First-Price Sealed-Bid Auctions

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We introduce the first-price, sealed-bid auction. This auction format requires auction winners to pay their bid. We go over the strategic consequences of this payment rule.

1 *The First-Price Sealed-Bid Auction*

We have $N = \{1, \ldots, n\}$ bidders, where each bidder has a private valuation v_i drawn from distribution F_i [1] with strictly positive density, $f_i : T_i \to \mathbb{R}_{>0}$, for a good up for sale by the auctioneer, agent 0. In a first-price, sealed-bid auction, each agent has a type $v_i \in T_i$, and submits a bid b_i to the auctioneer, without revealing what the contents of the bid are to any other bidder. Using the submitted bids $\mathbf{b} = (b_1, \ldots, b_n)$, the auctioneer then decides on the winner. In this setting, the winner of the auction is the bidder with the highest bid, $\max_{i \in N} b_i$. [2] Let $i^* \in \arg\max_N b_i$ denote the winner of the auction. Should there be a tie, i^* is randomly determined by the auctioneer. The auctioneer could, for example, select a bidder uniformly at random, or break ties using an alphanumeric schema. The winner is allocated with probability $x_{i^*}(b_{i^*}, \mathbf{b}_{-i^*}) = 1$, and all other bidders $i^* \neq i \in N$ are allocated with probability $x_i(b_i, \mathbf{b}_{-i}) = 0$. The winner, bidder i^*, pays her bid, b_{i^*}, and all other bidders pay nothing.

[1] Here, we assume that valuations follow the IPV model.

[2] This is the nth order statistic, $b_{(n)}$. The kth order statistic, $X_{(k)}$, is the kth smallest value amongst a set of n random variables $X_1, \ldots, X_n$, so $\min_{i \in N} = b_{(1)}$.

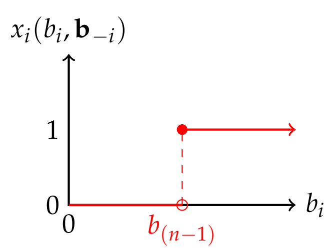

Figure 1: Bidder i's allocation function for a given $\mathbf{b}_{-i}$. Once bidder i submits a bid larger than the second highest bid, $b_{(n-1)}$, she is allocated with probability $x_i(b_i, \mathbf{b}_{-i}) = 1$.

1.1 *Computational Complexity*

With n bidders, determining the winner takes $O(n)$ time, as this is the complexity of an arg max function. Given the winner, we can determine payments in $O(1)$ time. Therefore, this auction can be run in polynomial time.

1.2 *Solving for the Bayes-Nash Equilibrium*

In this setting, we assume bidders have a risk-neutral, quasi-linear utility functions of the form

$$u_i(b_i, \mathbf{b}_{-i}) = v_i x_i(b_i, \mathbf{b}_{-i}) - p_i(b_i, \mathbf{b}_{-i}). \tag{1}$$

This means, that the resulting payoff each bidder receives is

$$u_i(b_i, \mathbf{b}_{-i}) = \begin{cases} v_i - b_i, & \text{if } i = i^* \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

If all bidders bid their valuation (i.e., they are truthful in their reports), then all bidders will receive zero payoff. Consequently, it is in each bidder's interest to shade their bid: each bidder will want to report a bid so that in expectation, they receive positive payoff. That is, bidders are solving for the following:

$$b_i = \max_{x \in \mathbb{R}} (v_i - x) \prod_{i \neq j \in N} \Pr\left(x \geq b_j\right). \tag{3}$$

In settings where there are many bidders, one natural thing to assume is that bidders shade less. When there are many bidders, it becomes more likely that someone with a high type is amongst them. Furthermore, we expect that bids increase as a bidder's type increases: it makes no sense for a bidder with high type to bid close to nothing, and vice-versa. We now show what a symmetric Bayes-Nash equilibrium looks like, and formally show how much each bidder should shade their bids by. Here, we assume that all $n > 1$ bidders are symmetric, meaning that $F_i = F_j$, for all $i, j \in N$, distributions are continuous, and, for simplicity of presentation, that the smallest valuation any bidder can have is 0. Finally, we abuse notation slightly, and express bids as functions.

Theorem 1.1. *In a symmetric Bayes-Nash equilibrium of a first-price, sealed bid auction with $n > 1$ symmetric bidders where valuations are iid random variables, each bidder i will bid*

$$b(v_i) = \begin{cases} v_i - \frac{\int_0^{v_i} F(x)^{n-1}\,\mathrm{d}x}{F(v_i)^{n-1}}, & \text{if } v_i > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Proof. To show that the given symmetric strategy is optimal, we have to show that it maximizes expected utility. Since we are looking for a symmetric Bayes-Nash equilibrium, each bidder must decide on how to use a bid function $b : \mathbb{R} \to \mathbb{R}$ based on her type v_i. That is, each bidder will place bid $b_i = b(v_i)$. The expected payoff of bidder i for using strategy b with type v_i, $u_i(b, v_i)$, is

$$u_i(b, v_i) = (v_i - b_i) \prod_{i \neq j \in N} \Pr\left(b_i \geq b_j\right) \tag{5}$$

$$= (v_i - b_i) \prod_{i \neq j \in N} \Pr\left(v_i \geq v_j\right)$$

$$= (v_i - b_i)\, F\left(b^{-1}(b_i)\right)^{n-1}.$$

Every bidder is using the same increasing bidding function.

$b^{-1}(b(v_i)) = v_i$. The probability of winning is a function of what one bids, which is why we explicitly write this out with the bid function.

We seek to maximize the expected utility, so we take derivatives and solve for the first order condition:

$$\frac{\mathrm{d}u_i(b, v_i)}{\mathrm{d}b_i} = \frac{\mathrm{d}\,(v_i - b_i)\, F\left(b^{-1}(b_i)\right)^{n-1}}{\mathrm{d}b_i}.$$

Notice that if we tried to start with $u_i(b, v_i) = (v_i - b_i)\, F(v_i)^{n-1}$, we would have $u_i'(b, v_i) = -F(v_i)^{n-1}$.

Recall that for two functions $f(x)$ and $g(x)$, chain rule tells us that

$$[f(g(x))]' = f'(g(x))g'(x), \tag{6}$$

and that for function inverse settings [3] ,

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}. \tag{7}$$

[3] By chain rule, $1 = [f(f^{-1}(x))]' = f'(f^{-1}(x))[f^{-1}(x)]'$.

This means

$$\frac{\mathrm{d}F\left(b^{-1}(b_i)\right)^{n-1}}{\mathrm{d}b_i} = (n-1)\, F\left(b^{-1}(b_i)\right)^{n-2} f\left(b^{-1}(b_i)\right) \frac{1}{b'(b^{-1}(b_i))}. \tag{8}$$

We simplify the notation, as $b^{-1}(b_i) = v_i$, leaving us with

$$\frac{\mathrm{d}u_i(b, v_i)}{\mathrm{d}b_i} = (v_i - b_i)\,(n-1)\, F\left(b^{-1}(b_i)\right)^{n-2} f\left(b^{-1}(b_i)\right) \frac{1}{b'(b^{-1}(b_i))} - F\left(b^{-1}(b_i)\right)^{n-1} \tag{9}$$

$$= (v_i - b(v_i))\,(n-1)\, F(v_i)^{n-2} f(v_i) \frac{1}{b'(v_i)} - F(v_i)^{n-1} \tag{10}$$

$$= 0. \tag{11}$$

What's left is to solve a differential equation of form

$$[b(v_i)]'\, F(v_i)^{n-1} = (v_i - b(v_i)) \left[F(v_i)^{n-1}\right]'. \tag{12}$$

One way to approach this is to observe that

$$\left[b(v_i) F(v_i)^{n-1}\right]' = b(v_i) \left[F(v_i)^{n-1}\right]' + [b(v_i)]'\, F(v_i)^{n-1}, \tag{13}$$

so we have

$$\left[b(v_i) F(v_i)^{n-1}\right]' = b(v_i) \left[F(v_i)^{n-1}\right]' + (v_i - b(v_i)) \left[F(v_i)^{n-1}\right]' \tag{14}$$

$$= v_i \left[F(v_i)^{n-1}\right]'. \tag{15}$$

We want to find a function that tells us how to bid for some bidder i. That is, something that bids $b(v_i)$. Currently, the notation tells us to take derivatives with respect to v_i, but we're going to want to integrate up to it, so change the notation to read

$$\left[b(x)F(x)^{n-1}\right]' = x\left[F(x)^{n-1}\right]'. \tag{16}$$

By the fundamental theorem of calculus[4],

$$\int_0^{v_i}\left[b(x)F(x)^{n-1}\right]' \, \mathrm{d}x = b(x)F(x)^{n-1}\Big|_0^{v_i} \tag{17}$$

$$= b(v_i)F(v_i)^{n-1}. \tag{18}$$

Integrate the right-hand side by parts[5]:

$$\int_0^{v_i} x\left[F(x)^{n-1}\right]' \, \mathrm{d}x = xF(x)^{n-1}\Big|_0^{v_i} - \int_0^{v_i} F(x)^{n-1}\, \mathrm{d}x \tag{19}$$

$$= v_iF(v_i)^{n-1} - \int_0^{v_i} F(x)^{n-1}\, \mathrm{d}x. \tag{20}$$

With the left-hand side and right-hand side, we now arrive at a much simpler equation to work with:

$$b(v_i)F(v_i)^{n-1} = v_iF(v_i)^{n-1} - \int_0^{v_i} F(x)^{n-1}\, \mathrm{d}x. \tag{21}$$

Dividing by $F(v_i)^{n-1}$ gets us a function of form

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{n-1}\,\mathrm{d}x}{F(v_i)^{n-1}}. \tag{22}$$

We know that our bid function evaluated at v_i is a critical point. What's left? The second order conditions, which tell us whether we are minimizing, or maximizing. Formally, we need to verify that

$$\frac{\mathrm{d}^2 u_i(b, v_i)}{\mathrm{d}b_i^2} < 0. \tag{23}$$

Rather than taking the straight-forward approach, and differentiating by b_i twice, we will approach this subject in a slightly different manner. Given that each bidder is using the same bidding function that we have derived, we can ask how should a bidder be using it? If bidder i submits bid $b(t_i)$, then it should be that utility, $u_i(b, v_i, t_i)$, is maximized when $t_i = v_i$. This means that the first derivative of the utility function with respect to a reported type t_i should be zero when $t_i = v_i$, and the second derivative should be strictly negative when $t_i = v_i$. Alternatively, if we find that the first derivative is positive when $t_i < v_i$, and negative when $t_i > v_i$, then we know bidding $b(v_i)$ maximizes expected utility.

The derivative of the bid function is

$$[b(v)]' = \left[v - \frac{\int_0^{v} F(x)^{n-1}\,\mathrm{d}x}{F(v)^{n-1}}\right]' \tag{24}$$

[4] $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$

[5] $\int_a^b u(x)v'(x)\,\mathrm{d}x = u(x)v(x)\big|_a^b - \int_a^b v(x)u'(x)\,\mathrm{d}x$

$$= 1 - \frac{\left[F(v)^{n-1}\right]\left[\int_0^v F(x)^{n-1}\,\mathrm{d}x\right]' - \left[F(v)^{n-1}\right]'\left[\int_0^v F(x)^{n-1}\,\mathrm{d}x\right]}{\left[F(v)^{n-1}\right]^2} \tag{25}$$

$$= 1 - \frac{\left[F(v)^{n-1}\right]\left[F(v)^{n-1}\right] - \left[(n-1)f(v)F(v)^{n-2}\right]\left[\int_0^v F(x)^{n-1}\,\mathrm{d}x\right]}{\left[F(v)^{n-1}\right]^2} \tag{26}$$

$$= \frac{\left[(n-1)f(v)F(v)^{n-2}\right]\left[\int_0^v F(x)^{n-1}\,\mathrm{d}x\right]}{\left[F(v)^{n-1}\right]^2}. \tag{27}$$

Plug this into the first derivative of the utility function for bidding with type $t_i \in T_i$:

$$\frac{\mathrm{d}u_i(b, v_i, t_i)}{\mathrm{d}b_i} = (v_i - b(t_i))\,(n-1)\,F(t_i)^{n-2}\,f(t_i)\,\frac{1}{b'(t_i)} - F(t_i)^{n-1} \tag{28}$$

$$= \left(v_i - \left(t_i - \frac{\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x}{F(t_i)^{n-1}}\right)\right)(n-1)\,F(t_i)^{n-2}\,f(t_i)\left(\frac{\left[F(t_i)^{n-1}\right]^2}{\left[(n-1)f(t_i)F(t_i)^{n-2}\right]\left[\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x\right]}\right) - F(t_i)^{n-1} \tag{29}$$

$$= \left((v_i - t_i) + \left(\frac{\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x}{F(t_i)^{n-1}}\right)\right)\left(\frac{\left[F(t_i)^{n-1}\right]^2}{\left[\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x\right]}\right) - F(t_i)^{n-1} \tag{30}$$

$$= (v_i - t_i)\left(\frac{\left[F(t_i)^{n-1}\right]^2}{\left[\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x\right]}\right) + \left(F(t_i)^{n-1}\right) - F(t_i)^{n-1} \tag{31}$$

$$= (v_i - t_i)\left(\frac{\left[F(t_i)^{n-1}\right]^2}{\left[\int_0^{t_i} F(x)^{n-1}\,\mathrm{d}x\right]}\right). \tag{32}$$

We see that setting $t = v_i$ is a critical point, and

$$\begin{cases} u_i'(b, v_i, t) > 0 & \text{if } t < v_i \\ u_i'(b, v_i, t) < 0 & \text{if } t > v_i. \end{cases} \tag{33}$$

Therefore, if everyone is using this same bidding function, it is optimal for all bidders to use it *truthfully*. □

Notice that the optimality of a bid is based on how other bidders act. Since strategies are based on how other bidders act, we say that there is no dominant strategy in a first-price, sealed-bid auction.

1.3 Properties of the Bid Function

Earlier we have argued that a bid function should be increasing with respect to types, and that bids should be shaded less as the number of bidders increase. Here we show formally that our solution does indeed have these attributes.

Shading The bid function we have derived, Equation (4), can be restated as

$$b(v_i) = \begin{cases} v_i - \int_0^{v_i} \left(\frac{F(x)}{F(v_i)}\right)^{n-1} \mathrm{d}x, & \text{if } v_i > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

We have said that the density function is strictly positive, so the distribution function is increasing. This means that for any $x \in (0, v_i]$, $\frac{F(x)}{F(v_i)} \leq 1$, so as the number of bidders grows, the quantity $\left(\frac{F(x)}{F(v_i)}\right)^{n-1}$ decreases. In the limit, as n goes to infinity, this term approaches zero.

Monotonicity We now observe that our bid function is monotonic (i.e., $b(x) \leq b(y)$ for $x \leq y$. Analytically, we can see this by taking the derivative and noticing that $[b(v)]' > 0$ when $v > 0$. Combined with the observation that $b(\epsilon) > b(0)$ for any $\epsilon > 0$, we have that bids are strictly increasing.

$$[b(v)]' = \frac{\left[(n-1)f(v)F(v)^{n-2}\right]\left[\int_0^v F(x)^{n-1}\,\mathrm{d}x\right]}{\left[F(v)^{n-1}\right]^2} \tag{35}$$

$$> 0. \tag{36}$$

1.4 An Alternative Solution Approach

Earlier we solved for the symmetric Bayes-Nash equilibrium directly. We now present an alternative method. Rather than go through the math, one can *guess* what the optimal strategy is, and then *check* that it is optimal. We give an example of this below, for n symmetric bidders, where valuations are drawn from a uniform $U(0,1)$ distribution.

Example 1.2. Assume that the optimal bidding strategy for each bidder is

$$b(v_i) = \frac{n-1}{n} v_i. \tag{37}$$

The probability that bidder i wins the auction is

$$\Pr\left(x_i(b_i, \mathbf{b}_{-i}) = 1\right) = \Pr\left(\max_{i \neq j \in N} b_j \leq b_i\right) \tag{38}$$

$$= \Pr\left(\max_{i \neq j \in N} \frac{n-1}{n} v_j \leq b_i\right) \tag{39}$$

$$= \Pr\left(\max_{i \neq j \in N} v_j \leq \frac{n}{n-1} b_i\right) \tag{40}$$

$$= \prod_{i \neq j \in N} F\left(\frac{n}{n-1} b_i\right) \tag{41}$$

$$= \prod_{i \neq j \in N} \left(\frac{n}{n-1} b_i\right) \tag{42}$$

$$= \left(\frac{n}{n-1} b_i\right)^{n-1}. \tag{43}$$

The expected utility of each bidder using this strategy is

$$u_i = (v_i - b_i) \Pr\left(x_i(b_i, \mathbf{b}_{-i}) = 1\right) \tag{44}$$

$$= (v_i - b_i) \left(\frac{n}{n-1} b_i\right)^{n-1} \tag{45}$$

$$= (v_i - b_i)\, b_i^{n-1} \left(\frac{n}{n-1}\right)^{n-1}. \tag{46}$$

To show that the bid function is optimal, we now differentiate with respect to b_i. It should be the case that the first derivative is zero when $b_i = \frac{n-1}{n} v_i$, and the second derivative is strictly negative. The first derivative is

$$\frac{\mathrm{d}u_i}{\mathrm{d}b_i} = \left(\frac{n}{n-1}\right)^{n-1} [(v_i - b_i)]' \left[b_i^{n-1}\right] + \left(\frac{n}{n-1}\right)^{n-1} [(v_i - b_i)] \left[b_i^{n-1}\right]' \tag{47}$$

$$= \left(\frac{n}{n-1}\right)^{n-1} [-1] \left[b_i^{n-1}\right] + \left(\frac{n}{n-1}\right)^{n-1} [(v_i - b_i)] \left[(n-1) b_i^{n-2}\right] \tag{48}$$

$$= 0. \tag{49}$$

This simplifies to

$$0 = [-1]' \left[b_i^{n-1}\right] + [(v_i - b_i)] \left[(n-1) b_i^{n-2}\right] \tag{50}$$

$$= -b_i + (v_i - b_i)(n-1) \tag{51}$$

$$= -b_i + v_i n - v_i - b_i n + b_i \tag{52}$$

$$= v_i n - v_i - b_i n \tag{53}$$

$$= v_i (n-1) - b_i n. \tag{54}$$

Plug in the assumed solution to get

$$v_i(n-1) - b_i n = v_i(n-1) - \left(\frac{n-1}{n} v_i\right) n \tag{55}$$

$$= 0. \tag{56}$$

The second derivative is

$$\frac{d^2 u_i}{d b_i^2} = \left(\frac{n}{n-1}\right)^{n-1} (n-1) b_i^{n-3} \left((n-2)v_i - n b_i\right). \tag{57}$$

If a bidder has valuation $v_i = 0$, then the second derivative is zero. This is utility maximizing for a bidder of this type. Notice that ties are zero-probability events, so any bidder with valuation zero is fated to lose. Bidding more than zero results in a probability of having negative utility. Thus, we focus on the setting where $v_i > 0$. The second order conditions are satisfied if

$$(n-2)v_i - n b_i < 0. \tag{58}$$

Plug in the assumed bid function to show that we satisfy the second order condition:

$$\begin{aligned}
(n-2)v_i - n b_i &= (n-2)v_i - n\left(\frac{n-1}{n} v_i\right) && (59)\\
&= (n-2)v_i - (n-1)\, v_i && (60)\\
&= -v_i && (61)\\
&\leq 0. && (62)
\end{aligned}$$

For all $v_i \in (0, 1]$, the first and second order conditions are satisfied, so we conclude that bidding $b_i = \frac{n-1}{n} v_i$ is optimal.

1.5 *Total Welfare*

Let the utility function of the auctioneer be

$$\sum_{i \in N} p_i\left(b_i, \mathbf{b}_{-i}\right). \tag{63}$$

Summing over the utility of the bidders and the auctioneer gives us the total welfare of our agents:

$$\left[\sum_{i \in N} v_i x_i(b_i, \mathbf{b}_{-i}) - p_i(b_i, \mathbf{b}_{-i})\right] + \left[\sum_{i \in N} p_i\left(b_i, \mathbf{b}_{-i}\right)\right] = \sum_{i \in N} v_i x_i(b_i, \mathbf{b}_{-i}). \tag{64}$$

In a Bayes-Nash equilibrium, the bidder with the highest bid also values the good the most:

$$x_i(b_i, \mathbf{b}_{-i}) = \begin{cases} 1 & \text{if } v_i \geq v_j, \forall j \in N \\ 0 & \text{otherwise.} \end{cases} \tag{65}$$

This allocation scheme maximizes total welfare, so in a Bayes-Nash equilibrium, the first-price, sealed-bid auction is a welfare maximizing auction.

1.6 The Auctioneer and Revenue

We now turn to the auctioneer, who has utility

$$u_0(b_i, \mathbf{b}_{-i}) = \sum_{i=1}^{n} p_i(b_i, \mathbf{b}_{-i}). \tag{66}$$

The utility of the auctioneer is based on the distribution of the nth order statistic of submitted bids, $b_{(n)}$. While you can immediately invoke the density function of the kth order statistic to arrive at the result,

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \left(F(x)\right)^{k-1} \left(1 - F(x)\right)^{n-k} f(x), \tag{67}$$

we will give a more intuitive explanation of the expected revenue of the first-price, sealed-bid auction, R_1, with the following Lemma:

Lemma 1.3. *In the symmetric bidder setting, where distributions have support $[0, \overline{v}]$, the total expected revenue generated by a a first-price, sealed-bid auction is:*

$$R_1 = \int_0^{\overline{v}} b(z) n f(z) F(z)^{n-1} \, \mathrm{d}z. \tag{68}$$

Proof. The expected revenue generated is based on the distribution of the highest type, $v_{(n)}$. The probability that type v is the highest type amongst the bidders is $F(v)^n$. This distribution has corresponding density

$$[F(v)^n]' = n f(v) F(v)^{n-1}. \tag{69}$$

Thus, the expected revenue is

$$R_1 = \int_0^{\overline{v}} b(z) n f(z) F(z)^{n-1} \, \mathrm{d}z. \tag{70}$$

□