Computational Game Theory (CIS 620/OPIM 952)
Notes on Nash Equilibria - Existence & Computation
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Notation & Definitions

- 2 players - "row" and "column"
- Payoff matrices M_1 and M_2
- Assume both players have m actions, so the M_i are $m \times m$ matrices
- If row player plays i and column player plays j, payoffs are $M_1(i,j)$ and $M_2(i,j)$
- *Mixed* strategies:
  - distributions P on rows and Q on columns
  - so $\Pr[\text{row}=i] \triangleq P_i \quad 1 \le i,j \le m$

    $\Pr[\text{col}=j] \triangleq Q_j$
- *Expected* payoffs under (P,Q):
  - row player:

    $$E_{i\sim P, j\sim Q}[M_1(i,j)] \triangleq \sum_{1\le i,j\le m} P_i Q_j M_1(i,j)$$
  - col player:

    $$E_{i\sim P, j\sim Q}[M_2(i,j)] \triangleq \sum_{1\le i,j\le m} P_i Q_j M_2(i,j)$$

- Pair (P, Q) is a <u>Nash Equilibrium</u> (NE) iff:
  - $\forall$ row distribution $P' \not\equiv P$,

    $$\underbrace{E_{i \sim P', j \sim Q}[M_1(i,j)]}_{\text{row player "cheating"}} \le E_{i \sim P, j \sim Q}[M_1(i,j)]$$

  - $\forall$ col distribution $Q' \not\equiv Q$

    $$\underbrace{E_{i \sim P, j \sim Q'}[M_2(i,j)]}_{\text{col player "cheating"}} \le E_{i \sim P, j \sim Q}[M_2(i,j)]$$

  - (etc. for more players)
  - say P is a <u>best response</u> to Q (& vice-versa)
  - Note: always $\exists$ a <u>deterministic</u> best response... but may not be NE!

Existence of NE - Crude Sketch (= Probably Wrong)

- Suppose (P,Q) is not a NE
- W.l.o.g., suppose P is not a best response to Q
- Let the row i^* be a deterministic best response to Q
- P must give some weight $\gneq 0$ to some i that is not a best response to Q
- So let's "shift" some of P from $i \to i^*$.
- For example:

$$P_{i^*} \leftarrow P_{i^*} + \min(1-P_{i^*}, P_i)$$

$$P_i \leftarrow P_i - \underbrace{\min(1-P_{i^*}, P_i)}_{\text{this makes sure we do not "overshoot" 1 in adding to } P_{i^*}}$$

- View this as a mapping $P \to \phi(P)$ or more generally $(P,Q) \to \phi(P,Q) = (P',Q')$
- (Details: which P_i to decrease, which P_{i^*} to increase, etc.)
- Note:
  - $(P',Q') = \phi(P,Q)$ are still distributions
  - $\phi(P,Q)$ is (or can be made) continuous
  - (P,Q) lies in a convex & compact space

• Important point:

(P,Q) is a NE $\Leftrightarrow \phi(P,Q) = (P,Q)$ (fixed point)

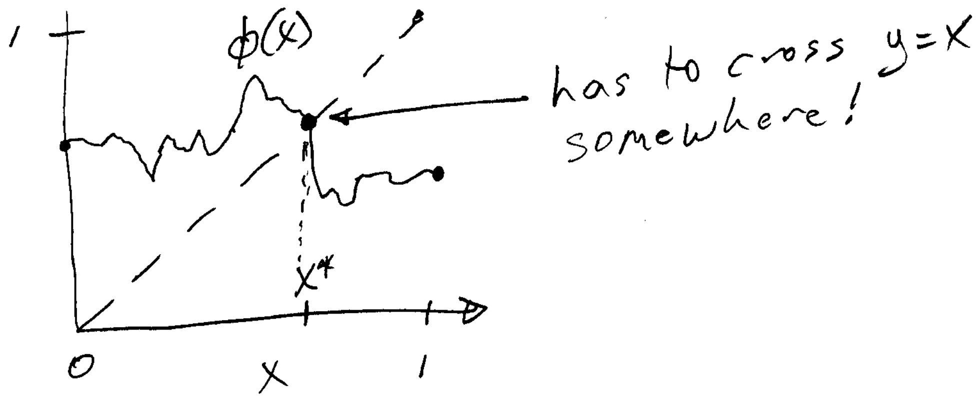
$\Rightarrow$: $\phi(P,Q)$ <u>defined</u> to have (P,Q) unchanged if P & Q are best responses to each other

$\Leftarrow$: if (P,Q) not NE, ϕ will move them

<u>Brouwer Fixed Point Theorem</u>

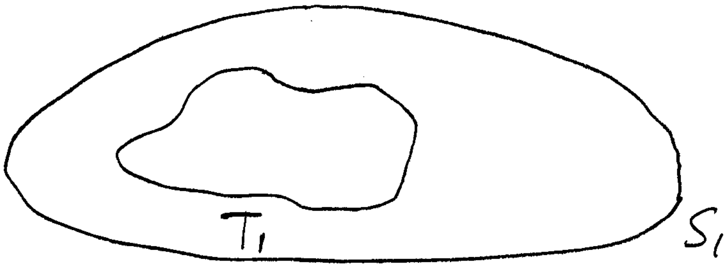
Let S be a convex, compact set.
Let $\phi: S \to S$ be continuous.
Then $\exists\, x^* \in S$ such that $\phi(x^*) = x^*$.
(Note: NE requires similar but stronger Kakutani Theorem, but Brouwer gives the flavor.)

<u>Brouwer Proof</u>: Hard, so we shall proceed "by picture". The case $S \subseteq \mathbb{R}$ is easy: say $S = [0,1]$, then:

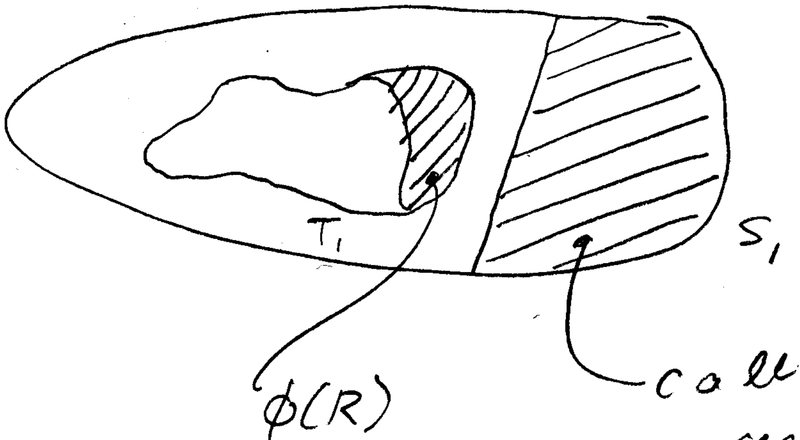


General case: suppose for simplicity that ϕ is actually a <u>contraction</u>: $\phi(S) \subsetneq S$.

Let $S_1 \triangleq S$; $T_1 \triangleq \phi(S_1)$

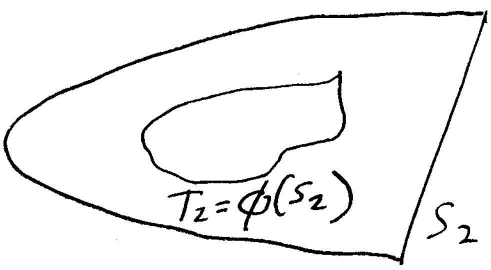


slice off a (convex) piece of S_1 <u>not</u> touching T_1



call shaded region R – <u>can't</u> have a fixed point of ϕ

toss out R,
let $S_2 = S_1 - R$
(still convex)



repeat...
the S_i keep getting smaller!

Computing NE

We'll start by considering the 2-player case - so the only thing that can be "large" or "complex" is m, the # of actions.

We want to compute a NE (P, Q). We begin with the zero-sum case $M_2 = -M_1$, and examine the problem from the column player's viewpoint.

Since anything row "wins", col "loses", col wants to choose Q s.t. no matter what row does, row cannot get "too much."

More formally: col wants a distribution Q and value U s.t.

$$\forall 1 \le i \le m \quad \underbrace{Q_1 M_1(i,1) + Q_2 M_1(i,2) + \cdots + Q_m M_1(i,m)}_{\text{exp. payoff to row player if row } i \text{ is chosen \& col player plays } Q} \le U \qquad (*)$$

Note that these m constraints imply no dist P can get more than U, either.

So, col player could solve the problem of <u>minimizing</u> U <u>subject to</u> the m constraints $(*)$.

This is a <u>linear</u> objective function in the variables $Q_1, \ldots Q_m$ and U. Also need the linear distributional constraints

$$\forall j \quad Q_j \geq 0$$

$$\sum_{j=1}^{m} Q_j = 1.$$

This is simply a linear programming (LP), solvable in time polynomial in m.

Further, from row players viewpoint, goal is to

<u>Minimize</u> V

<u>Subject to:</u>

$$\forall 1 \leq j \leq m \quad P_1 M_2(1,j) + P_2 M_2(2,j) + \cdots + P_m M_2(m,j) \leq V$$

$$\forall 1 \leq i \leq m \quad P_i \geq 0$$

$$\sum_{i=1}^{m} P_i = 1$$

<u>Note</u>: Solutions to these two LPs must obey $U = -V$! (Minimax Thm, Duality)

(Though <u>not</u> true that $P = Q$.)

Duality claim $U=-V$ special to zero-sum case.

For general sum, let's rewrite (for each i)

$$Q_1 M_1(i,1) + Q_2 M_1(i,2) + \cdots + Q_m(i,m) \le U$$

as

$$Q_1 M_1(i,1) + Q_2 M_1(i,2) + \cdots + Q_m(i,m) + X_i = U$$
$$X_i \ge 0$$

and rewrite (for each j)

$$P_1 M_2(1,j) + P_2 M_2(2,j) + \cdots + P_m M_2(m,j) \le V$$

as

$$P_1 M_2(1,j) + P_2 M_2(2,j) + \cdots + P_m M_2(m,j) + Y_j = V$$
$$Y_j \ge 0$$

Here the X_i and Y_j are "slack" variables.
Note that all constraints are still linear.
We can think of X_i (respectively, Y_i) as being the amount by which, at NE, row i (resp., col j) is "suboptimal" (less than U or V, resp.)

So now have 2 systems of linear constraints (in P, Q, U, V, X, Y). The equilibrium conditions relating them are:

$\forall i \quad P_i X_i = 0$ (either $P_i = 0$ or $X_i = 0$)

$\forall j \quad Q_j Y_j = 0$ (either $Q_j = 0$ or $Y_j = 0$)

These last constraints are not linear.

This is known as a

Linear Complementarity Problem (LCP)

and has no known polynomial time solution.

However, there is still a lot of structure here, and heuristics have been developed, most notably the Lemke-Howson algorithm.