گزارش هجدهمین کنفرانس ریاضی کشور

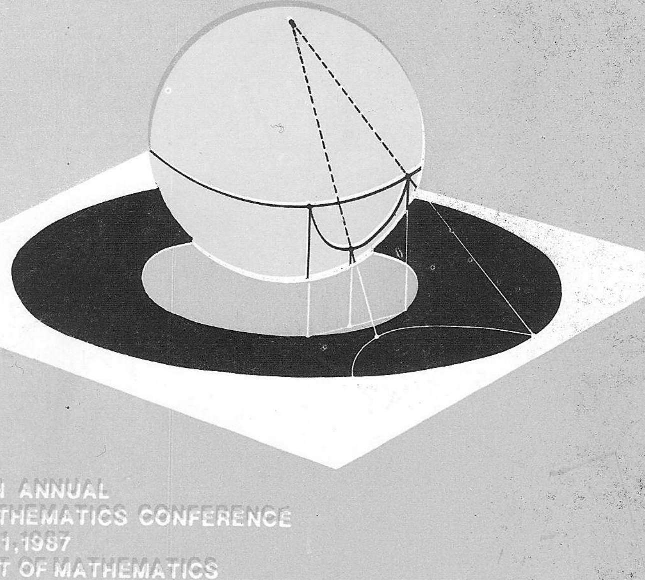
جمهوری اسلامی ایران

هجدهمین کنفرانس ریاضی کشور

۸ تا ۱۱ فروردین ۱۳۶۶

مجتمع آموزش عالی بیرجند

گروه ریاضی



EIGHTEENTH ANNUAL
IRANIAN MATHEMATICS CONFERENCE
MARCH 28-31, 1987
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BIRJAND
BIRJAND, IRAN

Generalization of Venn Diagram

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1. Introduction. Questions have been raised in mathematical literature, in generalizing the Venn Diagram to the case of more than three sets. For example, in [3, p. 55–56] the difficulties of doing this, is pointed out. In [2] a figure is drawn, showing the Venn Diagram for five sets, with ellipsis. In [1] in an article called "Is Venn Diagram Good Enough?", it has been shown that the maximum number of disjoint sets in the plane formed by n circles is $2 + n^2 - n$. Therefore, the article concludes that, for $n>4$ it is impossible to draw n circles showing all the possible intersections of n sets.

Now the question is: if we can not generalize Venn Diagram by using circles, then how about doing it with other geometric figures? To be precise, what we want to do is that: Given n sets $A_1, A_2, \ldots, A_n$, draw a diagram to show all 2^n regions, formed by $X_1 \cap \ldots \cap X_n$, where $X_i = A_i$ or A'_i (each of these regions will be called a minterm). We also want to have all advantages of Venn Diagram in general case, i.e. each minterm being represented in a simply connected region, and for each set A_i there is a simple closed curve whose inside region represents A_i, and its outside represents A'_i.

In section 2 of this note we present an algorithm for doing this. This algorithm is

* The work of these authors was supported by the Research Center of the Atomic Energy Organization of Iran.

programmed and we present some computer outputs for the cases n=4, 5, and 6 (see figures 2, 3, and 4). In section 3 we point out, some of the applications of this generalized diagram. Finally in section 4, we prove that the algorithm works!

2. The Algorithm. Suppose that the sets $A_1, A_2, ..., A_n$ are to be represented. First we show the universal set by the inside region of a rectangle (Fig. 1). Then draw two perpendicular lines one for A_1, and another for A_2. We represent A_3 by the inside region of a circle.

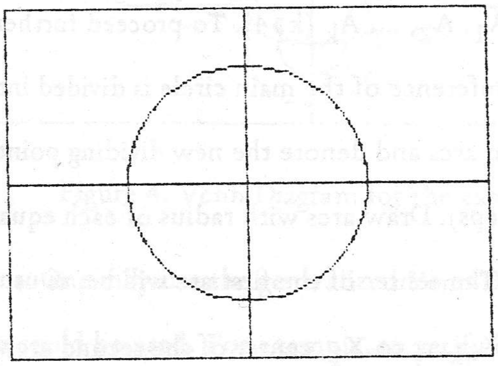

Figure 1. Venn Diagram for the case of three sets

Figure 1, may be used for a Venn Diagram for the case of three sets. To continue process, we note that at this stage the circumference of the circle (will be called the main circle hereafter) is divided into four equal arcs by the boundaries of A_1 and A_2. We divide each arc into two equal parts, and denote all the dividing points by $x_0, x_1, ..., x_7$ (Fig. 2).

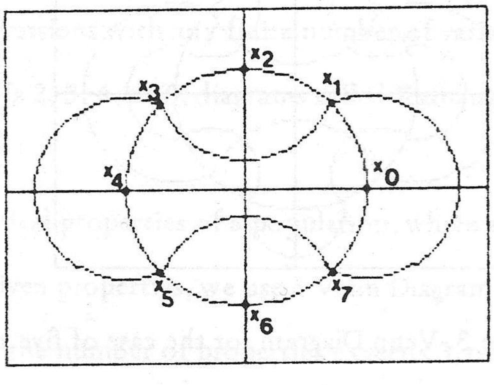

Figure 2. Venn Diagram for the case of four sets.

Now, we draw arcs of circles with all equal radii and the radius of each is equal to the distance between two succesive points, and the center of first one is at x_0 lying outside of the main circle from x_7 to x_1, the center of the second one is x_2 lying inside the main circle from x_1 to x_3, and so on. These arcs together make a closed ("bone" shaped) curve, whose inside region will represent A_4.

The same procedure as for A_4, will be continued to proceed further. In fact, suppose that we have presented the sets $A_1, A_2, \ldots, A_k$ ($k \geqslant 4$). To proceed further, we note that by induction, at this stage the circumference of the main circle is divided into 2^{k-1} equal arcs. We divide each arc into two equal arcs and denote the new dividing points by $X_0, X_1, \ldots, X_{2^k-1}$ (having fixed X_0 in all steps). Draw arcs with radius of each equal to the distance of two succesive new points (X_i's). The center of the first arc will be, as usual, at X_0 and lying outside of the main circle from X_{2^k-1} to X_1, center of the second arc will be at the point X_2 lying inside the main circle from X_1 to X_3, and so on. These arcs together make a closed curve whose inside will represent the set A_{k+1}. In figures 3 and 4 we have demonstrated a diagram for the case n=5 and n=6.

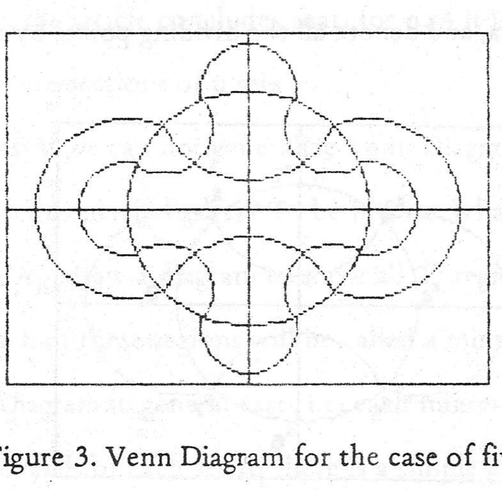

Figure 3. Venn Diagram for the case of five sets

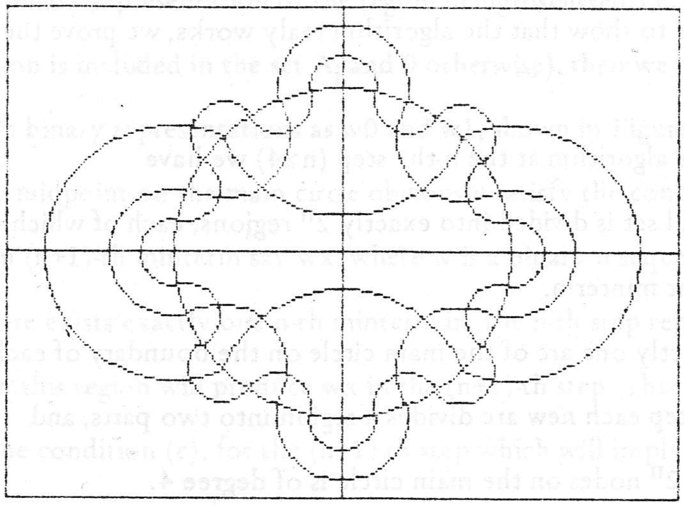

Figure 4. Venn Diagram for the case of six sets

3. Applications. One may use the generalized Venn Diagram for the similar purposes that Venn Diagram could be used. For example, to verify relations involving sets, or inclusion of a set theoretic formula into another formula, in which the number of involved letters in the formulae exceeds from 3, then one ought to use the generalized Venn Diagram.

Clearly, the same discussion may be used for the case of cheking the truth of the implication or equivalence of two propositional formulae, involving any finite number of variables.

Another application of generalized Venn Diagram may be thought of using it for simplifying boolean expressions with any finite number of variables. For the cases in which the number of variables is 2, 3, 4, or 5, diagrams called Karnaugh map is usually used in the textbooks.

To study the statistical properties of a population, where each member may or may not posses each of the given properties, we use a Venn Diagram, when there is not more than three properties. If the number of properties exceeds 3, one can use the generalized Venn Diagram.

4. Proof. In order to show that the algorithm realy works, we prove the following theorem:

Theorem. In the above algorithm at the n-th step ($n \geq 4$) we have

(a) The universal set is divided into exactly 2^n regions, each of which is correspondent to a different minterm,

(b) There is exactly one arc of the main circle on the boundary of each region,

(c) In the last step each new arc divides a region into two parts, and

(d) Each of the 2^n nodes on the main circle is of degree 4.

Proof. We proceed by induction on n. For n=4, statements are trivially true. Suppose that the statements hold true for n, and we have 2^n regions satisfying (a) through (d). To show the truth of the statement for n+1, consider one of the regions of the n-th step, and suppose that it lies, for example, outside of the main circle. (If the region is inside of the main circle the proof is similar). By (b) there is one arc of the main circle say $\widehat{AB}$, on the boundary of this region. Let C be the midpoint of this arc. In the (n+1)-th step two arcs with centers at A and B, and radii $\overline{AC} = \overline{BC}$ are drawn. One of these arcs is in the outside of the main circle and the other one is inside of it. Two possible cases which may occur, are represented in Fig. 5.

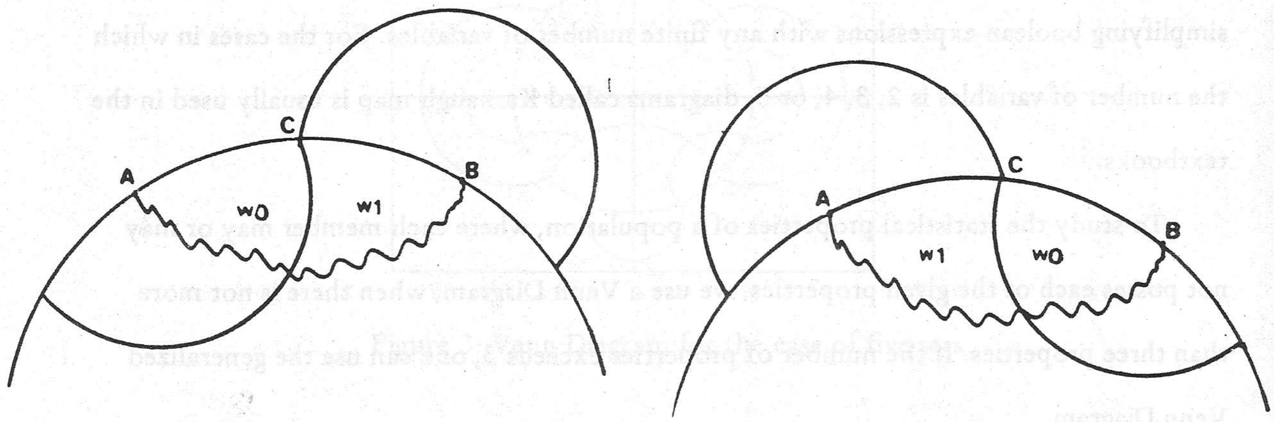

Fig. 5.

If w is the binary representation of the region being discussed (i.e. $w=x_1x_2..x_n$ where x_i is 1 if the region is included in the set A_i and 0 otherwise), then we obtain at least two new regions with binary representations as w0 and w1, shown in Figure 5. These two new regions and new midpoint on the main circle obviously satisfy the conditions (b) and (d). Now consider an (n+1)-th minterm say wx, where w is a binary n sequence and x is a binary digit. By (a), there exists exactly one n-th minterm in the n-th step representing w. And by above discussion this region will produce wx in the (n+1)-th step. Thus all we have to show is the truth of the condition (c), for the (n+1)-th step which will imply the truth of (a) for this step also.

Let A and B be two dividing points on the main circle, and C be the midpoint of A and B on the main circle. Let $\widehat{AXB}$ be an arc drawn from A to B with the center at C. We call $\widehat{ACB}$ (of the main circle) the support of arc $\widehat{AXB}$ (Fig. 6).

Now let $\widehat{AXB}$ be an arc which is drawn at the (n+1)-th step. To study the intersection points of this arc with the "old" arcs (drawn in the previous steps), we observe that: first, no arc on the outside of the main circle may intersects the arcs lying inside of it. We suppose that $\widehat{AXB}$ is an outside arc (the other case has similar proof). $\widehat{AXB}$ with any other arc in the outside say $\widehat{DYE}$, has either of the following three relations:

(i) the support of $\widehat{AXB}$ is disjoint from the support of $\widehat{DYE}$, (ii) the support of $\widehat{AXB}$ is contained in the support of $\widehat{DYE}$, and (iii) half of the support of $\widehat{AXB}$ is contained in the support of $\widehat{DYE}$. In the last case the point C must coincide either with D or with E. It is obvious that, in the case (i), $\widehat{AXB}$ has no intersection with $\widehat{DYE}$, and in the case (iii), $\widehat{AB}$ has exactly one point of intersection with $\widehat{DXE}$. In the following lemma, we show that in the case (ii), the arc $\widehat{AXB}$ has no intersection with $\widehat{DYE}$. This will stablish the fact that each new arc divides an old region exactly into two new regions.

Lemma. If the support of an outer (inner) arc $\widehat{DYE}$ contains the support of an outer (inner, respectively) arc $\widehat{AXB}$, then their circles do not have any intersection point.

Proof. We show the validity of the statement for the case in which the arcs lie outside of the main circle (Fig. 6). For the other case, proof is similar.

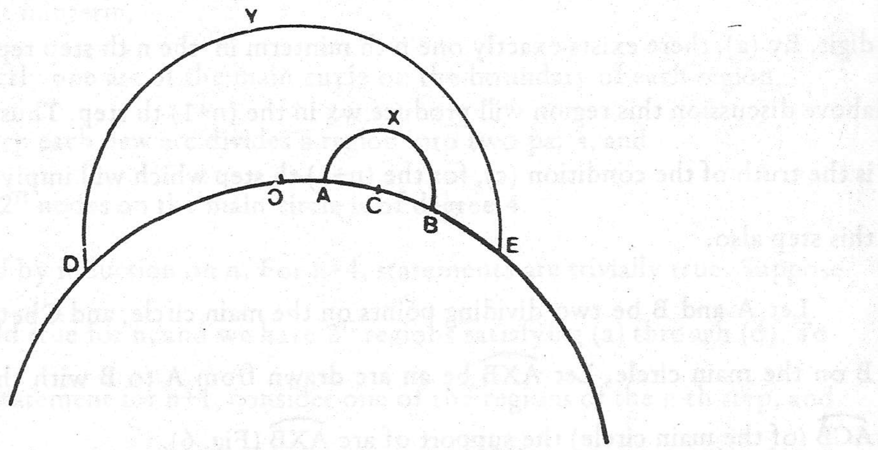

Fig. 6.

Suppose, that the center of $\widehat{DYE}$ is point O, and the center of $\widehat{AXB}$ is point C. Denote the size of $\widehat{AC}$ by θ. Then there exists natural numbers m and l (both greater than one) such that

$$\widehat{OC} = m\theta \text{ and } \widehat{CE} = l\theta.$$

Note that

$$0<(m+l)\,\theta = \widehat{OE} < \frac{\pi}{4}. \tag{1}$$

The circles X and Y do not have point of intersection iff $\overline{OC} + \overline{CB} < \overline{OE}$.

But by elementry trigonometry the last inequality in terms of θ is equivalent to

$$2\sin\frac{m\theta}{2} + 2\sin\frac{\theta}{2} < 2\sin\frac{m+l}{2}\,\theta. \tag{2}$$

We prove the following inequality which will follow (2) because of (1),

$$\sin\frac{m\theta}{2} + \sin\frac{\theta}{2} < \sin\frac{m+2}{2}\theta \qquad (3)$$

or equivalently

$$\sin\frac{\theta}{2} < \sin\frac{m+2}{2}\theta - \sin\frac{m\theta}{2}$$

$$\sin\frac{\theta}{2} < 2\sin\frac{\theta}{2}\cos\left(\frac{m\theta}{2}+\frac{\theta}{2}\right).$$

$$\sin\frac{\theta}{2}\left[1-2\cos\frac{m+1}{2}\theta\right] < 0 .$$

But since $\sin\frac{\theta}{2} > 0$, it will suffice to show that

$$1-2\cos\frac{m+1}{2}\theta < 0.$$

Since $0 < \frac{m+1}{2}\theta < \frac{m+l}{2}\theta \leqslant \frac{\pi}{8}$, and cosine is a decreasing function on this interval, therefore

$$\cos\frac{m+1}{2}\theta \geqslant \cos\frac{\pi}{8} > \frac{1}{2}.$$

Thus $1-2\cos\frac{m+1}{2}\theta < 0$, This stablishes the proof of the lemma, and therefore the theorem is proved.

REFERENCES

1. M. L. Kung and G. C. Harrison, Is the Venn Diagram Good Enough?, The College Mathematics Journal, 15 (1984) 48–50.
2. A. J. Schwenk, Venn Diagram for Five Sets, Mathematics Magazine, 57(1984) 297.
3. I. Stewart and D. Tall, The Foundations of Mathematices, Oxford University Press, Oxford, 1977.