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CENTROSYMMETRIC MATRICES: PROPERTIES AND AN ALTERNATIVE APPROACH

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ABSTRACT. We present a simple approach to deriving results/algorithms about centrosymmetric matrices. Also, we reveal new facts about centrosymmetric and skew-centrosymmetric matrices and we present a new characterization of centrosymmetric matrices and skew-centrosymmetric matrices.

1 Introduction. Centrosymmetric matrices have a rich eigenstructure that has been studied extensively in the literature (see [**26**], [**22**], [**5**], [**15**], [**6**], [**10**], [**7**], [**25**], [**23**], [**1**], [**2**]). Many results for centrosymmetric matrices have been generalized to wider classes of matrices that arise in a number of applications (see [**12**], [**13**], [**19**], [**17**], [**18**]). Most facts/algorithms about centrosymmetric matrices were derived using orthogonal similarity transformations. In this paper, we use an alternative simple approach to derive the most known important facts/algorithms about centrosymmetric matrices. We also use this approach to solve efficiently linear systems of equations involving centrosymmetric matrices and to reveal new properties of centrosymmetric matrices. We note that our approach is applicable in many cases to skew-centrosymmetric matrices. Chu [**8**] studied the class $\mathcal{M}$ of real orthogonal matrices such that if T is a real symmetric Toeplitz matrix (which implies it is real symmetric centrosymmetric) and $K \in \mathcal{M}$, then KTK^T is real symmetric Toeplitz. In this paper, we present a class $\mathcal{N}$ of orthogonal matrices such that if $Q \in \mathcal{N}$ and H is a centrosymmetric matrix, then $Q^T HQ$ is block-diagonal. This is important, because if H is a centrosymmetric matrix and $D = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$ is a block diagonal matrix similar to H via a member of $\mathcal{N}$, then H and D_1 and D_2 share many properties. We identify the class $\mathcal{L}$ of orthogonal matrices of even order such that if $Q \in \mathcal{L}$ and H

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is a centrosymmetric (resp. skew-centrosymmetric) matrix of even order, then QH is skew-centrosymmetric (resp. centrosymmetric). Thus, members of $\mathcal{L}$ are orthogonal transformations between centrosymmetric matrices of even order and skew-centrosymmetric matrices of even order. Hence, we can apply results/algorithms about centrosymmetric matrices to skew-centrosymmetric matrices and vice versa. We prove several theorems about centrosymmetric and skew-centrosymmetric matrices. But, we will focus on centrosymmetric matrices. In several cases, we will consider matrices of even order only (the case of odd order is either similar or it can not be put in a nice useful form).

2 Preliminaries. We employ the following notation. We denote the transpose of a matrix A by A^T, the Hermitian transpose by A^*, and the determinant of A by $\det(A)$. We use the notation $\lfloor x \rfloor$ for the largest integer less than or equal to x. As usual, I denotes the identity matrix and $\overline{k}$ denotes the complex conjugate of k. Throughout this paper, we let $\delta = \lfloor \frac{n}{2} \rfloor$, where n is a positive integer.

We mean by the time complexity the number of flops. When counting flops, we treat addition/subtraction the same as multiplication/division. By the *main counterdiagonal* (or simply *counterdiagonal*) of a square matrix we mean the positions which proceed diagonally from the last entry in the first row to the first entry in the last row.

Definition 2.1. The *counteridentity* matrix, denoted J, is the square matrix whose elements are all equal to zero except those on the counterdiagonal, which are all equal to 1.

We note that multiplying a matrix A by J from the left results in reversing the rows of A and multiplying A by J from the right results in reversing the columns of A. Throughout this paper, we will denote the counteridentity matrix by J.

A vector x is called *symmetric* if $Jx = x$ and *skew-symmetric* if $Jx = -x$. If x is an $n \times 1$ vector, then we let x^+ represent the symmetric part of x; i.e. $x^+ = \frac{1}{2}(x + Jx)$, where J is the $n \times n$ counteridentity matrix, and we let x^- represent the skew-symmetric part of x; i.e. $x^- = \frac{1}{2}(x - Jx)$.

Definition 2.2. A matrix A is *centrosymmetric* if $JAJ = A$, *skew-centrosymmetric* if $JAJ = -A$, and *persymmetric* if $JAJ = A^T$.

Centrosymmetric and skew-centrosymmetric matrices have applications in many fields including communication theory, statistics, physics, harmonic differential quadrature, differential equations, numerical analysis, engineering, sinc methods, magic squares, and pattern recognition. For applications of these matrices, see [**24**], [**14**], [**11**], [**20**], [**15**], [**7**], [**16**], [**4**]. Note that symmetric Toeplitz matrices are symmetric centrosymmetric and skew-symmetric Toeplitz matrices are skew-symmetric skew-centrosymmetric.

The following lemma can be found in many of the references listed at the end.

Lemma 2.3. *Let H be an $n \times n$ centrosymmetric matrix. If n is even, then H can be written as*

$$H = \begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix},$$

where A, J and C are $\delta \times \delta$ matrices. If n is odd, then H can be written as

$$\begin{bmatrix} A & x & JCJ \\ y^T & q & y^T J \\ C & Jx & JAJ \end{bmatrix},$$

where A, J and C are $\delta \times \delta$ matrices, x and y are $\delta \times 1$ vectors, and q is a number.

The following result, which can be found in several publications (see [**6**], [**10**], for example), is probably the most important known fact about centrosymmetric matrices.

Theorem 2.4. *Let H be an $n \times n$ centrosymmetric matrix and let H be decomposed as in the previous lemma. If n is even, then the eigenvalues of H are the eigenvalues of $F_1 = A - JC$ and the eigenvalues of $G_1 = A + JC$. Moreover, the eigenvectors corresponding to the eigenvalues of F_1 can be chosen to be skew-symmetric of the form $(u^T, -u^T J)^T$, where u is an eigenvector of F_1, while the eigenvectors corresponding to the eigenvalues of G_1 can be chosen to be symmetric of the form $(u^T, u^T J)^T$, where u is an eigenvector of G_1. Also,* $\det(H) = \det(F_1) \cdot \det(G_1)$, *and H is Hermitian (resp. skew-Hermitian, normal, positive-definite, positive-semidefinite) if and only if F_1 and G_1 are Hermitian (resp. skew-Hermitian, normal, positive-definite, positive-semidefinite).*

If n is odd, then the eigenvalues of H are the eigenvalues of F_1 and the eigenvalues of

$$G_2 = \begin{bmatrix} q & \sqrt{2} y^T \\ \sqrt{2} x & A + JC \end{bmatrix}.$$

Moreover, the eigenvectors corresponding to the eigenvalues of F_1 *can be chosen to be skew-symmetric of the form* $(u^T, 0, -u^T J)^T$*, where* u *is an eigenvector of* F_1*, while the eigenvectors corresponding to the eigenvalues of* G_2 *can be chosen to be symmetric of the form* $(u^T, \sqrt{2}\alpha, u^T J)^T$*, where* $(\alpha, u^T)^T$ *is an eigenvector of* G_2*. Also,* $\det(H) = \det(F_1) \cdot \det(G_2)$*, and* H *is Hermitian (resp. skew-Hermitian, normal, positive-definite, positive-semidefinite) if and only if* F_1 *and* G_2 *are Hermitian (resp. skew-Hermitian, normal, positive-definite, positive-semidefinite).*

If, in addition, H *is real symmetric, then (this is valid for both even and odd orders) we may choose* δ *orthonormal eigenvectors of* H *to be skew-symmetric and* $n - \delta$ *orthonormal eigenvectors of* H *to be symmetric.*

Fassbender and Ikramov [**9**] proposed an algorithm to compute Gx, where G is a centrosymmetric matrix and x is a vector. Most known results about centrosymmetric matrices (including Theorem 2.4 and Fassbender and Ikramov's algorithm) were derived using the fact that a centrosymmetric matrix is orthogonally similar to a block diagonal matrix via the following orthogonal matrices (the first is for even order and the second is for odd order):

$$Q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ I & J \end{bmatrix}, \quad Q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 & I \\ 0 & \sqrt{2} & 0 \\ -J & 0 & J \end{bmatrix},$$

where I and J are $\delta \times \delta$. (Note that the columns of Q_i, $i = 1, 2$, form an orthonormal basis for the eigenspace of J. We will show that Q_1 and Q_2 are not unique.) In this paper, we derive Theorem 2.4 and Fassbender and Ikramov's algorithm using a different approach.

The following theorem can be found in some of the references listed at the end.

Theorem 2.5. *Let* H *be an* $n \times n$ *nonsingular centrosymmetric matrix, where* n *is even, and let* H *be decomposed as in Lemma* 2.3*. Then*

$$H^{-1} = \frac{1}{2} \begin{bmatrix} V^{-1} + W^{-1} & (W^{-1} - V^{-1})J \\ J(W^{-1} - V^{-1}) & J(V^{-1} + W^{-1})J \end{bmatrix},$$

where $V = A - JC$ *and* $W = A + JC$*.*

The following lemma and theorem can be proved easily.

Lemma 2.6. *Let S be an $n\times n$ skew-centrosymmetric matrix. If n is even, then S can be written as*

$$S=\begin{bmatrix} A & -JCJ \\ C & -JAJ\end{bmatrix},$$

where A, J and C are $\delta\times\delta$. If n is odd, then S can be written as

$$S=\begin{bmatrix} A & z & -JCJ \\ y^T & 0 & -y^TJ \\ C & -Jz & -JAJ\end{bmatrix},$$

where A, J, and C are $\delta\times\delta$, and z and y are $\delta\times 1$.

Theorem 2.7. *Let S be an $n\times n$ nonsingular skew-centrosymmetric matrix, where n is even, and let S be decomposed as in Lemma 2.6. Then*

$$S^{-1}=\frac{1}{2}\begin{bmatrix} V^{-1}-W^{-1} & (V^{-1}+W^{-1})J \\ -J(V^{-1}+W^{-1}) & J(W^{-1}-V^{-1})J\end{bmatrix},$$

where $V=A+JC$ and $W=-A+JC$.

3 An alternative approach. First, we derive Theorem 2.4 using an alternative approach (this is perhaps the most important application of this approach because it shortens and simplifies the proof of Theorem 2.4). Then we derive Fassbender and Ikramov's algorithm using the same approach. Finally, we use the approach to derive an efficient algorithm to solve $Hx=b$, where H is centrosymmetric. The alternative approach is simply to replace a vector x by $x^+ + x^-$.

Let H be an $n\times n$ centrosymmetric matrix, where n is even, let H be decomposed as in Lemma 2.3, and let x be an $n\times 1$ vector. Then (λ, x) is an eigenpair of H if and only if $Hx^+ + Hx^- = \lambda x^+ + \lambda x^-$ and $Hx^+ - Hx^- = \lambda x^+ - \lambda x^-$ if and only if $Hx^+ = \lambda x^+$ and $Hx^- = \lambda x^-$. (Hence, if H has m linearly independent eigenvectors, then we can choose m linearly independent eigenvectors of H to be symmetric or skew-symmetric. Thus, there is no need to use the decomposition of H in Lemma 2.3 to reach this conclusion.) Now, since x^+ is symmetric, then it can be written as $x^+ = \left[\begin{smallmatrix} y \\ Jy\end{smallmatrix}\right]$, where y is $\delta\times 1$ and J is $\delta\times\delta$, and since x^- is skew-symmetric, then it can be written as $x^- = \left[\begin{smallmatrix} z \\ -Jz\end{smallmatrix}\right]$, where z is $\delta\times 1$ and J is $\delta\times\delta$. Thus, $Hx^+ = \lambda x^+$ if and only if

$$\begin{bmatrix} A & JCJ \\ C & JAJ\end{bmatrix}\begin{bmatrix} y \\ Jy\end{bmatrix}=\lambda\begin{bmatrix} y \\ Jy\end{bmatrix}$$

if and only if $(A+JC)y = \lambda y$. Similarly, $Hx^- = \lambda x^-$ if and only if

$$\begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix} \begin{bmatrix} z \\ -Jz \end{bmatrix} = \lambda \begin{bmatrix} z \\ -Jz \end{bmatrix}$$

if and only if $(A - JC)z = \lambda z$. This proves Theorem 2.4.

Now, we derive Fassbender and Ikramov's algorithm. Once again let H be an $n \times n$ centrosymmetric matrix, where n is even, let H be decomposed as in Lemma 2.3, and let x be an $n \times 1$ vector. And as before, let $x^+ = \left[\begin{smallmatrix} y \\ Jy \end{smallmatrix}\right]$ and $x^- = \left[\begin{smallmatrix} z \\ -Jz \end{smallmatrix}\right]$, where y and z are $\delta \times 1$ and J is $\delta \times \delta$. Then $Hx^+ = \left[\begin{smallmatrix} v \\ Jv \end{smallmatrix}\right]$, where $v = (A+JC)y$. Thus, if we find v by the traditional matrix-vector multiplication algorithm, then the time complexity of finding Hx^+ will be $\frac{3}{4}n^2 + O(n)$ (we need $\frac{n^2}{4}$ additions to find $A+JC$, and $\frac{n^2}{4}$ multiplications and $\frac{n^2}{4} - \frac{n}{2}$ additions to find v). Similarly, $Hx^- = \left[\begin{smallmatrix} w \\ -Jw \end{smallmatrix}\right]$, where $w = (A - JC)z$. If we find w by the traditional matrix-vector multiplication algorithm, then the time complexity of finding Hx^- will be $\frac{3}{4}n^2 + O(n)$. Thus, to find Hx, find v and w, then $v+w$ and $v-w$ (note that $Hx = \left[\begin{smallmatrix} v+w \\ J(v-w) \end{smallmatrix}\right]$). The time complexity of multiplying H by x using our method is $\frac{3}{2}n^2 + O(n)$. (If, in addition, H is symmetric, then the time complexity will be $\frac{5}{4}n^2 + O(n)$.) If $A+JC$ and $A-JC$ are stored and if r is an $n \times 1$ vector, then finding Hr or $H^T r$ by this method will cost $n^2 + O(n)$.

Remark To multiply a symmetric centrosymmetric matrix H by a vector x, Melman [**21**] replaced x by $x^+ + x^-$. But, his algorithm is different than the one we presented.

Now we describe an efficient method to solve the system $Hx = b$ (for x), where H is an $n \times n$ centrosymmetric matrix, x and b are $n \times 1$ vectors, and n is even. Now, let H be decomposed as in Lemma 2.3, and let

$$x^+ = \begin{bmatrix} y \\ Jy \end{bmatrix}, \quad x^- = \begin{bmatrix} z \\ -Jz \end{bmatrix}, \quad b^+ = \begin{bmatrix} d \\ Jd \end{bmatrix}, \quad b^- = \begin{bmatrix} e \\ -Je \end{bmatrix},$$

where y, z, d, and e, are $\delta \times 1$. First, note that $Hx = b$ if and only if $Hx^+ = b^+$ and $Hx^- = b^-$ if and only if $(A+JC)y = d$ and $(A-JC)z = e$. Therefore, to solve $Hx = b$ for x, solve instead $(A+JC)y = d$ for y and $(A-JC)z = e$ for z. Thus, instead of solving an $n \times n$ system, we end up solving two systems half the size. This results in a significant reduction in the time complexity. For example, if the original system is solved by Gaussian elimination, then the time complexity will be $\frac{2}{3}n^3 + O(n^2)$, while if Gaussian elimination is used to solve the two systems $(A + JC)y = d$ and $(A - JC)z = e$, then the time complexity of our method will be $\frac{1}{6}n^3 + O(n^2)$.

4 Block-diagonalization of centrosymmetric matrices. In this section, we present a class $\mathcal{N}_1$ of $n \times n$ orthogonal matrices such that if $Q \in \mathcal{N}_1$ and H is an $n \times n$ centrosymmetric matrix, then $Q^T HQ$ is block-diagonal, where n is even. We present a similar class $\mathcal{N}_2$ for odd n. It will be easy to see that $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) contains more than one element and it contains the matrix Q_1^T (resp. Q_2) defined in Section 2. Thus, although Q_1^T and Q_2 (see Section 2), or transformations of them, are the only orthogonal matrices used by researchers to block-diagonalize centrosymmetric matrices, they are not unique and they can be replaced by others. Now, we find $\mathcal{N}_1$. So, let n be even, let H be an $n \times n$ centrosymmetric matrix, let H be decomposed as in Lemma 2.3, let Q be an $n \times n$ orthogonal matrix such that $Q^T HQ = D$, where D is block-diagonal, and let

$$Q = \begin{bmatrix} \alpha & \beta \\ \gamma & \zeta \end{bmatrix},$$

where α, β, γ, and ζ, are $\delta \times \delta$ matrices. Thus, Q must satisfy

$$\alpha^T A\beta + \gamma^T C\beta + \alpha^T JCJ\zeta + \gamma^T JAJ\zeta = 0,$$

and

$$\beta^T A\alpha + \zeta^T C\alpha + \beta^T JCJ\gamma + \zeta^T JAJ\gamma = 0.$$

It is clear that if we choose $\alpha = -J\gamma$ and $\beta = J\zeta$, then the above two equations will be satisfied. Thus, we have the following lemma.

Lemma 4.1. *Let n be even, let H be an $n \times n$ centrosymmetric matrix, let H be decomposed as in Lemma* 2.3, *and let*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix},$$

where γ, ζ, and J, are $\delta \times \delta$ matrices. Then

$$Q^T HQ = 2\begin{bmatrix} \gamma^T (JAJ - CJ)\gamma & 0 \\ 0 & \zeta^T (JAJ + CJ)\zeta \end{bmatrix}.$$

Now, we need Q to be orthogonal. It is easy to see that if γ and ζ are invertible, then so is Q, and

$$Q^{-1} = \frac{1}{2}\begin{bmatrix} -\gamma^{-1}J & \gamma^{-1} \\ \zeta^{-1}J & \zeta^{-1} \end{bmatrix}.$$

Thus, we have the following lemma.

Lemma 4.2. *Let n be even and let*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix},$$

where γ, ζ, and J, are $\delta \times \delta$ matrices. If γ and ζ are invertible, and $\gamma^{-1} = 2\gamma^T$ and $\zeta^{-1} = 2\zeta^T$, then Q is orthogonal.

Now, we are ready to present the class $\mathcal{N}_1$.

Theorem 4.3. *Let n be even, let γ and ζ be invertible $\delta \times \delta$ matrices such that $\gamma^{-1} = 2\gamma^T$ and $\zeta^{-1} = 2\zeta^T$, let H be an $n \times n$ centrosymmetric matrix, and let H be decomposed as in Lemma* 2.3. *Then*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix}$$

is orthogonal and

$$Q^T H Q = 2\begin{bmatrix} \gamma^T(JAJ - CJ)\gamma & 0 \\ 0 & \zeta^T(JAJ + CJ)\zeta \end{bmatrix}.$$

Note that the columns of Q form δ linearly independent skew-symmetric vectors and δ linearly independent symmetric vectors, i.e., they form n linearly independent eigenvectors of J. Note also that $Q_1^T \in \mathcal{N}_1$ ($\gamma = -\frac{1}{\sqrt{2}}J$ and $\zeta = \frac{1}{\sqrt{2}}J$).

Lemma 4.4. *Let n be even, let R be an $n \times n$ matrix, and let*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix},$$

where γ and ζ are invertible $\delta \times \delta$ matrices and J is $\delta \times \delta$. If $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$, where D_1 and D_2 are $\delta \times \delta$ matrices, then R is centrosymmetric.

Proof. Write $R = \left[\begin{smallmatrix} R_1 & R_2 \\ R_3 & R_4 \end{smallmatrix}\right]$, where R_i, $i = 1, \dots, 4$, are $\delta \times \delta$. Then $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$ if and only if

$$\begin{aligned} R_3 J - J R_1 J + R_4 - J R_2 &= 0, \\ -R_3 J - J R_1 J + R_4 + J R_2 &= 0. \end{aligned}$$

Thus, $R_4 = J R_1 J$ and $R_2 = J R_3 J$. □

Theorem 4.5. *Let n be even, let R be an $n \times n$ matrix, and let $Q \in \mathcal{N}_1$. Then R is centrosymmetric if and only if $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$, where D_1 and D_2 are $\delta \times \delta$.*

We have similar results for skew-centrosymmetric matrices.

Theorem 4.6. *Let n be even, let γ and ζ be invertible $\delta \times \delta$ matrices such that $\gamma^{-1} = 2\gamma^T$ and $\zeta^{-1} = 2\zeta^T$, let S be an $n \times n$ skew-centrosymmetric matrix, and let S be decomposed as in Lemma* 2.6. *Then*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix}$$

is orthogonal and

$$Q^T S Q = 2 \begin{bmatrix} 0 & \gamma^T (CJ - JAJ)\zeta \\ -\zeta^T (JAJ + CJ)\gamma & 0 \end{bmatrix}.$$

Lemma 4.7. *Let n be even, let R be an $n \times n$ matrix, and let*

$$Q = \begin{bmatrix} -J\gamma & J\zeta \\ \gamma & \zeta \end{bmatrix},$$

where γ and ζ are invertible $\delta \times \delta$ matrices and J is $\delta \times \delta$. Then if $Q^T R Q = \left[\begin{smallmatrix} 0 & D_1 \\ D_2 & 0 \end{smallmatrix}\right]$, where D_1 and D_2 are $\delta \times \delta$ matrices, then R is skew-centrosymmetric.

Theorem 4.8. *Let n be even, let R be an $n \times n$ matrix, and let $Q \in \mathcal{N}_1$. Then R is skew-centrosymmetric if and only if $Q^T R Q = \left[\begin{smallmatrix} 0 & D_1 \\ D_2 & 0 \end{smallmatrix}\right]$, where D_1 and D_2 are $\delta \times \delta$.*

Similarly, we can present the class $\mathcal{N}_2$.

Theorem 4.9. *Let n be odd, let γ and ζ be invertible $\delta \times \delta$ matrices such that $\gamma^{-1} = 2\gamma^T$ and $\zeta^{-1} = 2\zeta^T$, let $k = \pm 1$, let H be an $n \times n$ centrosymmetric matrix, and let H be decomposed as in Lemma* 2.3. *Then*

$$Q = \begin{bmatrix} -J\gamma & 0 & J\zeta \\ 0 & k & 0 \\ \gamma & 0 & \zeta \end{bmatrix}$$

is orthogonal and

$$Q^T H Q = 2 \begin{bmatrix} \gamma^T (JA - C) J\gamma & 0 & 0 \\ 0 & \frac{1}{2} k^2 q & k y^T J\zeta \\ 0 & k\zeta^T J x & \zeta^T (JA + C) J\zeta \end{bmatrix}.$$

Note that the columns of Q form δ linearly independent skew-symmetric vectors and $n - \delta$ linearly independent symmetric vectors, i.e., they form n linearly independent eigenvectors of J. Note also that $Q_2 \in \mathcal{N}_2$ ($\gamma = -\frac{1}{\sqrt{2}} J$, $k = 1$, and $\zeta = \frac{1}{\sqrt{2}} J$).

Lemma 4.10. *Let n be odd, let $\omega = n - \delta$, let R be an $n \times n$ matrix, and let*

$$Q = \begin{bmatrix} -J\gamma & 0 & J\zeta \\ 0 & k & 0 \\ \gamma & 0 & \zeta \end{bmatrix}$$

where γ and ζ are invertible $\delta \times \delta$ matrices, k is a nonzero number, and J is $\delta \times \delta$. If $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$, where D_1 is $\delta \times \delta$ and D_2 is $\omega \times \omega$, then R is centrosymmetric.

Proof. Write $R = \left[\begin{smallmatrix} R_1 & x_1 & R_2 \\ x_2^T & q & x_3^T \\ R_3 & x_4 & R_4 \end{smallmatrix}\right]$, where R_i, $i = 1, \ldots, 4$, are $\delta \times \delta$, and x_i, $i = 1, \ldots, 4$, are $\delta \times 1$. Then $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$ if and only if

$$-\gamma^T J R_1 J\zeta + \gamma^T R_3 J\zeta - \gamma^T J R_2 \zeta + \gamma^T R_4 \zeta = 0,$$

$$-k\gamma^T J x_1 + \gamma^T x_4 = 0,$$

$$-k x_2^T J\gamma + k x_3^T \gamma = 0,$$

$$-\zeta^T J R_1 J\gamma - \zeta^T R_3 J\gamma + \zeta^T J R_2 \gamma + \zeta^T R_4 \gamma = 0.$$

Thus, $x_4 = Jx_1$, $x_3 = Jx_2$, $R_4 = JR_1J$ and $R_2 = JR_3J$. □

Theorem 4.11. *Let n be odd, let $\omega = n - \delta$, let R be an $n \times n$ matrix, and let $Q \in \mathcal{N}_2$. Then R is centrosymmetric if and only if $Q^T R Q = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$, where D_1 is $\delta \times \delta$ and D_2 is $\omega \times \omega$.*

5 Properties of centrosymmetric and skew-centrosymmetric matrices. In this section, we reveal new properties of centrosymmetric and skew-centrosymmetric matrices. Here, Q_1 and Q_2 refer to the orthogonal matrices defined in Section 2.

First, it is easy to prove that if H is an $n \times n$ centrosymmetric matrix, c is an $n \times 1$ symmetric vector, and s is an $n \times 1$ skew-symmetric vector, then $c^*Hs = 0$ and $s^*Hc = 0$. (If S is an $n \times n$ skew-centrosymmetric matrix, then $s^*Ss = 0$ and $c^*Sc = 0$. Hence, if (λ, x) is an eigenpair of S and $\lambda \neq 0$, then x cannot be symmetric or skew-symmetric.) Thus, if x is an $n \times 1$ vector and H is an $n \times n$ centrosymmetric matrix, then $x^*Hx = {x^+}^*Hx^+ + {x^-}^*Hx^-$.

Now, let M be an $n \times n$ matrix and let $M_c = \frac{1}{2}(M + JMJ)$ be the centrosymmetric part of M and $M_{sc} = \frac{1}{2}(M - JMJ)$ be the skew-centrosymmetric part of M. Then $Mx = \lambda x$ if and only if

$$M_c x^+ + M_c x^- + M_{sc} x^+ + M_{sc} x^- = \lambda x^+ + \lambda x^-$$

and

$$M_c x^+ - M_c x^- - M_{sc} x^+ + M_{sc} x^- = \lambda x^+ - \lambda x^-.$$

Thus, we have the following theorem:

Theorem 5.1. *Let M be an $n \times n$ matrix, let M_c and M_{sc} be as above, and let (λ, x) be an eigenpair of M. Then*

(1) $(\lambda, x^+ - x^-)$ *is an eigenpair of* $M_c - M_{sc}$.
(2) *If x is symmetric, then (λ, x^+) is an eigenpair of M_c and $(0, x^+)$ is an eigenpair of M_{sc}.*
(3) *If x is skew-centrosymmetric, then (λ, x^-) is an eigenpair of M_c and $(0, x^-)$ is an eigenpair of M_{sc}.*
(4) *If M is skew-centrosymmetric and x is not symmetric, then (λ^2, x^-) is an eigenpair of M_{sc}^2.*

Note that the case when M is centrosymmetric was handled in Section 3. Now, one of the most known properties of centrosymmetric matrices is that their eigenvectors can be chosen to be symmetric or skew-symmetric. Such a property does not hold for skew-centrosymmetric matrices. In fact, if S is an $n \times n$ skew-centrosymmetric matrix and $(\lambda \neq 0, z)$ is an eigenpair of S, then z can not be symmetric or skew-symmetric. But if, in addition, S is skew-symmetric, then we have the following proposition.

Proposition 5.2. *Let S be an $n \times n$ real skew-symmetric skew-centro-sym-metric matrix and let $(\lambda \neq 0\,, x+iy)$ be an eigenpair of S, where x and y are real. Then x is symmetric (resp. skew-symmetric) if and only if y is skew-symmetric (resp. symmetric).*

Theorem 5.3. *Let S be an $n \times n$ skew-centrosymmetric matrix, where n is even, let S be decomposed as in Lemma 2.6, and let $L = A - JC$ and $M = A + JC$. Then*

(1) *S is unitary (resp. orthogonal) if and only if L and M are unitary (resp. orthogonal).*
(2) *S is idempotent if and only if $S = 0$.*
(3) *S is symmetric if and only if $M^T = L$.*
(4) *S is skew-symmetric if and only if $M^T = -L$.*
(5) *S is normal if and only if $LL^* = M^*M$ and $L^*L = MM^*$.*
(6) *S is involutory if and only if $M^{-1} = L$.*
(7) $\|S\|_2 = \max\{\|L\|_2, \|M\|_2\}$,

$$\|S\|_\infty = \left\| \begin{bmatrix} A & JCJ \end{bmatrix} \right\|_\infty, \text{ and } \|S\|_1 = \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|_1.$$

Proof. It suffices to prove the first three parts.

(1)

$$Q_1^T S Q_1 = \begin{bmatrix} 0 & L \\ M & 0 \end{bmatrix}.$$

Thus,

$$S^{-1} = Q_1 \begin{bmatrix} 0 & M^{-1} \\ L^{-1} & 0 \end{bmatrix} Q_1^T$$

and

$$S^* = Q_1 \begin{bmatrix} 0 & M^* \\ L^* & 0 \end{bmatrix} Q_1^T.$$

(2) Note that S is skew-centrosymmetric while S^2 is centrosymmetric and note also that if a matrix P is centrosymmetric and skew-centrosymmetric, then $P = 0$.

(3) $S^T = S$ if and only if $Q_1 \left[\begin{smallmatrix} 0 & M^T \\ L^T & 0 \end{smallmatrix}\right] Q_1^T = Q_1 \left[\begin{smallmatrix} 0 & L \\ M & 0 \end{smallmatrix}\right] Q_1^T$. □

With the same notation as the previous theorem, note that if n is even, then $M^T = L$ if and only if A is symmetric and $JCJ = -C^T$, and $M^T = -L$ if and only if A is skew-symmetric and C is persymmetric.

Theorem 5.4. *Let S be an $n \times n$ skew-centrosymmetric matrix, where n is odd, let S be decomposed as in Lemma 2.6, and let $L = A - JC$ and $M = A + JC$. Then*

(1) *S is idempotent if and only if $S = 0$.*
(2) *S is symmetric if and only if $M^T = L$ and $z = y$.*
(3) *S is skew-symmetric if and only if $M^T = -L$ and $z = -y$.*
(4) *S is normal if and only if $2zz^* + LL^* = M^*M + 2\overline{y}y^T$, $y^T\overline{y} = z^*z$, $M\overline{y} = L^*z$, and $L^*L = MM^*$.*
(5)

$$\|S\|_\infty = \max\left\{ \left\| \begin{bmatrix} A & z & JCJ \end{bmatrix} \right\|_\infty,\ 2\|y\|_1 \right\},\ \textit{and}$$

$$\|S\|_1 = \max\left\{ \left\| \begin{bmatrix} A \\ y^T \\ C \end{bmatrix} \right\|_1,\ 2\|z\|_1 \right\}.$$

With the same notation as the previous theorem, note that if n is odd, then $M^T = L$ if and only if A is symmetric and $JCJ = -C^T$, and $M^T = -L$ if and only if A is skew-symmetric and C is persymmetric.

Theorem 5.5. *Let H be an $n \times n$ centrosymmetric matrix, let H be decomposed as in Lemma 2.3, let $L = A - JC$, and let $M = A + JC$ if n is even and $M = \begin{bmatrix} q & \sqrt{2}y^T \\ \sqrt{2}x & A + JC \end{bmatrix}$ if n is odd. Then*

(1) *H is unitary (resp. orthogonal) if and only if L and M are unitary (resp. orthogonal).*
(2) *H is idempotent (resp. nilpotent, involutory) if and only if L and M are idempotent (resp. nilpotent, involutory).*
(3) $\|H\|_2 = \max\{\|L\|_2, \|M\|_2\}$.
(4) *If n is even, then $\|H\|_\infty = \left\| \begin{bmatrix} A & JCJ \end{bmatrix} \right\|_\infty$, and $\|H\|_1 = \left\| \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \right\|_1$.*
(5) *If n is odd, then $\|H\|_\infty = \max\left\{ \left\| \begin{bmatrix} A & x & JCJ \end{bmatrix} \right\|_\infty,\ 2\|y\|_1 + |q| \right\}$, and $\|H\|_1 = \max\left\{ \left\| \left[\begin{smallmatrix} A \\ y^T \\ C \end{smallmatrix}\right] \right\|_1,\ 2\|x\|_1 + |q| \right\}$.*

Proof. It suffices to prove the first part. Let $P = Q_1$ if n is even and $P = Q_2$ if n is odd. Then

$$P^T H P = \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}.$$

Thus,

$$H^{-1} = P \begin{bmatrix} L^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} P^T$$

and

$$H^* = P \begin{bmatrix} L^* & 0 \\ 0 & M^* \end{bmatrix} P^T. \qquad \square$$

With the same notation as the previous theorem, note that if n is even, then H is symmetric if and only if L and M are symmetric if and only if A is symmetric and C is persymmetric, and H is skew-symmetric if and only if L and M are skew-symmetric if and only if A is skew-symmetric and $JCJ = -C^T$. If n is odd, then H is symmetric if and only if L and M are symmetric if and only if A is symmetric, $y = x$, and C is persymmetric, and H is skew-symmetric if and only if L and M are skew-symmetric if and only if A is skew-symmetric, $q = 0$, $y = -x$, and $JCJ = -C^T$.

Remarks (1) The previous theorems can be proved using any member of class $\mathcal{N}_1$ (see the previous section) instead of Q_1 and any member of class $\mathcal{N}_2$ instead of Q_2.

(2) Using the previous theorems to check if a centrosymmetric or a skew-centrosymmetric matrix is unitary, orthogonal, idempotent, etc., results in a significant reduction in the time complexity.

Proposition 5.6. *Let H_1 and H_2 be two $n \times n$ matrices. If n is even, let*

$$H_i = \begin{bmatrix} A_i & JC_iJ \\ C_i & JA_iJ \end{bmatrix}, \quad i = 1, 2,$$

where A_i, C_i, $i = 1, 2$, and J, are $\delta \times \delta$, and if n is odd, let

$$H_i = \begin{bmatrix} A_i & x_i & JC_iJ \\ y_i^T & q_i & y_i^T J \\ C_i & Jx_i & JA_iJ \end{bmatrix}, \quad i = 1, 2,$$

where: A_i, C_i, $i = 1, 2$, and J, are $\delta \times \delta$; x_i and y_i, $i = 1, 2$, are $\delta \times 1$; and q_i, $i = 1, 2$, are numbers. Let $L_i = A_i - JC_i$, $i = 1, 2$, and $M_i = A_i + JC_i$, $i = 1, 2$, if n is even, and $M_i = \left[\begin{smallmatrix} q_i & \sqrt{2}y_i^T \\ \sqrt{2}x_i & A_i + JC_i \end{smallmatrix}\right]$, if n is odd. Then H_1 and H_2 commute if and only if $L_1L_2 = L_2L_1$ and $M_1M_2 = M_2M_1$.

More properties (such as singular values) of centrosymmetric and skew-centrosymmetric matrices and regular magic squares are mentioned in [**3**].

6 Orthogonal transformations between centrosymmetric and skew-centrosymmetric matrices. In this section, we present the class $\mathcal{L}$ of even order orthogonal matrices such that if $Q \in \mathcal{L}$ and H is centrosymmetric (resp. skew-centrosymmetric) of even order, then QH is skew-centrosymmetric (resp. centrosymmetric).

Theorem 6.1. *Let n be even and let $\mathcal{L}$ be the class of $n \times n$ orthogonal matrices such that if $Q \in \mathcal{L}$ and H is an $n \times n$ centrosymmetric (resp. skew-centrosymmetric) matrix, then QH is skew-centrosymmetric (resp. centrosymmetric). Then $Q \in \mathcal{L}$ if and only if Q is an $n \times n$ orthogonal skew-centrosymmetric matrix.*

Proof. Let n be even. It is clear that if Q is an $n \times n$ orthogonal skew-centrosymmetric matrix and H is an $n \times n$ centrosymmetric (resp. skew-centrosymmetric) matrix, then QH is skew-centrosymmetric (resp. centrosymmetric). Conversely, if H is an $n \times n$ centrosymmetric (resp. skew-centrosymmetric) matrix and Q is an $n \times n$ orthogonal matrix such that QH is skew-centrosymmetric (resp. centrosymmetric), then $(JQJ + Q)H = 0$. This must hold for every $n \times n$ centrosymmetric (resp. skew-centrosymmetric) H. Now, choose H to be nonsingular to get $JQJ + Q = 0$. Thus, Q must be skew-centrosymmetric. $\square$

Corollary 6.2. *With the same notation as the previous theorem, $Q \in \mathcal{L}$ if and only if*

$$Q = \begin{bmatrix} \alpha & -J\gamma J \\ \gamma & -J\alpha J \end{bmatrix},$$

where α and γ are $\delta \times \delta$ matrices such that $\alpha - J\gamma$ and $\alpha + J\gamma$ are orthogonal.

Now let n be a positive integer. It is clear that $E = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$, where I is $\delta \times \delta$, is a member of $\mathcal{L}$ (note that $E^{-1} = E^T = E$). Similarly, it is easy to see that the following matrices are in $\mathcal{L}$

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -J \\ J & 0 \end{bmatrix}, \quad \begin{bmatrix} -J & 0 \\ 0 & J \end{bmatrix},$$

where I and J are $\delta \times \delta$.

The above transformations (e.g., E) are very useful. For example, we can use E with Theorem 2.5 to prove Theorem 2.7 and vice versa. Also, we can transform every skew-centrosymmetric singular value/determinant problem of even order to a centrosymmetric singular value/determinant problem of even order and vice versa. Moreover, we can transform every linear system in which the matrix of coefficients is centrosymmetric of even order to a linear system in which the matrix of coefficients is skew-centrosymmetric of even order, and vice versa.

Now, note that skew-centrosymmetric matrices of odd order are singular, while centrosymmetric matrices of odd order can be nonsingular. Thus, if H is an $n \times n$ nonsingular centrosymmetric matrix, where n is odd, and Q is an $n \times n$ orthogonal matrix such that QH is skew-centrosymmetric, then

$$0 = \det(QH) = \det(Q) \cdot \det(H) \neq 0,$$

which is a contradiction. Thus, no such Q exists. Moreover, if Q and H are $n \times n$ matrices, where H is centrosymmetric, such that QH is skew-centrosymmetric, then $(JQJ+Q)H = 0$. Thus, if H is nonsingular, then Q must be skew-centrosymmetric, and so if n is odd, then Q is singular, and hence, it can not be orthogonal. Therefore, there is no similar class (to $\mathcal{L}$) for centrosymmetric matrices of odd order.

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