EXPONENTIALS REITERATED

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Three questions motivate this article. When is x^y less than y^x? For what kind of numbers does $x^y = y^x$? And is there a formula for y as a function of x? The second question has attracted attention for some 250 years, since the time of Daniel Bernoulli's [28] interest in the integral solutions up to the present with the work of Sato [72] characterizing the algebraic numbers on the locus.

As regards the relationship between x^y and y^x, without apparent cause sometimes one, sometimes the other, is greater. For example,

$$1^2 < 2^1, \quad 2^2 = 2^2, \quad 3^2 > 2^3, \quad 4^2 = 2^4, \quad 5^2 < 2^5.$$

And recently Varner [76] proved a number of cases, for example, $x^e < e^x$ $(0 < x \neq e)$, among others. Is there a general pattern? Yes, but curiously many of the writers investigating this relation give no references whatsoever to previous work. It seems appropriate, therefore, in the hope of preventing further rediscovery, to review some of the most noteworthy facts about the commutativity of x^y, or the lack of it, and to cite the literature for the others that we don't touch on.

By graphing the equation

$$x^y = y^x, \tag{0.1}$$

we will split up the first quadrant into four regions according to the direction of inequality and thereby quickly answer any such questions about the relative magnitudes of x^y and y^x; this pictorial and historical approach even suggests methods of proof. Next we give the parametrization by Goldbach [29] of one of the curves separating these regions, on which Mahler & Breusch et al. [63] recently discovered the algebraic points.

We answer the third question by finding explicit expressions for y as a function of x. This leads us to the infinitely iterated exponential

$$x = h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}.$$

The first thing to settle about this "function" is whether it ever converges, and if so, where? Surprisingly, it converges for all z in the interval $[e^{-e}, e^{1/e}]$ and diverges for any other positive z. With convergence established, we go on to find two expressions for y in terms of x for our original equation.

These facts about $h(z)$ for real z were first discovered and proved by Euler [78], Eisenstein [44], and Seidel [73], and then rediscovered by others many times over. However, when one turns to complex values, not much is known. So we will close this article with several open problems on convergence, analytic continuation, bifurcation, and general recursion.

Before beginning, let us say clearly what we mean by x^y. In order to avoid complications in defining exponentials on negative or complex numbers, let us declare at the outset that for the first three sections of this paper all our variables x, y, z are positive real numbers. The function x^y has a standard definition for integral arguments; by the use of root extraction it may be extended to the rationals; and with continuity, its values for real numbers may be inferred. Alternatively, in a more modern mode of definition,

$$x^y = e^{y \ln x}.$$

In either case, exponentiation is jointly continuous in both arguments.

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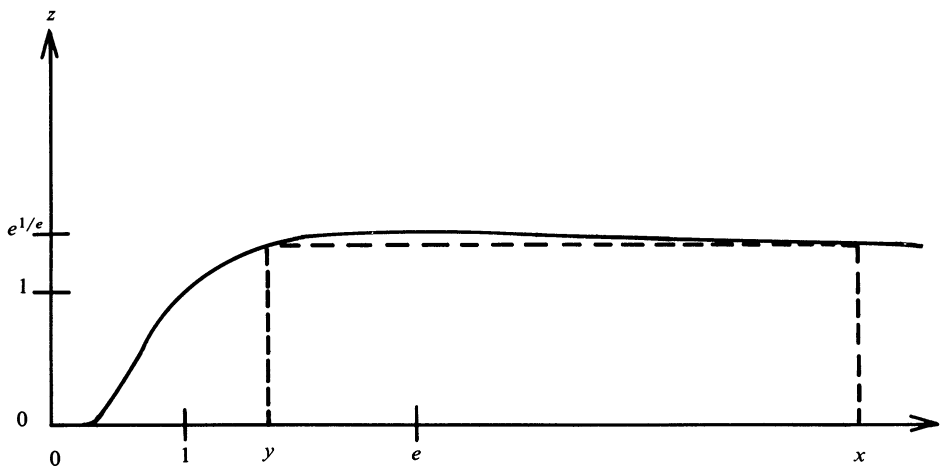

FIG. 1. $z = g(x) = x^{1/x}$

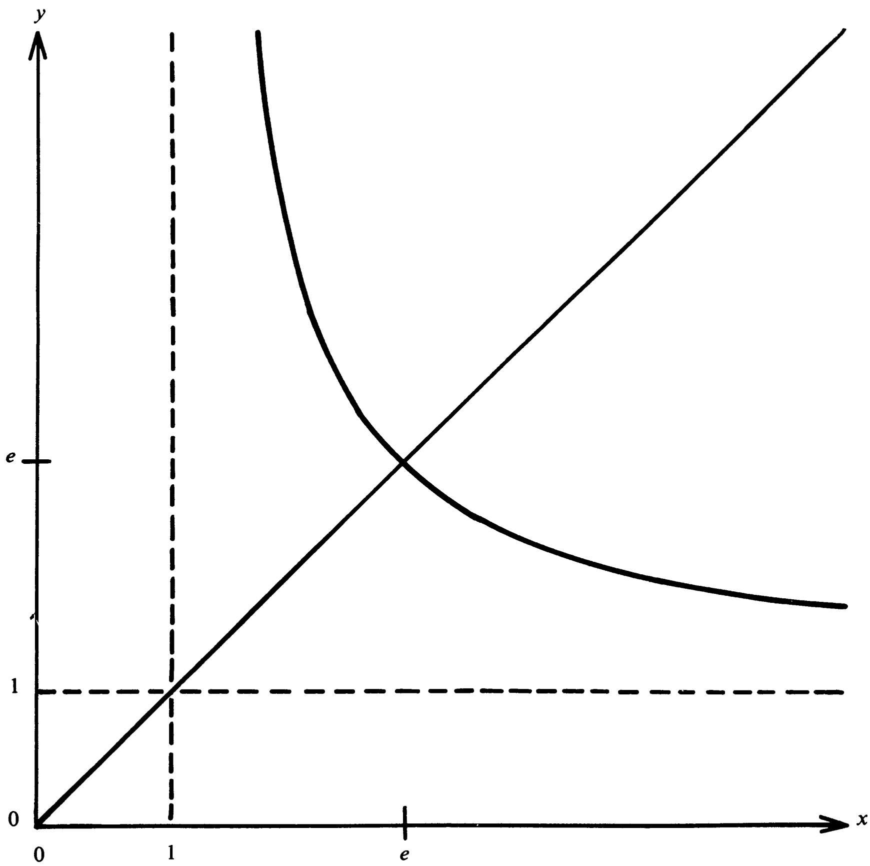

FIG. 2. $x^y = y^x$

For extending my expertise, many thanks are due to the referee, Ralph Boas, Carlton Evans, Michael Frese, Martha Gilmore, Hans Samelson, Ralph Wilkins, and many others who helped fill out the bibliography.

1. $x^y = y^x$. For convenience define

$$f(x,y) = x^y - y^x.$$

Now, for such values, $f(x,y)$ is a continuous function of both x and y. Therefore, to determine how its sign varies, we need only plot $f(x,y) = 0$ to find the boundary between regions of opposite sign. To this end, separate variables by taking xth and yth roots and obtain

$$x^{1/x} = y^{1/y}.$$

The auxiliary function $g(x) = x^{1/x}$ has the graph shown in Fig. 1. The salient points about this function, including the fact that $g(x)$ takes on its maximum $e^{1/e}$ at $x = e$, are given in Rotando & Plummer [77]. Now to plot $x^y = y^x$, simply find those abscissas which give equal values of $z = g(x)$. This yields Fig. 2 (Euler [48]). Notice that this locus divides the first quadrant into four regions alternating in the sign of $f(x,y)$.

The examples given earlier now emerge out of this picture. In particular, the fact that $x^e < e^x$ $(0 < x \neq e)$ corresponds in Fig. 2 to the fact that the line $y = e$ always manages to stay in the two negative regions. Any other line $y = y_0 \neq e$ would have to pass in part through a positive region.

The locus determined by $f(x,y) = 0$ has two branches intersecting at the point (e,e). The branch of equal points, $x = y$, we dismiss as trivial. Happily, the unequal branch can be parametrized

$$\begin{cases} x = s^{1/(s-1)}, \\ y = s^{s/(s-1)}, \end{cases}$$

by positive real numbers s (Goldbach [29]). Thus for each unequal pair (x,y) there is a unique $s = y/x$, the slope of the line from the origin to (x,y), which specifies that point. To check that these expressions really satisfy $x^y = y^x$ is a good exercise in the rules of exponentiation. Another exercise is to demonstrate Carlini's [89] relation, $xs = x^s$, which gives us simultaneously the solution of when the product and power functions are equal.

With this parametrization we can manufacture lots of simple pairs (x,y) such that $f(x,y) = 0$.

s	$1/2$	$2/3$	$4/3$	$3/2$	$5/3$	2	3	4
x	4	$27/8$	$64/27$	$9/4$	$(5/3)^{3/2}$	2	$3^{1/2}$	$4^{1/3}$
y	2	$9/4$	$256/81$	$27/8$	$(5/3)^{5/2}$	4	$3^{3/2}$	$4^{4/3}$

Note that if s gives the pair (x,y), then its reciprocal $1/s$ gives the converse pair (y,x). For $s = 1$, there is an apparent singularity, but it is only apparent. For if we take the limit $s \to 1$ and substitute $s = t + 1$, then we get the usual textbook expression defining e:

$$\lim_{t \to 0}(1 + t)^{1/t} = x = e = y = \lim_{t \to 0}(1 + t)^{1 + 1/t}$$

This parametrization easily characterizes certain important types of points. For example, as Euler [48] was the first to note, the points with rational coordinates are given by

$$s = \frac{m}{m+1} \quad \text{and} \quad s = \frac{n+1}{n}$$

for m and n positive integers.

The main point of the papers by Mahler & Breusch et al. [63] and Sato [72] is that the algebraic points (x,y) on the unequal branch are just those for which the parameter s is a rational m/n. They prove this by means of the Gel′fond-Schneider theorem (Baker [75]). Recall that a number is *algebraic* if it is the root of some polynomial with rational coefficients and a point is *algebraic* if both coordinates are algebraic numbers. In the case at hand, these algebraic numbers are especially simple: they are just roots of rational numbers; that is, the polynomial equations they satisfy are

$$x^{m-n} = \left(\frac{m}{n}\right)^n,$$

$$y^{m-n} = \left(\frac{m}{n}\right)^m.$$

In the case when s is the fraction $m/n > 1$, and when $m/n < 1$, the inverses of both sides of these equations must be taken.

A subclass of the algebraic points is the class of points whose coordinates are algebraic integers: the polynomial equations must have integral coefficients with the leading coefficients being 1. In this class the parameter s simplifies to

$$m \quad \text{or} \quad \frac{1}{n}.$$

We summarize these three paragraphs in a table.

Type of point	s
Rational	$\dfrac{m}{m+1}$, $\dfrac{n+1}{n}$
Algebraic	$\dfrac{m}{n}$ $(m \neq 1$ or $n \neq 1)$
Algebraically Integral	m, $\dfrac{1}{n}(m \neq 1$ and $n \neq 1)$

Notice that there are only two points which are both rational and algebraically integral, namely, (2, 4) and (4, 2) for $s = 2$ and $1/2$; of necessity these are the only points on the curve with integral coordinates.

A closely related equation is

$$\bar{x}^{\bar{x}} = \bar{y}^{\bar{y}}.$$

One sees that this is equivalent to (0.1) by substituting inverses:

$$\bar{x} = \frac{1}{x},$$

$$\bar{y} = \frac{1}{y};$$

Fig. 5 is its graph. On logarithmic graph paper, the curve of Fig. 5 would be just the reflection through the point (1, 1) of the curve of Fig. 2. The previous analysis can be routinely carried over to this new curve with no surprises, except for this one thing: there are no algebraically integral points.

2. An Application in Biochemistry. Lest the reader think that what we have done so far is all pure, recreational mathematics with no possibility of application, we dash this illusion with an example from the study of enzyme reactions in biochemistry. Consider the situation of two substances, called substrates in enzyme kinematics, that can react individually with a common enzyme, which is assumed to be much less in quantity than either of the substrates, and whose activity, therefore, must be divided between the substrates. To be specific, suppose that we are hydrolysing d-ethyl mandelate and l-ethyl mandelate with pig's liver lipase. The respective products are d-mandelic acid and l-mandelic acid. The letters "d-" and "l-" in these names stand for dextrorotatory and levorotatory, which mean that polarized light is rotated to the right or left when passed through the corresponding products (and to a lesser extent in the corresponding substrates). If only one substrate were present, the extent of hydrolysis could be measured by the degree of polarization of the solution. However, when both are initially present in equal quantities, only the difference in the magnitudes of the products can be so measured. Now there may be a differential action of the enzyme on the substrates with the result that one reaction races ahead of the other. This occurs at a maximum just when the difference of the products is maximal, and this in turn just when the optical rotation is a maximum. (Of course, if the reactions proceed at different rates, the substrates will also differ in concentration, but it is easy to see that this difference in substrate concentration is proportional to the difference in product concentration.)

Without going into the mathematical details (Haldane [30, pp. 85–88, 102–105] has an extensive exposition), the Michaelis-Menten theory tells us that this maximum occurs when

$$\bar{x} = s^{1/1-s}, \quad \bar{y} = s^{s/1-s};$$

where $\bar{x},\bar{y}$ are the current concentrations of the substrates, which are used up as the reaction proceeds, and s is a constant determined solely by the substrates and the enzyme. The reader will surely now recognize these expressions as the reciprocals $\bar{x} = 1/x, \bar{y} = 1/y$ of the parametrization given earlier of $x^y = y^x$. Thus the concentrations themselves satisfy

$$\bar{x}^{\bar{x}} = \bar{y}^{\bar{y}}.$$

For the example at hand, $s = 2$.

3. $z^{z^{\cdot^{\cdot^{\cdot}}}}$. Can one do better than this parametrization of the last section and express y in an explicit closed form as a function of x? Well, there are explicit, if not finite, formulas for y. After all, if we can find the inverse of $z = g(x) = x^{1/x}$, then we can solve for y in the equation $y^{1/y} = x^{1/x}$. Consider the function (Euler [78])

$$y = h(z) = z^{z^{\cdot^{\cdot^{\cdot}}}}$$

where the dots mean the limit of the sequence

$$z, z^z, z^{(z^z)}, \ldots$$

with the association of powers to the upper right. The function h is a partial but not complete inverse of g; even this much, though, will suffice to get a solution on one-half of the unequal branch. It surprises most people to learn that $h(z)$ converges for some $z > 1$. However, it does for all z running from e^{-e} up to $e^{1/e} \doteq 1.44$; see Fig. 3. (It is instructive for students to iterate powers on a pocket calculator and see the sequence slowly converge for $z = 1.4$ but blow up for $z = 1.5$ after an initial hesitation. In fact, I stumbled on this unexpected behavior of h while fooling around on a computer terminal using the interactive language APL, which invites just such experimenting because exponentiation is one primitive binary operation among lots of others and finite iteration is itself a primitive. See also Laidler & Landau [77], Wellen [78], and Wilson [77].)

Since the interval of convergence of h is somewhat unexpected (but was known by Euler [78]), we state this as a theorem. Two dichotomies split the statement and proof of the theorem into four parts according to which side of 1 z is on, and according to whether $h(z)$ converges or diverges.

For the convenience of the reader we give the numerical values of the various constants that will appear:

$$\begin{aligned} e^{1/e} &\doteq 1.444667861, & e^{-1/e} &\doteq .6922006276, \\ e &\doteq 2.718281828, & e^{-1} &\doteq .3678794412, \\ e^{e} &\doteq 15.15426224, & e^{-e} &\doteq .06598803585. \end{aligned}$$

It is also useful to have Maurer's [01] notation for "hyperpowers":

$${}^1z = z, \quad {}^2z = z^z, \quad {}^3z = z^{(z^z)}, \quad \ldots;$$

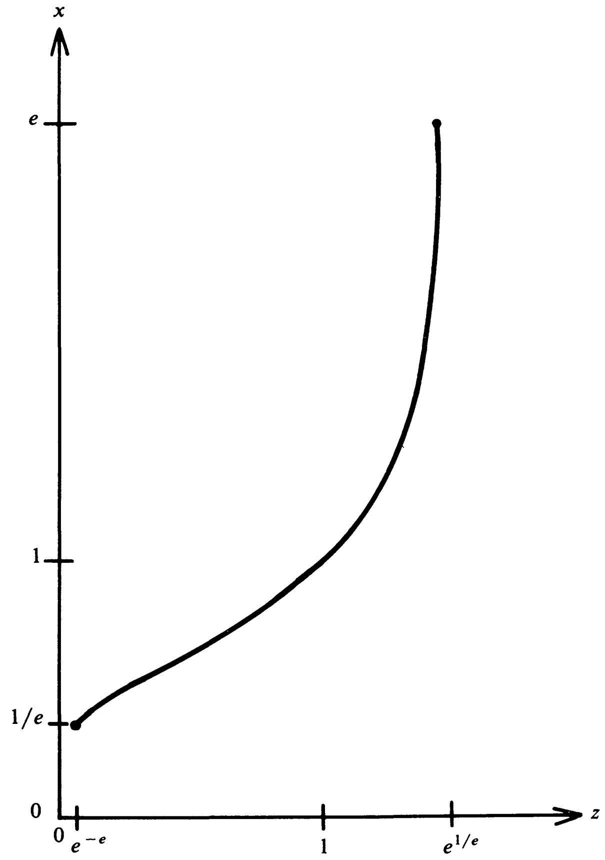

FIG. 3. $x = h(z) = z^{z^{\cdot^{\cdot^{\cdot}}}}$

that is,

$$^{n}z = \overbrace{z^{z^{\cdot^{\cdot^{\cdot^{z}}}}}}^{n \text{ times}}$$

THEOREM. *The function* $x = h(z) = z^{z^{\cdot^{\cdot^{\cdot}}}}$ *converges when* $e^{-e} \leqslant z \leqslant e^{1/e}$ *and diverges for all other positive* z *outside this interval. On this interval* h *is the partial inverse of* g, *that is,*

$$g(h(z)) = z \qquad (e^{-e} \leqslant z \leqslant e^{1/e}),$$
$$h(g(x)) = x \qquad (e^{-1} \leqslant x \leqslant e).$$

In particular, four nontrivial modes of convergence and divergence occur.

Case 1: $z > 1$: *The sequence of hyperpowers increases monotonically*:

$$z < {}^{2}z < {}^{3}z < \cdots. \tag{2.1}$$

Subcase 1c: $1 < z \leqslant e^{1/e}$. *The sequence* (2.1) *is bounded by* e, *and so* $h(z)$ *converges.*

Subcase 1d: $e^{1/e} < z$. *The sequence* (2.1) *increases without bound, and so* $h(z)$ *diverges.*

Case 2: $z < 1$: *The sequence of hyperpowers oscillates*:

$$z < {}^{2}z > {}^{3}z < {}^{4}z > \cdots,$$

and the two subsequences

$$z < {}^3z < {}^5z < \cdots$$

and

$${}^2z > {}^4z > {}^6z > \cdots$$

each converge.

Subcase 2c: $e^{-e} \leqslant z < 1$. *The preceding two subsequences of odd and even hyperpowers converge to the same value, and so $h(z)$ converges.*

Subcase 2d: $z < e^{-e}$. *The preceding two subsequences each converge separately to different values, and so $h(z)$ diverges.*

Proof. Before starting, we remind the reader of three pertinent inequalities for positive a, b, and c that will occur repeatedly throughout the proof:

$$\text{if } a < b, \text{ then } a^c < b^c;$$

$$\text{if } a < b \text{ and } c < 1, \text{ then } c^a > c^b;$$

$$\text{if } a < b \text{ and } c > 1, \text{ then } c^a < c^b.$$

When the sequence of hyperpowers converges, that is, when $h(z) = x$ holds, then by reason of the continuity of exponentiation $z^x = x$ or, what is the same,

$$z = x^{1/x} = g(x).$$

From this follows the equations stating that h and g are inverses over the interval of convergence.

Case 1: $z > 1$: With the inequalities above, we easily obtain

$$z < {}^2z < {}^3z < \cdots.$$

Subcase 1c: $1 < z \leqslant e^{1/e}$. We prove convergence by showing that the sequence of hyperpowers is bounded above by e. By induction on the index of the hyperpowers, it suffices to verify that

$$\text{if } 1 < z \leqslant e^{1/e} \text{ and } w \leqslant e, \text{ then } z^w \leqslant e.$$

And this last follows from the inequalities just mentioned:

$$z^w \leqslant (e^{1/e})^e = e.$$

Thus the sequence converges since it is bounded and monotonically increasing.

Subcase 1d: $e^{1/e} < z$. As already seen, if the sequence of hyperpowers were to converge to x, then $z = x^{1/x} = g(x)$. But we have already shown that g has a maximum at $x = e$; hence $z = x^{1/x} \leqslant e^{1/e}$, and therefore h would be double valued, which is impossible (look at the graph of g, Fig. 1).

Case 2: $z < 1$. This case requires a subtler proof than the first since the sequence of hyperpowers alternates between two monotonically converging subsequences. This oscillation follows directly from the preliminary inequalities:

$$z < {}^2z > {}^3z < {}^4z > \cdots;$$

and so do these chains:

$$z < {}^3z < {}^5z < \cdots,$$

and

$${}^2z > {}^4z > {}^6z \cdots.$$

Realizing that ${}^nz$ is bounded, that is, $0 < {}^nz < 1$ for $0 < z < 1$, and therefore that these last two sequences converge, we introduce two new functions, h_0 and h_e, defined for all positive $z \leqslant e^{1/e}$:

$$h_0(z) = \lim_{\substack{n\to\infty \\ n \text{ odd}}} {}^n z, \qquad h_e(z) = \lim_{\substack{n\to\infty \\ n \text{ even}}} {}^n z.$$

In case lc already considered, it's clear that

$$h_0(z) = h_e(z) = h(z) \qquad (1 \leqslant z \leqslant e^{1/e}).$$

We now have to consider the real possibility that $h_0(z) \neq h_e(z)$ for some $z < 1$.

To find out how and why this split may occur, it is convenient to introduce the more involved relationship

$$z^{z^x} = x.$$

It is easy to see that if the point (z, x) satisfies any one of the relations

$$\begin{aligned} x &= h(z), \\ x &= h_0(z), \\ x &= h_e(z), \\ z &= g(x) \ \ (\text{or equivalently } z^x = x), \end{aligned}$$

then (z, x) also must satisfy

$$z^{z^x} = x.$$

As we proceed through the proof of Case 2, we will also gradually accumulate all the evidence necessary to establish the converse: namely, that the locus of the last equation is the union of the loci of the preceding four. To both these ends, write $j(z,x) = z^{z^x} - x$ and calculate the differential

$$\begin{aligned} dj(z,x) &= \frac{\partial j}{\partial z} dz + \frac{\partial j}{\partial x} dx \\ &= z^{z^x} z^{x-1}(x \ln z + 1)\, dz + (z^{z^x} z^x \ln^2 z - 1)\, dx. \end{aligned}$$

The coefficients of dz and dx are both simultaneously zero in the first quadrant just when

$$z = e^{-e} \quad \text{and} \quad x = e^{-1}.$$

Through any other point, by the implicit function theorem, there is a unique trajectory or line satisfying $j(z, x) = 0$. With this observation we can now settle the first subcase.

Subcase 2c: $e^{-e} \leqslant z < 1$. Since $h_0(1) = h(1) = h_e(1)$, it must be that $h_0(z) = h_e(z)$ for z down to (but not necessarily including) the critical value e^{-e}. (Equality at this critical value will follow from the next case.) Hence $h(z)$ converges to the common value.

Subcase 2d: $z < e^{-e}$. Divergence in this case is arrived at in three steps. First, we show for n even that ${}^n z \geqslant e^{-1}$. Then, from this it follows for n odd that ${}^n z \leqslant e^{-1}$. And finally we return to n even and establish that ${}^n z > e$ strictly.

Eisenştein [44] showed that the function z^{z^β} (β a constant) has a minimum $e^{-1/\beta e}$ at $z = e^{-1/\beta}$ (this can be established by elementary calculus). In particular, if $x \geqslant e^{-1}$, then $z^{z^x} \geqslant e^{-1}$. By induction, it will follow that for n even, ${}^n z \geqslant e^{-1}$. Raising z to these powers reverses inequalities, and so for $z < e^{-e}$ and n odd, we have

$${}^n z = z^{({}^{n-1}z)} < (e^{-e})^{e^{-1}} = e^{-1}.$$

Hence for $z < e^{-e}$, $h_0(z) \leqslant e^{-1} \leqslant h_e(z)$. Using the differential of $j(z, x) = z^{z^x} - x$ again, we see that $dx/dz > 0$ on the open line

$$L = \{\langle z, x\rangle \mid 0 < z < e^{-e} \text{ and } x = e^{-1}\}.$$

It must be then that $h_e(z) \neq e^{-1}$ when $0 < z < e^{-e}$, for if it were otherwise then, for some z' in a neighborhood of this z, it would be that $h_e(z') < e^{-1}$, which is a contradiction. Q.E.D.

To fill out what was already mentioned in the proof about the union of all the loci being $j(z,x) = 0$, we add a corollary to the proof (see also Fig. 4).

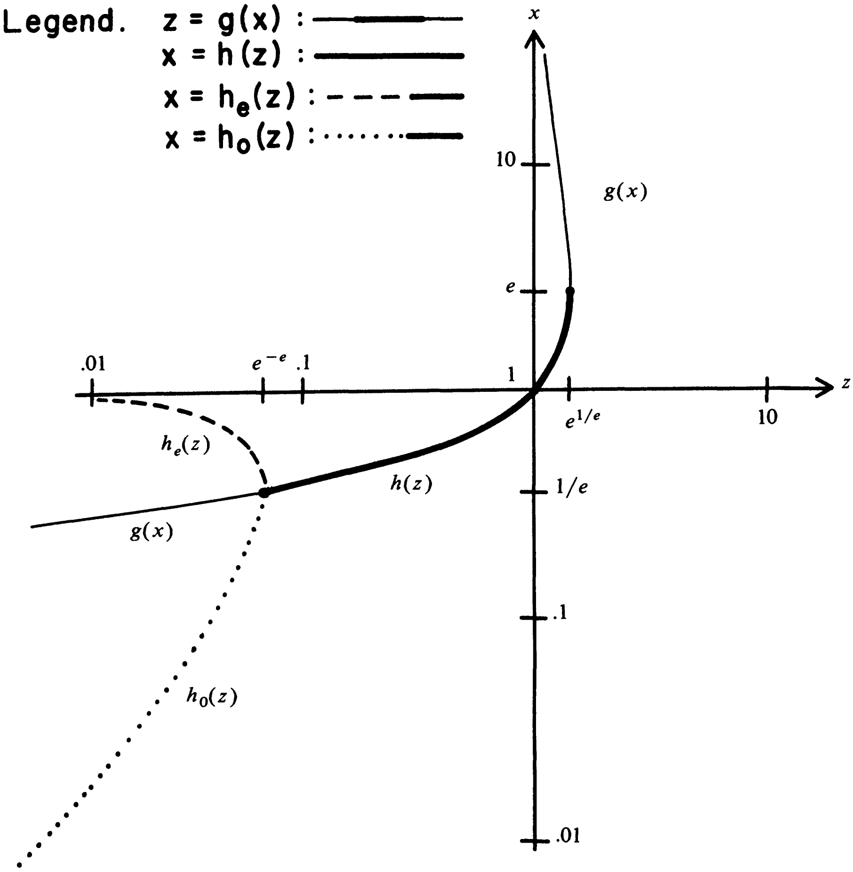

FIG. 4. $z^{z^x} = x$

COROLLARY. *The graph of the relation* $z^{z^x} = z$ *is exactly the union of the graphs of the functions* g, h, h_0, h_e *with the now established convention on the arguments*:

$$g(x) = z,$$
$$x = h(z),$$
$$x = h_0(z),$$
$$x = h_e(z).$$

Proof. As already noted, each of the individual loci satisfy $j(z,x) = 0$. For the other direction it suffices to show that there are no other points on $j(z,x) = 0$. And for this it suffices to establish that on each vertical line, i.e., $j(z_0,x)$ as a function of x alone, there are the right number of zeros. And this in turn will follow from the fact that the number of maxima and minima is exactly what it should be:

z_0	Number of extrema of $j(z_0,x)$	Number of zeros of $j(z_0,x)$
$0<z_0<e^{-e}$	2	3
$e^{-e}\leqslant z_0\leqslant 1$	0	1
$1<z_0<e^{1/e}$	1	2

And finally the number of extrema of $j(z_0,x)$ is found as usual by calculating $\partial j/\partial x$, which we have already done, and setting it equal to zero.

As an aside, there is a bit of numerology in these bounds on z:

$$e^{-e}=(e^e)^{-1},$$
$$e^{1/e}=e^{(e^{-1})}.$$

The forms on the right are e raised to the additive and multiplicative inverses of e, and those on the left are the two different ways of associating a triple power (exponentiation is neither associative nor commutative).

But I digress. We must get back to expressing y in terms of x on the unequal branch of the relation $x^y=y^x$. This relation is equivalent to

$$g(y)=y^{1/y}=x^{1/x}=g(x).$$

Since we have just shown that h is the partial inverse of g, we obtain the desired result

$$y=h(g(y))=h(x^{1/x})=(x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}}$$
$$=x^{x^{-1+x^{-1+x^{-1+\cdot^{\cdot^{\cdot}}}}}}\qquad (x\geqslant e).$$

It might appear that we have simply applied a function to its inverse and so would get nothing new, but one should bear in mind that on the one half of the unequal branch for which $x>e$ we are out of the range of h. Thus we have started with x, passed to $z=g(x)$, and dropped back down to $y=h(z)=h(g(x))$ as already shown in Fig. 1.

In a similar manner one obtains an expression for half of the unequal branch of the locus in Fig. 5 of $\bar{x}^{\bar{x}}=\bar{y}^{\bar{y}}$. Thus

$$\bar{y}=\bar{x}^{\bar{x}^{1-\bar{x}^{1-\bar{x}^{1-\cdot^{\cdot^{\cdot}}}}}}\qquad (\bar{x}\leqslant 1/e).$$

There is another explicit form (although still not closed) for y as a function of x. From the theorem and the papers of Eisenstein [44] or Wittstein [45], we find

$$h(z)=1+\ln z+\frac{3^2(\ln z)^2}{3!}+\frac{4^3(\ln z)^3}{4!}+\cdots\qquad (e^{-1/e}<z<e^{1/e}). \tag{2.2}$$

Therefore, with $z=x^{1/x}$ and $y=h(z)$, it follows that

$$y=1+\frac{\ln x}{x}+\frac{3^2}{3!}\left(\frac{\ln x}{x}\right)^2+\frac{4^3}{4!}\left(\frac{\ln x}{x}\right)^3+\cdots\qquad (x>e) \tag{2.3}$$

on the unequal branch (see also Carmichael [08]).

There is a curious historical note that gives an indication of how rigor in mathematics was evolving over the time Euler [78], Eisenstein [44], and Seidel [73] wrote their papers. Euler used mainly numerical examples together with some algebraic manipulation to convince himself of the interval of convergence and the existence of bifurcation. Eisenstein, unaware of Euler's paper, was simultaneously both careful and careless in questions about convergence. He explicitly assumes that $0<z<1$ (in that paper α plays the role of our z), and tacitly implies that $h(z)$ converges throughout this interval without so much as a hint of a proof. On the other hand, with reasonable care, he establishes that the right side of (2.2) converges for $e^{-1/e}<z<e^{1/e}$. With his previous assumptions about $h(z)$ converging for $0<z<1$, he concludes that (2.2) holds just when $e^{-1/e}<z\leqslant 1$. However, with the help of our theorem, we can establish the full interval of convergence of (2.2). With regard to $h(z)$, we now know that Eisenstein's

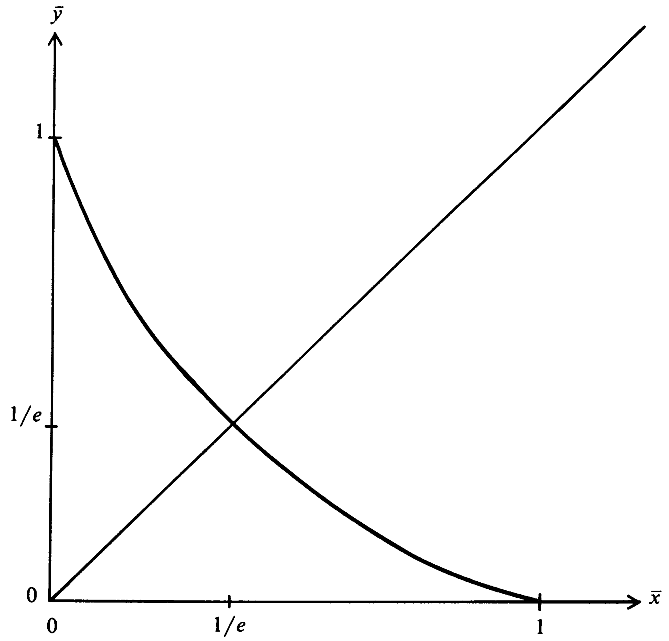

FIG. 5. $\bar{x}^{\bar{x}} = \bar{y}^{\bar{y}}$

implicit interval of convergence (0, 1) is wrong and must be stretched and moved over to the right to $[e^{-e}, e^{1/e}]$. To top this off, Seidel, also not knowing of Euler's paper, corrected Eisenstein's work and gave a complete proof for both convergence and bifurcation.

4. Open Problems. Despite the enormous literature on these topics when the variables are real numbers (see bibliography), once complex numbers are admitted, the literature becomes much, much smaller, and significant open problems appear. To limit the discussion, we look at only the iterated exponential. We describe four open problems about the region of convergence, the Riemann surface of analytic continuation, the nature of bifurcation, and higher levels of recursion.

Before proceeding we must cope with the fact that for complex values exponentiation is no longer single valued. By definition

$$z^w = e^{w \ln z};$$

its value depends on which sheet $\ln z = \operatorname{Log} r + i\theta$ is situated. Some authors get around this by working exclusively in terms of $t = \ln z$ as a new independent variable, but to maintain our present notation we agree to consider z as uniquely specified by its polar form $z = re^{i\theta}$. With this understanding, $\theta = 0$ and $\theta = 2\pi$ give distinct values of z.

Convergence. Where does $z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$ converge for complex z? On the one hand, Thron [57] established

$$|\ln z| \leqslant e^{-1}$$

as a region R_T of convergence. On the other hand, Carlsson [07] earlier showed that there is convergence only if $z = e^{\zeta e^{-\zeta}}$ for some ζ such that $|\zeta| \leqslant 1$; call this region R_C.

What about the points between R_T and R_C? Does the iterated exponential converge there? Shell [62] established several successively larger dumbbell-shaped regions of convergence which overlap that of Thron, leaving part of R_T uncovered along and near the real axis but extending well beyond R_T in other directions. However, the union of all these regions of established convergence fails to exhaust R_C itself. Thus our open problem is this: prove or disprove the conjecture of Shell [59], based on considerable computer calculation, that the sequence $z, {}^2z, {}^3z, \ldots$ converges for all $z \in R_C$.

Analytic continuation. An *analytic function* is one which can be expanded locally in a power series (see Churchill et al. [74]). For example, the series (2.2) shows that h is analytic in the interval $(e^{-1/e}, e^{1/e})$. It may be necessary to use more than one power series to represent an analytic function at various points. Specifically, we shall see shortly that h is analytic also in its whole domain of definition $(e^{-e}, e^{1/e})$ but the series (2.2) does not converge to the right of $e^{-1/e}$: there other power series are needed. The *analytic continuation* of a function along an arc going from its domain of definition and beyond it is a new analytic function defined on the whole arc and agreeing with the original function wherever it was defined. For our situation, we will show that h has an analytic continuation all the way down to (but not including) $z = 0$; it is in fact the inverse of g (see Fig. 4).

A fundamental result in the theory of complex variables is that analytic continuation is unique when it exists. But this is guaranteed only along the same arc: continuing the function $\sqrt{z}$ around the origin to the starting point will give the value $-\sqrt{z}$ of opposite sign. Thus analytic continuation may result in the complex domain being separated into two or more sheets. The analytic continuation of $\sqrt{z}$ has two sheets; the analytic continuation of $\ln z$ has an infinite number of sheets. This last fact will probably mean that the analytic continuation of h has many sheets (remember that $z^w = e^{w \ln z}$ is multivalued).

How much do we know about the analytic continuation $\hat{h}$ of h? We have already seen that if $\lim {}^nz$ converges to x then $z^x = x$ and so $z = x^{1/x} = g(x)$. (Here x is a complex variable and *not* the real part of z.) As an elementary function, $g(x)$ is analytic everywhere except at $x = 0$. It is also invertible everywhere except where $g'(x) = 0$, that is, at $x = e$. Hence its inverse is analytic except when $z = 0$ or $z = e^{1/e}$. Thus $h(z)$, as a segment of the inverse of $g(x)$, is itself analytic. There is an arc between any two x's other than 0 and e, so there is an analytic continuation between the corresponding z's. Hence all of the inverse of g is an analytic continuation of h.

In particular, we can travel from $x = 2$ to $x = 4$ along the real axis except at $x = e$, where we must take a slight detour around this point into the complex numbers because $g'(e) = 0$. This means that in the analytic continuation of h, we must be on different sheets:

$$\hat{h}(\sqrt{2}) = 2,$$

$$\hat{h}(\sqrt{2}) = 4.$$

This explains the anomaly of $2 = 4$ puzzled over by Etherington [38], Gottlieb [73], and Andrews & Lacher [77].

Analytic continuation explains another curiosity. From his famous equation $i = e^{i\pi/2} = (e^{\pi/2})^i$, Euler [78] also came across the solution $z = e^{\pi/2}$, $x = i$. Again this means that $\hat{h}(e^{\pi/2}) = i$ on some sheet in the analytic continuation. It is easy to see that on another sheet also $\hat{h}(e^{\pi/2}) = -i$.

If one is to solve this open problem of determining precisely the extent of the complete analytic continuation of h, one must check that the term $\ln z$ in the definition of h does actually introduce an infinite number of sheets, find all the additional sheets introduced by the lack of one-to-one-ness in its inverse g, and finally investigate the possibility that $x = h(z)$ might continue analytically beyond the relation $z^x = x$.

Bifurcation. Just as the branch g beyond the region of convergence of $\lim {}^nz$ in Fig. 4 is the germ for studying the analytic continuation of h, so the other two branches h_0 and h_e lead us to settle rather for convergence of subsequences in the complex plane. There are complex z with almost periodic behavior of the sequence

$$z, {}^2z, {}^3z, \ldots,$$

but the periods are often different from 2 as in the real case, and the behavior is much more complicated. These three values of $z = re^{i\theta}$ have the following near periods:

r	θ	near period
4	3	3
3	4	4
3.5	3.5	5

The reader is encouraged to try his hand at finding other periods with, say, a programmable pocket calculator or a computer terminal.

The author is unable and unqualified to make head or tail of this chaotic behavior. Hence the third open problem is to explain the bifurcation of iterated exponentials on complex numbers. To get started see Straffin [78] or Marsden [78].

Recursion. Finally, there is a circle of problems centered around the iteration of hyperpowers

$$z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}.$$

That this makes any sense at all depends on the extension of ${}^{n}m$ from positive integers to at least all positive real numbers (see the note and references to Wright [47] and others in the bibliography). For infinite iteration, we can repeat the three open problems above about convergence, analytic continuation, and chaotic behavior.

For finite iteration, we can "hype" the hyperpowers again,

$$z^{z^{\cdot^{\cdot^{\cdot^{z}}}}}\Big\}\, z \text{ times},$$

and then again, and away we go to any level of recursion. This has actually been done for the positive integers and the result is the three-argument Ackermann [28] function $\alpha(k,m,n)$, whose definition is given by

$$\begin{cases} \alpha(1,m,n) = m+n \\ \alpha(k+1,m,1) = m, \\ \alpha(k+1,m,n+1) = \alpha(k,m,\alpha(k+1,m,n)). \end{cases}$$

For the first four values of k, we get the familiar functions

$$\begin{aligned} \alpha(1,m,n) &= m+n \\ \alpha(2,m,n) &= m\cdot n, \\ \alpha(3,m,n) &= m^{n}, \\ \alpha(4,m,n) &= {}^{n}m. \end{aligned}$$

This indexing fits in well with the historical naming of hyperpowers as the "fourth natural algorithm" by Lémeray [98] and the "operation of the fourth kind" by Schubert [98].

The Ackermann function is the classic example in the theory of computation of a general recursive function which is not primitive recursive (see Hennie [77] and Calude, Marcus & Tevy [79]). The essential difference distinguishing these two kinds of algorithms is that a primitive recursive function, such as exponentiation, is defined by recursion on just one argument at a time, whereas a general recursive function may be defined by recursion on two or more arguments simultaneously, such as in the definition of Ackermann's function. This difference between double and single recursion is dramatically illustrated by the computer programs for α and any one of the binary arithmetical operations, say m^{n}: it takes incredibly more time and storage to compute, for example, $\alpha(3,5,5)$ as opposed to 5^{5}, even though the answer is the same.

General recursion goes well beyond the kind of definition which Wright [47] and others extended to the complex numbers. This leads to the final question: Can Ackermann's function be analytically continued to all numbers in the complex plane?

Bibliography.

Those who cannot remember the past are condemned to repeat it.—George Santayana

Here are some notes prefacing the references in order to help readers find their way through the literature. There are, first, additional references on the equation $x^y = y^x$, then some citations for the iterated exponential, and finally a miscellanea of notes about related papers. For the reader who wishes only to sample this extensive literature, the following accessible items are recommended: for the commutativity of exponentiation, Hausner [61], Hurwitz [67], Mahler & Breusch [63], and Moulton [16]; and for iterated exponentiation, Bender & Orszag [78], Barrow [36], Creutz & Sternheimer [?], Euler [78], Shell [62], Thron [57], and Wright [47].

Contrary to the initial observation that most articles in the area have few or no references, the relatively recent articles by Hausner [61] and Hurwitz [67] are exceptional in this respect and should be consulted for numerous citations on $x^y = y^x$. Likewise Dickson [19] and Archibald [21] should be consulted for older ones.

We have deliberately restricted ourselves to the case when both x and y are positive in this equation, since otherwise, to be modern, one should go over to complex numbers, which only Schwering [78] and Moulton [16] have completely done, so far as I know. The old-fashioned way of real root extraction, when it exists, is followed by Wittstein [45], Carmichael [08], Nesbitt [13], and Hurwitz [67]. Nesbitt [13] and Moulton [16] have nice graphs of $x^y = y^x$ for all real values of x and y, both positive and negative.

There is a related problem, which we have not mentioned, of when $x^y - y^x$ is not zero but equal to a prescribed integer. See Cassels [60], Krishnasastry & Perisastri [65], Schinzel [67], and Tijdeman [76] for special cases of this and also for additional references.

For the iterated exponential the most extensive surveys are to be found in Carlsson [07] and Shell [59]. Many authors have studied iterated exponentials with unequal exponents,

$$z_1^{z_2^{z_3^{\cdot^{\cdot^{\cdot}}}}},$$

for example, Barrow [36], Thron [57], and Creutz & Sternheimer [?]. This leads to the expansion of arbitrary numbers and functions by infinite exponential powers, which is discussed by Ditor [78] and Bender & Orszag [78].

Closely connected with the infinite iteration is the extension of the finite iterates to nonintegral values; in short, how should the hyperpower ${}^n x$ be defined for arbitrary n? For some selected references on this special problem as well as the more general one of iterating an arbitrary function and extending it to the reals, see Abel [?], Collins [38], Cayley [60], Schröder [71], Koenigs [84], Lemeray [95], Hadamard [44], Pincherle [06], Carlsson [07], and Wright [47], especially the last three for surveys and extensive bibliographies.

The mathematics of this article has been a fruitful source of problems: the equation $x^y = y^x$ appears in Bush [61], Mahler & Breusch [63], and Bryant et al. [65]; the iterated exponential in Lense [24], Bromwich [26], Francis & Littlewood [28], Chaundy [35], Knopp [51], Apostol [57], and Bryant et al. [65].

Just as the lack of commutativity in taking powers motivated this article, so the lack of associativity might also pique one's interest. However, the condition that

$$w^{(x^s)} = (w^x)^s,$$

when w, x, s are positive and $w \neq 1$, is equivalent to Carlini's [89] multiplicative relation $xs = x^s$, which in turn reduces to the trivial condition

$$x = s^{1/(s-1)},$$

already studied in §1. For the association of an arbitrary number of terms, see the contributions of Wöpcke [51], P. Goodstein [58], R. L. Goodstein [58], Göbel & Nederpelt [71], Riordan [73], and Gardner [73].

To see some of the equations of this paper solved in terms of ordinals and cardinals, consult G. Cantor [97], Sierpiński [58], and Hickman [76]. (The equation $z^x = x$ plays a significant role in the theory of ordinals.)

Lastly, we mention here some historical details not covered before about who did what when. The comparison of x^y and y^x apparently goes back to Hengel [88]. The finite iteration of the exponential can be traced to Condorcet [78] and the infinite to Euler [78]. Wittstein [45] was the first to use $g(x) = x^{1/x}$ in connection with $x^y = y^x$, and Euler [78] first applied it to the study of $h(z)$. Of course, the logarithm of $g(x)$ goes back to Euler's formulation of the prime number theorem.

The MR numbers at the end of some of the citations refer to abstracts in the Mathematical Reviews: those of JFM to the Jahrbuch über die Fortschritte der Mathematik, wherever known.

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