WEIGHTED LIMITS AND COLIMITS

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ABSTRACT. Assuming only a very rudimentary knowledge of enriched category theory — $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations for a closed, symmetric monoidal, complete and cocomplete category $\mathcal{V}$ — we introduce weighted limits and colimits, the appropriate sort of limits for the enriched setting. This "modern" approach was introduced to the author through talks by Mike Shulman at the Category Theory Seminar at the University of Chicago in Fall 2008. These notes were written in an attempt to understand and internalize the many wonderful things that he said.

1. Homs and Tensor Products of $\mathcal{V}$-functors

A one object category enriched in **Ab** is a ring, which we call R. A **Ab**-functor from R to **Ab** is a left R-module if it is covariant and a right R-module if it is contravariant. Let $M : R^{\text{op}} \to \mathbf{Ab}$ and $N : R \to \mathbf{Ab}$ be two such functors, and let M and N also denote the respective objects of **Ab** in their image. A slight modification of the usual functor tensor product to account for the fact that R and **Ab** are **Ab**-categories yields the following coequalizer in **Ab**:

$$M \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} N \underset{(m,r,n)\mapsto(mr,n)}{\overset{(m,r,n)\mapsto(m,rn)}{\rightrightarrows}} M \otimes_{\mathbb{Z}} N \longrightarrow M \otimes_R N.$$

This constructs the tensor product over R of a right R-module with a left R-module using the monoidal structure $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

More generally, let $(\mathcal{V}, \otimes, I)$ be a closed, symmetric monoidal category that is complete and cocomplete. Given a $\mathcal{V}$-category $\mathcal{C}$ and $\mathcal{V}$-functors $F : \mathcal{C}^{\text{op}} \to \mathcal{V}$ and $G : \mathcal{C} \to \mathcal{V}$ a generalization of the above construction yields the $\mathcal{V}$-tensor product of F and G, an object $F \otimes_{\mathcal{C}} G$ of $\mathcal{V}$:

$$\textstyle\coprod_{a,b\in\mathcal{C}} Fb \otimes \mathcal{C}(a,b) \otimes Ga \rightrightarrows \coprod_{c\in\mathcal{C}} Fc \otimes Gc \longrightarrow F \otimes_{\mathcal{C}} G. \text{ [1]}$$

The top map of the parallel pair is induced by the composite

$$Fb \otimes \mathcal{C}(a,b) \otimes Ga \xrightarrow{1\otimes F_{a,b}\otimes 1} Fb \otimes \mathcal{V}(Fb, Fa) \otimes Ga \xrightarrow{\text{ev}\otimes 1} Fa \otimes Ga \hookrightarrow \textstyle\coprod_c Fc \otimes Gc,$$

where $F_{a,b}$ is the arrow of $\mathcal{V}$ given because F is a V-functor and ev is the evaluation map, the counit of the adjunction on $\mathcal{V}$ of the monoidal product with the internal-hom. The bottom map is induced by a similar composite with G in place of F.

The dual notion gives an enriched hom of $\mathcal{V}$-functors $G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ (note that the codomain category need no longer

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[1]The $\mathcal{C}(a,b)$ that appears on the left is the internal-hom, an object of $\mathcal{V}$. This is the default; hom-*sets* will explicitly noted as such.

be $\mathcal{V}$). As with the tensor product, the hom is an object of $\mathcal{V}$, defined to be the equalizer

$$\mathrm{Hom}_{\mathcal{C}}(G,H) \longrightarrow \textstyle\prod_{c\in\mathcal{C}} \mathcal{D}(Gc,Hc) \rightrightarrows \textstyle\prod_{a,b\in\mathcal{C}} \mathcal{V}(\mathcal{C}(a,b),\mathcal{D}(Ga,Hb)).$$

The top arrow in the parallel pair is induced by the adjunct map to the composite

$$\left(\prod_{c\in\mathcal{C}} \mathcal{D}(Gc,Hc)\right) \otimes \mathcal{C}(a,b) \xrightarrow{\mathrm{proj}_b \otimes G_{a,b}} \mathcal{D}(Gb,Hb)\otimes\mathcal{D}(Ga,Gb) \xrightarrow{\circ} \mathcal{D}(Ga,Hb)$$

under the adjunction $-\otimes\mathcal{C}(a,b) \dashv \mathcal{V}(\mathcal{C}(a,b),-)$ in $\mathcal{V}$. The bottom arrow is defined similarly with H in place of G.

When $\mathcal{V}=\mathcal{D}=\mathbf{Ab}$ and $\mathcal{C}$ has one object, this gives the usual construction of the abelian group $\mathrm{Hom}_R(N,P)$ for two left R-modules $N : R \to \mathbf{Ab}$ and $P : R \to \mathbf{Ab}$ as the equalizer

$$\mathrm{Hom}_R(N,P) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(N,P) \underset{\phi\mapsto[r\mapsto r\phi(-)]}{\overset{\phi\mapsto[r\mapsto\phi(r-)]}{\rightrightarrows}} \mathrm{Hom}_{\mathbb{Z}}(R,\mathrm{Hom}_{\mathbb{Z}}(N,P)).$$

The hom of $\mathcal{V}$-functors constructed above gives the category $[\mathcal{C},\mathcal{D}]$ of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations the structure of a $\mathcal{V}$-category (modulo size issues).

2. Bimodules aka Profunctors aka Distributors

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories, with $\mathcal{V}$ as above. Note that $\mathcal{B}^{\mathrm{op}}$ and $\mathcal{A}\otimes\mathcal{B}$ are also $\mathcal{V}$-categories when $\mathcal{V}$ is symmetric, with $\mathcal{B}^{\mathrm{op}}(b,b') := \mathcal{B}(b',b)$ and

$$\mathcal{A}\otimes\mathcal{B}((a,b),(a',b')) := \mathcal{A}(a,a')\otimes\mathcal{B}(b,b'). \tag{2.1}$$

Definition 2.2. A $\mathcal{A}$-$\mathcal{B}$-*bimodule* or *profunctor* or *distributor* is a $\mathcal{V}$-functor

$$J : \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}} \to \mathcal{V}.$$

That is, an $\mathcal{A}$-$\mathcal{B}$-bimodule consists of an object function $(a,b)\mapsto J(a,b)$ together with arrows $\mathcal{A}\otimes\mathcal{B}((a,b'),(a',b)) \to \mathcal{V}(J(a,b),J(a',b'))$ in $\mathcal{V}$ that are compatible with identities and composition.

2.1. **Bimodule Tensor.** Given bimodules $J : \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{V}$ and $K : \mathcal{B}\otimes\mathcal{C}^{\mathrm{op}}\to\mathcal{V}$, we can form their tensor product

$$J\otimes_{\mathcal{B}} K : \mathcal{A}\otimes\mathcal{C}^{\mathrm{op}} \to \mathcal{V},$$

which is also a bimodule. For any object $(a,c)\in\mathcal{A}\otimes\mathcal{C}^{\mathrm{op}}$,

$$[J\otimes_{\mathcal{B}} K](a,c) := J(a,-)\otimes_{\mathcal{B}} K(-,c)$$

is defined to be the tensor product of the $\mathcal{V}$-functors $J(a,-)$ and $K(-,c)$ constructed in Section 1. Since we want $J\otimes_{\mathcal{B}} K$ to be a $\mathcal{V}$-functor, it remains to show that for all $(a,c),(a',c')\in\mathcal{A}\otimes\mathcal{C}^{\mathrm{op}}$ there is an arrow

$$\mathcal{A}\otimes\mathcal{C}^{\mathrm{op}}((a,c),(a',c')) \to \mathcal{V}([J\otimes_{\mathcal{B}} K](a,c),[J\otimes_{\mathcal{B}} K](a',c')) \tag{2.3}$$

in $\mathcal{V}$. Using the isomorphism (2.1), the arrow (2.3) is adjunct to a map

$$\mathcal{A}(a,a')\otimes[J\otimes_{\mathcal{B}} K](a,c)\otimes\mathcal{C}(c',c) \to [J\otimes_{\mathcal{B}} K](a',c'),$$

which we will construct below.

The functor $\mathcal{A}(a,a') \otimes - \otimes \mathcal{C}(c',c)$ is a left adjoint, so it preserves colimits. When we apply it to the coequalizer that defines $[J \otimes_{\mathcal{B}} K](a,c)$, we get the left column of

$$\begin{array}{ccc}
\coprod_{b,b'} \mathcal{A}(a,a') \otimes J(a,b') \otimes \mathcal{B}(b,b') \otimes K(b,c) \otimes \mathcal{C}(c',c) & \xrightarrow[u]{} & \coprod_{b,b'} J(a',b') \otimes \mathcal{B}(b,b') \otimes K(b,c') \\
\downarrow\downarrow & & \downarrow\downarrow \\
\coprod_{b} \mathcal{A}(a,a') \otimes J(a,b) \otimes K(b,c) \otimes \mathcal{C}(c',c) & \xrightarrow[v]{} & \coprod_{b} J(a',b) \otimes K(b,c') \\
\downarrow & & \downarrow \\
\mathcal{A}(a,a') \otimes [J \otimes_{\mathcal{B}} K](a,c) \otimes \mathcal{C}(c',c) & \dashrightarrow & [J \otimes_{\mathcal{B}} K](a',c')
\end{array}$$

We will define arrows u and v in $\mathcal{V}$ that make the two upper squares commute. This induces the desired arrow along the bottom. Actually, defining these arrows is easy[2]:

$$u = J(-,b')^{\#}_{a,a'} \otimes 1 \otimes K(b,-)^{\#}_{c',c} \quad \text{and} \quad v = J(-,b)^{\#}_{a,a'} \otimes K(b,-)^{\#}_{c',c}$$

ignoring the evident coproduct inclusions. Checking commutativity of the squares is relatively straightforward. The unique map induced from the left coequalizer is adjunct to the morphism that makes $J \otimes_{\mathcal{B}} K$ a $\mathcal{A}$-$\mathcal{C}$-bimodule.

2.2. **Bimodule Hom.** A dual construction takes bimodules $K : \mathcal{B} \otimes \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and $L : \mathcal{A} \otimes \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and produces a bimodule

$$\mathrm{Hom}_{\mathcal{C}}(K,L) : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{V}.$$

On objects,

$$[\mathrm{Hom}_{\mathcal{C}}(K,L)](a,b) := \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(K(b,-), L(a,-)),$$

using the definition of the enriched hom for $\mathcal{V}$-functors from Section 1. A similar construction is used to define the arrows in $\mathcal{V}$ that make $\mathrm{Hom}_{\mathcal{C}}(K,L)$ a $\mathcal{V}$-functor.

Note what happens to the indexing categories: a $\mathcal{B}$-$\mathcal{C}$-bimodule and a $\mathcal{A}$-$\mathcal{C}$-bimodule yield a $\mathcal{A}$-$\mathcal{B}$-bimodule.

2.3. **Bimodule Adjunction.** Together the bimodule tensor product and the bimodule hom form a $\mathcal{V}$-adjunction

$$- \otimes_{\mathcal{B}} K : [\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{V}] \underset{\perp}{\rightleftarrows} [\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}, \mathcal{V}] : \mathrm{Hom}_{\mathcal{C}}(K,-)\ .$$

In particular, there exist isomorphisms

$$[\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}, \mathcal{V}](J \otimes_{\mathcal{B}} K, L) \cong [\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{V}](J, \mathrm{Hom}_{\mathcal{C}}(K,L))$$

in $\mathcal{V}$, natural in J and L. We will use this adjunction later.

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[2]For those unfamiliar with this notation, the $\#$ means "the map that is the left adjunct to." So, for example, $J(-,b)_{a,a'} : \mathcal{A}(a,a') \to \mathcal{V}(J(a,b), J(a',b))$ is the map we get from $J(-,b)$ being a $\mathcal{V}$-functor (using the notation from §1) and $J(-,b)^{\#}_{a,a'} : \mathcal{A}(a,a') \otimes J(a,b) \to J(a',b)$ is its adjunct from the tensor-hom adjunction in $\mathcal{V}$. If we move the other way across the adjunction, the notation is $^{\flat}$.

3. J-weighted limits

All categories mentioned below are $\mathcal{V}$-categories, unless otherwise specified, where $(\mathcal{V}, \otimes, I)$ is closed, symmetric monoidal, complete, and cocomplete. All functors are $\mathcal{V}$-functors, and so forth.

Definition 3.1. For a diagram $D : \mathcal{A} \to \mathcal{M}$ and a bimodule $J : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ the *J-weighted limit* of D, if it exists, is a $\mathcal{V}$-functor $\lim^J D : \mathcal{B} \to \mathcal{M}$ with a $\mathcal{V}$-natural isomorphism of $\mathcal{B}$-$\mathcal{M}$-bimodules $\mathcal{M}(-, (\lim^J D)-)$ and $\mathrm{Hom}_{\mathcal{A}}(J(-,-), \mathcal{M}(-, D-))$. This means that levelwise we have isomorphisms

$$\mathcal{M}(m, (\lim^J D)b) \cong \mathrm{Hom}_{\mathcal{A}}(J(-,b), \mathcal{M}(m, D-)) = [\mathcal{A}, \mathcal{V}](J(-,b), \mathcal{M}(m, D-)) \tag{3.2}$$

in $\mathcal{V}$, natural in b and m.

Example 3.3. Suppose $\mathcal{V} = \mathbf{Set}$, $\mathcal{B} = \mathbb{1}$, the terminal category. Let $J : \mathcal{A} \to \mathbf{Set}$ be a functor that sends every object of $\mathcal{A}$ to a fixed terminal object $*$ of $\mathbf{Set}$. An element of the right hand side of (3.2) consists of maps $* \to \mathcal{M}(m, Da)$ in $\mathbf{Set}$ for each $a \in \mathcal{A}$. In other words, we get an arrow $m \to Da$ of $\mathcal{M}$ for each $a \in \mathcal{A}$, subject to naturality conditions that say that these arrows form a cone with summit m over the diagram D. So (3.2) expresses the usual universal property of an ordinary limit in this case.

Recall that the unit object of $\mathcal{V}$ can be used to define an *underlying category functor* $\mathcal{V}$-$\mathbf{Cat} \to \mathbf{Cat}$. Given a $\mathcal{V}$-category $\mathcal{C}$, we write $\mathcal{C}_0$ for its underlying category with $\mathrm{ob}\,\mathcal{C}_0 = \mathrm{ob}\,\mathcal{C}$ and

$$\mathcal{C}_0(x,y) := \mathcal{V}_0(I, \mathcal{C}(x,y)) \tag{3.4}$$

for all $x, y \in \mathrm{ob}\,\mathcal{C}$. In (3.4), the enriching category $\mathcal{V}$ is taken to be first an ordinary category with the additional closed symmetric monoidal structure that makes $\mathcal{V}$ into a $\mathcal{V}$-category. So $\mathcal{V}_0(-,-)$ is a hom-*set* while $\mathcal{V}(-,-)$ denotes the internal hom. With these conventions $\mathcal{V}_0(I,-)$ is a functor from $\mathcal{V}$ to $\mathbf{Set}$. Nice things happen when this functor is faithful or conservative, but this is often not the case.

The functor $\mathcal{V}_0(I,-)$ has a left adjoint which takes a set A to the copower $A \cdot I = \sqcup_A I$ in $\mathcal{V}$. This induces a left adjoint to the underlying category functor, called the *free $\mathcal{V}$-category functor* $\mathcal{V}[-] : \mathbf{Cat} \to \mathcal{V}$-$\mathbf{Cat}$. Given an ordinary category $\mathcal{C}$, $\mathcal{V}[\mathcal{C}]$ has the same objects as $\mathcal{C}$ with

$$\mathcal{V}[\mathcal{C}](x,y) := \mathcal{C}(x,y) \cdot I$$

for all $x, y \in \mathcal{C}$.[3]

Let $\mathcal{I}$ denote the $\mathcal{V}$-category $\mathcal{V}[\mathbb{1}]$ with one object $*$ and one hom-object I.

Example 3.5. Let $\mathcal{A}$ be an ordinary $\mathbf{Set}$-category, let $\mathcal{B} = \mathcal{I}$, and let $D : \mathcal{V}[\mathcal{A}] \to \mathcal{M}$ be a diagram in $\mathcal{M}$. Let $J : \mathcal{V}[\mathcal{A}] \otimes \mathcal{I}^{\mathrm{op}} \cong \mathcal{V}[\mathcal{A}] \to \mathcal{V}$ be the functor that sends each object to $I \in \mathcal{V}$ and with each $J_{a,a'} : \mathcal{A}(a,a') \cdot I \to I$ the "fold" map induced by identities on each component. Applying the functor $\mathcal{V}_0(I,-)$ to (3.2), we get

$$\mathcal{M}_0(m, \lim^J D) \cong [\mathcal{V}[\mathcal{A}], \mathcal{V}]_0(J-, \mathcal{M}(m, D-)) \cong [\mathcal{A}, \mathcal{V}_0](J-, \mathcal{M}(m, D-))$$

where the last isomorphism comes from the adjunction described above. Levelwise, the transformations on the right consist of arrows $I \to \mathcal{M}(m, Da)$, i.e., elements

[3]Note this construction only works when the tensor $\otimes$ preserves these colimits, e.g., when $\mathcal{V}$ is closed.

of $\mathcal{M}_0(m, Da)$. So a J-weighted limit is in particular a limit of $D^\flat : \mathcal{A} \to \mathcal{M}_0$. Weighted limits of this sort are called *conical* limits.

Example 3.6. Recall that a **Cat**-category is a (strict) 2-category. Let $\mathcal{V} = \mathbf{Cat}$ and let $\mathcal{A}$ be the ordinary **Set**-category with three objects - a, b, c - and two non-identity morphisms: $a \to c$ and $b \to c$. Let $\mathcal{B} = \mathcal{I}$ and define $J : \mathcal{V}[\mathcal{A}] \to \mathcal{V}$ to be the $\mathcal{V}$-functor adjunct to the ordinary functor $\mathcal{A} \to \mathcal{V}_0 = \mathbf{Cat}$ such that $a, b \mapsto \mathbf{1} = (\bullet)$, $c \mapsto \mathbf{2} = (\bullet \to \bullet)$, and $a \to c$, $b \to c$ map to the inclusions of $\mathbf{1}$ as the left and right objects of $\mathbf{2}$, respectively.

Given a 2-category $\mathcal{M}$ and a diagram $D : \mathbf{Cat}[\mathcal{A}] \to \mathcal{M}$, a J-weighted limit is an object of $\mathcal{M}$ called a *comma object* over D. The category $[\mathbf{Cat}[\mathcal{A}], \mathcal{M}](J-, \mathcal{M}(m, D-))$ is by definition the end

$$[\mathbf{Cat}[\mathcal{A}], \mathcal{M}](J-, \mathcal{M}(m, D-)) = \int_{x \in \mathbf{Cat}[\mathcal{A}]} \mathbf{Cat}(Jx, \mathcal{M}(m, Dx))$$
$$= \lim \left(\begin{array}{ccc} \mathbf{Cat}(Ja, \mathcal{M}(m, Da)) & & \\ & \searrow & \\ & & \mathbf{Cat}(Ja, \mathcal{M}(m, Dc)) \\ & \nearrow & \\ \mathbf{Cat}(Jc, \mathcal{M}(m, Dc)) & & \\ & \searrow & \\ & & \mathbf{Cat}(Jb, \mathcal{M}(m, Dc)) \\ & \nearrow & \\ \mathbf{Cat}(Jb, \mathcal{M}(m, Db)) & & \end{array} \right)$$

Concretely, the data of this category consists of 1-cells $m \to Da$ and $m \to Db$ in $\mathcal{M}$ such that

(3.7)
$$\begin{array}{ccc} m & \longrightarrow & Da \\ \downarrow & \swArrow & \downarrow \\ Db & \longrightarrow & Dc \end{array}$$

commutes and the 2-cell between the upper right and lower left composites as shown. The isomorphism (3.2) tells us that this data corresponds to that category $\mathcal{M}(m, \lim^J D)$ of 1-cells in $\mathcal{M}$ between m and the object $\lim^J D$ and 2-cells betwen these 1-cells. The enriched Yoneda lemma (see 6.1) tells us that a 2-cell of the form (3.7) factors as its corresponding 2-cell of $\mathcal{M}(m, \lim^J D)$ followed by the 2-cell adjunct to the identity 2-cell of the identity arrow at $\lim^J D$. This sounds complicated, but it is simply the universal property one would expect.

In particular, suppose $\mathcal{M}$ is the 2-category **Cat** of categories, functors, and natural transformations and let D be the diagram that takes the morphisms $a \to c$, $b \to c$ of $\mathcal{A}$ to functors f, g, respectively. Then the usual *comma category* $(f \downarrow g)$ is a J-weighted limit of D, and the isomorphism (3.2) expresses precisely its universal property.

3.1. **Powers.** When $\mathcal{A} = \mathcal{B} = \mathcal{I}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule $J : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ picks out an object $J \in \mathcal{V}$. The identity condition on the $\mathcal{V}$-functor map $I \to \mathcal{V}(J, J)$ requires that this just be the identity map in the $\mathcal{V}$-category structure on $\mathcal{V}$, i.e., the adjunct to the identity on J in $\mathcal{V}_0$. So the only data in the bimodule J is the object J of $\mathcal{V}$ in its image.

For the same reasons, a diagram $D : \mathcal{I} \to \mathcal{M}$ is just an object D of $\mathcal{M}$. In this case, the isomorphism (3.2) reduces to

$$\mathcal{M}(m, \lim^J D) \cong \mathcal{V}(J, \mathcal{M}(m, D)).$$

When $\mathcal{V} = \mathbf{Set}$ the right hand side is isomorphic to $\prod_J \mathcal{M}(m, D) = \mathcal{M}(m, D)^J$ and $\lim^J D = \prod_J D = D^J$ is called the J-th power of the object D. This gives us the terminology in the following definition.

Definition 3.8. A *power* or *cotensor* of $D \in \mathcal{M}$ by $J \in \mathcal{V}$ is an object $\pitchfork(J, D)$ of $\mathcal{M}$ such that there is a $\mathcal{V}$-natural isomorphism

$$\mathcal{M}(-, \pitchfork(J, D)) \cong \mathcal{V}(J, \mathcal{M}(-, D))$$

of $\mathcal{V}$-functors $\mathcal{M}^{\mathrm{op}} \to \mathcal{V}$. Other notation in use is $J \pitchfork D$ (in [1]), $F(J, D)$, or D^J.

The upshot is that powers are weighted limits.

4. J-weighted colimits

The following definition gives the dual notion. Note that in what follows J is now a $\mathcal{B}$-$\mathcal{A}$-bimodule.

Definition 4.1. For a diagram $D : \mathcal{A} \to \mathcal{M}$ and a bimodule $J : \mathcal{B} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ the *J-weighted colimit* of D, if it exists, is a $\mathcal{V}$-functor $\mathrm{colim}^J D : \mathcal{B} \to \mathcal{M}$ with a $\mathcal{V}$-natural isomorphism of $\mathcal{M}$-$\mathcal{B}$-bimodules $\mathcal{M}(\mathrm{colim}^J D)-, -)$ and $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(J(-, -), \mathcal{M}(D-, -))$. Levelwise, we have isomorphisms

(4.2)
$$\mathcal{M}(\mathrm{colim}^J D)b, m) \cong \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(J(b, -), \mathcal{M}(D-, m)) = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](J(b, -), \mathcal{M}(D-, m))$$

in $\mathcal{V}$, natural in b and m.

4.1. **Copowers.** When $\mathcal{A} = \mathcal{B} = \mathcal{I}$, $\mathrm{colim}^J D$ reduces to the notion of a copower of $D \in \mathcal{M}$ by $J \in \mathcal{V}$, defined below.

Definition 4.3. A *copower* or *tensor* of $D \in \mathcal{M}$ by $J \in \mathcal{V}$ is an object of $\mathcal{M}$ written variously as $J \otimes D$, $D \otimes J$, $J \odot D$, and $D \odot J$ (we choose $J \odot D$) such that there is a $\mathcal{V}$-natural isomorphism

$$\mathcal{M}(J \odot D, -) \cong \mathcal{V}(J, \mathcal{M}(D, -))$$

of $\mathcal{V}$-functors $\mathcal{M} \to \mathcal{V}$.

When $\mathcal{V} = \mathbf{Set}$, $J \odot D = \sqcup_J D = J \cdot D$ is the familiar notion of a copower of an object D of $\mathcal{M}$ by a set. Again, the upshot is that a copower is a weighted colimit.

Putting together the defining isomorphisms of the power and copower for $K, L \in \mathcal{M}$ and $J \in \mathcal{V}$, we see that

$$\mathcal{M}(J \odot K, L) \cong \mathcal{V}(J, \mathcal{M}(K, L)) \cong \mathcal{M}(K, \pitchfork(J, L)).$$

This defines an adjunction $J \odot - \dashv \pitchfork(J, -)$ on $\mathcal{M}$. A category $\mathcal{M}$ with powers is called *cotensored* and a category $\mathcal{M}$ with copowers is called *tensored*. Tensored and cotensored categories are particularly nice for homotopy theory.

5. The Adjunction

Let $J : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule and let $D : \mathcal{B} \to \mathcal{M}$ and $E : \mathcal{A} \to \mathcal{M}$ be diagrams. The defining isomorphisms for J-weighted limits and colimits give us the following adjunction

$$[\mathcal{A}, \mathcal{M}](\mathrm{colim}^J D, E) \cong [\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{V}](J, \mathcal{M}(D, E)) \cong [\mathcal{B}, \mathcal{M}](D, \lim{}^J E). \tag{5.1}$$

When we take $\mathcal{A} = \mathcal{B} = \mathcal{I}$, (5.1) reduces to the copower-power adjunction of Section 4.1, but it is vastly more general.

Example 5.2. Let $\mathcal{A} = \mathcal{I}$, $\mathcal{B} = \mathcal{V}[\mathcal{C}]$ for an ordinary **Set**-category $\mathcal{C}$ and let $J : \mathcal{V}[\mathcal{C}]^{\mathrm{op}} \to \mathcal{V}$ be the functor described in Example 3.5. As we saw above, a diagram $E : \mathcal{I} \to \mathcal{M}$ is just an object $E \in \mathcal{M}$, and accordingly the functor $\lim^J E : \mathcal{B} \to \mathcal{M}$ is constant at E everywhere. In other words, $\lim^J E = \Delta E$, where $\Delta : \mathcal{M} \to [\mathcal{B}, \mathcal{M}]$ is the diagonal functor. The isomorphism (5.1) gives us that

$$[\mathcal{B}, \mathcal{M}](D, \Delta E) \cong \mathcal{M}(\mathrm{colim}^J D, E)$$

for any diagram $D : \mathcal{B} \to \mathcal{M}$, which implies the familiar result that, when it exists, the colimit functor is left adjoint to the diagonal functor. If we switch the roles of $\mathcal{A}$ and $\mathcal{B}$, we obtain the dual statement for the limit functor.

6. All Concepts are Weighted Limits or Colimits[4]

6.1. **The Yoneda Lemma.** First we recall the two versions of the enriched Yoneda lemma:

Lemma 6.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-category and let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a $\mathcal{V}$-functor. Then for any $x \in \mathcal{C}$ we have isomorphisms*

V1. $[\mathcal{C}^{op}, \mathcal{V}]_0(\mathcal{C}(-, x), F) \cong \mathcal{V}_0(I, Fx)$ *in* **Set**

V2. $[\mathcal{C}^{op}, \mathcal{V}](\mathcal{C}(-, x), F) \cong Fx$ *in* $\mathcal{V}$

natural in x and F.

6.2. **All Concepts.** Given a $\mathcal{V}$-functor $j : \mathcal{B} \to \mathcal{A}$, we define an $\mathcal{A}$-$\mathcal{B}$-bimodule $j^\bullet$ and a $\mathcal{B}$-$\mathcal{A}$ bimodule $j_\bullet$ by

$$j^\bullet(a, b) := \mathcal{A}(jb, a), \qquad j_\bullet(b, a) := \mathcal{A}(a, jb).$$

A $j^\bullet$-weighted limit for $E : \mathcal{A} \to \mathcal{M}$ satisfies the isomorphisms

$$\begin{aligned} \mathcal{M}(m, \lim{}^J Eb) &\cong [\mathcal{A}, \mathcal{V}](j^\bullet(-, b), \mathcal{M}(m, E-)) \\ &\cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(jb, -), \mathcal{M}(m, E-)) \\ &\cong \mathcal{M}(m, Ejb) \end{aligned}$$

for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$, the last isomorphism given by the Yoneda lemma. Applying Yoneda again, we obtain an isomorphism

$$\lim{}^J E \cong Ej \text{ in } [\mathcal{B}, \mathcal{M}].$$

So precomposition is a $j^\bullet$-weighted limit. The isomorphism (5.1) is

$$[\mathcal{B}, \mathcal{M}](D, Ej) \cong [\mathcal{A}, \mathcal{M}](\mathrm{colim}^{j^\bullet} D, E)$$

in this case, which is the defining isomorphism of the left Kan extension $\mathrm{Lan}_j D$ of $D : \mathcal{B} \to \mathcal{M}$ along $j : \mathcal{B} \to \mathcal{A}$. So left Kan extensions are $j^\bullet$-weighted colimits.

[4]Assuming MacLane's famous assertion that "all concepts are Kan extensions" in [2, ch. 10]. See below.

Repeating this argument for $j_\bullet$, we see that precomposition is also a $j_\bullet$-weighted colimit and the right Kan extension $\mathrm{Ran}_j D$ of $D : \mathcal{B} \to \mathcal{M}$ along $j : \mathcal{B} \to \mathcal{A}$ is a $j_\bullet$-weighted limit. Hence, all concepts are weighted limits or colimits.

6.3. **Fubini and Applications.** Let $D : \mathcal{A} \to \mathcal{M}$ be a diagram and $J : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ and $K : \mathcal{B} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ be bimodules. For all $m \in \mathcal{M}$, we have isomorphisms

$$\begin{aligned}\mathcal{M}(m, \lim^K \lim^J D) &\cong \mathrm{Hom}_{\mathcal{B}}(K, \mathcal{M}(m, \lim^J D))\\ &\cong \mathrm{Hom}_{\mathcal{B}}(K, \mathrm{Hom}_{\mathcal{A}}(J, \mathcal{M}(m, D)))\\ &\cong \mathrm{Hom}_{\mathcal{A}}(J \otimes_{\mathcal{B}} K, \mathcal{M}(m, D))\\ &\cong \mathcal{M}(m, \lim^{J \otimes_{\mathcal{B}} K} D)\end{aligned}$$

by three applications of the defining isomorphism for weighted limits and one use of the adjunction of Section 2.3. By the Yoneda Lemma, we conclude that

$$\lim^K \lim^J D \cong \lim^{J \otimes_{\mathcal{B}} K} D,$$

a Fubini theorem of sorts for weighted limits.

There is a dual theorem for weighted colimits. Let J, K, D be as above. Again, for all $m \in \mathcal{M}$, we have isomorphisms

$$\begin{aligned}\mathcal{M}(\mathrm{colim}^J \mathrm{colim}^K D, m) &\cong \mathrm{Hom}_{\mathcal{B}^{\mathrm{op}}}(J, \mathcal{M}(\mathrm{colim}^K D, m))\\ &\cong \mathrm{Hom}_{\mathcal{B}^{\mathrm{op}}}(J, \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(K, \mathcal{M}(D, m)))\\ &\cong \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(J \otimes_{\mathcal{B}} K, \mathcal{M}(D, m))\\ &\cong \mathcal{M}(\mathrm{colim}^{J \otimes_{\mathcal{B}} K} D, m)\end{aligned}$$

and all isomorphisms are natural. By Yoneda, we conclude that

$$\mathrm{colim}^J \mathrm{colim}^K D \cong \mathrm{colim}^{J \otimes_{\mathcal{B}} K}.$$

In the special case of $\mathcal{A} = \mathcal{B} = \mathcal{I}$, the Fubini theorem for weighted limits gives us the following isomorphism for powers

$$\pitchfork(K, \pitchfork(J, D)) \cong \pitchfork(J \otimes K, D),$$

where $J, K \in \mathcal{V}$ and $D \in \mathcal{M}$. The dual statement for weighted colimits has the corollary that

$$K \odot (J \odot D) \cong (K \otimes J) \odot D.$$

For another important application, let $j : \mathcal{B} \to \mathcal{A}$ be a functor and let $a : \mathcal{B} \to \mathcal{A}$ be a constant functor at some object $a \in \mathcal{A}$. Let $D : \mathcal{B} \to \mathcal{M}$ be a diagram. As we saw above, $\mathrm{colim}^{j^\bullet} D$ is the left Kan extension $\mathrm{Lan}_j D$ of D along j. An $a_\bullet$ weighted colimit is precomposition, so

$$\mathrm{colim}^{a_\bullet} \mathrm{colim}^{j^\bullet} D \cong (\mathrm{Lan}_j D)(a),$$

the left Kan extension of D along j evaluated at $a \in \mathcal{A}$. The Fubini isomorphism

$$\mathrm{colim}^{a_\bullet} \mathrm{colim}^{j^\bullet} D \cong \mathrm{colim}^{j^\bullet \otimes a_\bullet} D$$

tells us that Kan extensions can be calculated pointwise.

References

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