TURBULENCE THEORIES

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Introduction

Turbulence has initially been defined as an irregular motion in fluids. The cloud formations in the atmosphere and the motion of water in rivers make this point clear. These are but a few readily available examples of a multitude of flows which display turbulent regimes. From the blood that flows in our veins and arteries to the motion of air within our lungs and around us. From the flow of water in creeks to the atmospheric and oceanic currents. From the flows past submarines, ships, automobiles, and aircrafts to the combustion processes propelling them. In the flow of gas, oil, and water, from the prospecting end to the entrails of the cities. The great majority of flows in nature and in engineering applications are somehow turbulent.

But turbulent flows are much more than simply irregular. More refined definitions were desirable and were later coined. A definitive and precise one, however, may only come when the phenomenon is fully understood. Nevertheless, several characteristic properties of a turbulent flow can be listed:

Irregularity and unpredictability: A turbulent flow is irregular both in space and time, displaying unpredictable, random patterns.

Statistical order: From the irregularity of a turbulent motion there emerges a certain statistical order. Mean quantities and correlation are regular and predictable (Figure 1).

Wide range of active scales: A wide range of scales of motion are active and display an irregular motion, yielding a large number of degrees of freedom.

Mixing and enhanced diffusivity: The fluid particles undergo complicated and convoluted paths, causing a large mixing of different parts of fluid. This mixing significantly enhances diffusion, increasing the transport of momentum, energy, heat, and other advected quantities.

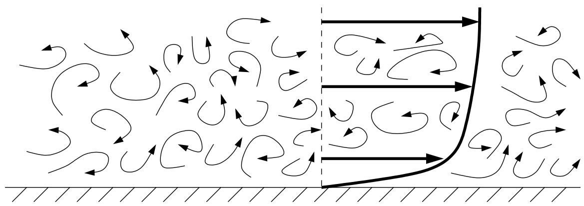

FIGURE 1. Illustration of the irregular motion of a turbulent flow over a flat plate (thin lines), and of the well-defined velocity profile of the mean flow (thick lines).

Vortex stretching: When a moving portion of fluid also rotates transversally to its motion an increase in speed causes it to rotate faster, a phenomenon called vortex stretching. This causes that portion of fluid to become thinner and elongated, and fold and intertwine with other such portions. This is an intrinsically three-dimensional mechanism which plays a fundamental role in turbulence and is associated with large fluctuations in the vorticity field.

Turbulent regimes

Turbulence is studied from many perspectives. The subject of *transition to turbulence* attempts to describe the initial mechanisms responsible for the generation of turbulence starting from a laminar motion in particular geometries. This transition can be followed with respect to position in space (e.g. the flow becomes more complicated as we look further downstream on a flow past an obstacle or over a flat plate) or to parameters (e.g. as we increase the angle of attack of a wing or the pressure gradient in a pipe). This subject is divided into two cases: wall-bounded and free-shear flows. In the former, the viscosity, which causes the fluid to adhere to the surface of the wall, is the primary cause of the instability in the transition process. In the latter, inviscid mechanisms such as mixing layers and jets are the main factors. The tools for studying the transition to turbulence include linearization of the equations of motion around the laminar solution, nonlinear amplitude equations, and bifurcation theory.

Fully-developed turbulence, on the other hand, concerns turbulence which evolves without imposed constraints, such as boundaries and external forces. This can be thought of turbulence in its "pure" form, and it is somewhat a theoretical framework for research due to its idealized nature. Hypotheses of homogeneity (when the mean quantities associated with the statistical order characterizing a turbulent flow are independent in space), stationarity (idem in time), and isotropy (idem with respect to rotations in space) concern fully-developed turbulent flows. The Kolmogorov theory was developed in this context and it is the most fundamental theory of turbulence. Current research is dedicated in great part to unveil the mechanisms behind a phenomenon called intermittency and how it affects the laws obtained from the conventional theory. Research is also dedicated to derive such laws as much from first principles as possible, minimizing the use of phenomenological and dimensional analysis.

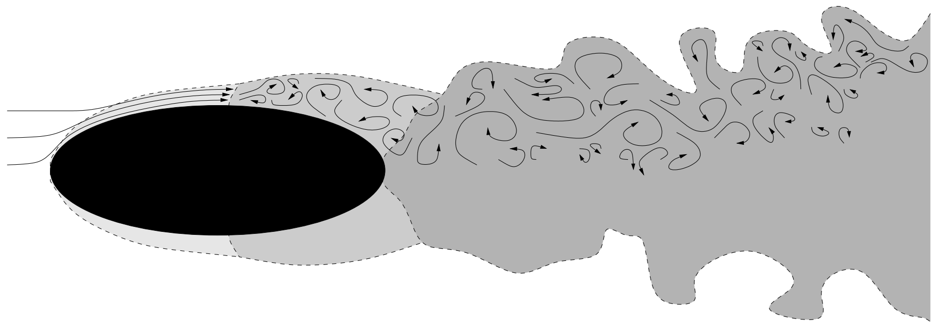

FIGURE 2. Illustration of a flow past an object, with a laminar boundary layer (light gray), a turbulent boundary layer (medium gray), and a turbulent wake (dark gray).

Real turbulent flows involve various regimes at once. A typical flow past a blunt object, for instance, displays laminar motion at its upstream edge, a turbulent boundary layer further downstream, and the formation of a turbulent wake (Figure 2). The subject of turbulent boundary layer is a world in itself with current research aiming to determine mean properties of flows over rough surfaces and varied topography. Convective turbulence involves coupling with active scalars such as large heat gradients, occurring in the atmosphere, and large salinity gradients, in the ocean. Geophysical turbulence involves also stratification and the anisotropy generated by the Earth rotation. Anisotropic turbulence is also crucial in astrophysics and plasma theory. Multiphase and multicomponent turbulence appear in flows with suspended particles or bubbles and in mixtures such as gas, water, and oil. Transonic and supersonic flows are also of great importance and fall into the category of compressible turbulence, much less explored than the incompressible case.

In all those real situations one would like, from the engineering point of view, to compute mean properties of the flow, such as drag and lift for more efficient designs of aircrafts, ships, and other vehicles. Knowledge of the drag coefficient is also of fundamental importance in the design of pipes and pumps, from pipelines to artificial human organs. Mean turbulent diffusion coefficients of heat and other passive scalars – quantities advected by the flow without interfering on it, such as chemical products, nutrients, moisture, and pollutants – are also of major importance in industry, ecology, meteorology, and climatology, for instance. And in most of those cases a large amount of research is dedicated to the *control of turbulence*, either to increase mixing or reduce drag, for instance. From a theoretical point of view, one would like to fully understand and characterize the mechanisms involved in turbulent flows, clarifying this fascinating phenomenon. This could also improve practical applications and lead to a better control of turbulence.

The concept of *two-dimensional turbulence* is controversial. A two-dimensional flow may be irregular and display mixing, statistical order, and a wide range of active scales but definitely it does not involve vortex stretching since the velocity field is always perpendicular to the vorticity field. For this reason many researchers discard two-dimensional turbulence altogether. It is also argued that real two-dimensional flows are unstable at complicated regimes and soon develop into a three-dimensional flow. Nevertheless, many believe that two-dimensional turbulence, even lacking vortex stretching, are of fundamental theoretical importance. It may shed some light into the three-dimensional theory and modeling, and it can serve as an approximation to some situations such as the motion of the atmosphere and oceans in the large and meso scales and some magneto-hydrodynamic flows. The relative shallowness of the atmosphere and oceans or the imposition of a strong uniform magnetic field may force the flow into two-dimensionality, at least for a certain range of scales.

Chaos serves as a paradigm for turbulence, in the sense that it is now accepted that turbulence is a dynamic processes in a sensitive deterministic system. But not all chaotic motions in fluids are termed turbulent for they may not display mixing and vortex stretching or involve a wide range of scales. An important such example appears in the dispersive, nonlinear interactions of waves.

The equations of motion

It is usually stressed that turbulence is a continuum phenomenon, in the sense that the active scales are much larger than the collision mean free path between molecules. For this reason, turbulence is believed to be fully accounted for by the Navier-Stokes equations.

In the case of incompressible homogeneous flows, the Navier-Stokes equations in the Eulerian form and in vector notation read

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}p = \mathbf{f}, \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{1b}$$

Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3)$ denotes the velocity vector of an idealized fluid particle located at position $\mathbf{x} = (x_1, x_2, x_3)$, at time t. The mass density in a homogeneous flow is constant, denoted ρ. The constant ν denotes the kinematic viscosity of the fluid, which is the molecular viscosity μ divided by ρ. The variable $p = p(\mathbf{x}, t)$ is the kinematic pressure, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1, f_2, f_3)$ denotes the mass density of volume forces.

Equation (1a) expresses the conservation of linear momentum. The term $\nu\Delta\mathbf{u}$ accounts for the dissipation of energy due to molecular viscosity, and the nonlinear term $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$, also called the *inertial term*, accounts for the redistribution of energy among different structures and scales of motion. Equation (1b) represents the incompressibility condition. In Einstein's summation convention these equations can be written as

$$\frac{\partial u_i}{\partial t} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i, \qquad \frac{\partial u_j}{\partial x_j} = 0.$$

The Reynolds number

The transition to turbulence was carefully studied by Reynolds in the late 19th century in a series of experiments in which water at rest in a tank was allowed to flow through a glass pipe. Starting with dimensional analysis Reynolds argued that it was likely to exist a critical value of a certain nondimensional quantity beyond which a laminar flow gives rise to a "sinuous" motion. This was followed by observations of the flow for tubes with different diameter L, different mean velocities U across the tube section, and with the kinematic viscosity $\nu = \rho/\mu$ being altered through changes in temperature. The experiments confirmed the existence of such a critical value for what is now called the *Reynolds number:*

$$\mathrm{Re} = \frac{LU}{\nu}.$$

The dimensional analysis argument can be reproduced in the following form: The physical dimension for the inertial term in (1a) is U^2/L, while that for the viscous term is $\nu U/L^2$. The ratio between them is precisely $\mathrm{Re} = LU/\nu$. For small values of Re viscosity dominates and the flow is laminar, while for large values of Re, the inertial term dominates, and the flow becomes more complicated and eventually turbulent. In applications, different types of Reynolds number can be used depending on the choice of the characteristic velocity and length, but in any case the larger the Reynolds number the more complicated the flow.

The Reynolds equations

Another advance put forward by Reynolds in a subsequent article was to decompose the flow into a mean component and the remaining fluctuations. In terms of the velocity and pressure fields this can be written as

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \qquad p = \bar{p} + p', \tag{2}$$

with $\bar{\mathbf{u}}$ and $\bar{p}$ representing the mean components and $\mathbf{u}'$ and p', the fluctuations. By substituting (2) into (1) one finds the *Reynolds-averaged Navier-Stokes* (RANS) equations for the mean flow:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} - \nu \Delta \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \boldsymbol{\nabla})\bar{\mathbf{u}} + \boldsymbol{\nabla}\bar{p} = \mathbf{f} + \boldsymbol{\nabla} \cdot \tau,$$
$$\boldsymbol{\nabla} \cdot \bar{\mathbf{u}} = 0.$$

It differs from (1) only by the addition of the *Reynolds stress tensor:*

$$\tau = -\overline{\mathbf{u}' \otimes \mathbf{u}'} = -\left(\overline{u_i' u_j'}\right)_{i,j=1}^{3}.$$

In a laminar flow, the fluctuations are negligible, otherwise this decomposition shows how they influence the mean flow through this additional turbulent stresses.

The closure problem and turbulence models

The RANS equations cannot be solved directly for the mean flow since the Reynolds stresses are unknown. Equations for these stress terms can be derived but they involve further unknown moments. This continues with equations for moments of a given order depending on new moments up to a higher order, leading to an infinite system of equations known as the *Friedman-Keller system.* For practical applications, approximations closing the system at some finite order are needed, in which is called the *closure problem*. Several ad-hoc approximations exists, the most famous being the Boussinesq *eddy-viscosity* approximation, in which the turbulent fluctuations are regarded as increasing the viscosity of the flow. Prandtl's *mixing-length* hypothesis yields a prescription for the computation of this eddy viscosity, and together they form the basis of the *algebraic models* of turbulence. Other models involve additional equations, such as the k-ϵ and k-ω models. Most of the practical computations of industrial flows are based on such lower-order models, and a large amount of research is done to determine appropriate values for the various ad-hoc parameters which appear in these models and which are highly dependent on the geometry of the flow. This dependency can be explained by the fact that the RANS is supposed to model the mean flow even at the large scales of motion, which are highly affected by the geometry.

Computational fluid dynamics (CFD) is indeed a fundamental tool in turbulence, both for research and engineering applications. From the theoretical side, *direct numerical simulations* (DNS), which attempt to resolve all the active scales of the flow, reveal some fundamental mechanisms involved in the transition to turbulence and in vortex-stretching. As for applications, DNS applies to flows up to low-Reynolds turbulence, with the current computational power not allowing for a full resolution of all the scales involved in high-Reynolds flows. And the current rate of evolution of computational power predicts that this will continue so for several decades.

An intermediate CFD method between RANS and DNS is the *large-eddy simulation* (LES), which attempts to fully resolve the large scales while modeling the turbulent motion at the smaller scales. Several models have been proposed which have their own advantages and limitations as compared to RANS and DNS. It is currently a subject of intense research, particularly for the development of suitable models for the structure functions near the boundary. Theoretical results on fully-developed turbulence play a fundamental role in the modeling process.

LES are a promising tool and they have been successfully applied to a number of situations. The choice of the best method for a given application, however, depends very much on the Reynolds number of the flow and the prior knowledge of similar situations for adjusting the parameters.

Elements of the statistical theory

Several types of averages can be used. The *ensemble average* is taken with respect to a number of experiments at nearly identical conditions. Despite the irregular motion of, say the velocity vector $\mathbf{u}^{(n)}(\mathbf{x},t)$ of each experiment $n=1,\ldots,N$, the average value

$$\bar{\mathbf{u}}(\mathbf{x},t)=\frac{1}{N}\sum_{n=1}^{N}\mathbf{u}^{(n)}(\mathbf{x},t)$$

is expected to behave in a more regular way. This type of averaging is usually denoted with the symbol $\langle\,\cdot\,\rangle$. This notion can be cast into the context of a probability space $(\mathcal{M},\Sigma,\mathcal{P})$, where $\mathcal{M}$ is a set, Σ is a σ-algebra of subsets of $\mathcal{M}$, and $\mathcal{P}$ is a probability measure on Σ. The velocity field is a random variable in the sense that it is a density function $\omega\mapsto\mathbf{u}(\mathbf{x},t,\omega)$ from $\mathcal{M}$ into the space of time-dependent divergence-free velocity fields. The mean velocity field in this context is regarded as

$$\langle\mathbf{u}(\mathbf{x},t)\rangle=\int_{\mathcal{M}}\mathbf{u}(\mathbf{x},t,\omega)\;\mathrm{d}\mathcal{P}(\omega).$$

Other flow quantities such as energy and correlations in space and time can be expressed by means of a function $\varphi=\varphi(\mathbf{u}(\cdot,\cdot))$ of the velocity field, with their mean value given by

$$\langle\varphi(\mathbf{u}(\cdot,\cdot))\rangle=\int_{\mathcal{M}}\varphi(\mathbf{u}(\cdot,\cdot,\omega)\;\mathrm{d}\mathcal{P}(\omega).$$

In general the statistics of the flow are allowed to change with time. A particular situation is when *statistical equilibrium* is reached, so that $\langle\mathbf{u}(\mathbf{x},t)\rangle$, and, more generally, $\langle\varphi(\mathbf{u}(\cdot,\cdot+t))\rangle$ are independent of t. In this case, an *ergodic assumption* is usually invoked, which means that for "most" individual flows $\mathbf{u}(\cdot,\cdot,\omega_0)$ (i.e. for almost all ω_0 with respect to the probability measure $\mathcal{P}$), the time averages along this flow converge to the mean ensemble value as the period of the average increases, to the mean value obtained by the ensemble average:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi(\mathbf{u}(\cdot,\cdot+s,\omega_0))\;\mathrm{d}s=\int_{\mathcal{M}}\varphi(\mathbf{u}(\cdot,\cdot,\omega))\;\mathrm{d}\mathcal{P}(\omega).$$

Based on this assumption, the averages may in practice be calculated as time averages over a sufficiently large period T. There is a related argument for substituting space averages by time averages and based on the mechanics of turbulence which is called the *Taylor hypothesis*.

Another fundamental concept in the statistical theory is that of homogeneity, which is the spatial analog of the statistical equilibrium in time. In *homogeneous turbulence* the statistical quantities of a flow are independent of translations in space, i.e.

$$\langle\varphi(\mathbf{u}(\cdot+\boldsymbol{\ell},\cdot)\rangle=\langle\varphi(\mathbf{u}(\cdot,\cdot)\rangle,$$

for all $\boldsymbol{\ell} \in \mathbb{R}^3$. The concept of *isotropic turbulence* assumes further independence with respect to rotations and reflections in the frame of reference, i.e.

$$\langle \varphi(Q^{\mathrm{t}}\mathbf{u}(Q\cdot,\cdot)\rangle = \langle \varphi(\mathbf{u}(\cdot,\cdot)\rangle,$$

for all orthogonal transformations Q in $\mathbb{R}^3$, with adjoint Q^{t}.

Under the homogeneity assumption, mean quantities can be defined independently of position in space, such as the *mean kinetic energy per unit mass*

$$e = \frac{1}{2}\langle |\mathbf{u}(\mathbf{x})|^2\rangle = \frac{1}{2}\sum_{j=1}^{3}\langle |u_i(\mathbf{x})|^2\rangle$$

and the *mean rate of viscous energy dissipation per unit mass and unit time*

$$\epsilon = \nu\sum_{i=1}^{3}\langle |\nabla u_i(\mathbf{x})|^2\rangle = \nu\sum_{i,j=1}^{3}\langle \left|\frac{\partial u_i(\mathbf{x})}{\partial x_j}\right|^2\rangle.$$

The mean kinetic energy can be written as $e = \operatorname{Tr} R(0)/2$, where $\operatorname{Tr} R(\boldsymbol{\ell})$ is the trace

$$\operatorname{Tr} R(\boldsymbol{\ell}) = R_{11}(\boldsymbol{\ell}) + R_{22}(\boldsymbol{\ell}) + R_{33}(\boldsymbol{\ell}), \qquad \boldsymbol{\ell} \in \mathbb{R}^3,$$

of the correlation tensor

$$R(\boldsymbol{\ell}) = \langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x}+\boldsymbol{\ell})\rangle = (R_{ij}(\boldsymbol{\ell}))_{i,j=1}^3 = (\langle u_i(\mathbf{x})u_j(\mathbf{x}+\boldsymbol{\ell})\rangle)_{i,j=1}^3,$$

which measures the correlation between the velocity components at different positions in space. From the homogeneity assumption, this tensor is a function only of the relative position $\boldsymbol{\ell}$. Then, assuming that the Fourier transform of $\operatorname{Tr} R(\boldsymbol{\ell})$ exists, and denoting it by $Q(\boldsymbol{\kappa})$, for $\boldsymbol{\kappa} \in \mathbb{R}^3$, we have

$$\operatorname{Tr} R(\boldsymbol{\ell}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} Q(\boldsymbol{\kappa})e^{i\boldsymbol{\ell}\cdot\boldsymbol{\kappa}}\,\mathrm{d}\boldsymbol{\kappa} = 2\int_0^\infty S(\kappa)e^{i\boldsymbol{\ell}\cdot\boldsymbol{\kappa}}\,\mathrm{d}\kappa,$$

where $\mathcal{S}(\kappa)$ is the *energy spectrum* defined by

$$\mathcal{S}(\kappa) = \frac{1}{2(2\pi)^{3/2}}\int_{|\boldsymbol{\kappa}|=\kappa} Q(\boldsymbol{\kappa})\,\mathrm{d}\Sigma(\boldsymbol{\kappa}), \qquad \forall\, \kappa > 0,$$

with $\mathrm{d}\Sigma(\boldsymbol{\kappa})$ denoting the area element of the 2-sphere of radius $|\boldsymbol{\kappa}|$. Then we can write

$$e = \frac{1}{2}\langle |\mathbf{u}(\mathbf{x})|^2\rangle = \frac{1}{2}\operatorname{Tr} R(0) = \int_0^\infty \mathcal{S}(\kappa)\,\mathrm{d}\kappa.$$

By expanding the velocity coordinates into Fourier modes $\exp(\boldsymbol{\ell}\cdot\boldsymbol{\kappa})$, with $\kappa \leq |\boldsymbol{\kappa}| \leq \kappa + \mathrm{d}\kappa$ and interpreting them as "eddies" with characteristic wavenumber $|\boldsymbol{\kappa}|$, the quantity $\mathcal{S}(\kappa)\mathrm{d}\kappa$ can be interpreted as the energy of the component of the flow formed by the "eddies" with characteristic wavenumber between κ and $\kappa + \mathrm{d}\kappa$.

Similarly,

$$\epsilon = 2\nu\int_0^\infty \kappa^2\mathcal{S}(\kappa)\,\mathrm{d}\kappa,$$

and we obtain the *dissipation spectrum* $2\nu\kappa^2\mathcal{S}(\kappa)$, which can be interpreted as the density of energy dissipation occurring at wavenumber κ.

In the previous arguments it is assumed that the flow extends to all the space $\mathbb{R}^3$. This avoids the presence of boundaries, addressing the idealized case of fully-developed turbulence. It is sometimes customary to assume as well that the flow is periodic in space, to avoid problems with unbounded domains such as infinite kinetic energy.

The random nature of turbulent flows was greatly explored by Taylor in the early 20th century, who introduced most of the concepts described above. Another important concept he introduced was the *Taylor microlength* ℓ_T, which is a characteristic length for the small scales based on the correlation tensor. A microscale Reynolds number based on the Taylor microlength is very often used in applications.

Kolmogorov theory

An inspiring concept in the theory of turbulence is Richardson's *energy cascade* process. For large Reynolds numbers the nonlinear term dominates the viscosity according to the dimensional analysis, but this is valid only for the large-scale structures. The small scales have their own characteristic length and velocity. In the cascade process, the inertial term is responsible for the transfer of energy to smaller and smaller scales until small enough scales are reached for which viscosity becomes important (Figure 3). At those smallest scales kinetic energy is finally dissipated into heat. It should be emphasized that turbulence is a dissipative process; no matter how large the Reynolds number is, viscosity plays a role in the smallest scales.

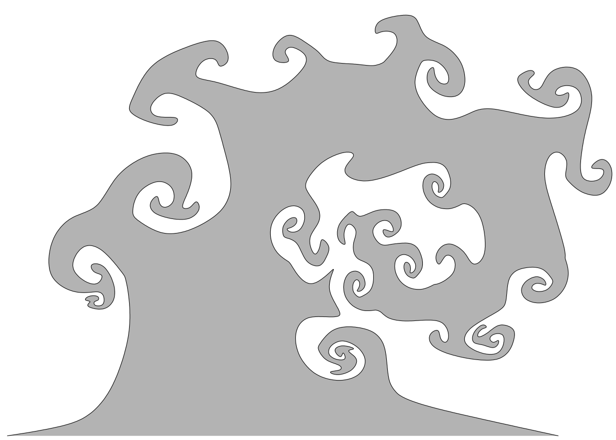

Figure 3. Illustration of the eddy breakdown process in which energy is transferred to smaller eddies and so on until the smallest scales are reached and the energy is dissipated by viscosity.

The Kolmogorov theory of *locally isotropic turbulence* allows for inhomogeneity and anisotropy in the large scales, which contain most of the energy, assuming that with the cascade transfer of energy to smaller scales, the orienting effects generated in the large scales become weaker and weaker so that for sufficiently small eddies the motion becomes statistically homogeneous, isotropic, and independent of the particular energy-productive mechanisms. He proposed that the statistical regime of the small-scale eddies is then universal and depends only on ν and ϵ. The *equilibrium range* is defined as the range of scales in which this universality holds.

Simple dimensional analysis shows that the only algebraic combination of ν and ϵ with dimension of length is $\ell_\epsilon = (\nu^3/\epsilon)^{1/4}$, which is then interpreted as that near which the viscous effect become important and hence most of the energy dissipation takes place. The scale ℓ_ϵ is known as *Kolmogorov dissipation length*.

Kolmogorov theory gives particular attention to moments involving differences of velocities, such as the *pth order structure function*

$$S_p(\ell) \stackrel{\text{def}}{=} \langle (\mathbf{u}(\mathbf{x} + \ell\mathbf{e}) \cdot \mathbf{e} - \mathbf{u}(\mathbf{x}) \cdot \mathbf{e})^p \rangle,$$

where $\mathbf{e}$ may be taken as an arbitrary unit vector thanks to the isotropy assumption. By restricting the search for universal laws for the structure functions only for small values of ℓ anisotropy and inhomogeneity are allowed in the large scales.

The theory assumes a wide separation between the energy-containing scales, of order say ℓ_0, and the energy-dissipative scales, of order ℓ_ϵ, so that the cascade process occurs within a wide range of scales ℓ such that $\ell_0 \gg \ell \gg \ell_\epsilon$. In this range, termed the *inertial range*, the viscous effects are still negligible and the statistical regime should depend only on ϵ. Then, the Kolmogorov *two-thirds law* asserts that within the inertial range the second-order correlations must be proportional to $(\epsilon\ell)^{2/3}$, i.e.

$$S_2(\ell) = C_K(\epsilon\ell)^{2/3},$$

for some constant C_K known as the *Kolmogorov constant* in physical space (there is a related constant in spectral space). The argument extends to higher-order structure functions, yielding

$$S_p(\ell) = C_p(\epsilon\ell)^{p/3}.$$

Kolmogorov's derivation of these results was not by dimensional analysis, it was in fact a more convincing self-similarity argument based on the universality assumed for the equilibrium range. A different and more sounded argument, however, was applied to the third-order structure function, yielding the more precise *four-fifths law:*

$$S_3(\ell) = -\frac{4}{5}\epsilon\ell.$$

The *Kolmogorov five-third law* concerns the energy spectrum $\mathcal{S}(\kappa)$ and is the spectral version of the two-third law, given by Obukhoff:

$$\mathcal{S}(\kappa) = C'_K \epsilon^{2/3} \kappa^{-5/3},$$

The constant C'_K is the Kolmogorov constant in spectral space. The spectral version of the dissipation length is the *Kolmogorov wavenumber* $\kappa_\epsilon = (\epsilon/\nu^3)^{1/4}$.

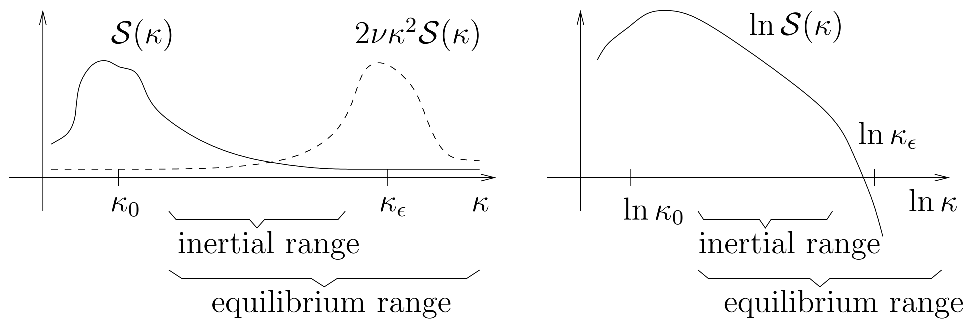

FIGURE 4. A typical distribution for the energy spectrum $\mathcal{S}(\kappa)$ and the dissipation spectrum $2\nu\kappa^2\mathcal{S}(\kappa)$ in spectral space in nonlogarithmic and logarithmic scales. The energy is mostly concentrated on the large scales while the dissipation is concentrated near the dissipation scale. In the logarithmic scale the four-fifths law for the energy spectrum stands out as a straight line with slope $-4/5$ over the inertial range.

A typical distribution of energy in a turbulent flow is depicted in Figure 4. The energy is concentrated on the large scales, while the dissipation is concentrated near the Kolmogorov scale ℓ_ϵ. The four-fifths law becomes visible as a straight line in the logarithmic scale.

A more precise mechanism for the energy cascade assumes that in the inertial range eddies with length scale ℓ transfer kinetic energy to smaller eddies during their characteristic time scale, also known as *circulation time*. If u_ℓ is their characteristic velocity, then $\tau_\ell = \ell/u_\ell$ is their circulation time, so that the kinetic energy transferred from these eddies during this time is

$$\epsilon_\ell \sim \frac{u_\ell^2}{\tau_\ell} = \frac{u_\ell^3}{\ell}.$$

In statistical equilibrium, the energy lost to the smaller scales equals the energy gained from the larger scales, and that should also equal the total kinetic energy dissipated by viscous effects. Hence, $\epsilon_\ell \equiv \epsilon$, and we find

$$\epsilon \sim \frac{u_\ell^3}{\ell}.$$

It also follows that $\tau_\ell = \ell/u_\ell = \ell(\epsilon\ell)^{-1/3} = \epsilon^{-1/3}\ell^{2/3}$ so that the circulation time decreases with the length scale and becomes of the order of the viscous dissipation time $(\nu/\epsilon)^{1/2}$ precisely when $\ell \sim \ell_\epsilon$.

A similar relation between ϵ and the large scales can also be obtained with heuristic arguments: Let e be the mean kinetic energy and ℓ_0, a characteristic length for the large scales. Then u_0 given by $e = u_0^2/2$ is a characteristic velocity for the large scales, and $\tau_0 = \ell_0/u_0$ is the large-scale circulation time. In statistical equilibrium the rate ϵ of kinetic energy dissipated per unit time and unit mass is expected to be of the order of e/τ_0, hence

$$\epsilon \sim \frac{u_0^3}{\ell_0},$$

which is called the *energy dissipation law.*

From the energy dissipation law several relations between characteristic quantities of turbulent flows can be obtained, such as $\ell_0/\ell_\epsilon \sim \mathrm{Re}^{3/4}$, for $\mathrm{Re} = \ell_0 u_0/\nu$.

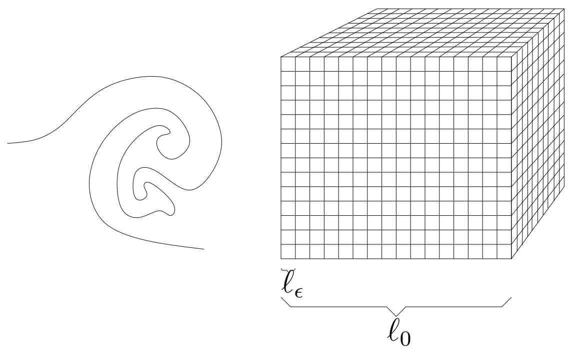

Figure 5. A schematic representation of a flow structure displaying a range of active scales and a three-dimensional grid with linear dimension ℓ_0 and mesh-length ℓ_ϵ, sufficient to represent all the active scales in a turbulent flow. The number of degrees of freedom is the number of blocks: $(\ell_0/\ell_\epsilon)^3$.

Now, assuming the active scales in a turbulent flow exist down to the Kolmogorov scale ℓ_ϵ, one needs a three-dimensional grid with mesh spacing ℓ_ϵ to resolve all the scales, which means that the number N of degrees of freedom of the system is of the order of $N \sim (\ell_0/\ell_\epsilon)^3$ (see Figure 5). This number can be estimated in terms of the Reynolds number by $N \sim \mathrm{Re}^{9/4}$. This relation is important in predicting the computational power needed to simulate all the active scales in turbulent flows.

Several such universal laws can be deduced and extended to other situations such as turbulent boundary layers, with the famous *logarithmic law* of the wall. They play a fundamental role in turbulence modeling and closure, for the calculation of the mean flow and other quantities.

Intermittency

The universality hypothesis based on a constant mean energy dissipation rate throughout the flow received some criticisms and was later modified by Kolmogorov in an attempt to account for observed large deviations on the mean rate of energy dissipation. Such phenomenon of *intermittency* is related to the vortex stretching and thinning mechanism, which leads to the formation of *coherent structures* of vortex filaments of high vorticity and low dissipation (Figure 6). These filaments have diameter as small as the Kolmogorov scale and longitudinal length extending from the

Taylor scale up to the large scales and with a lifetime of the order of the large-scale circulation time.

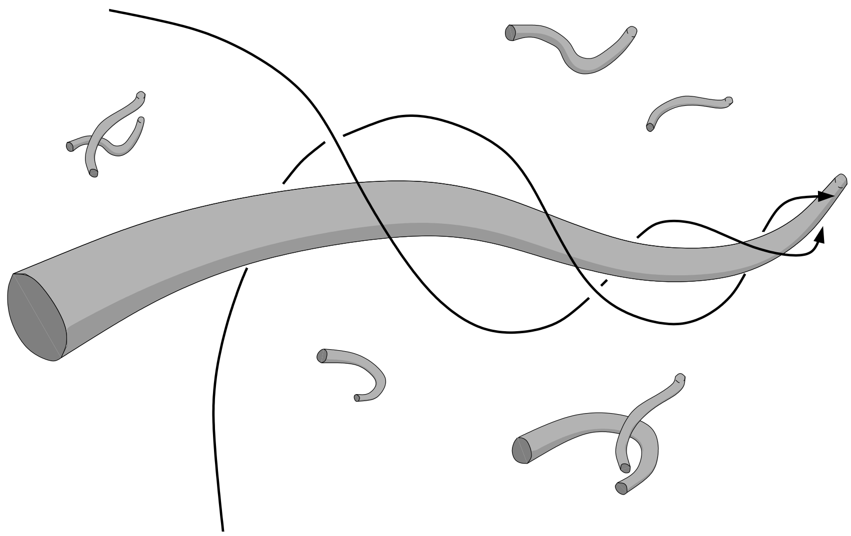

FIGURE 6. A portion of rotating fluid gets stretched and thinned as the flow speeds up, generating one of many coherent structures of high vorticity and low dissipation.

It has been argued based on experimental evidence that intermittency leads to modified power laws $S_p(\ell) \propto \ell^{\zeta(p)}$, $\zeta(p) < p/3$, for high-order ($p > 3$) structure functions. The issues of intermittency and coherent structures and whether and how they could affect the deductions of the universality theory such as the power laws for the structure functions are far from settled and are currently one of the major and most fascinating issues being addressed in turbulence theory. Several phenomenological theories attempt to adjust the universality theory to the existence of such coherent structures. *Multifractal models,* for instance, suppose that the eddies generated in the cascade process do not fill up the space and form multifractal structures. *Field-theoretic renormalization group* develops techniques based on quantum-field renormalization theory. *Intermediate-asymptotics* also exploits self-similar analysis and renormalization theory but with a somewhat different flavor. Detailed mathematical analysis of the vorticity equations are also playing a major role in the understanding of the dynamics of the vorticity field.

Mathematical aspects of turbulence theory

From a mathematical perspective it is fundamental to develop a rigorous background upon which to study the physical quantities of a turbulent flow. The first problem in the mathematical theory is related to the deterministic nature of chaotic systems assumed in dynamical system theory and believed to hold in turbulence.

This has actually not been proved for the Navier-Stokes equations. It is in fact one of the most outstanding open problems in mathematics to determine whether given an initial condition for the velocity field there exists, in some sense, a unique solution of the Navier-Stokes equations starting with this initial condition and valid for all later times. It has been proved that a global solution (i.e. valid for all later times) exist but which may not be unique, and it has been proved that unique solutions exist which may not be global (i.e. they are guaranteed to exist as unique solutions only for a finite time).

The difficulty here is the possible existence of singularities in the vorticity field (vorticity becoming infinite at some points in space and time). Depending on how large the singularity set is uniqueness may fail in strictly mathematical terms. The existence of singularities may not be a purely mathematical curiosity, it may in fact be related with the intermittency phenomenon. Rigorous studies of the vorticity equation may continue to reveal more fundamental aspects on vortex dynamics and coherent structures.

The statistical theory has also been put into a firm foundation with the notion of *statistical solution* of the NSE. It addresses the existence and regularity of the probability distribution assumed for turbulent flows and of the fundamental elements of the statistical theory such as correlation functions and spectra. Based on that, a number of relations between physical quantities of turbulent flows may be derived in a mathematically sounded and definitive way. This does not replace other theories, it is mostly a mathematical framework upon which other techniques can be applied to yield rigorous results.

Despite the difficulties in the mathematical theory of the NSE some successes have been collected such as estimates for the number of degrees of freedom in terms of fractal dimensions of suitable sets associated with the solutions of the NSE, and partial estimates of a number of relations derived in the statistical theory of fully-developed turbulence.

See also

Dynamical systems in fluid dynamics. Ergodic motions, chaos, and attractors. Geophysical fluid dynamics. Lyapunov exponents, strange attractors. Multi-scale approach and asymptotics. Newtonian fluids and thermohydraulics. Numerical methods in fluid dynamics.

Further Reading

Adzhemyan L T, Antonov N V, and Vasiliev A N (1999) *The field theoretic renormalization group in fully developed turbulence,* Gordon and Breach Science Publishers, Amsterdam.

Barenblatt G I and Chorin A J (1998) New perspectives in turbulence: scaling laws, asymptotics, and intermittency, *SIAM Review* 40, no. 2, 265–291

Batchelor G K (1953) *The Theory of Homogeneous Turbulence,* Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, New York.

Chorin A J (1994) *Vorticity and Turbulence,* Applied Mathematical Sciences 103, Springer-Verlag, New York.

Constantin P (1994) Geometric statistics in turbulence, *SIAM Review* 36, 73–98.

Foias C, Manley O P, Rosa R, and Temam R (2001) *Navier-Stokes Equations and Turbulence,* Encyclopedia of Mathematics and its Applications, vol. 83, Cambridge University Press.

Friedlander S and Topper L (1961) *Turbulence. Classic Papers on Statistical Theory,* Interscience Publisher, Inc., New York.

Frisch U (1995) *Turbulence. The legacy of A. N. Kolmogorov,* Cambridge University Press, Cambridge.

Hinze J O (1975) *Turbulence,* McGraw-Hill Series in Mechanical Engineering, McGraw-Hill Book Co., Inc..

Holmes P, Lumley J L, and Berkooz G (1996) *Turbulence, Coherent Structures, Dynamical Systems, and Symmetry,* Cambridge University Press.

Lesieur M (1997) *Turbulence in Fluids,* Third edition, Fluid Mechanics and its Applications, 40, Kluwer Academic Publishers Group, Dordrecht.

Monin A S and Yaglom A M (1975) *Statistical Fluid Mechanics: Mechanics of Turbulence* 2, MIT Press, Cambridge.

Sagaut P (2001) *Large Eddy Simulation for Incompressible Flows,* Springer-Verlag, Berlin Heidelberg New York.

Schlichting H and Gersten K (2000) *Boundary Layer Theory,* 8th Edition, Springer-Verlag, Berlin, Heidelberg, New York.

Tennekes H and Lumley J L (1972) *A First Course in Turbulence,* MIT Press, Cambridge, Mass..

Vishik M I and Fursikov A V (1988) *Mathematical Problems of Statistical Hydrodynamics,* Kluwer, Dordrecht.

Wilcox D C (2000) *Turbulence Modeling for CFD,* Second Edition, DCW Industries Inc..