *Analysis Tetragonistica*), which may be considered to contain the invention of the differential and integral calculus (*cf.* HOFMANN *1949*, 118–130).

One important step in the process of this invention was LEIBNIZ's decision to replace the symbol omn $\cdot\, l$, which he considered to denote the sum of all lines l, by $\int l$[117]. Thus, in these first studies, $\int l$ denoted a quadrature, not an infinitely long line. However, already soon afterwards LEIBNIZ became aware of the need to introduce the differentials along the axis in the symbol for the quadrature and to denote the quadrature by $\int y\, dx$.

6.3. LEIBNIZ has repeatedly stressed the importance of the fact that in his calculus quadratures are evaluated as sums of area-differentials rather than as sums or aggregates of lines. He emphasised that this aspect constitutes the fundamental difference between his calculus and CAVALIERI'S method of indivisibles. He asserted that CAVALIERI evaluated quadratures as $\int y$, the sum of the ordinates. If dx is supposed constant, there is, according to LEIBNIZ, only a formal difference between CAVALIERI'S $\int y$ and his own $\int y\, dx$; but if dx is no longer supposed constant, but arbitrary progressions of the variables are to be allowed, then the treatment of the quadratures as $\int y$ breaks down, whilst the use of $\int y\, dx$ is still acceptable; this because $\int y\, dx$ is independent of the progression of the variables. It is indeed essential that LEIBNIZ should allow arbitrary progressions of the variables in the study of quadratures, for otherwise transformations of the variables cannot be applied. For instance in the case of the transformation $\int n\, dx = \int y\, ds$ (n: normal to the curve, s: arclength), it is impossible to suppose both dx and ds constant, so that at least one of the integrals cannot be directly translated into CAVALIERIAN terminology and symbolism.

LEIBNIZ has appreciated this fact and hence, in his opinion, the evaluation of the quadrature as $\int y\, dx$ constitutes a great advantage of his calculus over CAVALIERI'S.

6.4. The views of LEIBNIZ summarised in the preceding section are expressed, for instance, in the following quotations:

> Before I finish, I add one warning, namely that one should not lightheartedly omit the dx in differential equations like the one discussed above $a = \int dx : \sqrt{1 - xx}$ because in the case in which the x are supposed to increase uniformly, the dx may be omitted. For this is the point where many have erred, and thus have closed for themselves the road to higher results, because they have not left to the indivisibles like the dx their universality (namely that the progression of the x can be assumed ad libitum) although from this alone innumerable transfigurations and equivalences of figures arise.[118]

> ... I denote the area of a figure in my calculus thus: $\int y dx$ or the sum of all the rectangles formed by the product of each y and its corresponding dx. Whereby,

[117] "Utile erit scribi $\int$ pro omn. ut $\int l$ pro omn $\cdot\, l$, id est summa ipsorum l." (LEIBNIZ *Analysis Tetragonistica* (29 oct. 1675)). $\int$ is the long script s, standing for "summa".

[118] "Antequam finiam, illud adhuc admoneo, ne quis in aequationibus differentialibus, qualis paulo ante erat $a = \int dx : \sqrt{1 - xx}$, ipsam dx temere negligat, quia in casu illo, quo ipsae x uniformiter crescentes assumuntur, negligi potest: nam in hoc ipso peccarunt plerique et sibi viam ad ulteriora praeclusere, quod indivisibilibus istiusmodi, velut dx, universalitatem suam (ut scilicet progressio ipsarum x assumi posset qualiscunque) non reliquerunt, cum tamen ex hoc uno innumerabiles figurarum transfigurationes et aequipotentiae oriantur." (LEIBNIZ *1686*; *Math. Schr.* V, p. 233.)

if the dx are supposed equal to each other, one has Cavalieri's method of indivisibles.[119]

And this indeed is also one of the advantages of my differential calculus, that one does not say, as was formerly customary, the sum of all y, but the sum of all ydx, or $\int \overline{ydx}$, for in this way I can make dx explicit and I can transform the given quadrature into others in an infinity of ways, and thus find the one by means of the other.[120]

But this [*i.e.* CAVALIERI's] method of indivisibles contained only the beginnings of the art (...). For whenever the space elements between parallel ordinates (straight lines or plane surfaces) are not equal to each other, then, in order to find the content of the figure, it is not allowed to add up the ordinates to one whole; but the infinitely small space elements between the ordinates have to be measured. (...) Indeed, this measurement of the infinitely small was beyond the power of the Cavalierian method.[121]

6.5. That quadratures did not introduce infinitely large quantities in the LEIBNIZIAN calculus, does not imply that these quantities were entirely absent. In fact, free manipulation with differentials in the formulas led sometimes to expressions which have to be interpreted as infinitely large quantities. Thus, for instance, LEIBNIZ asserted:

Surely we conceive in our analysis a straight line with infinite length, such as $aa:dx$.[122]

And JOHANN BERNOULLI wrote, in a passage already quoted above (§ 2.13), about the quantity $\frac{adx}{dddy}$ as "infinitely large of the second sort".

The infinitely large especially occurred in the studies which LEIBNIZ and JOHANN BERNOULLI, in letters exchanged in 1695, devoted to the analogy between powers and differentials in connection with LEIBNIZ's rule for the differentiation of a product. In these studies[123], on which I shall not digress here because they fall outside the scope of this appendix, positive integer powers of a line were compared with higher-order differentials of a variable, and, because of the reciprocity in both cases, negative integer powers with higher order sums. Here the reciprocity of the operators differentiation and summation made the infinitely large quantities, the sums, enter the investigations naturally.

[119] "... aream figurae calculo meo ita designo $\int ydx$, seu summam ex rectangulis cujusque y ducti in respondens sibi dx, ubi si dx ponantur se aequales, habetur Methodus indivisibilium Cavalerii." (LEIBNIZ *Elementa*, p. 150.)

[120] "Und das ist eben auch eines der avantagen meines calculi differentialis, dass man nicht sagt die summa aller y, wie sonst geschehen, sondern die summa aller ydx oder $\int \overline{ydx}$, denn so kan ich das dx expliciren und die gegebene quadratur in andere infinitis modis transformiren und also eine vermittelst der andern finden." (LEIBNIZ to VON BODENHAUSEN; *Math. Schr.* VII, p. 387.)

[121] "Sed haec Indivisibilium Methodus tantum initia quaedam ipsius artis continebat (...). Nam quoties ordinatim ductae inter se parallelae, nempe rectae lineae vel planae superficies (...) intercipiunt inaequalia quaedam elementa, non licet ipsas ordinatim applicatas in unum addere, ut contentum figurae prodeat, sed ipsa intercepta Elementa infinite parva sunt mensuranda; (...). Ea vero infinite parvorum aestimatio Cavalerianae methodi vires excedebat, ..." (LEIBNIZ *Scientiarum gradus* p. 597.)

[122] "Certe in nostra Analysi concipimus rectam infinitam modificatam, ut $aa:dx$, ..." (LEIBNIZ to GRANDI 6-IX-1713; *Math. Schr.* IV, p. 218.)

[123] The most important relevant texts are to be found in LEIBNIZ *Math. Schr.* III, pp. 175, 180–181, 199–200; compare also § 2.22.

As an example of the occurrence of the infinitely large in these studies I quote a characteristic formula:

$$\int n\,dz = nz - dn \int z + d^2n \int^2 z - d^3n \int^3 z \quad \text{etc.}^{124}$$

Appendix 2. The LEIBNIZIAN Calculus and Non-standard Analysis

7.0. In this Appendix I deal with the relation between the LEIBNIZIAN infinitesimal calculus and non-standard analysis. Non-standard analysis is an approach to analysis due to A. ROBINSON (*1966*). Its relevance to the LEIBNIZIAN infinitesimal calculus is stressed by ROBINSON himself and by others.

7.1. In non-standard analysis, certain concepts and formal tools of mathematical logic are used to provide a rigorous theory of infinitely small and infinitely large numbers. It is shown that the differential and integral calculus can be developed by means of these infinitely small and infinitely large numbers. That is, it is shown that it is possible to define the fundamental concepts of analysis (continuity, differentiation, integration, *etc.*) in terms of infinitesimals rather than in terms of limits.

Not only does non-standard analysis provide a new approach to the differential and integral calculus but also its methods yield interesting reformulations, more elegant proofs and new results in, for instance, differential geometry, topology, calculus of variations, in the theories of functions of a complex variable, of normed linear spaces, and of topological groups.

The infinitely small and infinitely large numbers are introduced in non-standard analysis by a method of mathematical logic which proves the existence of extensions of models of certain mathematical theories; these extensions are the so-called "non-standard" models of the theories. Applied to the field R of real numbers, considered as a model of the theory of real numbers, the method yields extensions R^* of R, such that statements about real numbers, if re-interpreted according to the rules for the extension of theories, are valid for elements of R^*. It is found, in particular, that the extension can be performed in such a way that R^* becomes a totally ordered field, which is non-ARCHIMEDEAN and which contains R as a proper subfield. This implies that R^* contains elements i, unequal to zero, with the property that, for every real number $a>0$,

$$-a < i < a.$$

These elements i are called *infinitesimals*, or infinitely small numbers; their reciprocals are called infinitely large numbers. An element a of R^*, which is not infinitely large, has a unique *standard part*, defined as the real number ${}^0a$, the difference of which from a is zero or an infinitesimal. Further, to every given function f, $R \to R$, is assigned a unique extension f^*, $R^* \to R^*$, which preserves certain properties of f.

The field R^* provides the framework for the development of the differential and integral calculus by means of infinitely small and infinitely large numbers.

[124] BERNOULLI to LEIBNIZ, 27-VII-1695; *Math. Schr.* III, p. 199.

To give one example, the derivative of a real function f can be defined as

$$f'(x) \underset{D}{=} {}^{0}\!\left(\frac{f^*(x+dx)-f^*(x)}{dx}\right),$$

in which dx is an arbitrary infinitesimal.[125]

7.2. Obviously, the use of infinitesimals in non-standard analysis is reminiscent of the LEIBNIZIAN infinitesimal calculus, and non-standard analysis might thus be considered by present-day mathematicians as a posthumous rehabilitation of LEIBNIZ'S use of infinitely small quantities. This view is strongly advocated by ROBINSON. He says that his book shows "that LEIBNIZ'S ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics" (*1966*, p. 2). The inconsistencies of LEIBNIZ'S infinitesimals are removed in non-standard analysis, and ROBINSON states that "LEIBNIZ'S theory of infinitely small and infinitely large numbers (...) in spite of its inconsistencies (...) may be regarded as a genuine precursor of the theory in the present book" (*1966*, p. 269). The creation of non-standard analysis makes it necessary, according to ROBINSON, to supplement and redraw the historical picture of the development of analysis (*1966*, pp. 260–261). This is so because history is usually written in the light of later developments, and non-standard analysis has to be considered as a fundamental change in these later developments, because "the theory of certain types of non-archimedean fields can indeed make a positive contribution to classical Analysis" (*1966*, p. 261).

7.3. It is indeed an interesting feature that, contrary to what was thought for a very long time, the LEIBNIZIAN use of infinitesimals can be incorporated (after some reinterpretations and readjustments) in a theory which is acceptable by present-day standards of mathematical rigor. Thus it is understandable that for mathematicians who believe that these present-day standards are final, non-standard analysis answers positively the question whether, after all, LEIBNIZ was right.

However, I do not think that being "right" in this sense is an important aspect of the appraisal of mathematical theories of the past. The founders, practitioners and critics of such theories judged with contemporary standards of acceptability, and these standards usually differed considerably from those of present-day mathematics.

Hence I disagree with ROBINSON'S opinion about the influence which the occurrence of non-standard analysis should have on the historical picture of the LEIBNIZIAN calculus, or of analysis in general. I do not think that the appraisal of a mathematical theory, such as LEIBNIZ'S calculus, should be influenced by the fact that two and three quarter centuries later the theory is "vindicated" in the sense that it is shown that the theory can be incorporated in a theory which is acceptable by present-day mathematical standards.

If the LEIBNIZIAN calculus needs a rehabilitation because of too severe treatment by historians in the past half century, as ROBINSON suggests (*1966*, p. 260), I feel that the legitimate grounds for such a rehabilitation are to be found in the

[125] The existence of non-standard models for the real numbers has been known since the 1930's (see ROBINSON *1966*, 48 & 88 for precise references), but ROBINSON was the first to use these non-standard models for the study of analysis in terms of infinitesimals.

LEIBNIZIAN theory itself. I believe that, in order to prove its value as a mathematical theory, LEIBNIZ's calculus does not need an adjustment to twentieth century requirements of acceptability through a reformulation in terms of non-standard analysis.

7.4. Apart from this general argument on the relevance of non-standard analysis for an appraisal of the LEIBNIZIAN infinitesimal calculus, I do not think that the two theories are so closely similar that historical insight in the latter can be much furthered by considering it as an early form of non-standard analysis. To substantiate this view, I mention some aspects in which non-standard analysis and LEIBNIZIAN infinitesimal analysis differ essentially.

Non-standard analysis provides a proof that there exists (in the usual modern mathematical sense of that term) a field R^* with the properties indicated in § 7.1, that is, that there exists a field including the real numbers and also infinitesimals. As ROBINSON indicates, LEIBNIZ and his followers were not able to give such a proof. Moreover, the many arguments in the later seventeenth and eighteenth century about the existence of infinitesimals, or about the acceptability of their use, did not in any way come close to the methods of the existence proof in non-standard analysis. ROBINSON quotes LEIBNIZ's argument "that what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa" (*1966*, p. 266; *cf.*, p. 262), but I cannot agree with him that this is "remarkably close to our transfer of statements from R to R^* and in the opposite direction", and in the context of this passage ROBINSON himself shows that LEIBNIZ did not, and could not have provided such a proof. Thus the most essential part of non-standard analysis, namely the proof of the existence of the entities it deals with, was entirely absent in the LEIBNIZIAN infinitesimal analysis, and this constitutes, in my view, so fundamental a difference between the theories that the LEIBNIZIAN analysis cannot be called an early form, or a precursor, of non-standard analysis.

7.5. Another aspect in which the two theories differ concerns the conception of the set of infinitesimals. LEIBNIZ and most of his followers (though not EULER; see below) conceived the set of infinitesimals to be made up of infinitesimals of successive positive integer "order of infinite smallness". Thus if dx was a first-order differential, then all other first-order differentials stood in finite ratio to dx, in general all n$^{\text{th}}$ order differentials stood in finite ratio to dx^n, and the set of infinitesimals consisted only of these classes of differentials.

However, in the set of infinitesimals in R^* of non-standard analysis, there is not a privileged subset of first-order differentials or infinitesimals. (In the definition of the derivative mentioned in § 7.1 any infinitesimal can be chosen for dx.) For a fixed infinitesimal h one might consider, as analogous to the LEIBNIZIAN classes of infinitesimals of successive orders of infinite smallness, classes I_n of infinitesimals, $i \in R^*$, of which ${}^0(i/h^n)$ exists and is unequal to zero. But it is immediately clear that the union of these I_n does not form the whole set of infinitesimals in R^* ($h^{\frac{1}{2}}$ is not included in any I_n).[126]

[126] ROBINSON defines (*1966*, pp. 79/80) higher-order differentials $d^n y$ for a function $y = f(x)$ with respect to an arbitrarily chosen positive infinitesimal dx; if we call $dx = h$, then the $d^n y$ so defined are elements of I_n.

Hence the two theories differ in a most important aspect, namely in the conception of the structure of the set of infinitesimals.

7.6. A third difference between the two theories lies in the fact that LEIBNIZIAN infinitesimal analysis deals with geometrical quantities, variables and differentials, while non-standard analysis, as well as modern real analysis in general, deals with real numbers, functions and (notwithstanding its acceptance of differentials) derivatives. The problems connected with higher-order differentiation of variable quantities (see §§ 2.16–2.21) therefore do not occur in non-standard analysis. ROBINSON does define higher-order differentials (*1966*, pp. 79/80), but these are differentials of a function f and they are defined by means of a *constant* differential dx.

7.7. For the reasons expounded above, I do not feel that the creation of non-standard analysis in itself requires that the history of analysis be re-appraised. But non-standard analysis certainly could suggest interesting historical questions about the early stages of analysis. As an example, I mention the question of the structure of the set of infinitesimals. As I indicated in § 7.5, non-standard analysis shows that if one requires the infinitesimals to be subject to the same operations as the real numbers, then the structure which LEIBNIZ thought the set of infinitesimals to have is insufficient. Therefore one may ask whether this problem did occur to mathematicians working with LEIBNIZ's conception of infinitesimals as divided into classes of successive orders of infinite smallness.

As I have indicated in § 2.15, I have found no trace of an awareness of this problem in LEIBNIZ's writings. EULER, however, was aware of it, and his attitude to the problem was that he let himself without hesitation be guided outside the LEIBNIZIAN orders of infinite smallness by the rules of the operations. His attitude is most clearly shown in his article *1778*, and I shall end this appendix with a summary of this piece.

7.8. In the first part of the article (§§ 1–22) EULER explored the different possible "degrees" ("gradus") of infinity or infinite smallness. Two infinitesimal quantities are of the same degree if their ratio is finite. EULER considered an infinitely large quantity x and remarked that x, x^2, x^3, *etc.* are of different degrees. He showed that, because $y = x^{1/1000}$ is also infinitely large, the degree of x is not the lowest degree and that between the successive degrees of x, x^2, x^3, *etc.* there are arbitrarily many intermediate degrees. The degrees of x^a, a positive, he called degrees of the first class.

Then EULER showed that there are degrees of infinity lower than all first class degrees. For this he considered $\log x$ and he asserted that $\log x$ is infinitely small with respect to $x^{1/n}$ for every n. Hence the degree of $\log x$, and of $(\log x)^a$ for positive a in general, is not of the first class, so that a wealth of new degrees is introduced by the logarithm, even interspersed between those of the first class, because $x^a \log x$ is infinitely large with respect to x^a, but infinitely small with respect to $x^{a+(1/n)}$ for every n.

A consideration of exponentials then led in a similar way to a class of degrees of infinity higher than all degrees of the first class.

These considerations of different classes of degrees of infinity were shown to apply, *mutatis mutandis*, to infinitely small quantities, "because these may be considered as reciprocals of infinitely large quantities".[127]

A remarkable aspect of EULER'S arguments is the use of L'HÔPITAL'S rule in the proofs of his assertions. Thus for instance the assertion that log x is infinitely small with respect to $x^{1/n}$ for every n, was proved as follows:

Call

$$\frac{x^{1/n}}{\log x} = v,$$

$$\frac{1}{\log x} = p$$

and

$$\frac{1}{x^{1/n}} = q,$$

so that

$$v = \frac{p}{q}.$$

Now for $x = \infty$, we have $p = 0$ and $q = 0$. Hence L'HÔPITAL'S rule is applicable and

$$v = \frac{dp}{dq}.$$

Now

$$dp = \frac{-dx}{x(\log x)^2},$$

and

$$dq = \frac{-dx}{n x^{(1/n)+1}},$$

so that

$$v = \frac{n x^{1/n}}{(\log x)^2}.$$

But we had

$$v = \frac{x^{1/n}}{\log x},$$

or

$$v^2 = \frac{x^{2/n}}{(\log x)^2}.$$

Hence

$$v = \frac{v^2}{v} = \left(\frac{x^{2/n}}{(\log x)^2}\right) \Big/ \left(\frac{n x^{1/n}}{(\log x)^2}\right) = \frac{x^{1/n}}{n}$$

(in fact EULER found $v = n x^{1/n}$, which must be an error in calculation), so that v is infinitely large, which proves the assertion.

The use of L'HÔPITAL'S rule in these proofs is very revealing, because it shows both EULER'S style and the difficulty caused by the absence of a clear definition of infinitesimals. Indeed, application of the rule implies the concept of the infinitely large x as a function tending to infinity (and $1/x$ tending to zero). Thus it is acceptable only in a theory which conceives infinitesimals as functions tending to zero or infinity, so that the orders of infinity correspond to the orders of

[127] "... quippe quae spectari possunt ut reciproca infinite magnorum." (EULER *1778*, § 14.)

approaching zero or infinity. However, nowhere did EULER indicate that he conceived the infinitesimals in this way; he took x as an actual infinitely large quantity, and he applied L'HÔPITAL'S rule purely as a formal rule.

In the second part of the article EULER considered functions like $y = c\,x^a$ and $y = c\,x^a(\log(1/x))^m$, for infinitely small values of x. He found, by formally applying differentiation and integration rules, that $\frac{dy}{dx}$ and $\int y\,dx$ are infinitely small and infinitely large, respectively, with respect to y. Applying the rules for discarding infinitesimals, he was able to compute the integral in some cases where this could not be done directly if x is supposed finite. He interpreted his results as assertions about the area under the relevant curve infinitely near the origin.

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Abbreviations used for seventeenth and eighteenth century sources:

Acta Erud. Acta Eruditorum, Leipzig, from 1682.

Mém. Trév. Mémoires pour servir a l'histoire des sciences et des arts ..., Trévoux and Paris, since 1701. (Known as *Journal de Trévoux* or *Mémoires de Trévoux*.)

Misc. Berol. Miscellanea Berolinensa ad Incrementum Scientiarum, ex scriptis Societatis Regiae Scientiarum exhibitis, I, Berlin, 1710.

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