SELECTED RUSSIAN PAPERS ON GAME THEORY

1959-1965

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PREFACE

Game theorists are aware of considerable interest in game theory that has developed in the Soviet Union over the last years. Language difficulties, however, have stood in the way of proper acquaintance with the original papers though some results have become known in summary form. In order to overcome some of this gap the following papers are being made available. They will immediately prove the high quality of work done in the Soviet Union and they should stimulate further publication of translations.

The papers I to XI were translated by Kiyoshi Takeuchi, and XII to XIV by Eugene Wesley.

Editing work was done by Louis Billera, Daniel Cohen and Richard Cornwall.

April 1968 — Oskar Morgenstern

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GAMES WITH FORBIDDEN SITUATIONS

N. N. Vorobjev
I. V. Romanovsky

Vestnik Leningradskogo Universiteta,
seria Matematiki, Mekaniki i Astronomy,
No. 2, 1959, pp. 50-54.

1. We recall the definition of a game in normal form (see, for example [1]. Let the following be given: the set $I = \{1,\ldots,n\}$, the elements of which will be called players; for each player $i\in I$, the set S_i, of the strategies of player i, and the vector-function $K(\mu) = (K_1(\mu),\ldots,K_n(\mu))$, defined for all $\mu\in S = S_1 x\ldots x S_n$, the values of which have real components.

The system $< I, \{S_i\}, K >$ is called an n-person game in normal form. The elements of S are called situations in this game. The number $K_i(\mu)$ is called the payoff of player i in the situation μ.

In some games encountered in practice the choice of certain strategies by one of the players narrows the choices of strategies of the other players. In addition the vector-function $K(\mu)$ mentioned in the definition of the game sometimes turns out not to be defined on all of S, but only on some of its subsets. In connection with this, we will find that we can benefit by a certain generalization of the concept of a game in normal form, i.e., the concept of a game with forbidden situations. The present note will be devoted to the establshment of certain properties of such games.

2. Let the following be given: the set $I = \{1,\ldots,n\}$; the set S_i for each $i\in I$; the set $M \subset S = S_1 x\ldots x S_n$ (the difference $S\setminus M$ is denoted by N); and the vector-function $K(\mu)$, defined on M.

The system $\Gamma = \langle I, \{S_i\}, M, K \rangle$ is called an n-person game in normal form with forbidden situations. The elements of the set N are called forbidden situations in this connection.

A game $\Gamma_\varphi = \langle I, \{S_i\}, K \rangle$ is said to be an extension of a game with forbidden situations $\Gamma = \langle I, \{S_i\}, M, K \rangle$ if

$$K_\varphi(\mu) = \begin{cases} K(\mu) & \text{if } \mu \in M \\ \varphi(\mu) & \text{if } \mu \in N \end{cases},$$

where φ is a vector valued function defined on N.

If $\varphi(\mu) \equiv c$ on N, then Γ_φ is denoted by Γ_c, and K by K_c.

A game Γ in normal form is said to be zero sum if for all μ in the domain of definition of $K(\mu) = (K_1(\mu), \ldots, K_n(\mu))$,

$$\sum_{i=1}^{\mu} K_i(\mu) = 0 .$$

A game in normal form is said to be finite if all the sets S_i are finite sets.

For the rest of this paper only zero sum finite two-person games in normal form (with forbidden situations) will be considered. We will refer to these simply as games (with forbidden situations). Such games in normal form, as is known (see, for example, [2]), can be described by matrices which have as the entry in the i-th row and j-th column the payoff of player 1 in the situation where he chooses his own i-th strategy and player 2 chooses his own j-th strategy. If the payoff matrix of such a game is A, we shall henceforth without confusion use $K(\{i,j\})$ to mean a_{ij} instead of the vector $(a_{ij}, -a_{ij})$.

Each probability measure on the set of strategies of a given player is called a mixed strategy for this player. We denote the set of all mixed strategies of player 1 (player 2) by P (Q). It is known [2] that the minimaxes

$$\min_{q\epsilon Q} \max_{p\epsilon P} \sum_{i,j} K((i,j)) \, p_i q_j$$

and

$$\max_{p\epsilon P} \min_{q\epsilon Q} \sum_{i,j} K((i,j)) \, p_i q_j$$

exist and are equal. Their common value is called the value of the game. The value of the game with matrix A is denoted by v_A .

We further denote the value of the game Γ_φ by v_φ .

3. In the following we shall use the simple assertions listed below:

Let the matrices $A = (a_{ij})$ and $B = (b_{ij})$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, be given. Then,

1) If $b_{ij} = a_{ij} + k$, then $v_B = v_A + k$.
2) If $b_{ij} \geq a_{ij}$, then $v_B \geq v_A$.
3) If $|a_{ij} - b_{ij}| < \varepsilon$, then $|v_A - v_B| < \varepsilon$.

4. <u>LEMMA:</u> <u>v_x is a non-decreasing and $v_x - x$ is a non-increasing continuous function of x</u> .

<u>PROOF</u>: It is clear that for $x_2 > x_1$

$$K_{x_2}(\mu) \geq K_{x_1}(\mu) .$$

Hence, from property 2, it follows that $v_{x_2} \geq v_{x_1}$.

Further, it is clear that $K_{x_2}(\mu) - K_{x_1}(\mu) \leq x_2 - x_1$. Therefore, from 3 it follows that $v_{x_2} - v_{x_1} \leq x_2 - x_1$, i.e., $v_{x_2} - x_2 \leq v_{x_1} - x_1$.

The continuity of these functions follows from property 3 .

5. We shall call a point x_0 which has the property that $v_{x_0} = x_0$ a value of the game Γ with forbidden situations. We denote the set of all values of Γ by $V[\Gamma]$.

THEOREM: The set $V[\Gamma]$ is non-empty, convex and closed.

PROOF: Assume

$$x_1 = \max_{\mu \in M} K(\mu) \quad .$$

In the game Γ_{x_1}

$$K_{x_1}(\mu) \leq x_1 \quad ,$$

and consequently, $v_{x_1} \leq x_1$, i.e., $v_{x_1} - x_1 \leq 0$.

Assuming

$$x_2 = \min_{\mu \in M} K(\mu) \ ,$$

we obtain

$$v_{x_2} - x_2 \geq 0 \ .$$

But v_x is a continuous function of x , so that by Cauchy's theorem there exists a point $x_0 \in [x_2, x_1]$ such that $v_{x_0} = x_0$. In this way, $V[\Gamma] \neq \emptyset$.

If x' and x'' are two values of Γ such that $x' < x''$, then for arbitrary $x \in [x', x'']$ we obtain from the lemma that

$$0 = v_{x'} - x' \geq v_x - x \geq v_{x''} - x'' = 0 \ ,$$

so that the set $V[\Gamma]$ is convex.

Finally, the closedness of the set $V[\Gamma]$ follows from the continuity of the function $v_x - x$.

COROLLARY: If $\varphi(\mu) \in V[\Gamma]$ for all $\mu \in N$, then $v_\varphi \in V[\Gamma]$.

6. <u>THEOREM:</u> *In order that $\inf V[\Gamma] = -\infty$, it is necessary and sufficient that N contain one of the columns of the matrix of the game. Similarly, in order that $\sup V[\Gamma] = \infty$, it is necessary and sufficient that N contain one of the rows of the matrix of the game.*

<u>PROOF:</u> We will prove only the first assertion. Let each column contain at least one element not belonging to N. We take $x < \min_{\mu\in M} K(\mu) - 1$ and consider the game Γ_x. Let $p = \{\frac{1}{m},\dots,\frac{1}{m}\}$. Clearly,

$$v_x \geq \min_{1\leq j\leq n} \sum_{i=1}^{m} \frac{1}{m} K_x(\{i,j\}) > x , \qquad \text{i.e.,}$$

$x \notin V[\Gamma]$. $V[\Gamma]$ is bounded below in this case, and the necessity is proved. The sufficiency is evident.

7. The mixed strategy ξ of player 1 in the game Γ_x is called optimal if

$$\sum_{i=1}^{m} K_x(\{i,j\})\ \xi_i \geq v_x , \qquad j=1,\dots,n .$$

The mixed strategy η of player 2 is called optimal if

$$\sum_{j=1}^{n} K_x(\{i,j\})\ \eta_j \leq v_x , \qquad i=1,\dots,m .$$

The set of all optimal strategies of player 1(2) is denoted by $T_x(1)$ (correspondingly, by $T_x(2)$).

<u>THEOREM:</u> *If $x_1, x_2 \in V[\Gamma]$ and $x_1 < x_2$, then*

$$T_{x_2}(1) \subset T_{x_1}(1) ,$$
$$T_{x_2}(2) \supset T_{x_1}(2) .$$

PROOF: Let $\xi \in T_{x_2}(1)$. Then

$$\sum_{j=1}^{m} K_{x_2}(\{i,j\})\, \xi_i \geq x_2 \quad \text{for} \quad j = 1,\ldots,n .$$

Further,

$$K_{x_1}(\{i,j\}) \geq x_1 - x_2 + K_{x_2}(\{i,j\}) ,$$

and on the basis of the previous inequality

$$\sum_{i=1}^{m} K_{x_1}(\{i,j\})\, \xi_i \geq x_1 - x_2 + \sum_{i=1}^{m} K_{x_2}(\{i,j\})\, \xi_i \geq x_1$$

for arbitrary j.

But since $x_1 \in V[\Gamma]$, we have $v_{x_1} = x_1$, and therefore $\xi \in T_{x_1}(1)$,

and the first part of the theorem is proved. The second part is proved symmetrically.

COROLLARY: For $\alpha = 1,2$

$$T_{V[\Gamma]}(\alpha) = \bigcap_{x \in V[\Gamma]} T_x(\alpha) \neq \emptyset .$$

For the proof (for definiteness we take $\alpha = 2$) it is sufficient in the case of $V[\Gamma]$ bounded below to let $x = \min V[\Gamma]$ and note that then

$$T_x(2) = T_{V[\Gamma]}(2) .$$

In the case, where $\inf V[\Gamma] = -\infty$, the result follows by the compactness of the $T_x(2)$.

8. THEOREM: Let $x_1, x_2 \in V[\Gamma]$, $x_1 < x_2$, $\xi \in T_{x_1}(1)$ and $\eta \in T_{x_2}(2)$. Then

$$\sum_{\{i,j\} \in M} \xi_i \eta_i = 0 . \qquad (*)$$

PROOF: We have:

$$x = \sum_{\{i,j\}\in M} K(\{i,j\})\,\xi_i\,\eta_j + x \sum_{\{i,j\}\in N} \xi_i\,\eta_j$$

for $x \in [x_1, x_2]$, i.e.,

$$x \sum_{\{i,j\}\in M} \xi_i\,\eta_j = \sum_{\{i,j\}\in M} K(\{i,j\})\,\xi_i\,\eta_j \quad .$$

Therefore $V[\Gamma]$ can have more than one element only in the case where (*) is fulfilled.

This theorem shows that if a game with forbidden situations has more than one value, then any play of the game, which is optimal simultaneously for any two distinct values, must actually take place in these forbidden situations. If the forbidden situations in a game are interpreted as the necessity to play another game, then the existence of more than one value means that in optimal play, this offer game will be played.

9. A sufficient (but not necessary) condition for the existence of more than one value in a game with forbidden situations is given in the following theorem.

THEOREM: If

$$\min_j \max_i K(\{i,j\}) < \max_i \min_j K(\{i,j\}) \quad ,$$

then $V[\Gamma]$ has more than one element.

PROOF: We have:

$$\max_i K_x(\{i,j\}) = \max\{\max_i K(\{i,j\}),\ x\} \quad .$$

Therefore

$$\min_j \max_i K_x(\{i,j\}) = \min_j \max\{\max_i K(\{i,j\}), x\} = \tag{1}$$

$$= \max\{\min_j \max_i K(\{i,j\}), x\}$$

and similarly

$$\max_i \min_j K_x(\{i,j\}) = \min \{\max_i \min_j K(\{i,j\}), x\} . \qquad (2)$$

If it is now assumed that

$$x_1 = \min_j \max_i K(\{i,j\}) ,$$

then by virtue of (1)

$$x_1 = \min_j \max_i K_{x_1}(\{i,j\}) . \qquad (3)$$

But under the conditions of the theorem $x_1 < \max_i \min_j K(\{i,j\})$; therefore the first part of (2) is equal to x_1, so that

$$x_1 = \max_i \min_j K_{x_1}(\{i,j\}) . \qquad (4)$$

From (3) and (4) follows that $x_1 = v_{x_1}$, i.e., $x_1 \in V[\Gamma]$. In just the same way we can establish that

$$x_2 = \max_i \min_j K(\{i,j\}) \in V[\Gamma] ,$$

and the theorem is proved.

That the above mentioned criterion for the non-uniqueness of the value of the game with forbidden situations is not necessary is seen in the example of the game with the non-complete matrix:

$$\begin{pmatrix} 7 & 8 & 2 \\ 1 & 9 & \cdot \\ 9 & 1 & \cdot \end{pmatrix}$$

Here minimaxes are equal, but at the same time the set of values of the game is $[2,5]$.(*)

SUMMARY

Let A be a certain non-complete matrix (i.e., a matrix in which some of the entries are vacuous. These are called forbidden situations). The first player chooses row i, and the second column j. If in A the entry (i,j) is occupied by the element a_{ij}, then the first player receives the payoff a_{ij}; in any other case another game is played.

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(Translated by Kiyoshi Takeuchi)

* Problems analogous with these examined in our article are considered in the articles of Milnor and Shapley, and also that of Everett [3].

II

A SOLUTION OF A GAME OF D. BLACKWELL

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Vestnik Leningradskogo Universiteta,
seria Matematiki, Mekaniki i Astronomy
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0. In this note we will give a solution for one of the particular cases introduced by David Blackwell in "a game of exhaustion" ([1], see also [2]), namely the so-called "game of women and cats versus men and mice." By an inductive argument, the formula for the value of such a game will be deduced. After that, we will consider some particular cases of this formula. Its asymptotic behavior, already partially obtained by Blackwell [1], will also be studied.

1. The game of women and cats versus men and mice consists of the following:

There are two commands. One consists of a women and b cats, the other of c men and d mice. Each of the commands dispatches a representative independently of the other. These representatives meet and one of them removes the other from the game according to the following rule: woman eliminates man, man-cat, cat-mouse, and mouse-woman. After this the game is continued by the same rule until one of the commands is completely eliminated.

In this way, this game consists of a sequence of matrix games; in each of these games each player has two pure strategies. The problem consists of finding the probability of the survival of the first command under the utilization of optimal strategies by both commands. It is easily seen, that if this probability is denoted by $f(a,b,c,d)$, then the following recurrence relations will hold:

$$f(a,b,c,d) = \mathrm{Val}\begin{pmatrix} f(a-1,b,c,d) & f(a,b,c-1,d) \\ f(a,b,c,d-1) & f(a,b-1,c,d) \end{pmatrix}, \qquad (1)$$

where

$$\begin{aligned} f(0,b,c,d) &= f(a,0,c,d) = 0, \\ f(a,b,0,d) &= f(a,b,c,0) = 1. \end{aligned} \qquad (2)$$

2. THEOREM 1:*

$$f(a,b,c,d) = \frac{\sum_{k=1}^{\infty} C_{a+c-1}^{a-k} \, C_{b+d-1}^{b-k}}{C_{a+b+c+d-2}^{a+d-1}} \qquad (3)$$

PROOF: We will prove this formula by induction. First of all the validity of equation (3) will be demonstrated when at least one of a,b,c,d is equal to zero. Clearly we have,

$$f(0,b,c,d) = f(a,0,c,d) = 0 .$$

The two other relations in (2) are derived from the well-known formula

$$\sum_{k=0}^{\infty} C_n^k \, C_m^{r-k} = C_{n+m}^r .$$

Let us now assume that we have proven the formula for all games with a given number of participants. Its validity will be established for a game whose number of participants is one larger. We now calculate the value of the matrix game

$$\begin{pmatrix} f(a-1,b,c,d) & f(a,b,c-1,d) \\ f(a,b,c,d-1) & f(a,b-1,c,d) \end{pmatrix} .$$

As is known, it is equal to (see [3])

*From here on we will sum from 1 to ∞ exclusively for the purpose of simplicity of notation. Unquestionably, in all these sums only a finite number of terms will be different from zero.

$$\frac{f(a,b,c-1,d)\cdot f(a,b,c,d-1)-f(a-1,b,c,d)\;f(a,b-1,c,d)}{f(a,b,c-1,d)+f(a,b,c,d-1)-f(a-1,b,c,d)-f(a,b-1,c,d)}\,. \qquad (4)$$

Let us substitute in place of f its value from formula (3) and calculate fraction (4). Performing a series of elementary transformations, we find that its denominator is equal to

$$\frac{\sum\limits_{k=1}^{\infty}\left[\,C_{a+c-2}^{a-k}\;C_{b+d-1}^{b-k}-C_{a+c-1}^{a-k}\;C_{b+d-2}^{b-k-1}\,\right]}{C_{a+b+c+d-3}^{a+d-1}}\;+$$

$$+\;\frac{\sum\limits_{k=1}^{\infty}\left[\,C_{a+c-1}^{a-k}\;C_{b+d-2}^{b-k}-C_{a+c-2}^{a-k-1}\;C_{b+d-1}^{b-k}\right.}{C_{a+b+c+d-3}^{a+d-2}}\;=$$

$$=\;\frac{a+b+c+d-2}{a+b+c+d-3}\cdot\frac{C_{a+c-2}^{a-1}\;C_{a+b+c+d-2}^{b-1}}{C_{a+b+c+d-4}^{a+d-2}}\,,$$

but the numerator is

$$\frac{\left(\sum\limits_{k=1}^{\infty}C_{a+c-2}^{a-k}\;C_{b+d-1}^{b-k}\right)\cdot\left(\sum\limits_{k=1}^{\infty}C_{a+c-1}^{a-k}\;C_{b+d-2}^{b-k}\right)}{C_{a+b+c+d-3}^{a+d-2}\;\;C_{a+b+c+d-3}^{a+d-1}}\;-$$

$$-\;\frac{\left(\sum\limits_{k=1}^{\infty}C_{a+c-1}^{a-k}\;C_{b+d-1}^{b-k-1}\right)\cdot\left(\sum\limits_{k=1}^{\infty}C_{a-c-2}^{a-k-1}\;C_{b+d-1}^{b-k}\right)}{C_{a+b+c+d-3}^{a+d-2}\qquad\qquad C_{a+b+c+d-3}^{a+d-1}}\;=$$

$$=\;\frac{a+b+c+d-2}{a+b+c+d-3}\cdot\frac{C_{a+c-2}^{a-1}\;C_{b+d-2}^{b-1}\;\sum\limits_{k=1}^{\infty}C_{a+c-1}^{a-k}\;C_{b+d-1}^{b-k}}{C_{a+b+c+d-4}^{a+d-2}\qquad C_{a+b+c+d-2}^{a+d-1}}\,.$$

This proves the theorem.

The optimal strategies for the players in this game are:

$$P_{wom} = \frac{a + b - 1}{a+b+c+d-2} , \qquad P_{cat} = \frac{b + c - 1}{a+b+c+d-2} ;$$

$$P_{man} = \frac{c + a - 1}{a+b+c+d-2} , \qquad P_{mou} = \frac{d + b - 1}{a+b+c+d-2} .$$

It is worthwhile to mention some special cases of formula (3). For example,

$$f(a,b, 1,1) = 1 - \frac{1}{C^{a}_{a+b}} ,$$

$$f(a,1, c,1) = \frac{a}{a+c} ,$$

$$f(a,b,c, 1) = 1 - \frac{C^{a}_{a+c-1}}{C^{a}_{a+b+c-1}} .$$

3. From various game-theoretic considerations the function f must have certain properties of symmetry, namely

$$f(a,b,c,d) = f(b,a,c,d) ,$$

and

$$f(a,b,c,d) = 1 - f(c,d,a,b).$$

These properties are easily obtained directly from (3).

4. In concluding we will find the asymptotic behavior of our function. It is easily seen, that $f(a,b,c,d)$ is the probability that a random variable, having the hypergeometric distribution with parameters $n = a + d - 1$, $N = a+b+c+d-2$ and $M = a + c - 1$, takes a value not greater than $a - 1$. From the convergence of the hypergeometric distribution to the normal distribution (c.f. [4]) we obtain:

THEOREM 2: If $M,N \to \infty$ and $0 < r \leq \frac{M}{N} \leq m < \infty$, where M,N are defined as above, then

$$f(a,b,c,d) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{G} e^{-\frac{x^2}{2}} dx \to 0 ,$$

where

$$G = \frac{a - 1 - \frac{Mn}{N}}{\sqrt{\frac{M(N-M)\, n\, (N-n)}{M^2(N-1)}}} =$$

$$\frac{(a-1)\,(b-1) - c\, d}{\sqrt{(a+c-1)\,(b+d-1)\,(a+d-1)\,(b+c-1)}} \sqrt{a+b+c+d-3} .$$

In particular, when $a = \alpha p$, $b = \beta p$, $c = \gamma p$, $d = \delta p$ and $p \to \infty$ we obtain the result of D. Blackwell:

If $\alpha\beta > \gamma\delta$, then $f(a,b,c,d) \to 1$ when $p \to \infty$.

SUMMARY

This note deals with D. Blackwell's game "Women and Cats Versus Men and Mice." In it we have proven that the value of the game is equal to

$$f(a,b,c,d) = \frac{\sum_{k=1}^{\infty} C_{a+c-1}^{a-k} \; C_{b+d-1}^{b-k}}{C_{a+b+c+d-2}^{a+d-1}} ,$$

where a,b,c,d are the number of women, cats, men and mice, respectively.

The probabilistic approach to the formula gives an asymptotic behavior for the value when a,b,c,d tend to infinity.

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(Translated by Kiyoshi Takeuchi)

III

A MULTI-DIMENSIONAL GAME-TYPE RANDOM WALK

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Vestnik Leningradskogo Universiteta,
seria, Matematiki, Mekaniki i Astronomy,
No. 2, 1962, pp. 89-95.

1. The purpose of this article is to study a multi-dimensional controlled random walk with discrete time intervals. Although basically the problem will be one of random walk controlled by two players with opposite interests, all results are applicable, of course, to the case where the control is carried out by one player, since in this case the other player has only one strategy.

The scheme being considered is naturally the general game of survival with random payoff (see [1]).

In section 2 the basic definitions and formulations of the problem will be given. In sections 4 and 5 we obtain theorems on the existence of the value of the game and on the functional equations which this value satisfies for the case of a finite domain. The necessity for the assumptions made will be seen from the example, mentioned in section 3.

2. Let the set D with boundary S be an open convex subset of E^R, the k-dimensional Euclidean space. With the help of the spherical mapping of S onto an arbitrary sphere R within D, we shall define a σ-algebra M of subsets of S as the σ-algebra of all inverse images under this map of the Borel sets of R. The determination of this σ-algebra evidently does not depend on the choice of the sphere.

Furthermore let μ be a probability measure on E^k. Let us denote by μ^x the measure which is determined by the relation

$$\mu^x(A) = \mu(x + A), \quad A \subset E^k, \quad x \in E^k,$$

and by $\tilde{\mu}^x(B)$ the measure given on S (not normalized) which is determined by the relation (see Figure 1)

$$\tilde{\mu}^x(B) = \mu^x(\{y \mid y \in D, \{x+\lambda(y-x)/\lambda \geq 0\} \cap B \neq \emptyset\}) . \qquad (1)$$

Finally, we shall denote by $\overline{\mu^x}$ the measure on the set $\overline{D}$, which on D is equal to μ^x, but on S is $\tilde{\mu}^x$.

The game being considered will have a general structure and be distinguished only in its own specific functions. The conditions of each game will be characterized by a point x of the set D. The transition from condition to condition will be given by a random vector, ξ_{ij}, determined by the probability measure, μ_{ij}, which depends on two indices, one of which (i) is chosen by the first player, and the other j is chosen by the second player. Each index takes a finite number of values. If the random vector ξ_{ij} is equal to y, then a game moves to the point $x + y$. If this point belongs to the set D, then the game is continued according to the same rule. (p. 90) If it does not belong to D, then the game ends at the point $x + \lambda y$, where λ is chosen so that $x + \lambda y \in S$.

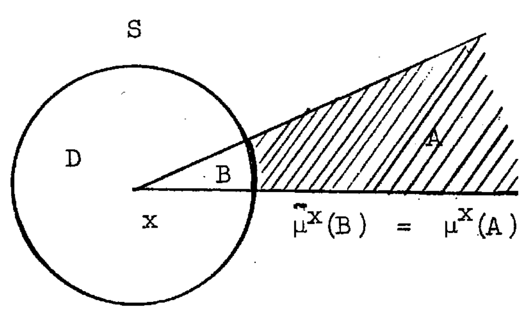

Figure 1.

Game Γ_1: A measurable bounded function $K(x)$, the payoff to the first player, is given on S. If the game ends at point x, the second player pays $K(x)$ to the first player.

Game Γ_2: $K(x) \equiv 0$. If the game does not end, the payoff of the first player is equal to 1.

Game Γ_3: The payoff of the first player is equal to the number of steps until the end of game.

In this way, the goal of the first player is: in game Γ_2 "survival", in game Γ_3 the maximum duration of the game, in game Γ_1 is the termination in the most profitable place.

Our description is complete only in the description of game Γ_2. In game Γ_3 we need to provide for rules of terminating the game; in Γ_1 the definition of the payoff for infinite games is required.

This definition is however artificial and it seems appropriate to isolate that class of games in which the value of the game does not depend on the assignment of payoff for infinite games.

The following example shows that none of the games considered satisfies this requirement.

3. Let us denote by μ the uniform distribution in the unit square $(0 \le x \le 1,\ 0 \le y \le 1)$ in space E^2. We shall consider game Γ_1 determined by the set

$$D = \{(x,y) \mid y \in (0,\ 5.04),\ \ x \in (0,\ 2.02)\},$$

by the payoff function

$$K(x,y) = [(3.02 - y)(y - 2.02)]^+ ,$$

given on the boundary of rectangle D , and <u>by the matrix</u>

$$\| \xi_{kl} \| \ , \quad k = 1, 2, \ldots, 8 \ ,$$

The random vector ξ_{kl} is determined by the measure μ^{c_k} and

$$\begin{aligned} c_1 &= (-1.00, \ 2.03), & c_5 &= (0, \ 2.03), \\ c_2 &= (-1.00, \ 3.03), & c_6 &= (0, \ 3.03), \\ c_3 &= (-1.00, -3.02), & c_7 &= (0, -3.02), \\ c_4 &= (-1.00, -4.02), & c_8 &= (0, -4.02) \ . \end{aligned}$$

It is easily seen that the player I at the cost of making the choice corresponding to the strategy can guarantee for himself an infinite walk in the unshaded zone of Figure 2. Namely, being found in zone A_k , he must choose the strategy k. Hitting the boundary from an unshaded zone will be possible only where the payoff K is equal to zero. Therefore, if in an infinite game the maximum payoff is given, the game will continue indefinitely and the value of the game will be equal to this payoff.

In this way we can see that in order for the value of the game not to depend on the determination of the payoff for infinite games, we will have to make some supplementary assumptions.

4. We will say that <u>the game satisfies the first condition of deflection</u>, if there exist $\epsilon_1, \epsilon_2 > 0$ and a vector $c \neq 0$, such that for all i, j

$$\mu_{ij}(c\, x \geq \epsilon_1) \geq \epsilon_2 \ .$$

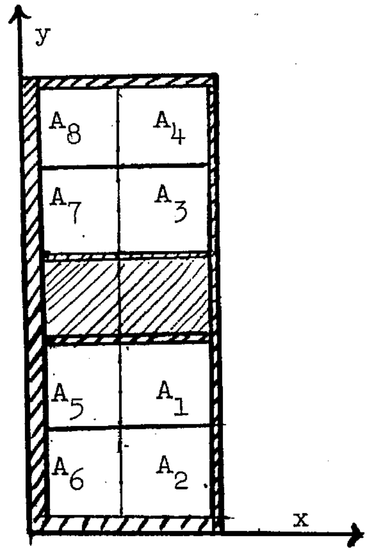

Figure 2.

LEMMA 1: If set D is bounded and the game satisfies the first condition of deflection, then no matter what strategies the players use, the game ends with probability 1.

This assertion is absolutely obvious. Further we will use the following theorem, the proof of which will be given in another work.

THEOREM 1: Let μ_x be the probability measure in E^k, depending on the k-dimensional real parameter $x \in D$. If a vector $c \neq 0$ and $\epsilon_1, \epsilon_2 > 0$ are found, such that for all x

$$\mu_x \,(c\, y \geq \epsilon_1) \geq \epsilon_2$$

and

$$\sup \; c\, y < \infty ,$$

then the functional equation*

$$\varphi(x) = \begin{cases} 0 , & x \notin D \\ \int \varphi(z)\; \mu_x^x(d\, z), & x \in D , \end{cases} \qquad (2)$$

*In notation μ_x^x the lower index denotes the dependence on x, the upper is understood according to the notations in section 2.

has a unique bounded solution, satisfying the condition $\varphi(x) \to 0$ when $\| x \| \to \infty$. This solution is $\varphi(x) \equiv 0$.

We will note, that this theorem is easily extended to the case of the functional "inequality"

$$\varphi(x) \leq \begin{cases} 0 , & x \notin D \\ \int \varphi(z)\, \mu_x^x(d\, z) , & x \in D \end{cases} \tag{3}$$

which also does not have a bounded non-negative solution converging to zero at infinity except the trivial one.

Hence from lemma 1 the theorem about the value of the game Γ_1 follows directly.

THEOREM 2: If the set D is bounded and the game satisfies the first condition of deflection, then the value of the game exists regardless of the starting point $x_0 \in D$, of the game, and is equal to $\varphi(x_0)$ where $\varphi(x)$ satisfies the functional equation

$$\varphi(x) = \begin{cases} K(x) , & x \in S , \\ \mathrm{Val} \; \| E\varphi(\xi_{ij}^x) \| , & x \in D \end{cases}$$

This equation has a unique bounded solution, and it may be obtained as the limit of the sequence of functions $\varphi_n(x)$

$$\varphi_{n+1}(x) = \begin{cases} K(x) , & x \in S \\ \mathrm{Val} \, \| E \;\; \varphi_n(\xi_{ij}^x) \| , & x \in D, \quad n=0, 1, \ldots , \end{cases}$$

where $\varphi_0(x)$ is an arbitrary measurable function, satisfying the conditions

$$\varphi_0(x) = K(x) \quad \text{for} \quad x \in S ,$$

$$\sup_D \varphi_0(x) \leq \sup_S \varphi_0(x) ,$$

$$\inf_D \varphi_0(x) \geq \inf_S \varphi_0(x) .$$

PROOF: As usual we shall consider two sequences of functions $\varphi_n(x)$, generated respectively by the function

$$\varphi_0(x) = \sup_{y \in D} \varphi_0(y) , \quad x \in D ,$$

and the function

$$\varphi_0(x) = \inf_{y \in S} \varphi_0(Y), \quad x \in D ,$$

and we prove that these sequences converge to one limit. Indeed, the difference of these limits (which exist by virtue of monotonicity and boundedness of these sequences) is non-negative and satisfies the relation (3), and consequently is equal to 0 .

The condition of deflection as introduced is, of course, too rigid. However, as we already said, in the game Γ_1 some condition of this type, guaranteeing the end of the game, is necessary. It would be interesting to weaken the condition of deflection and, in particular, to obtain a similar "guarantee", starting from the special form of the payoff function. (Certain results in this direction obtained by V. F. Kolichini are already known.)

5. For games Γ_2 and Γ_3 such rigid conditions are not required, and it is possible to manage with weaker conditions. Let p and q be probability vectors corresponding to the indicies. We shall denote by ξ_{pq} the random variable, which with probability $p_i q_j$ is equal to ξ_{ij} , and by $\mu_{pq} = \sum_{i,j} p_i q_j \mu_{ij}$ the corresponding probability measure.

We will say that <u>the game satisfies the second condition of deflection</u>, if there exist $\epsilon_1, \epsilon_2 > 0$, a vector $c \neq 0$ and strategy q_0 of the player II such that whichever strategy p of the player I was selected,

$$\mu_{pq_0}(c\,x \geq \epsilon_1) \geq \epsilon_2 .$$

<u>THEOREM 3: If the set D is bounded and the game satisfies the second condition of deflection, then the value of the game Γ_2 is equal to 0 and the functional equation (2), which this value must satisfy, does not have any bounded non-negative solution except the trivial one.</u>

<u>PROOF:</u> The first part of the theorem is evident. The second part follows from the fact that solutions of equation (2) do not exceed solutions of the equation

$$\psi(x) = \begin{cases} 0 , & x \in S , \\ \max\limits_{p} E\, \psi(\xi^{x}_{pq_0}) , & x \in D ; \end{cases}$$

whose only solution is equal to 0 by virtue of theorem 1.

<u>THEOREM 4: If the set D is bounded and the game satisfies the second condition of deflection, then the game Γ_3 ends with probability 1, regardless of the starting point x, and the value of the game $\varphi(x)$ as a function of the initial condition satisfies the functional equation</u>

$$\varphi(x) = \begin{cases} 0 , & x \notin D , \\ 1 + \mathrm{Val}\, \| E\, \varphi_n(\xi^{x}_{ij}) \| , & x \in D . \end{cases} \qquad (4)$$

<u>Functional equation (4) has a unique bounded solution, and this solution may be obtained as the limit of the sequence of the functions</u>

$$\varphi_{n+1}(x) = \begin{cases} 0 , & x \notin D , \\ 1 + \mathrm{Val}\, \| E\, \varphi_n(\xi^{x}_{ij}) \| , & x \in D \end{cases} .$$

PROOF: From the assumptions of the theorem it follows at once that the value of the game exists and is bounded. We now will prove that functional equation (4) has a unique solution.

We introduce the random variable $\xi_{ij}(\delta)$, which with probability $1 - \delta$ coincides with the corresponding variable ξ_{ij} and with probability δ guarantees that the outcome will be beyond the boundary of the domain from any point of the domain. We will denote the function which is equal to

$$1 + \operatorname{Val} \| E f(\xi_{ij}^{x}(\delta)) \|$$

for $x \in D$ and 0 otherwise by $T_\delta(f)$. Let us consider the sequences

$$g_0^\delta \equiv 0 ,$$

$$g_{n+1}^\delta = T_\delta(g_n^\delta)$$

and

$$h_0^\delta = \begin{cases} 0 , & x \notin D , \\ A , & x \in D , \end{cases}$$

$$h_{n+1}^\delta = T_\delta(h_n^\delta) .$$

It is clear that the sequence $\{g_n^\delta\}$ does not monotonically decrease and for all n $h_n^\delta \geq g_n^\delta$. We also have:

$$\sup_x (h_n^\delta(x) - g_n^\delta(x)) \leq \sup_x \max_{pq} E [h_{n-1}^\delta(\xi_{pq}(\delta)) - g_{n-1}^\delta(\xi_{pq}^{x}(\delta))] \leq$$

$$\leq (1-\delta) \sup_x (h_{n-1}^\delta(x) - g_{n-1}^\delta(x)) \leq A (1-\delta)^n .$$

Furthermore, that the sequence $\{g_n^\delta\}$ converges follows from its being bounded above by the constant $\frac{1}{1-\delta}$. Consequently, the sequences h_n^δ and g_n^δ converge to one limit f^δ.

Now let f be some bounded solution of equation (4). We will prove, that $f^{\delta} \to f$ when $\delta \to 0$. We have:

$$g_0^{\delta} \leq f ,$$

and therefore,

$$g_n^{\delta} \leq f \qquad \text{for all } n$$

and

$$f^{\delta} \leq f \quad .$$

Further, if $f(x) \leq K$, then for sufficiently large A we obtain $h_n^{\delta} \geq \frac{1}{1+K\delta} f$. Indeed we assume $A = K$. Then $h_0^{\delta} \geq \frac{1}{1+K\delta} f$. Assume the required inequality is true for n . We will prove it for $n+1$. We note, that

$$E[f(\xi_{ij}^{x}) - f(\xi_{ij}^{x}(\delta))] \leq K\delta .$$

In connection with this we have:

$$h_{n+1}^{\delta}(x) = 1 + \text{Val} \| E\, h_n^{\delta}(\xi_{ij}^{x}(\delta)) \| \geq 1 + \frac{1}{1+K\delta} \text{Val} \| E\, f(\xi_{ij}^{x}(\delta)) \| \geq$$

$$\geq 1 - \frac{K\delta}{1+K\delta} + \frac{1}{1+K\delta} \text{Val} \| E\, f(\xi_{ij}^{x}) \| = \frac{1}{1+K\delta} f(x) .$$

In this way,

$$\frac{1}{1+K\delta} f \leq f^{\delta} \leq f ,$$

from which follows the required convergence.

Since the sequence f^{δ} converges to any solution f , it follows that this solution is unique and our assertion is completely proven.

That the value of the game satisfies the functional equation (4) and that it may be obtained as the limit of successive approximations (5) is proved in the usual way.

6. The requirement of the boundedness of the domain D is not necessary; corresponding theorems can be obtained for infinite domains. We reproduce these theorems here without proofs.

We consider the convex domain D and its closure $\bar{D}$. It is known that for an arbitrary point $x \in \bar{D}$, there exists a maximum convex cone K_x with summit at point x, entirely lying in domain $\bar{D}$. Cones corresponding to different points x coincide under parallel translation. This permits us to consider <u>the cone K of infinite directions</u> of set $\bar{D}$, whose construction is possible by starting from point 0.

The polar cone K^* of the cone K is the convex cone which consists of vectors forming obtuse angles with all vectors in K.

<u>THEOREM 5:</u> <u>If for certain</u> $c \in K^*$

$$\mathrm{Val} \, \| E \, c \, \xi_{ij} \| > 0 ,$$

<u>then in game Γ_2 player II can choose a strategy under which the game ends with probability 1 independently of the behavior of player I, and the value of the game is equal to 0.</u>

<u>THEOREM 6:</u> <u>If for certain</u> $c \in K^*$

$$\mathrm{Val} \, \| E \, c \, \xi_{ij} \| = 0$$

<u>and</u> $0 < D \, c \, \xi_{ij} < \infty$ <u>for all</u> i and j, <u>then the assertion of theorem 5 holds</u>.

<u>THEOREM 7:</u> <u>If for arbitrary</u> $c \in K^*$

$$\mathrm{Val} \, \| E \, c \, \xi_{ij} \| > 0 ,$$

<u>then the value of game Γ_2 as a function of the initial conditions of the</u>

game x ($x \in D$) satisfies the functional equation

$$f(x) = \begin{cases} 0\,, & x \notin D\,, \\ \operatorname{Val} \| E\, f(\xi^{x}_{ij}) \|\,, & x \in D\,, \end{cases}$$

and is its unique solution converging to 1 in the cone K^* .

THEOREM 8: Under the hypotheses of theorem 5, the value of the game Γ_2 satisfies the functional equation

$$g(x) = \begin{cases} 0\,, & x \notin D \\ 1 + \operatorname{Val} \| E\, g(\xi^{x}_{ij}) \| & x \in D\,. \end{cases}$$

This solution may be obtained as the limit of the sequence of functions

$$g_0(x) \equiv 0\,,$$

$$g_{n+1}(x) = \begin{cases} 0\,, & x \notin D\,, \\ 1 + \operatorname{Val} \| E\, g_n(\xi^{x}_{ij}) \|\,, & x \in D\,. \end{cases}$$

The proof of theorems 5 - 7 is carried out in the same way as the corresponding theorems about the walk on the half-line (c.f. [1]). Theorem 8 is proved by the same method as theorem 4 of the present work.

It is possible to give certain a priori evaluations for the solution of the functional equations considered in this work. We shall treat them in a later paper.

SUMMARY

The paper contains some results on random walks in a convex set in Euclidean space. The surface of the set is an absorbing one. The walk is controlled by two opposite players who choose correspondingly a row and a column of the prescribed matrix of random vectors.

Two particular cases are discussed. In the first the payoff function is defined in terms of the absorbing point of the random walk and in the second it is defined in terms of the duration of the game (the number of steps until absorbtion.)

Values of these games (when they exist) satisfy certain functional equations which under some restrictions have unique solutions. These may be found by successive approximations.

Corresponding theorems are proved for the case of a bounded domain for walking and are formulated for an unbounded one.

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IV

THE THEORY OF THE CORE OF AN n-PERSON GAME

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We will call the following pair an n-person game: a set of players $I_n = \{1,2,\dots,n\}$ and a real valued function $V(S)$, defined on the subsets of this set

$$0 \leq V(S) \leq V(I_n) \ , \quad S \subset I_n; \quad V(\emptyset) = 0 \quad .$$

The function $V(S)$ is called the characteristic function of the game Γ .

The set of n-dimensional vectors $\alpha = (a_1,\dots,a_n)$ such that $a_i \geq V(\{i\})$, $i = 1,\dots,n$ and $\sum_{i=1}^{n} a_i = V(I_n) \equiv M$, is called the set of all imputations and is denoted by A . We shall assume from now on that the game is 0-1 normalized, i.e., $V(I_n) = 1$, $V(\{i\}) = 0 \quad i = 1,\dots,n$.

The following subset of the set A is called the core:

$$U = \{\alpha \in A: \ \sum_S a_i \geq V(S), \quad \text{for all } S \subset I_n\} \ .$$

Let $S_1, S_2,\dots,S_m$ be all those sets $S \subset I_n$, for which $V(S) > 0$ or $V(S) = 0$, but $|S| = 1$.

We shall associate with each set $S \subset I_n$ the vector $\bar{S} = (s^{(1)},\dots,s^{(n)})$, where

$$s^{(i)} = \begin{cases} 0 , & i \notin S , \\ 1 , & i \in S . \end{cases}$$

Let us call the set of real numbers $\lambda_1 \geq 0 ,\dots, \lambda_m \geq 0$ a q - θ - covering of the set I_n if the condition $\sum_{j=1}^{m} \lambda_j \bar{S}_j = I_n$ is satisfied, where q is the number of $\lambda_j > 0$, and is the system of vectors $\bar{S}_j$ corresponding to these positive λ_j .

The set of all $q - \theta$ coverings considered as points in m-dimensional Euclidean space is a bounded, closed, and convex set with a finite number of extreme points. The extreme points correspond to the so-called reduced coverings; these are easily found.

THEOREM 1: In order for the core to exist in the game Γ, it is necessary and sufficient that for any reduced $q - \theta$ - covering $(\lambda_1, \ldots, \lambda_m)$ the condition

$$\sum_{j=1}^{m} \lambda_j V(S_j) \leq 1$$

be satisfied.

The theorem is proved by using the necessary and sufficient condition for the solvability of a system of linear inequalities (see [1]).

We now investigate the relation between the core and the solution in the sense of von-Neumann-Morgenstern.

THEOREM 2: In order that the core of the game Γ be a solution, it is necessary that it have a non-empty intersection with each hyperplane $a_i = 0$, $i = 1, \ldots, n$.

If we extend the concept of covering by introducing $q - \theta_j$ - quasi - covering (as the "covering" sets we consider in addition the set $S_j' = I_n - S_j'$ where $|S_j| > 1$, with the characteristic function redefined as S_j' so that $V(S_j') = 1 - V(S_j)$ then it is possible in terms of these quasi-coverings to prove a sufficient condition for the existence of a unique solution in the game (coinciding with the core). It is possible to significantly weaken this condition.

<u>THEOREM 3: In order for a unique solution to exist in the game</u> Γ , <u>it is sufficient that the condition</u> $V(S) \leq \frac{1}{r}$ <u>be satisfied for all</u> $S \subset I_n$, <u>where</u> r <u>is the rank of the matrix composed of vectors</u> $\bar{S}_j$, <u>where</u> $V(S_j) > 0$, <u>and</u> $\bar{I}_n$.

As a corollary of this theorem we can derive the well-known theorem of D. Gillies in which the sufficient condition for the existence of a solution in the game is $V(S) \leq \frac{1}{n}$ for all $S \subset I_n$ (see [2]).

With the help of this established theorem it is possible to obtain new (and also some already known) results for quota games and the market game of Shapley. It is also possible to study 4-person games, writing out for these cases the conditions for the existence of a core and a solution.

SUMMARY

A necessary and sufficient condition is given for the existence of a core in an n-person game in terms of the characteristic function. Sufficient conditions for the existence of a unique von Neumann-Morgenstern solution (coinciding with the core) are given. Gillies existence theorem (see [2]) is shown to be a corollary of this theorem.

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V

A DISCRETE VARIANT OF MOSER'S PROBLEM

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Let n carefully mixed cards with the numbers $1,2,\ldots,n$ written on them be given. A person is presented with the problem choosing the card with the largest number possible. It is possible to throw out the first cards, and then to solve the problem of whether or not to select the $(k+1)^{st}$ card on the basis of an ordinal comparison of it with the k cards already drawn (without exact knowledge of the numbers written on these cards). The problem allows the following humorous interpretation. A fiancee has n bridegrooms. Let them be numbered in increasing order of their merits and propose marriage alternately. The fiancee can compare each proposer with the earlier persons who were rejected by her. What kind of behavior will maximize the mathematical expectation of the number of the chosen bridegroom and what numerical value is this maximum?

If in addition the assumption of knowledge of the numbers already drawn is introduced, then the problem becomes complicated. For the case of uniform distribution on $[0,1]$ this variant was considered by Moser [1]. The generalization of the problem of Moser can be found in [2]. The article of Rindung [3] is devoted to the problem of the maximization of the probability of the selection of the largest number.

1. Let us denote by $S(k,n-k)$ the largest average value of the chosen number under the condition that k numbers were already taken out. It is clear that $S(n,0) \equiv 0$. We are interested in $S(0,n)$. We will determine the recurrence relation connecting $S(k, n-k)$ and $S(k+1, n-k-1)$. We note that if k numbers were already removed, then in the following step it is only possible to use one of the $k+2$ strategies: either select as final choice the $(k+1)^{st}$ number if it is better than the i^{-th} one of the k preceeding $(0 \leq i \leq k)$, or in all cases to proceed to the following step. In fact, any strategy with "a zone of final selection" of measure i is dominated by the strategy of choosing the $(k+1)^{st}$ number if it is better than the $(k-i+1)^{th}$, of the numbers drawn earlier. Any strategy with a random selection of a zone is dominated by the better components of its pure strategies. Hence, the relation

$$S(k,\ n-k) \ = \max_{0 \leq i \leq k+1} \left[\frac{i}{k+1} S(k+1,\ n-k-1) + \frac{1}{k+1} \sum_{\ell=i}^{k} E(k,\ell,n) \right] \qquad (1)$$

holds, where $E(k,\ell,n)$ is the mathematical expectation of the $(k+1)^{st}$ number being the best choice if it is known that it is larger than ℓ of the k already possessed for comparison. We have now to find $E(k,\ell,n)$. The $(k+1)^{st}$ number can take values $\ell+1,\ldots,n$ depending on how many of $n-k-1$ numbers written down on the remaining cards prove to be smaller than it. The probability that the $(k+1)^{st}$ number is equal to $\ell+1+j$ $(0 \leq j \leq n-k-1)$ is equal to the probability that j of the remaining $n-k-1$ numbers fall into the $\ell+j$ cells up to the $(k+1)^{st}$ number, but $n-(k+j+1)$ numbers are in the remaining $n-(\ell+j+1)$ cells. Using the well-known formula for the number of allocations of indistinguishable objects into cells ([4], p. 60), we obtain, that the probability being sought is equal to

$C^{j}_{\ell+j}\, C^{k-\ell}_{n-\ell-j-1} \Big/ C^{k+1}_{n}$. From this we get,

$$E(k,\ell,n) = \sum_{j=0}^{n-k-1} (\ell+j+1) \frac{C^{j}_{\ell+j}\, C^{k-\ell}_{n-\ell-j-1}}{C^{k+1}_{n}} .$$

The last expression can be simplified, using the identity

$$\sum_{j=0}^{n-k-1} (\ell + j + 1) \frac{C^{j}_{\ell+j}\, C^{k-\ell}_{n-\ell-j-1}}{C^{k+1}_{n}} = (\ell + 1) \frac{n+1}{k+2} . \qquad (2)$$

Equation (2) can be proven more simply starting from theprobabilistic considerations. We will note at first that (2) is equivalent to

$$\sum_{j=1}^{n-k} C^{\ell+1}_{\ell+j}\; C^{k-\ell}_{n-\ell-j} = C^{k+2}_{n+1} . \qquad (3)$$

The right-hand side of (3) is equal to (to within a factor of $(\frac{1}{2})^{n+1}$) the probability of event A: the occurrence of $k+2$ favorable outcomes in $n+1$ Bernoulli trials with $p = \frac{1}{2}$. The left-hand side (to within the same factor) is equal to the sum of the probabilities of disjoint events A_j ($j=1,2,\ldots, n-k$), where A_j means the occurrence of $\ell+1$ favorable outcomes in the first $\ell+j$ trials, a favorable outcome in the $(\ell+j+1)^{st}$ trial and $k-\ell$ favorable outcomes in the remaining $n-\ell-j$ trials. It is clear that $A = \bigcup_{j=1}^{n-k} A_j$, and from this follows the validity of both (3) and (2).

Substituting the obtained value of $E(k,\ell,n)$ in (1) and summing over ℓ , we obtain finally:

$$S(k, n-k) = \max_{0 \leq i \leq k+1} \left[\frac{i}{k+1} S(k+1, n-k-1) + \frac{n+1}{2} - \frac{(n+1)}{2(k+1)} \frac{(i^2 + i)}{(k+2)}\right] . \qquad (4)$$

2. Formula (4) can probably be used with success for the computation of $S(0,n)$ for large values of n. Thus, for example, $S(0,12) = 10^{167/522}$. We will note that the maximum of the right-hand side of (4) is attained for the smallest number i, satisfying

$$S(k+1,\ n-k-1) < \frac{n+1}{k+2}(i+1) ,$$

i.e.,

$$i = \left[\frac{k+2}{n+1}\ S(k+1,\ n-k-1)\right] .$$

We now prove that

$$S(0,n) = n + 0(n) . \qquad (5)$$

The plan of the proof is as follows. We will establish that

$$S(n-k,\ k) = a_k n + 0(n) \qquad (6)$$

for fixed k and $n \to \infty$. Furthermore, it will be proved, that $\lim_{k\to\infty} a_k = 1$. The desired result will then be a consequence of the inequality $S(n-k,\ k) \leq S(0,n) \leq n$.

We will prove relation (6) by means of mathematical induction. Direct calculation from formula (4) will give

$$S(n-1,\ 1) = \frac{n+1}{2} ,$$

so that for $k=1$ relation (6) holds. We will assume, that it is valid for $k=K$, and prove its validity for $k = K+1$. We have

$$S(n-K-1,\ K+1) = \frac{S(n-K,\ K)}{n-K}\left[\frac{n-K+1}{n+1}\ S(n-K,K) + \theta_{nK}\right] +$$

$$+ \frac{n+1}{2} - \frac{n+1}{2(n-K)(n-K+1)}\left[\frac{n-K+1}{n+1} S(n-K,K) + \theta_{nK} + 1\right] \times$$

$$\times \left[\frac{n-K+1}{n+1}\ S(n-K,\ K) + \theta_{nK}\right] ,$$

where $0 \leq \theta_{nK} < 1$. Substituting in the right-hand side of the last equality for $S(n-K, K)$ its value (6), we obtain

$$S(n-K-1, K+1) = \frac{a_K^2 n}{2} + \frac{n}{2} + O(n) . \tag{7}$$

The validity of (6) for $k = K+1$ is then proven. Consequently, (6) is true in general. Furthermore, from (7) it follows that

$$a_{k+1} = \frac{a_k^2}{2} + \frac{1}{2} \tag{8}$$

with the initial condition that $a_1 = \frac{1}{2}$. It is easily seen that $\lim_{k \to \infty} a_k = 1$. The exact expression for a_k can be found in [1].

SUMMARY

In a box are placed n slips numbered from $1, \ldots, n$. The player draws one slip after another and stops after some number of such drawings. He does not know the number of the drawn slip but can compare it ordinally with the numbers on the slips drawn earlier. His gain is the number written on the last slip. A recurrance relation is obtained for the maximum of the expected gain. It is shown that this maximum is $n + O(n)$.

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VI

A SOLUTION OF A CERTAIN DISCRETE TWO-PERSON GAME WITH BLUFFING

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In the present article a particular modified model of the well-known card game called "believe it or not" is studied. This finite zero-sum two-person game is a typical example of a game with bluffing. In recent years a great deal of the articles have appeared devoted to the study of concrete games with bluffing, mostly dealing with a type of poker (see, for example, [1]), but the general theory of these games has not yet been worked out. Therefore their solutions have only been guessed at in most cases.

We will make certain assumptions on the form of solutions which will make it possible to find them and prove their optimality.

In section 1 the rules of the game are formalized and a description of the set of strategies of each player is given. In sections 2,3, and 4 the particular cases of this game are considered, and the optimal strategies of the players and the value of the game in these particular cases are found. Section 5 is devoted to the study of the general case. Here the optimal strategies of the players and the value of the game are also found. In section 6 relations, allowing the simplification of the computation of the values of the games, are considered. Then an application is given in which the optimal strategies of the players in some particular cases are mentioned.

1. Formulation of the problem. The game involves two players. Each player has a set of m cards numbered $1,2,\ldots,m$. To begin the play, player I names an arbitrary card $r \in (1,2,\ldots,m)$ and places a card $s \in (1,2,\ldots,m)$ face down on the table. It is now player II's turn. He can "believe" of "not believe" player I.

a) If II chooses to doubt, he must check the card on the table. If $s \neq r$, player I must take back the card s, and the game starts anew, with player II beginning the play by naming a card, etc. If $s = r$, then player II must pick up the card r and discard the pair (r,r), and the play reverts to player I; who begins the game (with $m-1$ cards) by naming a card r' and placing a card s' on the table.

b) If II believes I, then he plays a card $p \in (1,2,\ldots,m)$, asserting that it is r. Player I now has the option of believeing or not.

In this way the game is continued, each player asserting that he is playing card r until one of the players checks.

a') If the checking player reveals that an incorrect card was delivered on the last move by his opponent, then the opponent must gather up all the cards played on the table and throw out all the pairs he can make using the cards on the table and those already in his hand. The move goes to the one who checked, who begins the play by naming a new card r' and playing card s'.

b') If the checking player reveals that the correct card was played on his opponents last move, then the checking player must gather the played cards and throw out all pairs. The move now goes to the opponent, who begins the play as before with a new card r'.

When the two players have just one card each, then the player who moves second must check the move of the first. If the card is the one claimed, then the first wins the game. Otherwise the second wins.

<u>of using strategies</u> $b_{01}, b_{12}, b_{23},$ <u>then</u>

$$p_1^* = \frac{2\,V_4}{V_3 + V_4}\,, \qquad p_2^* = \frac{V_3 + V_4}{V_3 + V_2}\,, \qquad p_3^* = \frac{2\,V_4}{1 - V_3}\,, \tag{4}$$

$$q_1^* = \frac{V_4(V_3 + V_2)}{V_3(V_2 - V_3)}\,, \quad q_2^* = \frac{V_3 - V_4}{2\,V_3}\,, \qquad q_3^* = q_2^*\,\frac{V_3 + V_2}{1 + V_2}\,, \tag{5}$$

<u>where</u> V_4 <u>satisfies the equation</u>

$$V_4^2\,(V_3 + 2V_2 + 1) + V_4(-V_3^2 + V_2V_3 + 3V_3 + V_2) + V_3^2\,V_2 - V_3\,V_2 = 0\,. \tag{6}$$

<u>PROOF:</u> We shall first find the solution of the game in which player I may utilize only $a_{12}, a_{23}, a_{34},$ and player II only b_{01}, b_{12}, b_{23}. Afterwards we shall prove that this solution is equivalent to the solution of the original game. From matrix (*) we obtain the conditions to which p_i^*, q_j^* $(i,j = 1,2,3)$ are subject, i.e.:

$$\left.\begin{aligned} p_1^*\,V_3 - p_2^*\,V_4 - p_3^*\,V_4 &= V_4\,, \\ -p_1^*\,V_3 + p_2^*\,V_2 - p_3^*\,V_3 &= V_4\,, \\ p_1^*\,V_3 - p_2^*\,V_2 + p_3^* &= V_4\,, \end{aligned}\right\} \tag{7}$$

$$p_1^* + p_2^* + p_3^* = 1; \quad p_i^* \geq 0 \qquad (i = 1,2,3)\,,$$

$$\left.\begin{aligned} q_1^*\,V_3 - q_2^*\,V_3 + q_3^*\,V_3 &= V_4\,, \\ -q_1^*\,V_4 + q_2^*\,V_2 - q_3^*\,V_2 &= V_4\,, \\ -q_1^*\,V_4 - q_2^*\,V_3 + q_3^* &= V_4\,, \end{aligned}\right\} \tag{8}$$

$$q_1^* + q_2^* + q_3^* = 1\,; \quad q_j^* \geq 0 \qquad (j = 1,2,3).$$

Let us denote the value of the game by V_m (considering the payoff to player I). If in his own first move player II doubts the card and $s \neq r$, then the continuation can be considered as a new game with value V_m. If $s = r$, then it turns out that each of the players has $m - 1$ cards; it becomes player I's turn and the value of the continued game will be V_{m-1}. Similarly it is possible to find the value of the game which is given as a result of the check by player I or player II after the k^{th} move $(1 < k \leq m-1)$.

The following are the strategies for player I: 1) $a_{ok} (1 < k \leq m)$ is in the course of the first $k - 1$ moves to bluff*, and at the k^{th} move to check; 2) a_{io} $(1 \leq i \leq m)$ is to exhibit a correct** card at the i^{th} move, and to bluff at the remaining moves without checking; 3) a_{ik} is to deliver a correct card at the i^{th} move, and to check at the k^{th} move $(1 \leq i < k \leq m)$.

The following are the strategies for player II: 1) b_{oj} $(1 \leq j \leq m)$ is to check at the j^{th} move without exhibiting correct cards; 2) b_{ro} $(1 \leq r < m)$ is to play a correct card at the r^{th} move and never check; 3) b_{rj} $(1 \leq r < j < m)$ is to deliver a correct card at the r^{th} move, and to check at the j^{th} move.

It is easily seen that a_{ok} is dominated by the strategy*** $a_{k-1\ k} (1 < k \leq m)$. Indeed, against any strategy of player II in which he checks on the k^{th} move or later, strategies a_{ok} and $a_{k-1\ k}$ give the equal payoffs. However against a strategy in which player II checks on the $(k-1)^{st}$ move, $a_{k-1\ k}$ is seen to be better than a_{ok}. In addition, a_{io} is not better than a_{im} $(1 \leq i < m)$, since at the last move checking is more profitable than delivering an incorrect card.

*I.e., to play any card not coinciding with the one mentioned before the first move.

**I.e., to exhibit the card declared before the first move.

***On domination of strategies see [2].

Similarly for player II, b_{0j} is dominated by $b_{j-1,j}$ $(1 < j \leq m)$, and b_{r0} is dominated by $b_{r,m-1}$ $(1 \leq r < m-1)$.

From intuitive considerations strategies $a_{12}, a_{23}, \ldots, a_{m-1,m}, a_{m0}$ for player I and $b_{01}, b_{12}, \ldots, b_{m-2\ m-1}, b_{m-1\,0}$ for player II seem to be the most suitable for use in an optimal mixture. However, it is not the case that all of them need be used. It will be proved that for $m \geq 4$ the optimal strategy for player I is a mixture of a_{12}, a_{23}, a_{34}, and for player II a mixture of b_{01}, b_{12}, b_{23}; for $m = 2$ player I has to mix a_{12}, a_{20}, and player II b_{01}, b_{10}; for $m = 3$ player I mixes a_{12}, a_{23}, a_{30}, and player II b_{01}, b_{12}, b_{20}.

2. The Case $m = 2$.

We consider here a game in which each of the players has 2 cards.

There will be the following strategies for player I: a_{12} is at the first move to play a correct card, and at the second check; a_{10} is at the first move to play a correct card and at the second an incorrect one; a_{20} is at the first move to play an incorrect card, and at the second a correct one; a_{02} is at the first move to play an incorrect card and at the second to check.

For player II: b_{01} is not to believe at the first move; b_{10} is to play a correct card at the first move; b_{02} is to play an incorrect card at the first move. The matrix of this game is as follows:

	b_{01}	b_{10}	b_{02}
a_{12}	1	-1	1
a_{20}	$-V_2$	1	1
a_{10}	1	-1	-1
a_{02}	$-V_2$	-1	1

Clearly, strategies a_{10}, a_{02}, b_{02} are dominated, and the optimal strategies can be found in the matrix:

	b_{01}	b_{10}
a_{12}	1	-1
a_{20}	$-V_2$	1

Let p_1^*, p_2^*, q_1^*, q_2^* be probabilities with which strategies a_{12}, a_{20}, b_{01}, b_{10} are played respectively. They must satisfy the following conditions:

$$p_1^* - p_2^* V_2 = V_2 , \qquad q_1^* - q_2^* = V_2 ,$$

$$-p_1^* + p_2^* = V_2 , \qquad -q_1^* V_2 + q_2^* = V_2 ,$$

$$p_1^* + p_2^* = 1 \qquad q_1^* + q_2^* = 1 ,$$

$$p_i^* \geq 0 , \quad i = 1,2, \qquad q_j^* \geq 0 , \quad j = 1,2 .$$

Hence it follows that:

$$p_1^* = \frac{1 - V_2}{2} , \qquad p_2^* = \frac{1 - V_2}{2} ;$$

$$q_1^* = \frac{1 - V_2}{2} , \qquad q_2^* = \frac{1 - V_2}{2} ;$$

and V_2 is found from the equation

$$V_2^2 + 4 V_2 - 1 = 0 .$$

$V_2 = -\sqrt{5} - 2$ is an irrelevant root. Thus, $V_2 = \sqrt{5} - 2$.

<u>3. The case $m = 3$.</u> In this case player I, in accordance with the set of strategies described in section 1, has strategies:

a_{0k} $(1 < k \leq 3)$, a_{i0} $(1 \leq i \leq 3)$, a_{ik} $(1 \leq i < k \leq 3)$, where $a_{i0}(1 \leq i \leq 2)$ and $a_{0k}(1 < k \leq 3)$ are dominated and there remain only a_{12}, a_{13}, a_{23} , a_{30} ; player II has strategies: b_{12}, $b_{0j}(1 \leq j \leq 3)$, $b_{r0}(1 \leq r \leq 2)$, where b_{02}, b_{03} and b_{10} are dominated, and therefore, there remain only b_{01}, b_{12}, b_{20}.

From the matrix

	b_{01}	b_{12}	b_{20}
a_{12}	V_2	$-V_2$	V_2
a_{23}	$-V_3$	1	-1
a_{30}	$-V_3$	$-V_2$	1
a_{13}	V_2	$-V_2$	-1

we can see that a_{13} is dominated by strategy a_{12}. We will denote by p_1^*, p_2^*, p_3^* ; q_1^*, q_2^*, q_3^* the probabilities with which a_{12}, a_{23}, a_{30}, b_{01}, b_{12}, b_{20} are played respectively. They satisfy the following conditions:

$$\left.\begin{aligned} p_1^* V_2 - p_2^* V_3 - p_3^* V_3 &= V_3 , \\ -p_1^* V_2 + p_2^* \quad - p_3^* V_2 &= V_3 , \\ p_1^* V_2 - p_2^* + p_3^* &= V_3 , \end{aligned}\right\} \qquad (1)$$

$$p_1^* + p_2^* + p_3^* = 1 ; \qquad p_i^* \geq 0 \quad (i = 1,2,3)$$

$$\left.\begin{aligned} q_1^* V_2 - q_2^* V_2 + q_3^* V_2 &= V_3 , \\ -q_1^* V_3 + q_2^* - q_3^* &= V_3 , \\ -q_1^* V_3 - q_2^* V_2 + q_3^* &= V_3 , \end{aligned}\right\} \qquad (2)$$

$$q_1^* + q_2^* + q_3^* = 1 ; \quad q_j^* \geq 0 \quad (j=1,2,3) .$$

Substituting $1 - p_1^* - p_2^*$ for p_3^* and $1 - q_1^* - q_2^*$ for q_3^* , we have:

$$p_1^* = \frac{2 V_3}{V_2 + V_3} , \qquad p_2^* = \frac{V_3 + V_2}{1 + V_2} ;$$

$$q_1^* = \frac{V_3(V_2 + 1)}{V_2(1 - V_3)} , \qquad q_2^* = \frac{V_2 - V_3}{2 V_2} .$$

These values are obtained from the first and the second equations of systems (1) and (2) . Substituting p_1^* , p_2^* in the third equation of system (1) , we obtain the relation:

$$V_3^2(V_2 + 3) + V_3(-V_2^2 + 4 V_2 + 1) + V_2^2 - V_2 = 0 . \qquad (3)$$

Any negative root would be irrelevant to the given problem. Let us denote the coefficient of V_3^2 by a , the coefficient of V_3 by b . We will prove that $V_2 > V_3$.

$$V_3 = \frac{-b + \sqrt{b^2 + 4 V_2 a(1-V_2)}}{2 a}$$

$$V_2 - V_3 = \frac{2 V_2 a + b - \sqrt{b^2 + 4 V_2 a(1-V_2)}}{2 a} ,$$

but

$$(2 V_2 a + b)^2 > b^2 + 4 V_2 a(1-V_2) ,$$

or

$$4 V_2 a b > 4 V_2 a(1 - V_2) .$$

Therefore, $V_2 > V_3 > 0$. Then $p_1^* > 0$, $p_2^* > 0$, $q_1^* > 0$, $q_2^* > 0$. Adding the second and the third equations of system (1), we obtain $p_3^* = \dfrac{2 V_3}{1 - V_2} > 0$.

Subtracting the second equation from the third in system (2), we obtain

$$q_3^* = q_2^* \frac{1 + V_2}{2} > 0 .$$

<u>4. The Case $m = 4$.</u> Based on the solutions of the game for $m = 2$, $m = 3$ and from intuitive considerations it is possible to conjecture that in the case $m = 4$ the optimal strategies will be given by a mixture of the strategies a_{12}, a_{23}, a_{34}, a_{40} for player I and b_{01}, b_{12}, b_{23}, b_{30} for player II (a description of these strategies is given in section 1), i.e., that the game with matrix

$$\begin{array}{c|cccc} & b_{01} & b_{12} & b_{23} & b_{30} \\ \hline a_{12} & V_3 & -V_3 & V_3 & V_3 \\ a_{23} & -V_4 & V_2 & -V_2 & V_2 \\ a_{34} & -V_4 & -V_3 & 1 & -1 \\ a_{40} & -V_4 & -V_3 & -V_2 & 1 \end{array} \qquad (*)$$

containing matrices for $m = 2$ and $m = 3$ in the lower right corner, will be fully mixed (for fully mixed games, see, for example, [2]). However, this is not the case. It turns out that the following theorem is true.

<u>THEOREM 1:</u> <u>In the game under consideration, if each of the players has 4 cards then an optimal strategy for player I is represented by a mixture of</u> a_{12}, a_{23}, a_{34}, <u>and one for player II by a mixture of</u> b_{01}, b_{12}, b_{23}. <u>And if we denote by</u> p_1^*, p_2^*, p_3^* ($p_i^* \geq 0$, $i = 1,2,3$; $p_1^* + p_2^* + p_3^* = 1$) <u>the probabilities of using strategies</u> a_{12}, a_{23}, a_{34} <u>respectively, and by</u> q_1^*, q_2^*, q_3^* ($q_j^* \geq 0$, $j = 1,2,3$; $q_1^* + q_2^* + q_3^* = 1$) <u>the probabilities</u>

Substituting $1 - p_1^* - p_2^*$ in place of p_3^* and $1 - q_1^* - q_2^*$ in place of q_3^*, we find from the first two equations of systems (7) and (8) that

$$p_1^* = \frac{2\,V_4}{V_3 + V_4}\ , \qquad p_2^* = \frac{V_3 + V_4}{V_3 + V_2}\ ;$$

$$q_1^* = \frac{V_4(V_3 + V_2)}{V_3(V_2 - V_4)}\ , \qquad q_2^* = \frac{V_3 - V_4}{2\,V_3}\ .$$

Substituting the values found for p_1^*, p_2^* into the third equation of system (7), we obtain equation (6).

We shall prove that $V_3 > V_4 > 0$. Clearly any negative root of equation (6) is irrelevant, so that $V_4 > 0$. Let us denote the coefficient of V_4^2 by a, the coefficient of V_4 by b and consider the difference

$$V_3 - V_4 = \frac{2\,V_3\,a + b - \sqrt{b^2 + 4a\,V_3\,V_2\,(1 - V_3)}}{2\,a}$$

We shall show that $(2\,V_3\,a + b)^2 > b^2 + 4a\,V_3\,V_2\,(1 - V_3)$. Since $b = -V_3^2 + V_2\,V_3 + 3\,V_3 + V_2 > V_2\,(1-V_3)$, then $4\,V_3\,a\,b > 4\,V_3\,a\,V_2\,(1-V_3)$.

Therefore, $V_3 > V_4$, and hence, $p_1^* > 0$, $p_2^* > 0$, $q_1^* > 0$, $q_2^* > 0$.

Adding the second and the third equations of system (7), we obtain $p_3^* = \frac{2\,V_4}{1-V_3} > 0$; subtracting the second equation from the third one in system (8), we obtain

$$q_3^* = q_2^*\,\frac{V_3 + V_2}{1 + V_2} > 0\ .$$

It remains for us to show that these strategies are optimal for the original game.

The set of the strategies for player I consists of the elements $a_{i0} (1 \le i \le 4)$, $a_{0k} (1 < k \le 4)$, and $a_{ik} (1 \le i < k \le 4)$; the set for player II consists of

$b_{r0}(1 \le r \le 3)$, $b_{0j}(1 \le j \le 4)$, and $b_{rj}(1 \le r < j \le 3)$. Their description is given in section 1, and there it is explained why $a_{i0}(1 \le i \le 3)$, $a_{0k}(1 < k \le 4)$ and $b_{r0}(1 \le r < 3)$, $b_{0j}(1 < j \le 4)$ can not be included in the optimal mixture. The matrix, in which all the remaining strategies are included, has the following form:

$$\begin{array}{c|ccccc}
 & b_{01} & b_{12} & b_{23} & b_{30} & b_{13} \\
\hline
a_{12} & V_3 & -V_3 & V_3 & V_3 & -V_3 \\
a_{23} & -V_4 & V_2 & -V_2 & V_2 & V_2 \\
a_{34} & -V_4 & -V_3 & 1 & -1 & 1 \\
a_{13} & V_3 & -V_3 & -V_2 & V_2 & V_2 \\
a_{14} & V_3 & -V_3 & -V_2 & -1 & -V_2 \\
a_{24} & -V_4 & V_2 & -V_2 & -1 & -V_2 \\
a_{40} & -V_4 & -V_3 & -V_2 & 1 & -V_2
\end{array} \qquad \binom{*}{*}$$

We shall denote the vector (p_1^*, p_2^*, p_3^*) by p^*, and the vector (q_1^*, q_2^*, q_3^*) by q^* and denote by $R(p,q)$ the mathematical expectation of the payoff of player I, when he plays the mixed strategy p, and player II plays q. From matrix $\binom{*}{*}$ it is seen that $R(p,q^*) \le R(p^*, q^*)$ and that $R(p^*,q^*) < R(p^*, b_{13})$; it is then necessary only to prove that $R(p^*,q^*) < R(p^*,b_{30})$ i.e., $V_4 < p_1^* V_3 + p_2^* V_2 - p_3^*$.

Adding to both sides of this inequality the third equation of system (7), we obtain $2\, p_1^* V_3 > 2\, V_4$ or $p_1^* > \dfrac{V_4}{V_3}$, but $p_1^* = \dfrac{2\, V_4}{V_3 + V_4} > \dfrac{V_4}{V_3}$. Hence the above strategies are optimal, and the theorem is proved.

5. The Case $m \geq 4$. A description of the sets of strategies for the players in this case was given in section 1. There it was proven that the utilization of strategies $a_{i0} (1 \leq i \leq m-1)$, $a_{0k} (1 < k \leq m)$ in the optimal strategy of player I and b_{10}, b_{0k} in the optimal strategy of player II is impossible.

THEOREM 2: In the game under consideration when $m \geq 4$, the optimal strategy of player I is represented by a mixture of b_{01}, b_{12}, b_{23}. And if we denote by p_1^*, p_2^*, p_3^* the probabilities with which player I plays a_{12}, a_{23}, a_{34} respectively ($p_i^* \geq 0$, $i = 1,2,3$, $p_1^* + p_2^* + p_3^* = 1$), and by q_1^*, q_2^*, q_3^* the probabilities with which player II chooses b_{01}, b_{12} or b_{23} ($q_j^* \geq 0$, $j = 1,2,3$, $q_1^* + q_2^* + q_3^* = 1$), then

$$p_1^* = \frac{2V_m}{V_m + V_{m-1}} ; \quad p_2^* = \frac{V_m + V_{m-1}}{V_{m-1} + V_{m-2}} ; \quad p_3^* = \frac{2V_m}{V_{m-3} - V_{m-1}} ;$$

$$q_1^* = \frac{V_m (V_{m-1} + V_{m-2})}{V_{m-1}(V_{m-2} - V_m)} ; \quad q_2^* = \frac{V_{m-1} - V_m}{2V_{m-1}} ; \quad q_3^* = q_2^* \frac{V_{m-1} + V_{m-2}}{V_{m-2} + V_{m-3}} ,$$

where V_m is found from equation (12) introduced below.

PROOF: We shall first find the solution of the game in which player I may use only a_{12}, a_{23}, a_{34}, and player II b_{01}, b_{12}, b_{23}, and then prove that this solution is a solution for the original game. The matrix of the game corresponding to the strategies is as follows:

	b_{01}	b_{12}	b_{23}
a_{12}	V_{m-1}	$-V_{m-1}$	V_{m-1}
a_{23}	$-V_m$	V_{m-2}	$-V_{m-2}$
a_{34}	$-V_m$	$-V_{m-1}$	V_{m-3}

Consequently, $p_i^*, q_j^* (i,j = 1,2,3)$ are subject to the following conditions:

$$\left.\begin{aligned} p_1^* V_{m-1} - p_2^* V_m - p_3^* V_m &= V_m , \\ -p_1^* V_{m-1} + p_2^* V_{m-2} - p_3^* V_{m-1} &= V_m , \\ p_1^* V_{m-1} - p_2^* V_{m-2} + p_3^* V_{m-3} &= V_m , \end{aligned}\right\} \qquad (9)$$

$$p_1^* + p_2^* + p_3^* = 1; \quad p_i^* \geq 0 \quad (i=1,2,3)$$

$$\left.\begin{aligned} q_1^* V_{m-1} - q_2^* V_{m-1} + q_3^* V_{m-1} &= V_m , \\ -q_1^* V_m + q_2^* V_{m-2} - q_3^* V_{m-2} &= V_m , \\ -q_1^* V_m - q_2^* V_{m-2} + q_3^* V_{m-3} &= V_m , \end{aligned}\right\} \qquad (10)$$

$$q_1^* + q_2^* + q_3^* = 1 ; \quad q_j^* \geq 0 \quad (j = 1,2,3).$$

Substituting $1 - p_1^* - p_2^*$ for p_3^* in system (9) and $1 - q_1^* - q_2^*$ for q_3^* in system (10), we obtain from the first two equations of each system

$$\begin{aligned} p_1^* &= \frac{2 V_m}{V_m + V_{m-1}} ; \qquad p_2^* = \frac{V_m + V_{m-1}}{V_{m-1} + V_{m-2}} ; \\ q_1^* &= \frac{(V_{m-1} + V_{m-2})}{V_{m-1} (V_{m-2} - V_m)} ; \quad q_2^* = \frac{V_{m-1} - V_m}{2 V_{m-1}} . \end{aligned} \qquad (11)$$

Substituting these values for p_1^*, p_2^* into the third equation of system (9), we come to the following equation for V_m:

$$\begin{aligned} V_m^2 (V_{m-1} + 2 V_{m-2} + V_{m-3}) + V_m (-V_{m-1}^2 + V_{m-1} V_{m-2} + 3 V_{m-1} V_{m-3} + V_{m-2} V_{m-3}) + \\ + V_{m-1} V_{m-2} (V_{m-1} - V_{m-3}) = 0 . \end{aligned} \qquad (12)$$

Let us denote the coefficient of V_m^2 by a, and the coefficient of V_m by b. We shall prove by induction that $V_{m-1} > V_m > 0$. We have $V_2 > V_3 > V_4 > 0$. Suppose that $V_2 > V_3 > \ldots > V_{m-2} > V_{m-1} > 0$, then

$$V_m = \frac{-b + \sqrt{b^2 - 4\,a\,V_{m-1}\,V_{m-2}\,(V_{m-1} - V_{m-3})}}{2\,a}$$

(any negative root is irrelevant). Consider the difference

$$V_{m-1} - V_m = V_{m-1} - \frac{-b + \sqrt{b^2 + 4\,a\,V_{m-1}\,V_{m-2}(V_{m-3} - V_{m-1})}}{2\,a} =$$

$$= \frac{2\,a\,V_{m-1} + b - \sqrt{b^2 + 4\,a\,V_{m-1}\,V_{m-2}(V_{m-3} - V_{m-1})}}{2\,a}$$

It is not difficult to show that

$$(2a\,V_{m-1} + b)^2 > b^2 + 4\,a\,V_{m-1}\,V_{m-2}(V_{m-3} - V_{m-1}) .$$

Indeed, it is sufficient to convince ourselves that

$$4\,a\,V_{m-1}\,b > 4\,a\,V_{m-1}\,V_{m-2}(V_{m-3} - V_{m-1}) ,$$

but this is valid, since

$$b = - V_{m-1}^2 + V_{m-1}\,V_{m-2} + 3\,V_{m-1}\,V_{m-3} + V_{m-2}\,V_{m-3} > V_{m-2}(V_{m-3} - V_{m-1}) .$$

Consequently, $V_{m-1} > V_m > 0$. Clearly, $p_1^* > 0$, $p_2^* > 0$, $q_1^* > 0$, $q_2^* > 0$. Adding the second and the third equations of system (9), we obtain $p_3^* = \dfrac{2\,V_m}{V_{m-3} - V_{m-1}} > 0$. Subtracting from the third equation of system (10) the second one, we have $q_3^* = q_2^* \dfrac{V_{m-1} + V_{m-2}}{V_{m-2} + V_{m-3}} > 0$.

Let us denote the vector (p_1^*, p_2^*, p_3^*) by p^*, and (q_1^*, q_2^*, q_3^*) by q^*; we shall denote by p an arbitrary mixed strategy for player I ; and an arbitrary one for player II by q ; $R(p,q)$ is the mathematical expectation of the payoff of player I, when he plays strategy p and his opponent plays q . We will prove that p^* and q^* are optimal strategies of player I and player II respectively. For this it is necessary to prove that $R(p,q^*) \leq R(p^*,q^*) \leq R(p^*,q)$, for any p,q .

a) $R(p,q^*) \leq R(p^*, q^*)$.

Player I can use only strategies a_{m0}, $a_{ik}(1 \leq i < k \leq m)$. The remaining ones, as was proved in section 1, are dominated. Player II, using q^* , checks at least by the third move; therefore, any strategy of player I under which he bluffs in the course of the first three moves will be dominated by a strategy prescribing at one of these moves to exhibit a correct card. Hence for player I there remain only strategies $a_{ik}(i \leq 3,\ 1 \leq i < k \leq m)$. In addition, it is evident that a_{ik} is not better than a_{i4} , when $i \leq 3$, $k > 4$. In this way, it is necessary to show only that among the strategies $a_{ik}(1 \leq i < k \leq 4)$ only the $a_{i\ i+1}(i = 1,2,3)$ are not dominated. This is seen from the matrix:

	b_{01}	b_{12}	b_{23}
a_{12}	V_{m-1}	$-V_{m-1}$	V_{m-1}
a_{23}	$-V_m$	V_{m-2}	$-V_{m-2}$
a_{34}	$-V_m$	$-V_{m-1}$	V_{m-3}
a_{13}	V_{m-1}	$-V_{m-1}$	$-V_{m-2}$
a_{14}	V_{m-1}	$-V_{m-1}$	$-V_{m-2}$
a_{24}	$-V_m$	V_{m-2}	$-V_{m-2}$

b) $R(p^*,q^*) \leq R(p^*,q)$

As was proved in section 1, player II can use only strategies b_{01}, $b_{m-1\ 0}$, $b_{rj}(1 \leq r < j \leq m-1)$. But player I playing p^*, checks at least by the fourth move. Therefore, all strategies, by which player II bluffs in the course of the first three moves, are dominated by those, under which he plays a correct card at one of these moves. Hence, if $m > 4$, player II can use only b_{01}, $b_{rj}(r \leq 3,$ $1 \leq r < j \leq m-1)$. In addition, b_{r4} dominates b_{rj} for $j > 4$ and $r \leq 3$. We will write down the matrix of the game, in which player I uses a_{12}, a_{23}, a_{34}, and player II uses b_{01}, $b_{rj}(1 \leq r < j \leq 4)$:

	b_{01}	b_{12}	b_{23}	b_{34}	b_{13}	b_{14}	b_{24}
a_{12}	V_{m-1}	$-V_{m-1}$	V_{m-1}	V_{m-1}	$-V_{m-1}$	$-V_{m-1}$	V_{m-1}
a_{23}	$-V_m$	V_{m-2}	$-V_{m-2}$	V_{m-2}	V_{m-2}	V_{m-2}	$-V_{m-2}$
a_{34}	$-V_m$	$-V_{m-1}$	V_{m-3}	$-V_{m-3}$	V_{m-3}	V_{m-3}	V_{m-3}

Obviously, strategies b_{13}, b_{14}, b_{24} are dominated. We shall prove that b_{34} will not be included in an optimal strategy, i.e., that

$$R(p^*, b_{34}) = V_{m-1}\, p_1^* + V_{m-2}\, p_2^* - V_{m-3}\, p_3^* > V_m.$$

Let us add the third equation of system (9) to both sides of this inequality. We obtain

$$2\, p_1^*\, V_{m-1} > 2\, V_m \quad \text{or} \quad p_1^* > \frac{V_m}{V_{m-1}} \quad ,$$

which is obvious, since $p_1^* = \dfrac{2\, V_m}{V_m + V_{m-1}} > \dfrac{V_m}{V_{m-1}}$.

Therefore, the solution (p^*, q^*) is an optimal solution for the entire game.

6. The Asymptotic Behavior of the Values of the Game.

To compute the values of the game and optimal strategies for the players for different m , it was convenient to consider the quantity $\alpha_m = \dfrac{V_m}{V_{m-1}}$. Dividing equation (12) by $V_m\, V_{m-1}\, V_{m-2}$, we obtain the following relation for α_m :

$$\alpha_m^2\, \alpha_{m-1}(\alpha_{m-1}\, \alpha_{m-2} + 2\, \alpha_{m-2} + 1) + \alpha_m(-\alpha_{m-1}^2\, \alpha_{m-2} + \alpha_{m-1}\, \alpha_{m-2} + + 3\, \alpha_{m-1} + 1) + \alpha_{m-1}\, \alpha_{m-2} - 1 = 0 . \tag{13}$$

<u>THEOREM 3:</u> <u>The sequence</u> $\{\alpha_m\}$ <u>converges</u>.

<u>PROOF:</u> We will denote the explicit expression for α_m derived from equation (13) by $f(\alpha_{m-1}, \alpha_{m-2})$. Consider the difference

$$\alpha_{m'} - \alpha_m = f(\alpha_{m'-1}, \alpha_{m'-2}) - f(\alpha_{m-1}, \alpha_{m-2}).$$

By the formula of finite increments

$$\alpha_{m'} - \alpha_m = \frac{\partial f(\bar{\alpha}_{m-1}, \alpha_{m-2})}{\partial \alpha_{m-1}} (\alpha_{m'-1} - \alpha_{m-1}) + + \frac{\partial f(\alpha_{m-1}, \bar{\alpha}_{m-2})}{\partial \alpha_{m-2}} (\alpha_{m'-2} - \alpha_{m-2}) \quad , \tag{14}$$

where $\bar{\alpha}_{m-1}$ is between $\alpha_{m'-1}$ and α_{m-1} ; and $\bar{\alpha}_{m-2}$ is between $\alpha_{m'-2}$ and α_{m-2} . From equation (10)

$$\frac{\partial f}{\partial \alpha_{m-1}} = \frac{-\alpha_m^2(2\alpha_{m-1}\alpha_{m-2} + 2\alpha_{m-2} + 1) - \alpha_m(-2\alpha_{m-1}\alpha_{m-2} + \alpha_{m-2} + 3) - \alpha_{m-2}}{2\alpha_m(\alpha^2_{m-1}\alpha_{m-2} + 2\alpha_{m-1}\alpha_{m-2} + \alpha_{m-1}) - \alpha^2_{m-1}\alpha_{m-2} + \alpha_{m-1}\alpha_{m-2} + 3\alpha_{m-1} + 1},$$

$$\frac{\partial f}{\partial \alpha_{m-2}} = \frac{-\alpha_m(-\alpha^2_{m-1} + \alpha_{m-1}) - \alpha_{m-1} - \alpha_m^2\alpha_{m-1}(\alpha_{m-1} + 2)}{2\alpha_m\alpha_{m-1}(\alpha_{m-1}\alpha_{m-2} + 2\alpha_{m-2} + 1) - \alpha^2_{m-1}\alpha_{m-2} + \alpha_{m-1}\alpha_{m-2} + 3\alpha_{m-1} + 1}.$$

As is seen from the supplement (see below) and by virtue of the continuity of $f(\alpha_{m-1}; \alpha_{m-2})$ there exists an m_0 such that for $m - 2 > m_0$

$$0.361 < \alpha_m < 0.362 .$$

By calculations it is shown that for these m

$$\left|\frac{\partial f}{\partial \alpha_{m-1}}\right| \leq 0.6 \ , \quad \left|\frac{\partial f}{\partial \alpha_{m-2}}\right| \leq 0.3 \ ,$$

and then from (14) we have

$$|\alpha_{m'} - \alpha_m| \leq 0.6\,|\alpha_{m'-1} - \alpha_{m-1}| + 0.3\,|\alpha_{m'-2} - \alpha_{m-2}| .$$

Let us denote $|\alpha_{m'-i} - \alpha_{m-i}|$ by ρ_{m-i} . Then

$$\rho_m \leq b_1\rho_{m-1} + c_1\rho_{m-2} \leq b_2\rho_{m-2} + c_2\rho_{m-3} \leq \cdots \leq$$

$$\leq b_{k-1}\rho_{m-k+1} + c_{k-1}\rho_{m-1} \leq b_k\rho_{m-k} + c_k\rho_{m-k-1} \leq \cdots ,$$

where

$$b_1 = 0.6, \quad c_1 = 0.3\ , \quad b_k = b_{k-1}b_1 + c_{k-1}\ , \quad c_k = b_{k-1}c_1\ ,$$
$$k = 2,3\ldots\ ; \quad m-k > k_0 \ . \tag{15}$$

It converges, since it is bounded below by zero and monotonically decreases. Then $b_k \to 0$, and together with b_k, by virtue of (15), $c_k \to 0$ for $k \to \infty$. Consequently, $|\alpha_{m'} - \alpha_m| \to 0$ as $m, m' \to \infty$; i.e., the sequence $\{\alpha_m\}$ converges.

Going to the limit as $m \to \infty$ in equation (13) and reducing the similar terms, we have

$$\alpha^5 + \alpha^4 + 2\alpha^3 + 4\alpha^2 + \alpha - 1 = 0$$

or, dividing both sides by $(\alpha + 1)^2$, we obtain

$$\alpha^3 - \alpha^2 + 3\alpha - 1 = 0 ,$$

from which we get that $\alpha \doteq 0.36110308$.

The above theorem allows us to find out the asymptotic behavior of the optimal strategies of the players.

From (11) for $m \to \infty$, we have:

$$p_1^* = \frac{2\alpha}{1+\alpha}, \quad p_2^* = \alpha ; \quad q_1^* = \frac{\alpha}{1-\alpha} , \quad q_2^* = \frac{1-\alpha}{2} .$$

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SUMMARY

A concrete zero-sum two-person game is considered. The game is a simplified model of the card game called "verish - ne - verish" (believe it or not).

Optimal strategies for the game and recurrence relations for its value as a function of m are found, where m is the number of cards that each player holds.

The asymptotic behavior of the value of the games is also investigated.

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(Translated by Kiyoshi Takeuchi)

Supplement

m	α_m	V_m	p_1	p_2	p_3	q_1	q_2	q_3
1		1.0000	1					0
2	0.23607	0.2361	0.38	0.62	0	0.62	0.38	0.20
3	0.35386	$0.0^{1}8353$	0.52	0.26	0.22	0.48	0.32	0.08
4	0.39301	$0.0^{1}3283$	0.56	0.37	0.07	0.62	0.30	0.12
5	0.34270	$0.0^{1}1125$	0.51	0.38	0.11	0.55	0.33	0.12
6	0.36694	$0.0^{2}4128$	0.54	0.35	0.11	0.56	0.32	0.11
7	0.36119	$0.0^{2}1491$	0.53	0.37	0.10	0.57	0.32	0.12
8	0.35982	$0.0^{3}5366$	0.53	0.36	0.11	0.56	0.32	0.11
9	0.36192	$0.0^{3}1942$	0.53	0.36	0.11	0.57	0.32	0.11
10	0.36083	$0.0^{4}7007$	0.53	0.36	0.11	0.57	0.32	0.11
11	0.36111	$0.0^{4}2530$	0.53	0.36	0.11	0.57	0.32	0.11
12	0.36115	$0.0^{5}9138$	0.53	0.36	0.11	0.57	0.32	0.11
13	0.36107	$0.0^{5}3300$	0.53	0.36	0.11	0.57	0.32	0.11
14	0.36111	$0.0^{5}1192$	0.53	0.36	0.11	0.57	0.32	0.11
15	0.36111	$0.0^{6}4303$	0.53	0.36	0.11	0.57	0.32	0.11
16	0.36111	$0.0^{6}1554$	0.53	0.36	0.11	0.57	0.32	0.11
17	0.36111	$0.0^{7}5611$	0.53	0.36	0.11	0.57	0.32	0.11
18	0.36111	$0.0^{7}2026$	0.53	0.36	0.11	0.57	0.32	0.11

Note: $0.0^{5}9138 = .9138 \times 10^{-5}$.

VII

ON THE DURATION OF A GAME OF SURVIVAL

I. V. Romanovsky

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1. We will consider a two-person game, in which player I has a certain finite amount of capital x and player II has infinite capital. This game consists of separate groups, in each of which the players play a matrix game with random payoffs ξ_{ij}, and the game is continued up to the point when the first player becomes bankrupt. In the case in which the value of this matrix game is negative player I becomes bankrupt with probability 1. Here we shall consider the problem of determining the duration of the game, where the first player endeavors to prolong the game, while the second player endeavors to make the first bankrupt as promptly as possible.

We will prove that when optimal strategies are used by each player, the average time, considered as a function of the initial capital x of player I, satisfies the functional equation

$$f(x) = \begin{cases} 0, & x \leq 0, \\ 1 + \mathrm{Val}\,\|Ef(x+\xi_{ij})\|, & x > 0, \end{cases} \tag{1}$$

and has an order $f(x) \approx x/c$, where c is the average payoff of the first player in each of the separate groups.

The problem of determining the optimal strategies for the players is not considered in this article.

2. The following zero-sum two-person game will be studied. Let the matrix $\| \xi_{ij} \|$ be given, the elements of which are random variables having a finite mathematical expectation, and let there be given a number x, characterizing the initial condition of the game. Player I chooses row i of the matrix, player II chooses column j. After this, player II pays a unit to player I and the condition of the game becomes $x + y$, where y is the realization of the random variable ξ_{ij}. (The mixed strategies of the players are determined in the usual way.) The game is repeated until the condition of the game ceases to be a positive quantity. By the end of the game the second player has payed a quantity to the first player proportionate to the duration of the game.

We note that for the game to end with probability 1, it is sufficient that

$$\operatorname{Val} \| E\, \xi_{ij} \| = -c < 0 ,$$

where
$$\operatorname{Val} \|a_{ij}\| = \min_q \max_p \sum_{i,j} a_{ij}\, p_i\, q_j = \max_p \min_q \sum_{i,j} a_{ij}\, p_i\, q_j .$$

(For the above notation and terminology, see, for example, [1], [2]). Later on we will assume that this condition is satisfied.

3. Before proceeding to the study of the game itself, we shall prove certain auxiliary assertions.

<u>LEMMA 1:</u> <u>Let the function</u> $\varphi(x)$ <u>be determined for</u> $x \geq 0$ <u>and satisfy the following conditions</u>:

1) <u>for arbitrary</u> x_0 <u>there exists a</u> $y > x_0$ <u>such that</u>
$$\sup_{x \leq x_0} \varphi(x) < \varphi(y) ; \qquad \text{and}$$

2) <u>there exist an</u> $a > 0$ <u>and</u> b, <u>such that for all</u> $x \geq 0$
$$\varphi(x) \leq a\,x + b .$$

Then for arbitrary $L > 0$ and $\epsilon > 0$ there exist $r > 0$, d, and x_0 such that

$$\varphi(x) \leq r x + d \quad \text{for} \quad x \geq x_0 - L$$

and

$$\varphi(x_0) \geq r x_0 + d - r\epsilon .$$

PROOF: Let

$$M = \{ (x,y) | x \geq 0, \ y \leq \varphi(x) \} ,$$

and detonote by $\bar{M}$ the closure of M. Denote by N the convex hull of $\bar{M}$, and by $y = p(x)$ the boundary of N.

If for $\bar{x} > L$ there exists a point $(\bar{x},\bar{y}) \in \bar{M}$, lying on the boundary of N, then the assertion of the lemma holds. In fact, let $\bar{y} = p(\bar{x})$. Let us construct at $(\bar{x},\bar{y})$ the tangent to the concave curve $p(x)$. It is evident that its slope will be larger than zero. It will be the line being sought, since the first condition is evidently satisfied. The second requirement is satisfied by virtue of the fact that, from the definition, a point can be found in M which is as close as is desired to $(\bar{x},\bar{y})$. In fact, we can choose a point of the form $(x, \varphi(x))$, for which the second requirement will be fulfilled.

We shall assume now that for $s > L$ such points do not exist. Let

$$x_0 = \max \{ x | \ (x, p(x)) \in \bar{M}\} \qquad (x_0 \leq L) .$$

It is easily seen that for $x > x_0$ $p(x)$ is a straight line. Let $p(x) = r x + q$ (where $r > 0$). From the points of $\bar{M}$ in the interval $[x_0, L]$ we choose one which maximizes the expression $y - (r - \frac{r\epsilon}{L}) x$. Let us denote this point by $(\tilde{x}, \tilde{y})$ and construct through it the straight line $y = (r - \frac{r\epsilon}{L}) x + s$. As a result of the convexity of the set N there exists a point $(x,y) \in \bar{M}$ such that $x > L$ and $y > (r - \frac{r\,\epsilon}{L}) x + s$. We shall consider the set of all such points and find $\sup(y - (r x + q)) = \alpha$ on this set. Choosing from this set a point

$(\bar{x},\bar{y})$ such that $\bar{y} - r\,\bar{x} - q \geq \alpha - \frac{r\,\epsilon}{2}$, we can now prove without difficulty that in the neighborhood of $(\bar{x},\bar{y})$ a point (x,y) can be found satisfying the requirement of the lemma, if $y - r\,\bar{x} - q + \frac{r\,\epsilon}{L} - \alpha$ is chosen as d. And so the lemma is proven.

It is possible to exclude condition (1) if it is required that the function φ be determined and be equal to zero for $x < 0$. When condition (1) is not satisfied, the straight line sought is the supporting line to N, constructed through the point $(-L, 0)$. We shall refer to this fact as well as to lemma 1.

Let ξ_{ij} be a random variable such that for certain $L > 0$ and for all i,j and

$$P(\xi_{ij} > -L) = 1$$

$$\mathrm{Val}\,\| E\,\xi_{ij}\| = -c < 0 .$$

We shall now consider the sequence

$$\bar{f}_1(x) = \begin{cases} x/c , & x \leq 0 , \\ 0 , & x > 0 , \end{cases}$$

$$\bar{f}_{n+1}(x) = \begin{cases} x/c & x \leq 0 , \\ \max\, \{\bar{f}_n(x) ,\ 1 + \mathrm{Val}\,\| E\,\bar{f}_n(x + \xi_{ij})\|\} , & x > 0 . \end{cases} \qquad (2)$$

<u>LEMMA 2:</u> $\bar{f}_n(x) \leq \bar{f}_{n+1}(x) \leq \frac{x}{c}$.

This lemma is completely obvious. From this result it should be equally clear that the sequence $\{\bar{f}_n\}$ converges to some function f_0 and this function satisfies the functional equation

$$\bar{f}_0(x) = \begin{cases} \frac{x}{c} & x \leq 0 , \\ \max\ \{\bar{f}_0(x),\ 1 + \mathrm{Val}\| E\,\bar{f}_0(x + \xi_{ij})\|\} & x > 0 . \end{cases}$$

<u>LEMMA 3:</u> <u>The functions</u> $\bar{f}_n$ <u>and</u> $\bar{f}_0$ <u>are monotonically non-decreasing</u>.

This is obvious.

Before proceeding further, we shall note that the set of points in which $\bar{f}_0(x) > 1 + \mathrm{Val}\|E\,\bar{f}_0(x + \xi_{ij})\|$ is clearly an interval (a,b) and for

$x \in (0,b)$ we have $\bar{f}_0(x) = 0$. This follows from the monotonicity of the sequence $\{\bar{f}_n\}$ and from the monotonicity of the functions $\bar{f}_n(x)$. We will also note that the function $\bar{f}_0$ will not increase too slowly. In fact, for $x > 0$ $\bar{f}_0(x) \geq [\frac{x}{L}]$.

<u>LEMMA 4:</u> $\bar{f}_0(x) = \frac{x}{c}$.

<u>PROOF:</u> Let the lemma be assumed false. We then consider the function $\varphi(x) = \frac{x}{c} - \bar{f}_0(x)$. It satisfies condition (2) of lemma 1, and according to the remark made after lemma 1 for arbitrary ϵ there exist $r > 0$, d, and $x_0 > b$ such that

$$\varphi(x) \leq r\,x + d \qquad \text{for } x \geq x_0 - L$$

$$\varphi(x_0) \geq r\,x_0 + d - r\,\epsilon .$$

In connection with $\bar{f}_0(x)$ we obtain

$$\bar{f}_0(x) \geq \left(\frac{1}{c} - r\right) x - d \qquad \text{for } x \geq x_0 - L$$

and

$$\bar{f}_0(x_0) \leq \left(\frac{1}{c} - r\right) x_0 - d + r\,\epsilon ,$$

where we can assume that $\frac{1}{c} - r > 0$. We have now

$$\bar{f}_0(x_0) = 1 + \mathrm{Val}\, \| E\, \bar{f}_0(x_0 + \xi_{ij}) \| \geq$$

$$\geq 1 - d + \left(\frac{1}{c} - r\right) x_0 + \left(\frac{1}{c} - r\right) \mathrm{Val}\, \| E\, \xi_{ij} \| = \left(\frac{1}{c} - r\right) x_0 - d + c\,r ,$$

which for $\epsilon < c$ leads to a contradiction.

In this way, we have proven the following:

<u>THEOREM 1:</u> <u>Let</u> $\mathrm{Val}\, \| E\, \xi_{ij} \| = -c < 0$ <u>and for some</u> $L > 0$ <u>and for all</u> i,j $P(\xi_{ij} > -L) = 1$. <u>Then the sequence</u> $\{\bar{f}_n\}$ <u>converges to the function</u>

$\bar{f}_0(x) = x/c$, satisfying the equation

$$\bar{f}_0(x) = \begin{cases} x/c , & x \leq 0 , \\ 1 + \mathrm{Val} \, \| \mathrm{E} \, \bar{f}_0(x + \xi_{ij}) \| , & x > 0 . \end{cases} \tag{3}$$

COROLLARY: Under the hypothesis of theorem 1 , the sequence

$$\tilde{f}_1(x) = \begin{cases} \frac{x}{c} + a , & x \leq 0 , \\ a , & x > 0 , \end{cases}$$

$$\tilde{f}_{n+1}(x) = \begin{cases} x/c + a , & x \leq 0 , \\ \max \{\tilde{f}_n(x) , \; 1 + \mathrm{Val} \, \| \mathrm{E} \, \tilde{f}_n(x + \xi_{ij}) \| \} , & x > 0 , \end{cases} \tag{4}$$

converges to the function $\tilde{f}_0(x) = \frac{x}{c} + a$.

Theorem 1 is valid without the assumption of boundedness from below on the random variable ξ_{ij} . However, we shall not prove it in the general case. The weaker result is sufficient for our purpose, although it leads to some difficulties in the proof of theorem 2.

4. Let us now turn to the study of our game.

THEOREM 2: The value (duration) of the game $f_0(x)$ (as a function of the initial condition of the game x) is equal to the limit of the sequence of functions

$$f_1(x) \equiv 0$$

$$f_{n+1}(x) = \begin{cases} 0 , & x \leq 0 , \\ 1 + \mathrm{Val} \, \| \mathrm{E} \, f_n(x + \xi_{ij}) \| , & x > 0 . \end{cases} \tag{5}$$

$f_0(x)$ satisfies functional equation (1) and has as order

$f_0(x) = \frac{x}{c}(1 + o(1))$ and $o(1) \geq 0$.

PROOF: Before proceeding to the proof we shall introduce two lemmas.

LEMMA 5: *Let* $\varphi(x) \le \psi(x)$ for $x \le 0$. *Assume*

$$f_1'(x) = \begin{cases} \varphi(x) , & x \le 0 , \\ 0 , & x > 0 , \end{cases}$$

$$f_{n+1}'(x) = \begin{cases} \varphi(x) , & x \le 0 , \\ \max \{f_n'(x) , 1 + \mathrm{Val} \| E f_n'(x + \xi_{ij})\|\} , & x > 0 , \end{cases}$$

and

$$f_1''(x) = \begin{cases} \psi(x) , & x \le 0 \\ 0 , & x > 0 , \end{cases}$$

$$f_{n+1}''(x) = \begin{cases} \psi(x) , & x \le 0 , \\ \max \{f_n''(x) , 1 + \mathrm{Val} \| E f_n''(x + \xi_{ij})\|\} , & x > 0 . \end{cases}$$

If the sequence $\{f_n''\}$ *converges to a function* f_0'' , *then the sequence* $\{f_n'\}$ *also converges to some function* f_0' *and* $f_0'(x) \le f_0''(x)$. *In addition if for some* $L > 0$ *and for all* i,j $P(\xi_{ij}, \ge - L) = 1$, *then it is sufficient that* $\varphi(x) \le \psi(x)$ *only for* $- L \le x \le 0$.

The proof follows from the monotonicity of the sequences $\{f_n'(x)\}$ and $\{f_n''(x)\}$ and from the fact that $f_n'(x) \le f_n''(x)$.

We adopt the notation $\xi_{ij}^L = \max (\xi_{ij}, - L)$.

LEMMA 6: *Let*

$$f_1^L(x) \equiv 0 ,$$

$$f_{N+1}^L(x) = \begin{cases} 0 , & x \le 0 , \\ 1 + \mathrm{Val} \| E f_n^L(x + \xi_{ij}^L) \| , & x > 0 . \end{cases} \quad (6)$$

<u>If the sequence</u> $\{f_n^L\}$ <u>converges to a function</u> f_0^L , <u>then the sequence</u> $\{f_n\}$ <u>also converges to some function</u>. f_0 <u>and</u> $f_0 < f_0^L$.

The proof follows from the monotomicity of the sequences $\{f_n\}$ and $\{f_n^L\}$ and from the fact that $f_n(x) \leq f_n^L(x)$.

Let us now turn to the proof of the theorem. That the value of the game is equal to the limit of the sequence of equations (5) and satisfies functional equation (1) follows from the principle of optimality of R. Bellman [2] . We must prove that this limit exists, and find the order of the limiting function.

Let us choose L such that $\operatorname{Val} \| E\, \xi_{ij}^L \| = - c^L < 0$. By the corollary to theorem 1 there exists a limit for the sequence

$$\tilde{f}_1^L(x) = \begin{cases} \dfrac{x}{c^L} + \dfrac{L}{c^L} , & x \leq 0 , \\ \dfrac{L}{c^L} ; & x > 0 , \end{cases}$$

$$\tilde{f}_{n+1}^L(x) = \begin{cases} \dfrac{x}{c^L} + \dfrac{L}{c^L} , & x \leq 0 , \\ \max \{\tilde{f}_n^L(x),\ 1 + \operatorname{Val} \| E\, \tilde{f}_n^L(x + \xi_{ij}^L) \| \} , & x > 0 , \end{cases}$$

which is equal to $\tilde{f}_0^L := \dfrac{x}{c^L} + \dfrac{L}{c^L}$. By lemma 5 there exists a limit f_0^L of of the sequence $\{f_n^L\}$ in (6), where $\bar{f}_0 \leq f_0^L \leq \tilde{f}_0^L$: By lemma 6 it follows that the sequence (5) also has a limit f_0 , where

$$f_0(a) \leq f_0^L(x) \leq \tilde{f}_0^L = \frac{x}{c^L} + \frac{L}{c^L} . \tag{7}$$

We have now proved that the theorem is valid in case of a random variable bounded from below, but for the general case there still remains for us to prove that $f_0(x) = \frac{x}{c}(1 + O(1))$. For this it is necessary for us to prove that $f_0(x) \geq \frac{x}{c}$. We use that $f_0^L(x) > \dfrac{x}{c^L}$, and that $f_0^L(x) \to f_0(x)$ for $L \to \infty$. The latter needs to be proved.

LEMMA 7: For arbitrary x

$$\lim_{L \to \infty} f_0^L(x) = f_0(x) .$$

PROOF of Lemma: We shall divide the proof into a sequence of steps.

I. For arbitrary $k > 0$ the sequence

$$g_1^k(x) = \begin{cases} f_0(k + x) - f_0(k) , & x \leq 0 , \\ 0 , & x > 0 , \end{cases}$$

$$g_{n+1}^k(x) = \begin{cases} f_0(k + x) - f_0(k) , & x \leq 0 , \\ \max\{g_n^k(x), 1 + \mathrm{Val} \, \| E \, g_n^k(x + \xi_{ij}) \|\} , & x > 0 , \end{cases}$$

converges to function $g_0^k(x) = f_0(x + k) - f_0(k)$. Indeed, $g_n^k(x) \geq f_n(x + k) - f_0(k)$ and $g_n^k(x) \leq f_0(x + k) - f_0(k)$.

II. There exist an a and a d such that for all $k > 0$ $f_0(x+k) - f_0(k) \leq ax + d$. This follows (by lemma 5) from the fact that $g_n^k(x) \leq f_n(x)$, and from the existence of the limit $f_0(x)$ for the sequence $\{f_n\}$, which is bounded by a linear function.

III. When $L \to \infty$ for all i,j

$$\max_x \; E(f_0(x + \xi_{ij}^L) - f_0(x + \xi_{ij})) \to + 0 .$$

This assertion follows from II and from the existence of $E \, \xi_{ij}$.

IV. If $\max_x E \, [f_0(x + \xi_{ij}^L) - f_0(x + \xi_{ij})] \leq b < 1$, then

$$f_n^L(x) \leq \frac{1}{1 - b} f_0(x) .$$

We will prove that $f_n^L(x) \leq \frac{1}{1 - b} f_0(x)$.

For $n = 1$ this holds. Further, if it is valid for n, then

$$f^L_{n+1}(x) = 1 + \mathrm{Val} \, \| E \, f^L_n(x + \xi^L_{ij}) \| \leq$$

$$\leq 1 + \frac{b}{1-b} + \frac{1}{1-b} \mathrm{Val} \, \| E \, f_0(x + \xi_{ij}) \| = \frac{1}{1-b} f_0(x) .$$

Now the assertion of the lemma follows from the fact that according to III $b \to 0$ as $L \to \infty$.

From this theorem follows without difficulty.

5. In the proof of uniqueness we will restrict ourselves (for the sake of simplicity) only to the case where, for all L, $\mathrm{Val} \, \| E \, \xi^L_{ij} \| > \mathrm{Val} \, \| E \, \xi_{ij} \|$. In the general case the proof also holds but instead of "reduction" of random variables a more complex discussion is necessary.

<u>THEOREM 3: Functional equation (1) has a unique solution in the class of non-negative non-decreasing functions, increasing not faster than linearly.</u>*

<u>PROOF:</u> We shall prove this theorem in several steps. First we still prove the uniqueness (in the case of sufficiently large L) of the solution of the equation

$$g^L(x) = \begin{cases} 0 , & x \leq 0 , \\ 1 + \max\limits_p p \, \| E \, g^L(x + \xi^L_{ij}) \| \, q_0 & x > 0 , \end{cases} \tag{8}$$

where q_0 is an optimal strategy of player II in the game $\| E \, \xi^L_{ij} \|$. Then we still prove that if f is a solution of equation (1), and g is a solution of equation (8), then $f \leq g$. Finally, we still prove that the solution of the equation

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* After finding the solution we shall understand the function of this class.

$$f^L(x) = \begin{cases} 0\,, & x \le 0\,, \\ 1 + \mathrm{Val}\,\| E\, f^L(x + \xi^L_{ij})\|\,, & x > 0\,, \end{cases} \tag{9}$$

always exceeds the solution of equation (1) . From Lemma 7 we then obtain the uniqueness of the solution.

I. Uniqueness of the solution of (8). Suppose that equation (8) has two solutions $g(x)$ and $h(x)$. We will prove that $g(x) \le h(x)$. Assume not. We have

$$k(x) = g(x) - h(x) = \max_p p\| E\, g(x + \xi^L_{ij})\|\, q_0 - \max_p p\, \| E\, h(x + \xi^L_{ij})\|\, q_0 \le$$

$$\le \max_p p\, \| E\, k\,(x + \xi^L_{ij})\, \|\, q_0 \,.$$

The function $k(x)$ satisfies the conditions of lemma 1, by virtue of which for given L and $\epsilon < \frac{c}{3}$ there exist an $r > 0$ and a d and an x_0 such that $k(x) \le r\,x + d$ for $x \ge x_0 - L$, and $k(x) \ge r\,x_0 + d - r\,\epsilon$. Hence

$$\max_p p\, \| E\, k\,(x_0 + \xi^L_{ij})\, \|\, q_0 \le r\,x_0 + d - r\,c^L \le r\,x_0 + d - \frac{2rc}{3}\,,$$

which leads to a contradiction. Therefore, $k(x) \le 0$, and hence $k(x) \equiv 0$.

II. Let f be a solution of the equation (1). We prove that $f \le g$. Indeed,

$$k(x) = f(x) - g(x) = \mathrm{Val}\, \| E\, f(x + \xi_{ij})\| - \max_p p\|E\, g(x + \xi^L_{ij}\,)\|\, q_0 \le$$

$$\le \mathrm{Val}\, \| E\, f(x + \xi^L_{ij})\| - \max_p p\|\, E\, g(x + \xi^L_{ij})\, \|\, q_0 =$$

$$= \min_q \max_p p\|\, E\, f(x + \xi^L_{ij})\|\, q - \max_p p\|\, E\, g(x + \xi^L_{ij})\|\, q_0 \le$$

$$\le \max_p p\, \|E\, f(x + \xi^L_{ij})\|\, q_0 - \max_p p\|E\, g(x+\xi^L_{ij})\|\, q_0 \le \max_p p\|E\, k(x+\xi^L_{ij})\|\, q_0 \,.$$

The rest of the proof is similar to the one given in section I.

III. From I and II follows that each solution of equation (1) has an order x/c. We shall prove that any solution f of equation (1) does not exceed a solution f^L of equation (9). Indeed, from the fact that $c^L < c$, it follows that f^L has an order $x/_{c^L}$ and $\sup_x (f(x) - f^L(x)) < \infty$. Setting $k(x) = f(x) - f^L(x)$, we have $k(x) \leq \max_p p \| E\ k(x+\xi^L_{ij}) \| q_0(x)$; where $q_0(x)$ is an optimal strategy for player II in the game $\| E\ f^L(x + \xi^L_{ij}) \|$.

Taking into account that $b < \infty$, $k(x) = 0$ for $x \leq 0$ and that there exists the sequence $\{x_n\}$, for which $x_n \to x_0 < \infty$ and $k(x_n) \to b$, we can construct a contradiction to the assumption of the positivity of quantity b. (Roughly speaking, the contradiction consists of the fact that the maximum value of function L is found to be smaller than its average, in which the trivially smaller values are given positive weight.

IV. Now from lemma 7 follows the uniqueness of the solution to equation (1), since for its arbitrary solution f we have $f(x) \geq f_n(x)$ and $f(x) \leq f^L(x)$. But, $f_n(x) \to f_0(x)$ for $n \to \infty$ and $f^L(x) \to f_0(x)$ for $L \to \infty$. And so the theorem is proven.

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VIII

THE EXISTENCE OF A SOLUTION, COINCIDING WITH THE CORE, IN AN n-PERSON GAME

O. N. Bondareva

Trudi VI Vsesoyuznogo Soveshchaniya
po Teorii Veroyatnosti i
Matematicheskoy Statistiki, 1962, p. 337.

SUMMARY

An n-person game with an arbitrary characteristic function is considered. Sufficient conditions are given for the existence of a solution coinciding with the core (making the solution unique) and having the same dimensionality as the set of all imputations. If we represent the coalition S, $v(S) > 0$ by the vectors $\bar{S} = (s^{(1)},\ldots,s^{(n)})$, where $s^{(i)} = \begin{cases} 0 & i \notin S \\ 1 & i \in S \end{cases}$, then the aforementioned sufficient conditions will be: $v(S) \leq \frac{1}{r}$, where $v(S)$ is the characteristic function, and r is the rank of the matrix consisting of these vectors.

(Translated by Kiyoshi Takeuchi)

IX

MINIMAX PROBLEMS IN THE THEORY OF DIFFUSION PROCESSES

I. V. Girsanov

Trudi VI Vsesoyuznogo Soveshchaniya
po Teorii Veroyatnosti
i Matematicheskoy Statistiki, 1962, pp. 339-340.

Summary of a part of the joint lecture with V. A. Kolchin (p. 359) and I. V. Romanovsky (p. 365).[1]

The following game is considered. Let U be the domain in the $(n+1)$-dimensional space of variables $(x_1, x_2, \ldots, x_n, t)$ with the boundary γ. A class of diffusion processes $X(m^1, m^2)$ is given in U with the diffusion matrix $A = \|a_{ij}(t,x)\|$ and the velocity vector $b = \{b_i(t,x)\}$. Moreover $b = b^0 + m^1 + m^2$, where the vector field m^i belongs to some admissible set M' of the vector field in U. If $\ell(x(\cdot))$ is a functional on the continuous curves in U, and $u(\mu,m^1,m^2) = E_\mu[\ell(x(\cdot)) \,|m^1,m^2]$ is the mathematical expectation of the value of this functional on the trajectory of the process $X(m^1,m^2)$ with the initial distribution μ, then taking M^i as the set of strategies of the i^{th} player, and $u(\mu,m^1,m^2)$ as the payoff function, we obtain some game $G(u,M^1,M^2)$ in normal form.

We assume that M^i consists of the vector fields, satisfying the condition

$$\sum_{k=1}^{n} m_k^i(t,x)^2 \leq c^i(t,x), \tag{1}$$

but $\ell(x(\cdot))$ is given by the formula

[1] A more detailed presentation of the results is contained in [2].

$$\ell(x(\cdot)) = f(\tau_\gamma, x_{\tau_\gamma}) \exp\left[\int_0^{\tau_\gamma} h(s,x_s)\,ds\right] + \int_0^{\tau_\gamma} g(s,x_s) \exp\left[\int_0^{s} h(u,x_u)\,du\right] ds , \tag{2}$$

where τ_γ is the moment of the outcome on γ. The following theorem holds:

<u>THEOREM:</u> <u>Let</u> $X(m^1,m^2)$ <u>be the diffusion process,</u> $A = \|a_{ij}(t,x)\|$ <u>its diffusion matrix</u> <u>and</u> $b = b^0 + m^1 + m^2$ <u>be the velocity vector. Let</u> M^i <u>be given by condition (1), and</u> ℓ <u>by formula (2). Let the equation</u>

$$\frac{\partial u}{\partial t} + a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i^0(t,x) \frac{\partial u}{\partial x_i} + h(t,x)\,u + [c^1(t,x) - c^2(t,x)] \left[\Sigma\left(\frac{\partial u}{\partial x_i} \right)^2 \right]^{\frac{1}{2}} = g(t,x) , \tag{3}$$

<u>have a continuous general solution in</u> $U \cup \gamma$, <u>taking on</u> γ <u>the value of</u> $f(t,x)$.

<u>Then</u> $\int u(0,x)\,d\mu$ <u>is the value of the game</u> G, <u>and the vector field</u>

$$\bar{m}^i = \begin{cases} (-1)^{i+1} c_i \dfrac{\nabla u}{\left[\Sigma \left(\frac{\partial u}{\partial x_i}\right)^2\right]^{\frac{1}{2}}} , & \text{if } \nabla u \text{ exists,} \\ 0 , & \text{if } \nabla u \text{ is not determined,} \end{cases}$$

<u>gives the optimal strategy of</u> i^{th} <u>player.</u>

A special case of the theorem is interesting. If the coefficients of the right side of equation (3) do not depend on t, and U is a cylinder with the base U_0, then the problem is reduced to the determination of a minimax point for the elliptic equation, which is reduced to the investigation of a quasi-linear equation:

$$a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x)\frac{\partial u}{\partial x_i} + h(x)\,u + [c^1(x) - c^2(x)]\,[\Sigma(\frac{\partial u}{\partial x_i})^2]^{\frac{1}{2}} = g(x) .$$

If one of the M^i contains only one point, we obtain the problem of determining the extreme.

The existence of the general solution for equation (3) is proved by sufficiently broad assumptions with respect to A, b, h, f, g, c^i and U. In case of small c^i Friedman [1] proved the existence of the classical solution. Using stronger a priori evaluation and the theorem of Lere-Schauder on the index of the solution, it becomes possible to remove this restriction.

Similar problems concerning the variations of the leading coefficients reduce to equations with strong nonlinearity of the form:

$$a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + (d^1 - d^2)\sqrt{\Sigma(\frac{\partial^2 u}{\partial x_i \partial x_j})^2} \quad .$$

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[1] A. Friedman, Journal Mathematics and Mechanics, 5(1958), 793-809.

[2] I. V. Girsanov, DAN, 136, No. 4.

(Translated by Kiyoshi Takeuchi)

X

SOME PROBLEMS IN THE THEORY OF DYNAMIC GAMES

V. F. Kolchin

Trudi VI Vsesoyuznogo Soveshchaniya
po Teorii Veroyatnosti
i Matematicheskoy Statistiki, 1962, pp. 359-361.

A part of the joint lecture with I.V. Girsanov (p. 339) and I.V. Romanovsky (p.365).

We shall consider the following process. Let the state of the process be given by the n-dimensional vector $x(t) = [x_1(t),\dots,x_n(t)]$, where $t = 0, 1,\dots j$ and let the possible increments of the process form the $r \times s$ matrix $A = \|a(i,j)\|$, each element of which is an n-dimensional vector $a(i,j) = [a_1(i,j),\dots,a_n(i,j)]$. We shall call the function $f(x_1,\dots,x_n) = [f_1(x_1,\dots,x_n),\dots,f_r(s_1,\dots,x_n)]$, $f_i \geq 0$, $\sum_{i=1}^{r} f_i = 1$, (a probability distribution on the set $[1,2,\dots,r]$) the strategy of player I. The strategy of player II is given by the function $g(x_1,\dots,x_n) = [g,(x_1,\dots,x_n),\dots,g_s(x_1,\dots,x_n)]$, $g_i \geq 0$, $\sum_{i=1}^{s} g_i = 1$, (a probability distribution on the set $[1,2,\dots,s]$).

Let player I chose the strategy $f(x_1,\dots,x_n)$, and player II $g(x_1,\dots,x_n)$, and let the process for $t = 0$ be found in the state $x(0) = [x_1(0),\dots,x_n(0)]$. The process is then developed in the following way. In the first step, from the probability distributions $f[x_1(0),\dots,x_n(0)]$ and $g[x_1(0),\dots,x_n(0)]$, are chosen indexes i_0 and j_0, and the process obtains the increment $a(i_0, j_0)$, passing into the state $x(1) = x(0) + a(i_0,j_0)$, and so on. In this way, under the given strategies of the players we obtain a Markov random process. With respect to this process it is possible to consider the following problems:

1. To study the average behavior of $x(t)/t$ depending on the strategies of the players. To determine whether or not there exists an n_0 for arbitrary $\epsilon > 0$, such that player I independently of player II can attain $P\{\rho(\frac{x(m)}{n}, S) < \epsilon\} > 1-\epsilon$ for all $n > n_0$, where S is some subset of the space of the values of the process, and $\rho(x,y)$ is the distance between x and y.

For arbitrary S the answer is known only for the univariate case. For the multivariate case it is known that each convex set possesses the property indicated above for one of the players. Sufficient conditions are given in [1].

2. It is possible to study the probability that the outcome of the process exceeds certain boundaries depending on the strategies of the players. This approach generalizes the game of survival, considered by Milnor and Shapley [2].

Let us denote by K the n-dimensional cube $\{x: 0 < x_i < 1, \ i = 1,\ldots,n\}$, by Γ_1 the domain $\bigcup_{i=1}^{n} \{x: x_i \geq 1\}$ and by Γ_0 the domain $\bigcup_{i=1}^{n} \{x: x_i \leq 1\} - \Gamma_1$. We shall define a functional on the realizations of the process $x(t)$: if the process goes out of the domain Γ_0, the functional is equal to 0, if the process goes into the domain $\Gamma_0 - K$, then it is equal to 1. In the case of infinite wandering within the cube, the functional is equal to some Q, $0 \leq Q \leq 1$, depending on the realization of the process. The game is called the game of wandering. The strategies are those described above, and the mathematical expectation of the functional determined above is called the payoff under the given strategies. The following theorem is proved:

<u>THEOREM 1:</u> <u>If</u> $A = \|a(i,j)\|$ <u>consists of vectors, having for fixed</u> i,j <u>either non-positive, or non-negative components, but not all being equal to zero, then the value of the game of wandering exists and does not depend on</u> Q.

The value of the game is equal to a unique root of the equation $f(x) = \mathrm{Val}\, \|f(x + a(i,j))\|$, where Val denotes the operator of taking the value of the game with the matrix indicated previously, and $f(x) = \begin{cases} 0 & x \in \Gamma_1 , \\ 1 & x \in \Gamma_0 - K \end{cases}$.

The proof of this theorem is similar to the proof of the theorem for the univariate case [2]. Since the solution of this functional equation by the method of successive approximations represents a difficult problem, we consider the limit behavior of the value of the game of wandering, where the elements of the matrix converge to zero.

<u>THEOREM 2:</u> <u>Let</u> λ_0 <u>be a root of the equation</u>

$$\varphi(\lambda) = \mathrm{Val}\, \|a_1(i,j)+\dots+a_m(i,j) + \lambda[a_{m+1}(i,j)+\dots+a_n(i,j)]\| = 0 ,$$

$$\underline{\text{where}} \quad \varphi(\lambda) < 0 \quad \underline{\text{for}} \quad \lambda > \lambda_0 ,$$
$$\varphi(\lambda) > 0 \quad \underline{\text{for}} \quad \lambda < \lambda_0 .$$

<u>If the conditions of the preceding theorem are satisfied, then in the game of wandering with matrix</u> $A_t = \|t \cdot a(i,j)\|$, <u>as</u> $t \to 0$, <u>the value of the game converges to</u> 0 <u>in the domain</u> $x_1 +\dots+ x_m + \lambda_0(x_{m+1}+\dots+x_n) < 1$ <u>and to</u> 1 <u>in the domain</u> $x_1 +\dots+ x_m + \lambda_0(x_{m+1} +\dots+ x_n) > m + \lambda_0(n-m-1)$.

The proof of this theorem is based on the fact that

$$\frac{e^{\lambda[x_1 +\dots+ x_m + (\lambda_0 + \epsilon)(x_{m+1} +\dots+ x_n)]}}{\lambda}$$

is the solution of the functional equation for some λ.

We will apply the results of theorem 2 to the game of exhaustion [3]. By the game of exhaustion we mean the following subclass of the game of wandering: the elements of matrix A possess the following properties;

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(Translated by Kiyoshi Takeuchi)

XI

ITERATION METHODS IN THE SOLUTION OF BIMATRIX GAMES

E. B. Yanovskaya

Trudi VI Vsesoyuznogo Soveshchaniya
po Teorii Veroyatnosti
i Matematicheskoy Statistiki, 1962, p. 371.

SUMMARY

Iteration methods are applied to the solution of bimatrix games. These are similar to the Brown-Robinson method for the solution of matrix games.

The convergence has an order which is equal at least to $O(t^{-\frac{1}{m+n-2}})$, where m and n denote the corresponding number of the rows and columns in the matrix games.

(Translated by Kiyoshi Takeuchi)

XII

SEVERAL APPLICATIONS OF LINEAR PROGRAMMING METHODS TO THE THEORY OF COOPERATIVE GAMES

O. N. Bondareva

Nekotoriye primeneniya metodov lineynogo programmirovaniya
k teorii Kooperativnikh igr

Problemi Kibernetiki
Moscow, 1963. Tenth Issue pp. 119-139.

This article is devoted to the application of a theorem on linear inequalities to the existence problem concerning solutions of n-person cooperative games.

In the first section of the paper we shall introduce the concept of coverings, which characterize a game's coalition structure, and shall study their properties.

In the second section we shall investigate the core, a set which is always contained in the solution and, when it exists, in a certain sense replaces the solution. Necessary and sufficient conditions for the core's existence (expressed in terms of coverings) will be set forth.

In the third section, quota games will be studied with the aid of the same methods used in the second section.

In the fourth section we will investigate the link between the core and the solution. We shall point out some necessary conditions that must be present in order for the core to coincide with the solution. Several sufficient conditions will also be indicated.

In the last section we shall present some examples.

§1. Basic Concepts. Definitions and Notations

A cooperative game Γ, given in the form of a characteristic function (see [1] and [2]) is a pair, consisting of:

1) A set $I_n = \{1,2,\ldots,n\}$, called the set of players, and

2) A real function $v(S)$, defined on the subsets S of this set and having the properties:

$$v(\Lambda) = 0$$
$$v(I_n) = M,$$

where M is some positive number;

$$0 \leq v(S) \leq M \qquad \text{for any } S \subset I_n .$$

The subsets S of the set I_n are called coalitions, and the function $v(S)$ is called a characteristic function.

If $M = 1$ and $v(\{i\}) = 0$, $i=1,\ldots,n$, i.e. the characteristic function receives the value zero on single-element sets, then we say that the game is given in (0-1)-reduced form.

In what follows, we shall, unless otherwise stated, assume that the games under discussion are given in (0-1)-reduced form.

Consider the systems of real numbers

$$\alpha = (a_1,\ldots,a_n), \text{ where } a_i \geq v(\{i\}) = 0 \text{ and } \sum_{i=1}^{n} a_i = 1 .$$

We shall call any such system an imputation. Denote the set of all such systems - the set of all imputations - by the letter A. We shall, from here on, look upon α as an n-dimensional vector, and on A as a subset of n-dimensional Euclidean space.

We shall now define a dominance relation on the set A. We shall say that the imputation $\alpha = (a_1,\ldots,a_n)$ <u>dominates</u> the imputation $\beta = (b_1,\ldots,b_n)$ (this is written $\alpha \succ \beta$), if:

1). There exists a set $S \subset I_n$ such that $\sum_{i\in S} a_i \leq v(S)$ (The set S is then called <u>effective</u> for α),

2) $a_i > b_i$ for all $i\in S$.

Occasionally, in order that it be clear with respect to which set the dominance takes place, we shall write $\alpha \underset{S}{\succ} \beta$.

Note that any such β must fulfill the condition $\sum_{i\in S} b_i < v(S)$, i.e., dominance may take place only with respect to those sets S for which $\sum_{i\in S} b_i < v(S)$.

We shall call the set S <u>essential</u> for α, if:

1) S is effective for α, i.e. $\sum_{i\in S} a_i \leq v(S)$;

2) There exists no set $T \subset S$, $T \neq S$, effective for α; i.e. for any set $T \subset S$, $T \neq S$

$$\sum_{i\in T} a_i > v(T) .$$

Note 1: It is easy to show (see, for example, [2]), that the dominance relation may take place only with respect to sets which are essential for the dominating imputation.

The dominance relation does not constitute a partial ordering; in fact, every logically conceivable possibility may take place between two imputations (it is possible for example that, with respect to non-intersecting sets, both $\alpha \succ \beta$ and $\beta \succ \alpha$ are realized).

Let $P \subset A$. We shall denote by dom P the set of all imputations in A that are dominated by some imputation in P.

The set $U = A \setminus \text{dom } A$ is called the core.

The set V is a solution, if $V = A \setminus \text{dom } V$, i.e., a solution is a set of imputations such that

1) No two imputations in V dominate one another.

2) For every $\upsilon \notin V$, there exists an imputation $\alpha \in V$ that dominates it, $a \succ \upsilon$ (see [1] and [2]).

Obviously $U \subset V$, since the core consists of all imputations not dominated by any imputation in A $(U = A \setminus \text{dom } A)$. Simple examples show that the solution is in general not unique (see, for example, [1]).

LEMMA 1.1: If a solution coincides with the core, then the solution is unique.

The lemma is trivial; we formulated it so that it may be easily referred to further on.

For any set $S \subset I_n$, we denote the number of elements in S by $|S|$.

We denote by $\mathfrak{N} = \{S_1, \dots, S_m\}$ the system consisting of all $S_j \subset I_n$ for which either $v(S_j) > 0$, or, if $v(S_j) = 0$, then $|S_j| = 1$.

LEMMA 1.2: In order for an imputation $\alpha = (a_1, \dots, a_n)$ to belong to the core U, it is necessary and sufficient that the inequality

$$\sum_{i \in S} a_i \geq v(S)$$

be satisfied for all $S \in \mathfrak{N}$.

PROOF: Necessity. Let $\alpha \in U$ and suppose that the condition of the lemma is not valid, i.e., that there exists an S_0, such that $\sum_{i \in S_0} a_i < v(S_0)$. Consider

$\beta = (b_1,\ldots,b_n)$, where

$$b_i = a_i + \epsilon\,, \quad \text{if } i \in S_0$$

$$b_i \geq 0\,, \quad \text{if } i \notin S_0\,.$$

We stipulate that

$$\epsilon = \frac{v(S_0) - \sum_{i\in S_0} a_i}{|S_0|}$$

$$\sum_{i \notin S_0} b_i = 1 - v(S_0)\,.$$

Such a vector exists, since $1-v(S_0) \geq 0$. Since $b_i \geq 0$, $i=1,\ldots,n$, and

$$\sum_{i=1}^{n} b_i = \sum_{i\in S_0} a_i + |S_0| \frac{v(S_0) - \sum_{S_0} a_i}{|S_0|} + 1 - v(S_0) = 1\,,$$

then $\beta \in A$. But since $\beta \succ \alpha$, $\alpha \in \text{dom } A$. However, by supposition $\alpha \in U = A\backslash\text{dom } A$. We have then proven the necessity of the condition by way of contradiction.

Sufficiency. If for some $\alpha = (a_1,\ldots,a_n)$ the condition of the lemma is fulfilled, i.e. that $\sum_S a_i \geq v(S)$ for all $S \subset I_n$ (for $S \notin \mathcal{N}$, $\sum_S a_i \geq 0 = v(S)$), then (by definition of dominance) $\alpha \notin \text{dom } A$, i.e. $a \in A\backslash\text{dom } A$.

<u>COROLLARY:</u> The core constitutes a closed, bounded, convex subset of n-dimensional space with a finite number of extreme points. (This is because it consists of the intersection between the hyperplane $\sum_{i=1}^{n} a_i = 1$ and the convex polyhedral region $\sum_{i\in S_j} a_i \geq v(S_j)$, $S_j \in \mathcal{N}$).

We correspond to each $S_j \in \mathcal{N}$ and to I_n the vectors $\underline{S_j}$, $j=1,\ldots,n$, and $\underline{I_n}$. Here,

$$\underline{S_j} = (s_j^{(1)},\ldots,s_j^{(n)})\,, \quad \text{where} \quad s_j^{(i)} = \begin{cases} 0, & \text{if } i \notin S_j \\ 1, & \text{if } i \in S_j \end{cases}, \text{ and}$$

$$\underline{I_n} = (1,\ldots,1)\,.$$

We denote the zero vector by $\underline{0}$.

We define a $(q\text{-}\theta)$-covering of the set I_n to be a system of non-negative real numbers $(\lambda_1,\ldots,\lambda_m)$, such that

$$\sum_{j=1}^{m} \lambda_j \underline{S_j} = \underline{I_n} .$$

Here q is the number of λ_j's such that $\lambda_j > 0$, and θ is the system of subsets corresponding to these λ_j's, $\theta = \{ S_{j_1},\ldots, S_{j_q} : \lambda_{j_\ell} > 0 \}$.

We shall say that a $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ is <u>reduced</u>, if for any other $(q\text{-}\theta)$-covering $(\lambda'_1,\ldots,\lambda'_m)$ the equation $\lambda'_\ell = \lambda_\ell$ holds for all ℓ .

<u>LEMMA 1.3:</u> A necessary and sufficient condition for a $(q\text{-}\theta)$-covering to be a reduced $(q\text{-}\theta)$-covering is that the system θ consists of linearly independent vectors.

<u>PROOF:</u> We may assume, with no loss in generality, that $\lambda_1 > 0 ,\ldots,\lambda_q > 0$, $\lambda_{q+1} = \cdots = \lambda_m = 0$; then θ consists of the vectors $\underline{S_1} ,\ldots, \underline{S_q}$. Consider the corresponding system of equations

$$\sum_{j=1}^{q} \lambda_j s_j^{(i)} = 1 , \quad i=1,2,\ldots,n . \qquad (1.1)$$

The system is feasible, for $\lambda_1,\ldots,\lambda_q$ constitutes its solution. The requirement that the $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ be reduced is equivalent to the requirement that this solution be unique. When the lemma is formulated in this way, it is seen to be trivial.

<u>COROLLARY 1.1:</u> A necessary and sufficient condition for a $(q\text{-}\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$ to be reduced is that the rank of the matrix $\| \underline{S_j} \|_{\lambda_j > 0}$ be equal to q .

COROLLARY 1.2: For all reduced $(q\text{-}\theta)$ coverings, $q \leq n$.

COROLLARY 1.3: The number of reduced coverings is finite.

COROLLARY 1.4: Let: 1) $(\lambda_1,\dots,\lambda_m)$ be a reduced $(q\text{-}\theta)$-covering; 2) q_1 be the number of sets $S_j \in \theta$ such that $|S_j| > 1$. (We may assume that these sets are $S_1,\dots,S_{q_1}$); 3) T be the set of components of I_n "completely covered" by the sets $S_1,\dots,S_{q_1}$, i.e.

$$T = \{i: \ i \in I_n; \ \sum_{j=1}^{q_1} \lambda_j s_j^{(i)} = 1\} .$$

Then $|T| \geq q_1$.

PROOF: Consider the given $(q\text{-}\theta)$-covering. Eliminate all single-element sets from the system of "covering" sets. The number of such sets is equal to $q-q_1$, each of which takes part in covering exactly one element; therefore, $n-q+q_1$ components now remain "completely covered". The set of these components was denoted in the conditions of the assertion by T. Hence $|T| = n-q+q_1$. Since $n-q \geq 0$, then $|T| \geq q_1$.

LEMMA 1.4: If a $(q\text{-}\theta)$-covering is regarded as a point in n-dimensional Euclidean space, then the set of all $(q\text{-}\theta)$-coverings, Ξ , is a closed, bounded, and convex point set. A member of the set is a reduced $(q\text{-}\theta)$-covering if and only if it is an extreme point of the set.

PROOF: The closedness of the set is trivially true. The boundedness of Ξ is due to the fact that $0 \leq \lambda_i \leq 1$. Convexity follows from the linearity of the conditions defining a covering; the last assertion is true by definition of reduced covering.

Thus, the set of all coverings is described by the set of reduced coverings, which are finite in number. It follows from Lemma 1.3 and its corollaries that the reduced coverings can be determined quite easily.

§2. Basic Theorems of Core Theory

In this section we shall demonstrate necessary and sufficient conditions for the existence of the core * for n-person cooperative games.

We will first prove a lemma dealing with linear inequalities of a certain type.

LEMMA 2.1: Let $\underline{A_1}, \ldots, \underline{A_m}$ be a system of n-dimensional vectors with non-negative coefficients and let $\underline{I} = (1, \ldots, 1)$. Then

1) the system

$$\begin{cases} \underline{A_j}\ \underline{X} \geq v_j, & j=1,\ldots,m, \\ \underline{I}\ \underline{X} = 1 \end{cases} \tag{2.1}$$

has a solution if and only if for all systems of real numbers $\lambda_j \geq 0$, $j=1,\ldots,m$, for which

$$\sum_{j=1}^{m} \lambda_j \underline{A_j} = \underline{I}, \tag{2.2}$$

the inequality $\sum_{j=1}^{m} \lambda_j v_j \leq 1$ is fulfilled;

2) the system

$$\begin{cases} \underline{A_j}\ \underline{X} > v_j & j=1,\ldots,m, \\ \underline{I}\ \underline{X} = 1 \end{cases} \tag{2.1'}$$

*Here and further on, when we speak of the core's existence, we mean the existence of a non-empty core.

has a solution if and only if every system of real numbers $\lambda_j \geq 0$, $j=1,\dots,m$, satisfying (2.2), fulfills the inequality $\sum_{j=1}^{m} \lambda_j v_j < 1$.

PROOF: Let us first note, that in order for (2.1) to be solvable, it is necessary and sufficient that the system

$$\begin{cases} \underline{A_i}\ \underline{X} \geq v_i \\ \underline{I}\ \underline{X} \leq 1 \end{cases} \qquad (2.3)$$

be solvable, or equivalently, that

$$\begin{cases} \underline{A_i}\ \underline{X} \geq v_i \\ -I\,X \geq -1 . \end{cases}$$

The necessity of this condition is trivially true. For proof of sufficiency, note that if for some $\underline{X}$ the strict inequality $\underline{I}\ \underline{X} < 1$ is fulfilled, then increasing the components of X so that $\underline{I}\,\underline{X} = 1$, we receive a solution to (2.1). This is because the inequalities are thus only strengthened, in view of the non-negativeness of the system's coefficients.

According to a theorem in [3] dealing with solvability conditions for systems of linear inequalities, a necessary and sufficient condition for (2.3) to have a solution is that for any system of real numbers $\lambda'_0, \lambda'_1, \dots, \lambda'_m$ for which

$$\sum_{j=1}^{m} \lambda'_j\, \underline{A_j} - \lambda'_0\, \underline{I} = \underline{0}\,, \qquad (2.4')$$

the condition

$$\sum_{j=1}^{m} \lambda'_j v_j \leq \lambda'_0 \qquad (2.5')$$

be fulfilled.

Note, that if $\lambda'_0 = 0$, then in view of the non-negativeness of the components of $\underline{A_j}$, $j=1,\dots,m$, and in view of (2.4'), all the remaining λ'_j's, $j=1,\dots,m$, are also equal to zero, and hence (2.5') is fulfilled trivially. We may therefore assume that $\lambda'_0 > 0$. Dividing both sides of (2.4') and (2.5') by λ'_0 and substituting:

$$\lambda_j = \frac{\lambda'_j}{\lambda'_0} , \quad j=1,\dots,m ,$$

we perceive that the fulfillment of (2.4') and (2.5') is equivalent to the respective conditions:

$$\sum_{j=1}^{m} \lambda_j \underline{A_j} = \underline{I} \tag{2.4}$$

and

$$\sum_{j=1} \lambda_j v_j \leq 1 . \tag{2.5}$$

In this manner, the first assertion is proven.

For proof of the second assertion we first show that in order that the system

$$\begin{cases} \underline{A_j}\,\underline{X} > v_j \\ \underline{I}\,\underline{X} = 1 \end{cases} \tag{2.6}$$

be solvable, it is necessary and sufficient that the system

$$\begin{cases} \underline{A_j}\,\underline{X} > v_j \\ \underline{-I}\,\underline{X} > -1 \end{cases} \tag{2.7}$$

also be solvable. The sufficiency of the condition is proven in the same manner as was done when (2.1) was followed through. We shall now prove the condition's necessity. Suppose that $\underline{X_0} = (x_1^0,\dots,x_n^0)$ is a solution of (2.6) whereby there exists an $x_k^0 > 0$, since $\underline{I}\,\underline{X_0} = \sum_{i=1}^{n} x_i^0 = 1$. Suppose, further, that

$$\epsilon < \min_j (\underline{A_j}\,\underline{X} - v_j)$$

and

$$\epsilon < x_k^0 .$$

Since in view of (2.6) $\underline{A_j}\ \underline{X} - v_j > 0$, $j=1,\ldots,m$, and $x_k^0 > 0$, then $\epsilon > 0$.

Consider $X' = (x_1',\ldots,x_n')$, where

$$x_i' = x_i^0 , \quad i \neq k$$

$$x_k' = x_k^0 - \epsilon$$

By proper choice of ϵ we have $\underline{A_j}\ \underline{X'} > v_j$ and $\underline{I}\ \underline{X'} < 1$, i.e., $\underline{X'}$ is a solution of (2.7), whereupon the necessity of the condition is proven.

We shall now make use of a theorem in [3], dealing with the solvability of systems of strict inequalities, in connection with (2.7). We receive that in order for (2.7) to be solvable, it is necessary and sufficient that for any arbitrary system $\lambda_0' \geq ,\ldots,\lambda_m' \geq 0$ for which (2.4') is fulfilled, the strict inequality

$$\sum_{j=1}^{m} \lambda_j' v_j < \lambda_0' \qquad (2.8)$$

be also fulfilled. If $\lambda_0' = 0$, then, as above, $\lambda_j' = 0$, $j=1,\ldots,m$, but then (2.8) is not fulfilled; this means that $\lambda'_0 > 0$. Dividing both sides of (2.4') and (2.8) by λ_0' and introducing the same notation as were introduced when (2.1) was investigated, we receive that for any system of numbers $\lambda_1 \geq 0,\ldots,\lambda_m \geq 0$, satisfying the condition $\sum_{j=1}^{m} \lambda_j \underline{A_j} = \underline{I}$, the inequality $\sum_{j=1}^{m} \lambda_j v_j < 1$ must be satisfied.

We shall now prove some fundamental theorems.

THEOREM 2.1: A necessary and sufficient condition for a game Γ to have a core is that for any arbitrary reduced $(q-\theta)$-covering $(\lambda_1,\dots,\lambda_m)$, the inequality

$$\sum_{j=1}^{m} \lambda_j \, v(S_j) \leq 1 \tag{2.9}$$

be fulfilled.

PROOF: By lemma 1.2 the core is equal to the set of solutions α of the system

$$\begin{cases} \alpha \, \underline{S_j} \geq v(S_j) , & S_j \in \mathcal{N} \\ \alpha \, \underline{I_n} = 1 \end{cases} \tag{2.10}$$

This system satisfies the conditions of lemma 2.1, when $\underline{A_j} = \underline{S_j}$ and $v_j = v(S_j)$.

Applying this lemma, we receive that (2.10) has a solution if and only if any arbitrary system of real numbers, $\lambda_1 \geq 0,\dots,\lambda_m \geq 0$, for which $\sum_{j=1}^{m} \lambda_j \underline{\bar{S}_j} = \underline{\bar{I}_n}$, satisfies the inequality $\sum_{j=1}^{m} \lambda_j \, v(S_j) \leq 1$, i.e. inequality (2.9) must be fulfilled for every $(q-\theta)$-covering. But since the left hand side of (2.9) is a linear function of $(\lambda_1,\dots,\lambda_m)$, and since the set of all coverings is convex (see lemma 1.5), it is therefore sufficient to require the fulfillment of (2.9) for every reduced $(q-\theta)$-covering.

THEOREM 2.2: In order that a game Γ have a core of maximum dimension (i.e. of dimension $n-1$, the dimension of the set of all imputations A) it is necessary and sufficient that the inequality

$$\sum_{j=1}^{m} \lambda_j \, v(S_j) < 1$$

be fulfilled for any arbitrary reduced $(q-\theta)$-covering $(\lambda_1,\dots,\lambda_m)$.

<u>PROOF:</u> As was stated above, the core consists of the set of solutions to the system

$$\begin{cases} \alpha\ \underline{\bar{S}_j} \geq v(S_j)\ , & S_j \in \mathcal{N} \\ \alpha\ \underline{\bar{I}_n} = 1 \end{cases}$$

This is a convex polyhedral region within the hyperplane $\alpha\ \underline{\bar{I}_n} = 1$. In order for this region to have maximal dimension, i.e. $n-1$, it is necessary and sufficient that it contain relative interior points of the hyperplane $\alpha\ \underline{\bar{I}_n} = 1$, i.e., there must exist α's for which

$$\alpha\ \underline{\bar{S}_j} > v(S_j)\ , \quad j=1,\ldots,m\ ,$$

$$\alpha\ \underline{\bar{I}_n} = 1$$

By lemma 2.1, in order for such a system to have a solution it is necessary and sufficient that the inequality $\sum_{j=1}^{m} \lambda_j\, v(S_j) < 1$ be fulfilled for any $\lambda_1 \geq 0, \ldots \lambda_m \geq 0$ satisfying the condition

$$\sum_{j=1}^{m} \lambda_j\ \underline{\bar{S}_j} = \underline{\bar{I}_n} \qquad (2.11)$$

As in the proof of theorem 2.1, note that $(\lambda_1,\ldots,\lambda_m)$ is a $(q-\theta)$-covering and that it is sufficient to require the fulfillment of the conditions of the lemma for reduced $(q-\theta)$-covering.

Note: Consider the linear-programming problem consisting of the determination of the numbers $\lambda_1 \geq 0,\ldots,\lambda_m \geq 0$, satisfying the system (2.11) of constraints and minimizing the linear form $\sum_{j=1}^{m} \lambda_j\, v(S_j)$. Then the reduced coverings constitute admissible basic solutions. If $(\lambda_1^0,\ldots,\lambda_m^0)$ is an optimal solution, then for theorems 2.1 and 2.2 to be valid, it is necessary and sufficient to require that $\sum_{j=1}^{m} \lambda_j^0\, v(S_j) \leq 1$ and $\sum_{j=1}^{n} \lambda_j^0\, v(S_j) < 1$, respectively.

The problem of directly verifying the existence of the core may be regarded, as well, as a linear-programming problem; the problem of finding the "maximal" covering then turns out to be a dual problem.

This analogy can, evidently, be extended.

This interrelationship allows us to use numerical methods of linear programming to determine whether a given game has a core. Note, however, that in view of the specific character of the problems appearing here, these methods may possibly lend themselves to modification.

From the fundamental theorems just proven, the following assertions ensue.

<u>COROLLARY 2.1:</u> In order that a game Γ not have a core, it is necessary and sufficient that there exist a reduced $(q\text{-}\theta)$-covering $(\lambda_1^0, \ldots, \lambda_m^0)$ such that

$$\sum_{j=1}^{m} \lambda_j^0 \, v(S_j) > 1 .$$

<u>THEOREM 2.3:</u> In order that the dimension of the core be less than n-1, it is necessary and sufficient that there exist a reduced $(q\text{-}\theta)$ core $(\lambda_1^0, \ldots, \lambda_m^0)$ for which $\sum_{j=1}^{m} \lambda_j^0 \, v(S_j) = 1$, the dimension τ of the core then obeys the inequality $\tau \leq n-q$.

<u>PROOF:</u> The first part of the assertion follows directly from theorem 2.1 and 2.2. We shall prove the validity of the evaluation for τ. First let us note that by corollary 1.2 $q \leq n$. By definition of covering

$$\sum_{j=1}^{m} \lambda_j \, \underline{S_j} \, \alpha = \underline{I_n} \, \alpha = 1 .$$

For any $\alpha \in U$, $\underline{S_j} \, \alpha \geq v(S_j)$, therefore

$$1 = \sum_{j=1}^{m} \lambda_j \, \underline{S_j} \, \alpha = \sum_{\lambda_j > 0} \lambda_j \, \underline{S_j} \, \alpha \geq \sum_{\lambda_j > 0} \lambda_j \, v(S_j) = 1 ,$$

i.e. $\underline{S_j} \, \alpha = v(S_j)$ for all j for which $\lambda_j > 0$. This means that q linearly independent constraints (the covering is reduced) of the form $\underline{S_j} \, \alpha = v(S_j)$ are imposed on α. Since the constraint $\underline{I_n} \, \alpha = \sum_{i=1}^{n} a_i = 1$ is a consequence of the former constraints $(\Sigma \lambda_j \; \underline{S_j} = \underline{I_n})$, then $\tau \leq n-q$.

As examples we shall point out conditions under which the dimension of the core is n-2 or n-3.

COROLLARY 2.2: If the core is of dimension n-2, then there exist two sets S_{j_1} and S_{j_2} such that

$$S_{j_1} \cup S_{j_2} = I_n , \quad S_{j_1} \cap S_{j_2} = \Lambda , \quad \text{and} \quad v(S_{j_1}) + v(S_{j_2}) = 1 .$$

COROLLARY 2.3: If the core is of dimension n-3, then one of the following conditions is fulfilled:

1) There exist sets $S_{j_1}, S_{j_2}, S_{j_3}$ such that

$$\underline{\bar{S}_{j_1}} + \underline{\bar{S}_{j_2}} + \underline{\bar{S}_{j_3}} = \underline{\bar{I}_n}$$

and

$$v(S_{j_1}) + v(S_{j_2}) + v(S_{j_3}) = 1$$

2) There exist sets $S_{j_1}, S_{j_2}, S_{j_3}$ such that

$$\underline{\bar{S}_{j_1}} + \underline{\bar{S}_{j_2}} + \underline{\bar{S}_{j_3}} = 2\,\underline{\bar{I}_n}$$

and

$$v(S_{j_1}) + v(S_{j_2}) + v(S_{j_3}) = 2$$

3) There exists sets $S_{j_1}, S_{j_2}, S_{j_3}, S_{j_4}$ such that

$$\bar{S}_{j_1} + \bar{S}_{j_2} = \bar{I}_n , \quad \bar{S}_{j_3} + \bar{S}_{j_4} = \bar{I}_n$$

and

$$v(S_{j_1}) + v(S_{j_2}) = 1 , \quad v(S_{j_3}) + v(S_{j_4}) = 1 .$$

These assertions are proven by direct use of theorem 2.3.

We shall demonstrate a simple sufficiency condition for the existence of the core.

THEOREM 2.4: In order for a game Γ to have a core it is sufficient that $v(S)$ fulfill the condition

$$v(S_j) \leq \frac{|S_j|}{n} \quad , \quad S_j \in \mathcal{N} \quad .$$

PROOF: Consider the imputation $\alpha = (\frac{1}{n},\ldots,\frac{1}{n})$; since $\underline{S}_j \, \alpha = \frac{|S_j|}{n} \geq v(S_j)$, then $\alpha \in U$, i.e. $U \neq \Lambda$, which is what had to be proven.

It turns out that for a certain class of games this condition is also necessary.

We recall the a symmetric game is a game whose characteristic function satisfies the condition $v(S) = \varphi(|S|)$.

THEOREM 2.5: A necessary and sufficient condition for a symmetric game to have a core is that

$$v(S) \leq \frac{|S|}{n} \quad \text{for any} \quad S \subset I_n \quad .$$

PROOF: The sufficiency of the condition follows from Theorem 2.4. We shall prove necessity. Let $|S| = t$. Consider $t \, \underline{I}_n$; we may "cover" it with n vectors of "length" t . Let these vectors be $\underline{S}_1,\ldots,\underline{S}_n$; then $\frac{1}{t}\sum_{j=1}^{n} \underline{S}_j = \underline{I}_n$ and consequently $(\frac{1}{t},\ldots,\frac{1}{t}, 0,\ldots,0)$ is a covering (possibly even a reduced covering). But by Theorem 2.2, the condition

$$\frac{1}{t} \sum_{j=1}^{n} v(S_j) \leq 1$$

must necessarily be fulfilled for any arbitrary $(q\text{-}\theta)$-covering, and since $v(S_j) = v(S_\ell)$, if $|S_j| = |S_\ell|$ then $\frac{n}{t} v(S_j) \leq 1$, or $v(S_j) \leq \frac{|S_j|}{n}$.

§ 3. Quota Games

We shall now consider the so called quota game and shall make use of the methods developed in the preceding section as tools for their investigation.

We recall that a quota game is a game for which there exists a system of real numbers $\omega_1,\ldots,\omega_n$ (not necessarily positive) such that $\omega_1+\ldots+\omega_n = 1$, and such that $v(S) = \sum_{i\in S}\omega_i$ for any $S \subset I_n$ for which $|S| = 2$. This definition was given by Shapley in [4] (see also [5]); the concept was extended by Kalish (see [5]) in the following manner: a game is called an ℓ-quota game, if there exists a system of real numbers $\omega_1,\ldots,\omega_n$ such that $\omega_1+\ldots+\omega_n = 1$ and $v(S) = \sum_{i\in S}\omega_i$ for any $S \subset I_n$ for which $|S| = \ell$.

We shall now investigate the question of the existence of an ℓ-quota (resting on the case of games with Shapely quotas, where $\ell = 2$).

THEOREM 3.1: In order that a game have an ℓ-quota, it is necessary and sufficient that the equation

$$\sum_{j=1}^{m} \lambda_j v(S_j) = 1$$

be fulfilled for any arbitrary $(q-\theta)$-covering $(\lambda_1,\ldots,\lambda_m)$, consisting only of ℓ-element sets.

PROOF: If a quota exists, it must satisfy the conditions

$$\begin{cases} \sum_{i\in S_j} \omega_i \geq v(S_j) & \text{for all } S_j \text{ such that } |S_j| = \ell \\ \sum_{i=1}^{n} \omega_i = 1 \end{cases} \qquad (3.1)$$

and

$$\begin{cases} \sum_{i\in S_j} \omega_i \leq v(S_j) & \text{for all } S_j \text{ such that } |S_j| = \ell \\ \sum_{i=1}^{n} \omega_i = 1 \end{cases} \qquad (3.2) .$$

By lemma 2.1 system (3.1) is solvable if and only if the condition $\sum_{j=1}^{m} \lambda_j v(S_j) \leq 1$ is fulfilled for any non-negative $\lambda_1,\ldots,\lambda_m$ for which $\sum_{j=1}^{m}\lambda_j \underline{S_j} = \underline{I_n}$. We derive analogously that (3.2) is solvable if and only if the condition

$$\sum_{j=1}^{m}\lambda_j v(S_j) \geq 1$$

is fulfilled for any system of non-negative numbers $\lambda_1,\ldots,\lambda_m$ for which $\sum_{j=1}^{m} \lambda_j \underline{\ddot{S}_j} = \underline{\dot{I}_n}$. Hence, for any such system of numbers, the equation $\sum_{j=1}^{m}\lambda_j v(S_j) = 1$ must therefore be fulfilled. Noting that all such systems of numbers constitute coverings consisting of ℓ-element sets, we obtain the desired result.

<u>THEOREM 3.2:</u> If an ℓ-quota exists, it is unique.

<u>PROOF:</u> By definition, an ℓ-quota must satisfy the equation $\sum_{i\in S} \omega_i = v(S)$ for all S such that $|S| = \ell$. It is therefore sufficient to prove that among the vectors $\underline{\dot{S}}$ for which $|S| = \ell$ there exist n linearly independent ones. We shall point out n such vectors.

$$\underline{\bar{S}_1} = (\underbrace{1,\ldots,1}_{\ell}\ 0,\ldots,0),$$

$$\underline{\dot{S}_2} = (0,\underbrace{1,\ldots,1}_{\ell}\ 0,\ldots,0)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\underline{\bar{S}_{n-1}} = (\underbrace{1,\ldots,1}_{\ell-2},\ 0,\ldots,0,1,1)$$

$$\underline{\dot{S}_n} = (\underbrace{1,\ldots,1}_{\ell-1},\ 0,\ldots,0,\ 1)$$

<u>THEOREM 3.3:</u> The quota belongs to the set of imputations if and only if $\sum_{j=1}^{m} \lambda_j v(S_j) = 1$ for all reduced $(q\text{-}\theta)$-coverings $(\lambda_1,\ldots,\lambda_m)$ such that $|S| = \ell$ or $|S| = \dot{1}$ for $S \in \theta$.

PROOF: Note that aside from the conditions imposed on the quota by systems (3.1) and (3.2), the inequality $\omega_i \geq 0$, $i=1,\ldots,n$, must, in this case, also hold. Applying, as above, lemma 2.1, we receive what was required.

Note: Theorem 1 in [4] is equivalent to theorem 3.1 for games with $\ell = 2$.

It is easy to prove conditions analogous to those given in [4], for ℓ-quota games.

THEOREM 3.4: A necessary and sufficient condition for a game to have an ℓ-quota is that $\Sigma\, v(S_j) = \dfrac{(n-1)\ldots(n-\ell+1)}{(\ell-1)\,!}$, where the summation is taken over the S_j's for which $|S_j| = \ell$.

PROOF: Consider the sets S_j for which $|S_j| = \ell$ (including those sets for which $v(S_j) = 0$. We shall assume that they are numbered $1,2,\ldots,N$. The total number of such sets is:

$$N = C_n^{\ell} = \frac{n!}{\ell!\,(n-\ell)!}$$

Let us construct a covering from just those sets. Each member of I_n will in this manner be "covered" exactly $\dfrac{(n-1)\ldots(n-\ell+1)}{(\ell-1)!} = a$ times, i.e. $\sum_{j=1}^{N} \frac{1}{a} \cdot \underline{\bar{S}_j} = \ddot{I}_n$, and consequently, $(\underbrace{\frac{1}{a},\ldots,\frac{1}{a}}_{N}, 0,\ldots,0)$ is a $(q-\theta)$ covering (as a rule, not reduced). By the conditions for an ℓ-quota's existence we have:

$$\Sigma\, v(S_j) = a$$

which is equivalent to the assertion of the theorem.

§4. The Relation Between the Core and the Von-Neumann-Morgenstern Solution.

Existence Theorems

We consider the conditions under which the core is a solution. By lemma 1.1 the solution is in this case unique.

THEOREM 4.1: If the core is a solution then it intersects each of the hyperplanes $a_i = 0$, $i=1,2,\dots,n$ (i.e. the core has "enough" points lying on the border of the set A of all imputations).

PROOF: Suppose, on the contrary, that there exists a hyperplane $a_{i_0} = 0$ such that $a_{i_0} > 0$ for all $\alpha \in U$. Set $a^*_{i_0} = \min_{\alpha \in U} a_{i_0}$; the minimum is attained, because the core is a closed set. Denote by $\alpha^* = (a^*_1,\dots,a^*_n)$ an imputation for which the minimum $a^*_{i_0}$ is attained. Let $0 < \epsilon < a^*_{i_0}$. Consider the imputation $\alpha^0 = (a^0_1,\dots,a^0_n)$, where

$$a^0_1 = a^*_1 + \epsilon ;$$
$$a^0_i = a^*_i \quad \text{for } i \neq 1, \quad \text{and} \quad i \neq i_0 ;$$
$$a^0_{i_0} = a^*_{i_0} - \epsilon .$$

Since $a^0_{i_0} < \min_{\alpha \in U} a_{i_0}$, $\alpha^0 \notin U$; this means that there exists an $S_0 \in \eta$ (if there exist many such sets we may pick a n y one of them arbitrarily, and label it S_0) for which $\sum_{i \in S_0} a^0_i < v(S_0)$. Suppose S_0 does not contain i_0. Then

$$\sum_{i \in S_0} a^0_i \geq \sum_{i \in S_0} a^*_i \geq v(S_0) .$$

It follows from this and from the preceding inequality that $i_0 \in S_0$.

In order that an imputation $\beta \in U$ dominate α^0, it is necessary that the equation $\sum_{i\in S_0} b_i = v(S_0)$ be fulfilled.

In fact, $\sum_{i\in S_0} b_i \geq v(S_0)$ for all $\beta \in U$, but dominance may take place only with respect to an effective set, i.e. only when $\sum_{i\in S_0} b_i \leq v(S_0)$. But $\alpha^* \in U$. Hence $\sum_{i\in S_0} a^*_i \geq v(S_0)$ and

$$\sum_{i\in S_0} b_i = v(S_0) \leq \sum_{i\in S_0} a^*_i = a^*_{i_0} + \sum_{i\in S\setminus i_0} a^0_i \ .$$

Taking into account that $b_{i_0} \geq a^*_{i_0}$, we receive:

$$\sum_{i\in S_0\setminus i_0} b_i \leq \sum_{i\in S_0\setminus i_0} a^0_i \ .$$

This means that for some i_1, $b_{i_1} \leq a^0_{i_1}$. Hence no imputation β in U may dominate α^0 with respect to S_0. Since S_0 was arbitrarily chosen from the sets with respect to which dominance could take place, U therefore is not a solution.

<u>COROLLARY 4.1:</u> A core of dimension 0 cannot be a solution.

<u>PROOF:</u> Since the core is a convex set, a core of dimension 0 necessarily consists of one imputation. By theorem 4.1, this imputation can only equal the vector $(0,\ldots,0)$. This vector, however, does not belong to the set of imputations.

Let us examine the sets $S'_j = I_n \setminus S_j$, where $|S_j| > 1$. Denote the system of all such sets by $\mathfrak{N}'$. Certain subsets of I_n may now be regarded as members of the system $\mathfrak{N}$ or of the system $\mathfrak{N}'$. Extend the characteristic function $v(S)$ onto the system $\mathfrak{N}'$ of subsets of I_n, setting $v(S'_j) = 1 - v(S_j)$.

Denote by $\mathcal{N}_j$ the system of sets $\mathcal{N}_j = \{\mathcal{N}, S'_j\}$ $(S'_j \in \mathcal{N}'$, i.e. $v(S'_j) = 1 - v(S_j))$, and consider the $(q-\theta_j)$-coverings $(\lambda_1, \ldots, \lambda_m, \nu_j)$ of the system of sets $\theta_j \subset \mathcal{N}_j$ (ν_j corresponds to S'_j). We shall call such a covering a $(q-\theta_j)$-quasi-covering; we shall also occasionally refer to it simply as a covering, since a covering is a special case of a quasi-covering, i.e. a quasi-covering becomes a covering when $\nu_j = 0$.

Since in many questions concerning the solution, single-element sets play a special role (they take no part in dominance), it would be convenient in the discussions that lie ahead, to single out the components of coverings corresponding to these sets. We shall therefore denote quasi-coverings, and thus coverings as well, by $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n, \nu_j)$ where $k + n = m$ and μ_p is the component corresponding to the single-element set consisting of the element p . We shall assume that the sets S_j are denumerated so that $S_{k+p} = (\underbrace{0, \ldots, 0, 1}_{p}, 0, \ldots, 0)$.

Quasi-coverings may, just as coverings, be reduced in form. Lemma 3.1 and its corollaries are valid for quasi-coverings. Lemma 1.4 takes the form:

LEMMA 4.1: For any fixed j , $0 \leq j \leq k$, the $(q-\theta_j)$-quasi-coverings when regarded as points of $m + 1$-dimensional Euclidean space, form a closed, bounded, convex set; the extremal points of this set, and only they, all constitute reduced quasi-coverings.

THEOREM 4.2: A sufficient condition for a game Γ to have a unique solution is that the inequality

$$\sum_{\ell=1}^{k} \lambda_\ell v(S_\ell) + \nu_j(1-v(S_j)) + \mu^{(j)} v(S_j) \leq 1 , \qquad (4.1)$$

where $\mu^{(j)} = \max_{i \in S_j} \mu_i$, be fulfilled for any arbitrary $(q-\theta_j)$-quasi-covering $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n, \nu_j)$.

PROOF: We shall prove that under these conditions there exists a solution which coincides with the core; then, by lemma 1.1, the solution is unique.

Since the core is always included within the solution, it is sufficient to prove that when the pre-conditions of the theorem are fulfilled, the solution is included within the core, i.e., that for any $\gamma \in A \setminus U$ there exists some $\alpha \in U$, such that $\alpha \succ \gamma$. Let $\alpha = (a_1, \ldots, a_n)$, and $\gamma = (c_1, \ldots, c_n) \in A \setminus U$.

Since $\gamma \notin U$, the conditions of lemma 1.2 are not fulfilled; this means there exists some S_{j_0} for which $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$. In order that $\alpha \succ \gamma$, it is necessary and sufficient that for at least one such S_{j_0}, the system of inequalities

$$\alpha \underline{\ddot{S}_j} \geq v(S_j), \quad j=1,\ldots,m; \tag{4.2}$$

$$\alpha \underline{\ddot{S}_j} \leq v(S_{j_0}); \tag{4.2'}$$

$$a_i > c_i, \quad i \in S_{j_0} \tag{4.2''}$$

be satisfied. The fulfillment of (4.2) is equivalent, by lemma 1.2, to the condition that $\alpha \in U$. The fulfillment of (4.2') and (4.2") is equivalent to the condition that $\alpha \succ \gamma$ with respect to S_{j_0}.

Because of this, in order that the solution be included within the core and that U be consequently a solution, it is sufficient to require that for any $\gamma \in A \setminus U$, the system of inequalities (4.2 - 4.2") have a solution for any S_{j_0} such that $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$.

Making use of the relation $\alpha(\underline{S}'_{j_0} + \underline{S}_{j_0}) = \alpha I_n = 1$, let us rewrite (4.2') in the form $\alpha\,\underline{S}'_{j_0} \geq 1 - v(S_{j_0})$. Note that the requirement that the system of inequalities be fulfilled for any γ in $A \backslash U$ and for every S_{j_0} corresponding to γ such that $\sum_{i \in S_{j_0}} c_i < v(S_{j_0})$ may be replaced by the equivalent requirement that the system of inequalities be fulfilled for each S_j, $j=1,\ldots,m$, and for every γ corresponding to S_j such that $\sum_{i \in S_{j_0}} c_i \leq v(S_{j_0})$. It is clear that this does not weaken (nor does it strengthen) the requirement.

Thus, in order that U be a solution it is sufficient that for all $S_{j_0} \in \mathfrak{N}$ and for every imputation $\beta = (b_1,\ldots,b_n)$ such that $\sum_{i \in S_{j_0}} b_i \leq v(S_{j_0})$ the system

$$
\begin{aligned}
&\underline{S_j}\,\alpha \geq v(S_j)\,, \quad j=1,\ldots,k; \\
&a_i \geq 0, \ i \in I_n \backslash S_{j_0}\ ; \\
&a_i \geq b_i, \ \ i \in S_{j_0}\ ; \qquad\qquad (4.3) \\
&\underline{S_{j_0}}\,\alpha \geq 1 - v(S_{j_0})\ ; \\
&\underline{I_n}\,\alpha = 1\,.
\end{aligned}
$$

(4.3) satisfies the conditions of lemma 2.1. A solution to (4.3) exists, according to this lemma, if for all systems of numbers $\lambda_1 \geq 0,\ldots,\lambda_k \geq 0$, $\mu_1 \geq 0,\ldots,\mu_n \geq 0$, $\nu'_{j_0} \geq 0$, for which

$$\sum_{j=1}^{k} \lambda_j \underline{S_j} + \sum_{p=1}^{n} \mu_p \underline{S_{k+p}} + \nu_{j_0} \underline{S_{j_0}} = \underline{I_n} \qquad (4.4)$$

the condition

$$\varphi(\lambda, \mu, \nu_{j_0}) = \sum_{j=1}^{k} \lambda_j v(S_j) + \sum_{p \in S_{j_0}} \mu_p b_p + \nu'_{j_0}(1 - v(S_{j_0})) \leq 1 \quad (4.5)$$

be fulfilled. Note, that (4.4) obviously implies that the system $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n, \nu_{j_0})$ form a $(q-\theta_{j_0})$ quasi-covering. Inasmuch as (4.3) must be fulfilled for all S_{j_0} and for every system of numbers $b_p \geq 0$ satisfying the condition

$$\sum_{p \in S_{j_0}} b_p \leq v(S_{j_0}) ,$$

we must therefore investigate the behavior of the function $\varphi(\lambda, \mu, \nu_{j_0})$ on the set of all $(q-\theta_{j_0})$-quasi-coverings. Since this set is convex (see lemma 4.1) and since the function is linear, then in order for the inequality $\varphi(\lambda, \mu, \nu_{j_0}) \leq 1$ to be fulfilled it is sufficient that it be fulfilled for the extremal points of the set of $(q-\theta_{j_0})$-quasi coverings, i.e. for the reduced quasi-coverings.

If we substitute $\mu^{(j_0)} = \max_{p \in S_{j_0}} \mu_p$ for every μ_p appearing in (4.5), then, bearing in mind that $\sum_{i \in S_{j_0}} b_i \leq v(S_{j_0})$, we obtain the conditions of the theorem.

Note: Condition (4.1) can often be weakened. For example, if there exists an $S_{j_1} \subset S_{j_0} \cap T$, where, as always, $T = \{i: \mu_i = 0\}$, then condition (4.1) takes the form

$$\sum_{j=1}^{k} \lambda_j v(S_j) + \nu_{j_0}(1-v(S_{j_0})) + \mu^{(j)}(v(S_{j_0}) - v(S_{j_1})) \leq 1 .$$

In fact, in (4.5)

$$\sum_{p \in S_{j_0}} \mu_p b_p = \sum_{p \in S_{j_0} \setminus T} \mu_p b_p \leq \mu^{(j)} \sum_{p \in S_{j_0} \setminus T} b_p .$$

But since dominance can be considered only with respect to essential sets (see § 1) and $S_{j_1} \subset S_{j_0}$, then $\sum_{p \in S_{j_1}} b_p \geq v(S_{j_1})$; therefore $\sum_{p \in S_{j_0} \setminus T} b_p \leq v(S_{j_0}) - v(S_{j_1})$, because $S_{j_1} \subset T$.

Simpler though more restrictive conditions, whose fulfillment is sufficient for the existence of a unique solution, may be expressed in the form of an evaluation for $v(S)$. Let us first introduce some new notations. Denote by D the matrix formed by the vectors $\underline{S_1}, \ldots, \underline{S_k}$ and $\underline{I_n}$, i.e.

$$D = \left\| \begin{matrix} s_1^{(1)} & s_2^{(1)} & \ldots\ldots\ldots & s_k^{(1)} & 1 \\ & & \ldots\ldots\ldots & & \\ s_1^{(n)} & s_2^{(n)} & \ldots\ldots\ldots & s_k^{(n)} & 1 \end{matrix} \right\| .$$

Denote by r the rank of this matrix.

<u>THEOREM 4.3:</u> In order that a game Γ have a unique solution it is sufficient that

$$v(S) \leq \frac{1}{r}, \qquad S \subset I_n .$$

<u>PROOF:</u> It is sufficient to prove that if $v(S_j) \leq \frac{1}{r}$, $j=1,\ldots,k$, then the condition of theorem 4.2 is fulfilled, i.e. that for any arbitrary $(q-\theta_j)$-quasi-covering $(\lambda_1,\ldots,\lambda_k, \mu_1,\ldots,\mu_n, \nu_j)$, the condition (4.1), or (4.5), is satisfied. In other words, it is enough to prove that

$$\varphi(\lambda,\mu, \nu_j) = \sum_{\ell=1}^{k} \lambda_j\, v(S_\ell) + \nu_j(1-v(S_j)) + \mu^{(j)}\, v(S_j) \leq 1,$$

or

$$\varphi(\lambda,\mu, \nu_j) = \sum_{\ell=1}^{k} \lambda_\ell v(S_\ell) + \nu_j(1-v(S_j)) + \sum_{p\in S_j} \mu_p\, b_p \leq 1 .$$

1. Let us first consider the case where $\nu_j = 0$; assume that the numbers are so denumerated, that $\lambda_1 > 0,\ldots \lambda_{r'} > 0$, $\lambda_{r'+1} = \ldots = \lambda_k = 0$ and $\mu_1 > 0,\ldots,\mu_{n'} > 0$, $\mu_{n'+1} = \ldots = \mu_n = 0$ $(r' + n' = q)$. Then the condition

(4.1) becomes

$$\varphi_1(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{r'} \lambda_\ell \; v(S_\ell) + \mu^{(j)} v(S_j) \leq 1 .$$

There are two possibilities:

a) $r' < r$; we then receive the following evaluation for $\varphi_1(\lambda,\mu,\nu_j)$:

$$\varphi_1(\lambda,\mu,\nu_j) \leq \sum_{\ell=1}^{r'} v(S_\ell) + v(S_j) \leq \frac{r'}{r} + \frac{1}{r} \leq 1 .$$

This is because $\lambda_\ell \leq 1$ and $\mu^{(j)} \leq 1$.

b) $r' = r$. In this case consider the vectors $\underline{S_1},\ldots,\underline{\ddot{S}_r}$. They are linearly independent (the covering is reduced). But since the rank of D is equal to r , $\underline{I_n}$ is a linear combination of the vectors, i.e.

$$\sum_{\ell=1}^{r} \eta_\ell \, \underline{S_\ell} = \underline{I_n}$$

By definition of $(q\text{-}\theta)$-covering

$$\sum_{\ell=1}^{r} \lambda_\ell \underline{\ddot{S}_\ell} + \sum_{p=1}^{n'} \mu_p \; \underline{\ddot{S}_{k+p}} = \underline{\ddot{I}_n} , \qquad r + n' = q .$$

Substituting for $\underline{I_n}$ the expression $\sum_{\ell=1}^{r} \eta_\ell \underline{\ddot{S}_\ell}$ we receive:

$$\sum_{\ell=1}^{r} (\lambda_\ell - \eta_\ell) \, \underline{S_\ell} + \sum_{p=1}^{n'} \mu_p \; \underline{\ddot{S}_{k+p}} = \underline{0} ;$$

but $\underline{\ddot{S}_1},\ldots,\underline{\ddot{S}_r}, \underline{\ddot{S}_{k+1}},\ldots,\underline{\ddot{S}_{k+n}}$ are linearly independent, since the covering is reduced; hence

$$\lambda_\ell - \eta_\ell = 0 , \; \ell = 1,\ldots,r, \quad \text{and} \quad \mu_p = 0, \; p=1,\ldots,n' ,$$

and since, aside from this, $\mu_{n'+1} = \ldots. = \mu_n = 0$, it follows that $\mu_p = 0$, $p = 1,\ldots,n$, and

$$\varphi_1(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{r} \lambda_\ell \, v(S_\ell) \leq \frac{1}{r} \cdot r = 1 .$$

2. Let us now assume that $\nu_j > 0$.

We shall examine the corresponding $(q-\theta_j)$-quasi-covering $(\lambda_1,\ldots,\lambda_k,\ \mu_1,\ldots,\mu_n,\ \nu_j)$. We shall assume the sets to be renumbered so that $\lambda_1 > 0,\ldots,\lambda_{\tau-1} > 0$, $\lambda_\tau = \ldots = \lambda_n = 0$, $\mu_1 = \ldots = \mu_{\bar{n}} = 0$, $\mu_{\bar{n}+1} > 0,\ldots,\mu_n > 0$. Since by assumption $\nu_j \neq 0$,

$$n-\bar{n} + \tau-1 + 1 = n-\bar{n} + \tau = q .$$

Since, furthermore, $q \leq n$, then $\bar{n} \geq \tau$.

Let us write the vectorial equation with the new numbering:

$$\sum_{\ell=1}^{\tau-1} \lambda_\ell \underline{S_\ell} + \sum_{p=\bar{n}+1}^{n} \mu_p \underline{\ddot{S}_{k+p}} + \nu_j \underline{S'_j} = \underline{\dot{I}_n} .$$

(we recall that $\underline{\ddot{S}_{k+p}} \doteq (\underbrace{0,\ldots,0}_{p}, 1, 0,\ldots,0))$. Expressed by coordinates this reads:

$$\begin{aligned} &\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j s_j'^{(i)} = 1, \quad i=1,2,\ldots,\bar{n} . \\ &\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j s_j'^{(i)} + \mu_i = 1, \quad i=\bar{n}+1,\ldots, n. \end{aligned} \qquad (4.6)$$

Since the vectors corresponding to the non-zero components of the covering are linearly independent and since the equations whose indices exceed $\bar{n}$ are $n - \bar{n} = q - \tau$ in number, this means that among the first $\bar{n}$ there exist τ linearly independent ones. Hence, the set of covering components appearing within them comprises a unique solution (see lemma 1.3). Consider the system consisting of these equations. The system may be split up into two parts:

$$\sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} = 1 \quad (i \in S_j) \quad \text{and} \quad \sum_{\ell=1}^{\tau-1} \lambda_\ell s_\ell^{(i)} + \nu_j = 1 \quad (i \notin S_j). \qquad (4.7)$$

Let us evaluate, in this case, the function

$$\varphi(\lambda,\mu,\nu_j) = \sum_{\ell=1}^{\tau-1} \lambda_\ell v(S_\ell) + \sum_{p\in S_j} \mu_p b_p + \nu_j(1-v(S_j)) .$$

Split the summation $\sum_{\ell=1}^{\tau-1} \lambda_\ell v(S_\ell)$ into two parts: Let $\Sigma' \lambda_\ell v(S_\ell)$ be taken over those S_ℓ which are included within S_j, and let $\Sigma'' \lambda_\ell v(S_\ell)$ be taken over those S_ℓ for which $S_\ell \cap (I_n \setminus S_j) \neq \Lambda$. (In the first case λ_ℓ does not appear together with ν_j in a single equation in (4.7), since ν_j is a coefficient for $\underline{I_n \setminus S_j}$; in the second case each λ_ℓ appears together with ν_j in at least one equation in (4.7).

Let us first evaluate the function

$$\psi(\lambda,\mu,\nu_j) = \Sigma' \lambda_q v(S_q) + \sum_{p\in S_j} \mu_p b_p$$

Since $S_q \subset S_j$, then by the essentiality of S_j (see §1), it follows that $\sum_{i\in S_q} b_i \geq v(S_q)$; therefore

$$\psi(\lambda,\mu,\nu_j) \leq \Sigma' \lambda_q \sum_{i\in S_q} b_i + \sum_{p\in S_j} \mu_p b_p .$$

Reducing similar terms, we notice that λ_q is a coefficient of b_i if $i \in S_q$; therefore

$$\psi(\lambda,\mu,\nu_j) = \sum_{i\in S_q} b_i \Big(\sum_{q:S_q \subset S_j} \lambda_q s_q^{(i)} + \mu_i \Big) + \sum_{p\in S_j \setminus S_q} \mu_p b_p$$

where the $s_q^{(i)}$'s are the coordinates of S_q, i.e.

$$s_q^{(i)} = \begin{cases} 0, & \text{if } i \notin S_q , \\ 1, & \text{if } i \in S_q . \end{cases}$$

(According to Gillies' formulation $v(S) < \frac{1}{n}$).

Theorem 4.3 turns out to be a proof of the following known fact.

COROLLARY 4.4: The set of n-person games having a (unique) solution has the same dimension as the set of all n-person games.

For this reason, the probability that an arbitrarily chosen n-person game have a solution is positive.

§ 5. Examples

In the way of example let us first consider a game with a core of dimension 1.

THEOREM 5.1: A game may have a core of dimension 1 that also turns out to be a solution only if the following conditions are fulfilled.

1) There exist sets

$$M = \{1,2,\dots,k\} \subset I_n, \quad M \neq \Lambda$$

$$N = \{k+1,\dots,\ell\} \subset I_n, \quad N \neq \Lambda\,,$$

and numbers

$$a_1 > 0,\dots,a_k > 0, \quad \sum_{i=1}^{k} a_i = 1$$

$$b_1 > 0,\dots, b_\ell > 0, \quad \sum_{i=1}^{\ell} b_i = 1\,,$$

such that
$$v(S_j) \leq \min \Big(\sum_{S_j \cap M} a_i ; \sum_{S_j \cap N} b_i \Big) ;$$

2) for any $0 \leq i \leq \ell$ there exist at least two sets $S_{i_1}, S_{i_2} \subset M \cup N$ such that $i \in S_{i_1} \cap S_{i_2}$ and $v(S_{i_1}) = v(S_{i_2}) = a_i$, if $i \leq k$, and $v(S_{i_1}) = v(S_{i_2}) = b_i$, if $i > k$.

Since

$$\sum_{q:S_q \subset S_j} s_q^{(i)} \lambda_q + \mu_i \le \sum_{q=1}^{\tau-1} s_q^{(i)} \lambda_q + \mu_i \le 1 \qquad \text{and} \qquad \mu_p \le 1 \text{ , therefore}$$

$$\psi(\lambda,\mu,\ \nu_j) \le \sum_{i\in \cup S_q} b_i + \sum_{p\in S_j \setminus S_q} b_p = \sum_{i\in S_j} b_i \le v(S_j) \quad .$$

Thus

$$\varphi(\lambda,\mu,\nu_j) \le \Sigma'' \ \lambda_\ell v(S_\ell) + v(S_j) + \nu_j(1-v(S_j)) =$$

$$= \Sigma'' \ \lambda_\ell v(S_\ell) + \nu_j + (1-\nu_j) v(S_j) \ ,$$

where Σ'' as formerly stated, is a summation over the ℓ's for which λ_ℓ appears in at least one equation of (4.7) containing ν_j and hence $\lambda_\ell \le 1-\ \nu_j$. Since the number of different λ_ℓ's does not exceed $\tau-1$ and since $v(S_\ell) \le \frac{1}{r}$, then

$$\Sigma'' \ \lambda_\ell v(S_\ell) \le (1-\nu_j) \ \Sigma'' \ v(S_\ell) \le (1-\nu_j) \ \frac{\tau-1}{r} \quad .$$

Thus,

$$\varphi(\lambda,\mu,\nu_j) \le (1-\nu_j) \ \frac{\tau-1}{r} + \nu_j + \frac{1-\nu_j}{r} = \frac{(1-\nu_j)\tau}{r} + \nu_j \ ,$$

and since $\frac{\tau}{r} \le 1$ and $1-\ \nu_j > 0$, therefore $\varphi(\lambda,\mu,\ \nu_j) \le 1-\nu_j + \nu_j = 1$.

<u>COROLLARY 4.2:</u> If the number k of coalitions S, for which $v(S) > 0$, is less than n, then a sufficient condition for the existence of a unique solution is the fulfillment of the inequality

$$v(S) \le \frac{1}{k} \ .$$

From Theorem 4.3 and the inequality $r \le n$ the following corollary ensues.

<u>COROLLARY 4.3:</u> (Gillies' theorem, see [2]). In order that a game Γ have a unique solution (coinciding with the core) it is sufficient that $v(S) \le \frac{1}{n}$.

The proof is carried through with the aid of Theorem 4.1. Since no difficulties present themselves when this procedure is performed, and since the theorem itself is of no particular importance, the proof will be omitted.

THEOREM 5.2: In order that a game Γ have a core of dimension 1 that is also a solution, it is sufficient that there exist sets $M, N \subset I_n$ of a single cardinality k, such that $v(S) \leq \frac{\min(|S \cap M|, |S \cap N|)}{k}$ and $v(S) = \frac{\min(|S \cap M|, |S \cap N|)}{k}$, for $|S| = 2$.

PROOF: Since the imputations $\alpha = (\underbrace{\tfrac{1}{k}, \ldots, \tfrac{1}{k}}_{k}, 0, \ldots, 0)$ and $\beta = (\underbrace{0, \ldots 0}_{k}, \underbrace{\tfrac{1}{k}, \ldots, \tfrac{1}{k}}_{k}, 0, \ldots,$ are contained in the core, the core therefore exists and is of dimension $\tau \geq 1$. We shall now show that when the conditions of the theorem are fulfilled the core is of dimension 1 and turns out to be a solution.

We construct the following chain of two-member sets:

$$S_1 = \{1, k+1\}, \ S_2 = \{k+1, 2\}, \ S_3 = \{2, k+2\}, \ldots, \ S_{2k-1} = \{k-1, 2k\}$$

$$S_{2k} = \{2k, 1\} .$$

Let

$$\lambda_j^0 = \frac{1}{2}, \quad \text{if } 0 \leq j \leq 2k;$$

$$\lambda_j^0 = 1, \quad \text{if } |S_j| = 1, \ j \in I_n \setminus (M \cup N),$$

$$\lambda_j^0 = 0 \quad \text{for all remaining } S_j .$$

Then $(\lambda_1^0, \ldots, \lambda_m^0)$ is an $(n-\theta)$-covering. Since $\Sigma\, \lambda_j^0\, v(S_j) = 1$ and since the number of linearly independent vectors in the system θ is equal to n-1, then $\tau \leq n-n-1 = 1$. This, together with the previously derived inequality $\tau \geq 1$ gives us $\tau = 1$.

Let us examine the conditions imposed on the imputations in the core by the existence of the covering λ^o. Let $\gamma = (c_1,\dots,c_n) \in U$. Then

$$c_1 + c_{k+1} = \frac{1}{k} ,$$

$$c_{2k} + c_1 = \frac{1}{k} ,$$

$$c_{2k+1} = \dots = c_n = 0 .$$

Denote c_{k+1} by t ; then

$$c_i = \frac{1}{k} - t , \qquad i = 1,\dots,k;$$

$$c_i = t , \qquad i = k+1,\dots, 2k;$$

$$c_i = 0, \qquad i = 2k+1,\dots,n .$$

It is easy to show that

$$U = \{u(t) = \underbrace{\frac{1}{k} - t,\dots,\frac{1}{k} - t}_{k}, \underbrace{t,\dots,t}_{k}, 0,\dots,0), 0 \le t \le \frac{1}{k} \}$$

is the core. We will show that U is a solution. In fact, let $\delta = (d_1,\dots,d_n)$. If for all $S_1,\dots,S_{2k}$ $\sum_{S_\ell} d_i = v(S_\ell) = \frac{1}{k}$, then $\delta \in U$. If, however, for some S_{ℓ_0}

$$\sum_{S_{\ell_0}} d_i = d_{i\ell_0} + d_{j\ell_0} < v(S_{\ell_0}) = \frac{1}{k} ,$$

then there exists a t' such that

$$d_{i\ell_0} < \frac{1}{k} - t'$$

$$d_{j\ell_0} < t'$$

(for example, $t' = d_{j\ell_0} + \epsilon$, where $\epsilon < \frac{1}{k} - d_{i\ell_0} - d_{j\ell_0}$), i.e. $\delta_{S_{\ell_0}} \prec u(t')$.
The theorem is thus proven.

<u>Example</u>. Symmetric Shapley market games.

<u>A symmetric market game</u> (see [6]) is a game with the characteristic function

$$v(S) = \min(|S \cap M|, |S \cap N|) .$$

where $M \cup N = I_n$. The game is not normal, and $v(I_n) = \min(|M|, |N|)$. Let $|M| \leq |N|$ and $|M| = k$; then $v(I_n) = k$. In (0-1) reduced form

$$v(S) = \frac{\min(|S \cap M|, |S \cap N|)}{k} .$$

The game has a non-empty core, since

$$(\underbrace{\tfrac{1}{k},\ldots,\tfrac{1}{k}}_{k}, 0,\ldots,0) \in U .$$

If $|M| = |N| = k$, then by theorem 5.2 the game has a core of dimension 1 , which is also a solution; this core was investigated by Shapley also. If $|M| < |N|$, then by examination of the coverings consisting of two-member sets, we immediately receive that the core is of dimension 0 . In this case, as shown in [6], a solution exists but is necessarily not unique. Hence, the following assertion is valid: <u>The solution of a symmetric bargaining game is unique if and only if it coincides with the core</u>. It is possible that an analogous assertion is true in more general cases as well.

As an application of the general theory let us consider four-person games.

Example: Investigation of four-person games.

Let $I_n = \{1,2,3,4\}$. We shall denote the coalitions by S_i, S_{ij}, S_{ijk}, where the lower index is the enumeration of the coalition's members; for this reason i,j,k,ℓ shall henceforth always be different.

Using corollary 1.4, we enumerate all reduced $(q\text{-}\theta)$-coverings:

$$\text{I.}\quad \underline{S_{ij}} + \underline{S_{k\ell}} = \underline{I_n}\ , \quad \{i,j,k,\ell\} = I_n$$

$$\text{II.}\quad \tfrac{1}{3}\,(\underline{S_{123}} + \underline{S_{124}} + \underline{S_{134}} + \underline{S_{234}}) = \underline{I_n}$$

$$\text{III.}\quad \tfrac{1}{2}\,(\underline{S_{ij}} + \underline{S_{i\ell}} + \underline{S_{j\ell k}} + \underline{S_{k}}) = \underline{I_n}$$

$$\text{IV.}\quad \tfrac{1}{3}\,(\underline{S_{ij}} + \underline{S_{ik}} + \underline{S_{i\ell}}) + \underline{\tfrac{2}{3}\,S_{jk\ell}} = \underline{I_n}$$

$$\text{V.}\quad \tfrac{1}{2}\,(\underline{S_{ijk}} + \underline{S_{ij\ell}} + \underline{S_{k\ell}}) = \underline{I_n}$$

These are all the reduced $(q\text{-}\theta)$-coverings with the exception of the "trivial" ones (the coverings that yield trivial evaluations of the characteristic function.

By theorem 4.2, for the core to exist it is necessary and sufficient that:

$$\text{I.}\quad v(S_{ij}) + v(S_{k\ell}) \le 1\ .$$

$$\text{II.}\quad v(S_{123}) + v(S_{124}) + v(S_{134}) + v(S_{234}) \le 3\ .$$

$$\text{III.}\quad v(S_{ij}) + v(S_{ik}) + v(S_{jk\ell}) \le 2\ .$$

$$\text{IV.}\quad v(S_{ij}) + v(S_{ik}) + v(S_{i\ell}) + 2v(S_{jk\ell}) \le 3\ .$$

$$\text{V.}\quad v(S_{ijk}) + v(S_{ij\ell}) \le 2\ .$$

In order to write the necessary conditions for the existence of the solution it is necessary, for each and every S to examine S, in its second quality, i.e. with the characteristic function $1-v\,(I_n \setminus S)$. We must do this for each of the conditions I-V.

In view of the large number of such conditions, let us write them for the special case of a symmetric game. We receive that for any $0 \le \epsilon \le \frac{1}{6}$, any symmetric four-person game with a characteristic function satisfying the conditions

$$v(S_{ij}) \leq \frac{1}{3} + \epsilon ; \quad v(S_{ijk}) = \frac{2}{3} - \epsilon ,$$

has a unique solution (coinciding with the core).

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XIII

SOME THEOREMS ON ψ-STABILITY IN COOPERATIVE GAMES

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(Nekotoriye teoremi teorii ψ-ustoychivocti v kooperativnykh igrakh)
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Let us consider a cooperative game, determined by a characteristic function in 0-1-reduced form (see [1]). We shall examine the set of imputations

$$A = \{\alpha = (a_1, \dots, a_n): \quad a_i \geq 0, \quad \sum_{i=1}^{n} a_i = 1\} \quad .$$

We shall set in correspondence with each coalition $S \subset I_n$ a vector $\underline{S} = \{s_1, \dots, s_n\}$, where $s_i = 1$ if $i \epsilon S$ and $s_i = 0$ if $i \notin S$.

A q-θ covering of I_n is defined as a system of numbers $(\lambda_j, \dots, \lambda_m)$, $\lambda_j \geq 0$, such that $\sum_{j=1}^{m} \lambda_j \underline{S}_j = \underline{I}_n$, $S_j \subset I_n$. Here q is the number of positive λ_j's and θ is the set of corresponding coalitions S_j (see (3)). The extremal points of the set of coverings are called reduced coverings; the number of reduced points is finite.

A subset U of the set A is called the core, if for any arbitrary $\alpha \epsilon U$ the condition

$$\sum_{i \epsilon S} a_i = \underline{S} \cdot \alpha \geq v(S) \qquad \text{for all } S \subset I_n$$

is satisfied.

Any partition τ of the set I_n into non-intersecting coalitions is called a coalition structure.

Let $\psi(\tau)$ be a mapping of each τ into the set of all coalitions $S \subset I_n$, such that $\tau \subset \psi(\tau)$. The pair $[\alpha,\tau]$ $(\alpha \in A)$ is called ψ-stable, if the following conditions are fulfilled:

1) $\underline{S} \cdot \alpha \geq v(S)$ for all $S \in \psi(\tau)$

2) If $a_i = 0$ $(= v(\{i\}))$, then $\{i\} \in \tau$.

The concept of ψ-stability, as well as the concept of k-stability, was introduced by Luce (see, for example, [1], [4].) The well known k-stability theorems, concerning classes of symmetric games and quota games are also due to him.

In this paper ψ-stability will be studied with the aid of linear programming methods.

LEMMA: In order for a system of inequalities of the form

$$\underline{S} \cdot \alpha \geq v(S), \quad S \in \Xi$$

$$\underline{I_n} \cdot \alpha = 1$$

to have a solution (here Ξ is some set of coalitions), it is necessary and sufficient that for any arbitrary q-θ-covering for which $\theta \subset \Xi$, the inequality

$$\sum_{j=1}^{m} \lambda_j v(S_j) \leq 1$$

be fulfilled. Here, in order for even one of the inequalities to be strict, $\sum_{j=1}^{m} \lambda_j v(S_j)$ must be less than 1 for coverings for which, correspondingly to this inequality, $\lambda_j > 0$.

The proof is based on a theorem (see [2]) about the solvability of systems of linear inequalities.

THEOREM 1: In order that there exist, for some τ , a ψ stable pair $[\alpha,\tau]$, it is necessary and sufficient that for any arbitrary reduced q- θ-covering $(\lambda_1,\ldots,\lambda_m)$ for which $\theta \subset \{\psi(\tau),\{1\},\ldots,\{n\}\}$, the inequality $\sum_{j=1}^{m}\lambda_j\ v(S_j) \leq 1$ be fulfilled. Here, the inequality must be strict for coverings containing single-element sets, not included in τ .

PROOF: In order for the pair $[\alpha,\tau]$ to be ψ-stable, it is necessary and sufficient that α be a solution of the system of inequalities

$$\underline{S} \cdot \alpha \geq v(S)\ , \quad S \in \psi(\tau) \qquad \text{(1st condition)}$$

$$a_i > 0, \quad i \in S \in \tau \ \text{ and } \ |S| \geq 2 \qquad \text{(2nd condition)}$$

$$\left.\begin{array}{ll} a_i \geq 0 & \text{for the remaining terms} \\ I_n \cdot \alpha = 1 & \end{array}\right\} \quad \alpha \in A \quad .$$

To complete the proof we merely make use of the above formulated lemma.

Let us denote by $\bar{\Gamma}$ a game for which no q-θ-covering $(\lambda_1,\ldots,\lambda_m)$ exists such that

$$\sum_{j=1}^{m} \lambda_j\ v(S_j) \leq 1 .$$

COROLLARY 1: In the game $\bar{\Gamma}$ no ψ-stable pairs exist for any mapping $\psi(\tau)$ whatsoever.

COROLLARY 2: For definite choice of ψ there exist ψ-stable pairs for super-additive payoff functions.

The assertion is true if, for example, the mapping $\psi(\tau)$ is a "subdivision" of the coalitions in τ .

COROLLARY 3: In a given game, let $(\lambda_1^{(1)},\ldots,\lambda_m^{(1)}),\ldots,(\lambda_1^{(t)},\ldots,\lambda_m^{(t)})$ be all the reduced q_i- θ_i-coverings for which $\sum_{j=1}^{m} \lambda_j^{(i)}\ v(S_j) \leq 1$; then in order for there to exist a ψ-stable pair $[\alpha,\tau]$, it is necessary that $\psi(\tau) \subset \bigcup_{i=1}^{t} \theta_i$.

COROLLARY 4: (Theorem 1 in [3]). In order that the game Γ have a kernel, it is necessary and sufficient that for any q-θ-covering $(\lambda_1,\dots,\lambda_m)$, the inequality $\sum_{j=1}^{m} \lambda_j\, v(S_j) \leq 1$ be satisfied.

The proof follows from the fact that if, for any arbitrary τ, $\psi(\tau)$ constitutes a mapping onto the set of all coalitions, then any pair $[\alpha,\tau]$, α being in U, is ψ-stable. Conversely, for any arbitrary ψ-stable pair, $\alpha \in U$.

We shall now consider a function $\psi(\tau)$ of a somewhat more special type (see [1]): We shall assume that $\psi(\tau)$ consists of all the coalitions T for which an $S \in \tau$ exists such that $|T\setminus S| + |S \setminus T| \leq k$ (The modulus sign associated with a set refers to the number of elements in the set). k here is a given integer. The ψ-stability present in this case is called k-stability.

We recall that a game is said to be symmetric, if $v(S) = v(|S|)$.

THEOREM 2 (Theorem 1 in [4]): A necessary and sufficient condition for a symmetric game to have a kernel is that $v(S) \leq \frac{|S|}{n}$ for $|S| = 2,\dots,k+1$.

PROOF: Sufficiency. If $v(S)$ satisfies the condition of the Theorem, then the pair $[(\frac{1}{n},\dots,\frac{1}{n}),\ (\{1\},\dots,\{n\})]$ is k-stable by definition.

Necessity. Let $[\alpha,\tau]$ be k-stable and let $\tau = (S_1,\dots S_\ell)$. We set r equal to some fixed value, $2 \leq r \leq k+1$. Consider $rI_n = \{1,\dots,n,\ 1,\dots,n,\dots,1,\dots,n\}$. We redistribute the players in rI_n, producing a partition of rI_n into sets $S'_1,\dots,S'_m$, such that $|S'_j| = a_j r$ (the a_j's being integers). The redistribution can be carried forth in a way such that the smallest negative (positive) remainder resulting from division by r would be added to (subtracted from) each set in τ. Since $r \leq k+1$, the remainders would not exceed k. This means that the sets $S'_1,\dots,S'_m \in \psi(\tau)$. They form m-$\theta$-coverings, because $\sum_{j=1}^{m} \underline{S}'_j = r\underline{I}_n$, or

$\sum_{j=1}^{m} \frac{1}{r} \underline{S}_j = \underline{I}_n$. By Theorem 1 , it is necessary that $\sum_{j=1}^{m} \frac{1}{r} v(S'_j) \leq 1$. Since $|S'_j| = a_j r$, and $v(S)$ is super-additive, we receive $v(S'_j) \geq a_j\, v(r)$, $j=1,\ldots,m$. Thus,

$$1 \geq \sum_{j=1}^{m} \frac{1}{r} v(S'_j) \geq \frac{v(r)}{r} \sum_{j=1}^{m} a_j = v(r)\ \frac{n}{r} ,$$

i.e., $v(r) \leq \frac{r}{n}$ for any $r = 2,\ldots, k+1$, which is precisely what was necessary to prove.

COROLLARY: In order for a symmetric game to have a core, it is necessary and sufficient that $v(S) \leq \frac{|S|}{n}$ for any $S \subset I_n$.

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XIV

STABILITY IN m-QUOTA GAMES

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Following the terminology used in [1], an n-person cooperative game is a pair $\langle N,v \rangle$, where 1) $N = (1,2,\ldots,n)$ is the set of players, 2) $v(B)$ is a real characteristic function defined on some system $\mathfrak{N} = \{B : B \subset N\}$ of subsets. (We shall always assume that $\{i\} \in \mathfrak{N}$). We shall presuppose that $v(B)$ is normalized so that

$$v(\{i\}) = 0 , \quad i \in N ,$$
$$v(B) \geq 0 , \quad B \in \mathfrak{N} .$$

If we were to allow further that $v(S) = 0$ for $S \subset N$ and $S \notin \mathfrak{N}$, then the given definition would cease to differ from the classical one. ([3]) .

Instead of imputations we shall consider payoff vectors together with a coalition structure $\mathfrak{B}$

$$(x ; \mathfrak{B}) = (x_1 \ldots, x_n ; B_1, \ldots, B_m):$$

$$B_1 \cup \ldots \cup B_m = N ; \quad B_i \cap B_j = \Lambda, \quad i \neq j ; \quad \sum_{j \in B_i} x_j = v(B_i) , \quad i=1,\ldots,m.$$

We shall refer to the pair $(x; \mathfrak{B})$ as a payoff configuration.

The set $P[K; (x; \mathcal{B})] = \bigcup_j \{i \mid i \in B_j, B_j \cap K \neq \Lambda\}$ is called the <u>set of partners in $(x; \mathcal{B})$</u> of the set K.

A payoff configuration $(x; \mathcal{B})$ is called <u>coalitionally rational</u> if $\sum_{i \in B} x_i \geq v(B)$ for all $B \subset B_j$.

Consider the coalitionally rational payoff configuration $(x; \mathcal{B})$ and let $K \cap L = \Lambda$, $K, L \subset B_j$; then a threat of K against L is a coalitionally rational payoff configuration

$$(y, \mathcal{C}) = (y_1, \ldots, y_n; C_1, \ldots, C_p)$$

for which

$$L \cap P[K; (y; \mathcal{C})] = \Lambda$$

$$y_i > x_i, \quad i \in K$$

$$y_i \geq x_i, \quad i \in P[K; (y; \mathcal{C})];$$

a counter-threat of L against K is a coalitionally rational payoff configuration

$$(z, \mathcal{D}) = (z_1, \ldots, z_n; D_1, \ldots, D_q)$$

such that $K \not\subset P[L; (z, \mathcal{D})]$ and

$$z_i \geq x_i, \quad i \in P[L; (z, \mathcal{D})],$$

$$z_i \geq y_i, \quad i \in P[L; (z, \mathcal{D})] \cap P[K; (y; \mathcal{C})].$$

A coalitionally rational payoff configuration $(x, \mathcal{C})$ in Γ is said to be <u>stable</u> if for any arbitrary threat of some set K against L there exists a counter threat of L against K. The set M of all stable configurations is called the <u>set of agreements</u>; $M \neq \Lambda$, since the configuration $(0, \ldots, 0; \{1\}, \ldots, \{n\})$ is always stable.

§1. <u>Effective coalitions</u>. A coalition B^* is said to be <u>effective</u> if there exists a payoff vector $\{x_i\}, i \in B^*$, such that

$$\sum_{i \in B^*} x_i = v(B^*) \; ; \; \sum_{i \in B} x_i \geq v(B), \quad B \subset B^*, \quad B \in \mathfrak{N}$$

$(x_i \geq 0$, since $\{i\} \in \mathfrak{N}$).

Let Γ_{B^*} consist of the following truncation of Γ : $N^* = B^*$ and $v^*(B) = v(B)$, $B \subset B^*$, i.e. $\mathfrak{N}^* = B^* \cap \mathfrak{N}$. Obviously for B^* to be effective, it is necessary and sufficient that Γ_{B^*} have a core.

Let $\Omega = (B_1, \ldots, B_k)$. As in [4] , a <u>covering</u> of set N is defined as a system of real numbers $(\lambda_1 \geq 0, \ldots, \lambda_k \geq 0)$, such that $\sum_{j=1}^{k} \lambda_j \cdot \bar{B}_j = \bar{N}$, where $\bar{B}_j$ and $\bar{N}$ are the characteristic functions of the corresponding sets, i.e.,

$$\bar{B}_j = (\delta_{1j}, \ldots, \delta_{nj}), \; \delta_{ij} = \begin{cases} 1, & i \in B_j \\ 0, & i \notin B_j \end{cases}, \quad \bar{N} = (1, \ldots, 1) .$$

A covering is said to be <u>reduced</u>, if the vectors $\{\bar{B}_j\}$ corresponding to $\lambda_j > 0$ are linearly independent. The reduced coverings are finite in number. In [4] it is shown that in order for a game Γ to have a core it is necessary and sufficient that the inequality

$$\sum_{j=1}^{k} \lambda_j \cdot v(B_j) \leq v(N)$$

be fulfilled for any arbitrary reduced covering $(\lambda_1, \ldots, \lambda_k)$.

<u>THEOREM 1</u>: In order that B^* be an effective coalition, it is necessary and sufficient that for any arbitrarily chosen $B_1 \subset B^*, \ldots, B_\ell \subset B^*$ such that the $\bar{B}_j$'s are linearly independent and such that $\sum_{j=1}^{\ell} \lambda_j \bar{B}_j = \bar{B}^*$, the inequality $\sum_{j=1}^{\ell} \lambda_j v(B_j) \leq v(B^*)$ is satisfied.

Theorem 3.1 in [1] is a special case of this theorem, since if every admissible set $B_j \subset B^*$ has the form $B^*\setminus\{i\}$, then the coverings of B^* may be constructed only in two ways: 1) from the sets $B^*\setminus\{i\}$, $i\in B^*$

$$\sum_{i\in B^*} \frac{1}{k-1}\, \overline{B^*\setminus\{i\}} = \bar{B} \qquad (k = |B^*|) ,$$

or 2) from the sets $B^*\setminus\{i\}$, $i\in B^*$ and $\{i\}$.

$$\overline{B^*\setminus\{i\}} + \overline{\{i\}} = \bar{B}^* ;$$

For coverings of the first type we obtain the condition

$$\sum_{i\in B^*} v(B^*\setminus\{i\}) \leq (k-1)\, v(B^*) ,$$

and for those of the second type

$$v(B^*\setminus\{i\}) \leq v(B^*) , \quad i\in B^* .$$

§2. <u>M-Games</u>. We shall say that a game Γ is an M-game if the only permissible coalitions are those which consist of one, M, or n players.

A system of real numbers $\omega_1, \ldots, \omega_n$ is defined (in [1]) to be an <u>M-quota</u> if $v(B) = \sum_{i\in B} \omega_i$ for all $B: |B| = M$ (this definition differs from the corresponding definition of Shapley and Calish in that the requirement that $\sum_{i=1}^{n} \omega_i = v(N)$ is not present here).

Let us now investigate stable configurations in M-games. Note that in general $(x_1, \ldots, x_n; N)$ can be stable only if N is effective. The effectiveness of an arbitrary $B_i \neq N$ in an M-game is trivial since only for N do subsets differing from single-element sets exist.

<u>THEOREM 2:</u> If for any arbitrary $B_{ik} \in \mathfrak{N}$,

$$\sum_{k=1}^{n} v(B_{ik}) \leq M\, v(N) ,$$

then N is effective in an M-game. For $M = n-1$ these conditions are necessary.

The theorem follows directly from Theorem 1, if we note that any arbitrary reduced covering containing n components "covers" each element of N no less than M times. For coverings containing less than n components, however, the fulfilment of somewhat weaker conditions is necessary.

THEOREM 3: If an M-game has an M-quota, then it has a stable configuration of the form

$$(x_1,\dots,x_n;B,\{i_1\},\dots,\{i_k\})\ (x_{i_1} = \dots x_{i_k} = 0,\ |B| = M)\ .$$

PROOF: Consider the M-quota such that $\omega_1 \geq \omega_2 \geq \dots \geq \omega_n$. Assume that $B = \{1,2,\dots,M\}$. Examine the separate cases where $\omega_M \geq 0$ and $\omega_M < 0$.

1. $\omega_M \geq 0$, i.e. all $\omega_i \geq 0$, $i \epsilon B$. We shall show that in this case the configuration $(\omega_1,\dots,\omega_M, 0,\dots,0; B, \{M+1\},\dots,\{n\})$ is stable. Suppose that, for some $K \cap L = \Lambda$, $K \cup L \subset B$, K has a threat against L, i.e. there exists a configuration $(y_1,\dots,y_n;\ C_1,\dots,C_p)$ such that $L \cap P\,[K;\ (y, \mathcal{C})] = \Lambda$ (if we denote by $\tilde{C}_j$ those C_j's for which $C_j \cap K \neq \Lambda$ then we may with greater facility write $\cup\, \tilde{C}_j$ instead of $P[K;\ (y;\ \mathcal{C}\ )])$.

$$\begin{aligned} y_i &> \omega_i\ , && i \in K \\ y_i &\geq \omega_i\ , && i \in \cup\, \tilde{C}_j \cap B\ , \\ y_i &\geq 0\ , && i \in \cup\, \tilde{C}_j \setminus B\ , \end{aligned}$$

$$\sum_{\tilde{C}_j} y_i = \sum_{\tilde{C}_j} \omega_i\ , \quad \text{i.e.} \quad \sum_{\cup \tilde{C}_j} \omega_i = \sum_{\cup \tilde{C}_j} y_i$$

Reenumerate the players in $\cup\, \tilde{C}_j$ in the following manner. Suppose $i_1,\dots i_\ell \epsilon K$. For the remaining players we have $\sum_{\cup \tilde{C}_j \setminus K} y_i < \sum_{\cup \tilde{C}_j \setminus K} \omega_i$; furthermore a) if all

$y_i < \omega_i$ then reenumerate the players arbitrarily, b) if there exists i: $y_i > \omega_i$ then call it $i_{\ell+1}$. For the remaining players we receive $\Sigma y_i < \Sigma \omega_i$. Repeating this process we develop a sequence $i_1,\dots,i_q$, with the following characteristics:

$$\sum_{k=r}^{q} y_{i_k} < \sum_{k=r}^{q} \omega_{i_k}, \quad r > 1 \quad . \qquad (*)$$

Let

$$D = L \cup \{i_q\} \cup \{i_{q-1}\} \cup \dots \cup \{i_{q-|L|+1}\} \ .$$

Since $|L| < M$ $(L \subset B \cup C_1)$, then $q - |L| + 1 > 1$ and $K \not\subset D$.

We now show that L has a counter threat against K of the form

$$(z,\mathfrak{D}) = (z_1,\dots,z_n;\ D, \{i_{k_1}\},\dots,\{i_{k_{n-m}}\}) \ .$$

In fact, in order that $(z,\mathfrak{D})$ be a counterthreat, it is necessary and sufficient that the conditions

$$z_i \geq \omega_i \ , \quad i \in L$$

$$z_i \geq y_i, \quad i \in D \setminus L \ (D \setminus L = D \cap \cup \tilde{C}_j) \ ,$$

$$\sum_{i \in D} z_i = \sum_{i \in D} \omega_i \quad .$$

For this system, in its turn, to be solvable the conditions

$$\sum_{i \in D \setminus L} y_i \leq \sum_{i \in D \setminus L} \omega_i$$

must be fulfilled. The fulfillment of these conditions follows from (*), by construction of D.

2) $\omega_M < 0$. Suppose $\omega_1,\dots,\omega_k \geq 0$, $\omega_{k+1} < 0$. Set

$$\Delta = \Big| \sum_{i=k+1}^{M} \omega_i \Big| \text{ and } S = \{1,2,\dots,k\} \ .$$

We shall show that the payoff structure

$$(\omega_1 - \epsilon_i\Delta, \ldots, \omega_k - \epsilon_k\Delta, 0, \ldots, 0; \ B, \{M+1\}, \ldots, \{n\})$$

is stable $(\epsilon_i \geq 0 \ , \ \sum_{i=1}^{k} \epsilon_i = 1 \ \ \epsilon_i\Delta \leq \omega_i$; we may take, for example, $\epsilon_i = \dfrac{\omega_i}{\sum_{i=1}^{k} \omega_i}$.
Since $\Delta \leq \sum_{i=1}^{k} \omega_i$, hence $\epsilon_i\Delta \leq \omega_i)$.

We shall prove that no set $K \subset B$ can have a threat against any L whatsoever.

In order for K to have a threat against L it is necessary that there exist a configuration $(y_1, \ldots, y_n; \ C \ , \ \{i_1\}, \ldots, \ \{i_{n-M}\})$ such that (by definition of M-quota $M > \frac{n}{2}$ in our case; therefore all payoff configurations are of this form)

$$y_i > \omega_i - \epsilon_i\Delta \ , \qquad i \in (S \cap C) \cap K \ ,$$

$$y_i \geq \omega_i - \epsilon_i\Delta \ , \qquad i \in (S \cap C) \setminus K$$

$$y_i \geq 0 \ , \qquad i \in C \setminus S \ ,$$

$$\sum_{i \in C} y_i = \sum_{i \in C} \omega_i \ .$$

In order for these inequalities to be solvable, it is necessary that

$$\sum_{S \cap C} (\omega_i - \epsilon_i\Delta) < \sum_{C} \omega_i \ .$$

Note that $-\Delta = \sum_{i=k+1}^{M} \omega_i \geq \sum_{C \setminus S} \omega_i$,

since $|C \setminus S| \geq M-k$ ($|C| = M$ and $|S| = k$), all $\omega_i \geq \omega_j \quad i < j$,

and $\omega_i \leq 0 \ , \ i \notin S$.

Therefore

$$\sum_{i\in S\cap C} \omega_i - \Delta \sum_{i\in S\cap C} \epsilon_i \geq \sum_{i\in S\cap C} \omega_i - \Delta \geq \sum_{i\in S\cap C} \omega_i + \sum_{i\in C\setminus S} \omega_i = \sum_{C} \omega_i ,$$

i.e. the system does not have a solution.

Note that for case 1) the inequalities $\omega_1 \geq \cdots \geq \omega_n$ are of no significance. All that is important is that $\omega_i \geq 0$ for $i\in B$.

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