derivation of geometric mean as the limit of the power mean*

Mathprof[†]

2013-03-21 17:16:47

Fix $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$. Then let

$$\mu(r) := \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r}.$$

For $r \neq 0$, by definition $\mu(r)$ is the rth power mean of the x_i. It is also clear that $\mu(r)$ is a differentiable function for $r \neq 0$. What is $\lim_{r\to 0} \mu(r)$?

We will first calculate $\lim_{r\to 0} \log \mu(r)$ using l'Hôpital's rule.

$$\begin{aligned}
\lim_{r\to 0} \log \mu(r) &= \lim_{r\to 0} \frac{\log\left(\frac{x_1^r + \cdots + x_n^r}{n}\right)}{r} \\
&= \lim_{r\to 0} \frac{\left(\frac{x_1^r \log x_1 + \cdots + x_n^r \log x_n}{n}\right)}{\left(\frac{x_1^r + \cdots + x_n^r}{n}\right)} \\
&= \lim_{r\to 0} \frac{x_1^r \log x_1 + \cdots + x_n^r \log x_n}{x_1^r + \cdots + x_n^r} \\
&= \frac{\log x_1 + \cdots + \log x_n}{n} \\
&= \log \sqrt[n]{x_1 \cdots x_n}.
\end{aligned}$$

It follows immediately that

$$\lim_{r\to 0} \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{1/r} = \sqrt[n]{x_1 \cdots x_n}.$$

*⟨*DerivationOfGeometricMeanAsTheLimitOfThePowerMean*⟩ created: ⟨*2013-03-21*⟩ by: ⟨*Mathprof*⟩ version: ⟨*35741*⟩ Privacy setting: ⟨*1*⟩ ⟨*Derivation*⟩ ⟨*26D15*⟩