

## On Absolute Units, I: Choices

Frank Wilczek



In 1899 Max Planck proposed what he called a system of absolute units, based on the speed of light  $c$ , Newton's gravitational constant  $G$ , and his newly minted quantum of action  $\hbar$ . From these, by taking powers and ratios, one can manufacture units of mass  $M_{\text{Planck}} \equiv \sqrt{\hbar c/G} = 2.2 \times 10^{-5}$  gram, length  $L_{\text{Planck}} \equiv \sqrt{\hbar G/c^3} = 1.6 \times 10^{-33}$  centimeter, and time  $T_{\text{Planck}} \equiv \sqrt{\hbar G/c^5} = 5.4 \times 10^{-44}$  second. Upon adopting those units as standards, one can express any physical quantity in purely numerical terms. For example, the near-Earth acceleration due to gravity is  $g = 9.8 \times 10^2$  cm/sec<sup>2</sup> =  $1.8 \times 10^{-51} L_{\text{Planck}}/T_{\text{Planck}}^2$ .

Obviously Planck units are not very handy for practical purposes. But Planck admired the fact that they are based on quantities that appear in (presumably) universal physical laws. By using them, one could convey the values of, say, your mass (or weight!), height, and age to a friend in Andromeda purely by transmitting information. You wouldn't need to send a standard kilogram, a meterstick, or a solar system to standardize astronomical time, much less your physical body. Just numbers. Your friend would be able to reconstruct the units by consulting the laws of physics. That's why Planck called them absolute units.

During the 20th century, as physics developed, Planck's construction took on ever greater significance. We physicists came to understand that each of the quantities  $c$ ,  $G$ , and  $\hbar$  that he used plays the role of a conversion factor that is essential to implementing a profound physical concept. For example, special relativity postulates symmetry operations (Lorentz transformations) that mix space and time. However, space and time are measured in different units, so for this symmetry concept to make sense, there must be a conversion factor between them. That role is fulfilled by  $c$ . Similarly, general relativity postu-

lates that spacetime curvature is induced by energy-momentum density; but curvature and energy are measured in different units, and  $G$  must be brought in as a conversion factor. Quantum theory postulates an inverse relation between wavelength and momentum and proportionality between frequency and energy, as aspects of wave-particle duality; but the pairs of quantities are measured in different units, and  $\hbar$  must be brought in as a conversion factor. Within this circle of ideas  $c$ ,  $G$ , and  $\hbar$  attain a new status. They make possible profound principles of physics that couldn't be formulated meaningfully without them.

More than 25 centuries ago, Pythagoras proclaimed "All things are number." It's hard to be sure exactly what he had in mind; probably part of his thinking was a form of atomism, based on the idea that you could build shapes from numbers, a concept the still-used terminology of square, cube, and triangular numbers recalls. Taken literally the assertion surely goes too far, since numbers by themselves can't provide physical units. But if we loosen up just a bit and use Planck's construction, then Pythagoras's vision becomes a logically coherent possibility. In earlier columns (PHYSICS TODAY, June 2001, page 12; November 2001, page 12; August 2002, page 10) I've discussed our partial, yet impressive, progress toward realizing it. Here I will take a very different perspective by asking: Should we let Planck have the last word on absolute units?

### Alternatives

You might choose to construct absolute units in many other ways. In place of one or two of  $c$ ,  $G$ , and  $\hbar$  you might consider using the quantum of charge  $e$ , or either of the masses  $m_e$  or  $m_p$  of the electron and proton. Surely each of these quantities qualifies as a fundamental parameter of physics. You might object that  $m_p$  is a secondary parameter, since we now appreciate that the proton is a complex entity, but that objection is superficial, as I'll discuss presently. The quantum of charge  $e$  appears as the conversion factor between the energy (or, more

accurately, amplitude) in a field and the force it exerts, while masses appear as conversion factors in the oldest law of all,  $F = ma$ —and if my preceding columns have made you leery of that one, take  $E = mc^2$ . (For sophisticates:  $e$  mediates the conversion between particle number and gauge field curvature, so in electromagnetism it plays a role closely analogous to that of  $G$  in general relativity.)

Tasked with choosing three entries from the menu ( $c$ ,  $G$ ,  $\hbar$ ,  $e$ ,  $m_e$ ,  $m_p$ ), Chinese-restaurant style, we discover six candidate unit systems including  $m_e$ , an equal number including  $m_p$ , and four, including Planck's, that don't contain either mass. A few of these candidate systems don't work: The choice  $c$ ,  $\hbar$ ,  $e$  fails to generate three independent units, because there is a purely numerical relationship among those parameters. Specifically, we have  $e^2/\hbar c = .092$ ; indeed, this is nothing but the fine-structure constant of atomic theory (times  $4\pi$ ). The choice  $G$ ,  $e$ ,  $m_e$  fails likewise, because  $Gm_e^2/e^2 = 1.9 \times 10^{-44}$ . That leaves five systems for either choice of mass, and three mass-free systems. I'll now elaborate on three of the most interesting.

The classical system  $G$ ,  $c$ ,  $e$  of absolute units might very well have been proposed instead of Planck's choice  $G$ ,  $c$ ,  $\hbar$  near the beginning of the 20th century, or even earlier. J. J. Thomson discovered the electron in 1897 and measured  $e/m_e$ . A direct separate measurement of  $e$  awaited Robert Millikan's oil-drop experiment in 1912, but its value might have been inferred much earlier from the value of Avogadro's number,  $N_A$ , which had been estimated—though, to be sure, not with great accuracy, from the kinetic theory of gases—and the measured value of Faraday's electrochemical constant,  $N_A e$ .

Viewed concretely, the classical system is not very different from Planck's. Indeed, due to the above-mentioned numerical connection among  $c$ ,  $\hbar$ , and  $e$ , once you have  $c$  available, it is a matter of simple algebra to trade  $e$  for  $\hbar$  or vice versa. The existence of the classical system raises an extremely interesting conceptual point, however. If we take the

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Pythagorean program at face value, taking off from the classical system of absolute units, then we will strive to derive the numerical value of  $\hbar$ , and thereby presumably of quantum behavior, from quantities that can be defined entirely classically!

### Atomic and strong units

A very important system of absolute units, which is based on  $\hbar$ ,  $e$ ,  $m_e$ , defines what are often called atomic units. (A fine point: I've used the reduced Planck constant and the rationalized charge, which is somewhat unconventional in this context.) We have  $L_{\text{atomic}} \equiv \hbar^2/m_e e^2 = 4.2 \times 10^{-10}$  centimeter,  $T_{\text{atomic}} \equiv \hbar^3/m_e e^4 = 1.5 \times 10^{-19}$  second, and of course  $M_{\text{atomic}} \equiv m_e = 9.1 \times 10^{-28}$  gram. In these units, the Schrödinger equation for interacting electrons subject to the electric fields of infinitely massive nuclei becomes purely numerical. By focusing on stable configurations, in which the force at the nuclei vanishes, one can also build models of molecular states. All those physical quantities, which form the core of structural chemistry, are then determined as pure numbers by perfectly definite algorithms. Thus the  $\hbar$ ,  $e$ ,  $m_e$  system provides the natural units in which to express the sizes, shapes, and electronic spectra of atoms and molecules. We have thereby defined an ideally Pythagorean theory, which also happens to provide a splendid model for important aspects of the real world.

I'm particularly fond of the system based on  $\hbar$ ,  $c$ , and  $m_p$ . We might call these "strong units," because they are so natural for the description of the strong interaction. Indeed,  $L_{\text{strong}} \equiv \hbar/m_p c = 2.1 \times 10^{-14}$  centimeter is nothing but the proton's Compton radius, its intrinsic fuzziness in position due to quantum fluctuations, while  $T_{\text{strong}} \equiv \hbar/m_p c^2 = 7.2 \times 10^{-25}$  second is a characteristic strong interaction time scale, and of course  $M_{\text{strong}} \equiv m_p = 1.7 \times 10^{-24}$  gram speaks for itself. From a theoretical perspective, we should of course view these units in the light of quantum chromodynamics, the fundamental theory of the strong interaction. Quantum mechanics and special relativity are so thoroughly woven into the fabric of QCD that  $\hbar$  and  $c$  are inexorable choices. The proton mass  $m_p$  is more negotiable. Its great virtues are that it is precisely defined, accurately measured, and useful in such applications as nuclear physics. On the other hand, the position of  $m_p$  in fundamental theory is not exactly central. The proton mass  $m_p$  does not appear directly in the basic equations that de-

fine QCD. Not even  $p$  does! We understand that the proton  $p$  is a complex entity assembled from quarks and gluons, and that the value  $m_p$  of its mass includes, in addition to the dominant contribution from intrinsic QCD dynamics, small contributions from the masses of quarks and from electromagnetism. We could sidestep these issues by using  $\Lambda_{\text{QCD}}$ , a parameter with dimensions of mass that characterizes the rate of approach to asymptotic freedom, in place of  $m_p$ . But  $\Lambda_{\text{QCD}}$ , for the value it brings in theoretical purity, extracts a high price: It lacks intuitive appeal, and it is both much more difficult to measure precisely than  $m_p$  and less useful for applications. Fortunately, the numerical difference between  $m_p$  and  $\Lambda_{\text{QCD}}$  is fairly modest, and on balance it seems wiser to use  $m_p$ .

An appealing feature of atomic and strong units, in contrast to Planck units, is that the characteristic length, time, and mass can be constructed without taking square roots. It is disconcerting to imagine that we must extract roots in order to express the basic units in terms of fundamental parameters. (Sophisticates will recognize that extracting roots is a nonanalytic procedure, in the technical sense.) The fact that  $G$ ,  $\hbar$ ,  $c$  can be expressed in terms of  $m_p$ ,  $\hbar$ ,  $c$  without extracting roots, but not vice versa, on the face of it suggests that the strong units are more fundamental than Planck units. (I find it remarkable that a similar conclusion is suggested by string theory, where the closed-string gravitational coupling naturally appears as the square of the open-string gauge field coupling.)

### Admitting a weight problem

The usual premise of dimensional analysis is that if naturally defined

quantities are calculated in naturally defined units, the values will be of order unity. Upon surveying the various units we've defined, we discover that we've got some serious explaining to do, because the factors relating our various "natural" units are often very far from unity. The source of the problem is that some very large pure numbers can be constructed from the various parameters we've taken to be fundamental.

Specifically, the ratio  $M_{\text{Planck}}/M_{\text{strong}} = 1.3 \times 10^{19}$ , which is very far from unity. I think we have a remarkably compelling idea about how this particular large number arises. The essence of the matter is that the change of the effective coupling with energy (or alternatively with mass, or momentum, or inverse distance—by applying appropriate powers of  $\hbar$  and  $c$ , these measures become fungible) is essential for generating the proton mass. But this change with energy is only logarithmic. So it can require an enormous, exponentially large change in scale to produce the required change. The (natural) logarithm  $\ln M_{\text{Planck}}/M_{\text{strong}}$  is only about  $(2\pi)^2$ , so dimensional analysis is not mocked. These considerations can be made much more precise and quantitative, as I spelled out in earlier columns.

In contrast, we have no comparably compelling idea about the origin of the enormous number  $M_{\text{Planck}}/M_{\text{atomic}} = 2.4 \times 10^{22}$ . If you'd like to humble someone who talks glibly about the Theory of Everything, just ask about it, and watch 'em squirm. To double the fun, ask why  $M_{\text{strong}}/M_{\text{atomic}} = 1.8 \times 10^3$  is not nearly so far from unity. And to triple the fun, ask why that ratio isn't very close to unity, either.

With these problems, we've sighted the visible tip of an iceberg, which I'll steam toward next time. ■

