Computational Game Theory (CIS 620/OPIM 952)
Notes on Correlated Equilibria
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Recall NE definition: a <u>product</u> distribution $\bar{p}$ (so p_i is a distribution over i's actions) such that $\forall$ player i $\forall$ action a

$$E_{\bar{x} \sim \bar{p}}[M_i(\bar{x})] \geq E_{\bar{x} \sim \bar{p}}[M_i(\bar{x}[i:a])]$$

Now let P be an <u>arbitrary</u> distribution over the population action $\bar{x}$

Now actions (components of $\bar{x}$) may be <u>correlated</u> in P

<u>Notation</u>: let $\bar{x}_{-i}$ denote all components of $\bar{x}$ <u>except</u> x_i

Say P is a <u>correlated equilibrium</u> (CE) if $\forall$ player i $\forall$ actions a, a':

$$\underbrace{\sum_{\bar{x}_{-i}} P(\bar{x}_{-i} \mid x_i = a)\, M_i(\bar{x}_{-i}, a)}_{\substack{E_P[M_i(\bar{x}_{-i}, a) \mid x_i = a] \\ (i \text{ is "honest"})}} \geq \underbrace{\sum_{\bar{x}_{-i}} P(\bar{x}_{-i} \mid x_i = a)\, M_i(\bar{x}_{-i}, a')}_{\substack{E_P[M_i(\bar{x}_{-i}, a') \mid x_i = a] \\ (i \text{ "cheats"})}}$$

<u>Example</u>: traffic intersection

$$\begin{array}{cc} & \begin{array}{cc} \text{go} & \text{yield} \end{array} \\ \begin{array}{c} \text{go} \\ \text{yield} \end{array} & \begin{pmatrix} (-10,-10) & (5,0) \\ (0,5) & (-1,-1) \end{pmatrix} \end{array}$$

<u>NE</u>: (go, yield); (yield, go); $\underbrace{(\Pr[\text{go}] = 1/16, \Pr[\text{go}] = 1/16)}_{E[\text{payoff}] = -5/8}$

<u>CE</u>: traffic signal!

Let $\Pr[\text{row sees red, col sees green}] = 1/2$

$\Pr[\text{row sees green, col sees red}] = 1/2$

CE for both: go if green, yield if red

$E[\text{payoff}]: \frac{1}{2}(5) + \frac{1}{2}(-1) = 2$

In this example, actions are <u>perfectly</u> correlated & CE is a <u>mixture</u> of NE

However, consider modified signal:

$$Pr[\text{red}, \text{green}] = \tfrac{1}{2} - \varepsilon$$

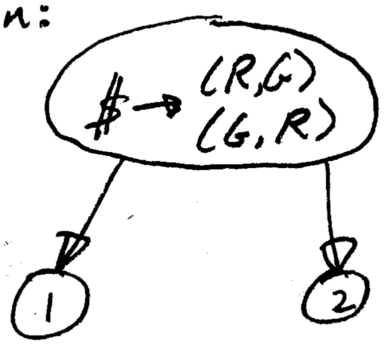
$$Pr[\text{green}, \text{red}] = \tfrac{1}{2} - \varepsilon \quad \text{for some small } \varepsilon \geq 0.$$

$$Pr[\text{red}, \text{red}] = 2\varepsilon$$

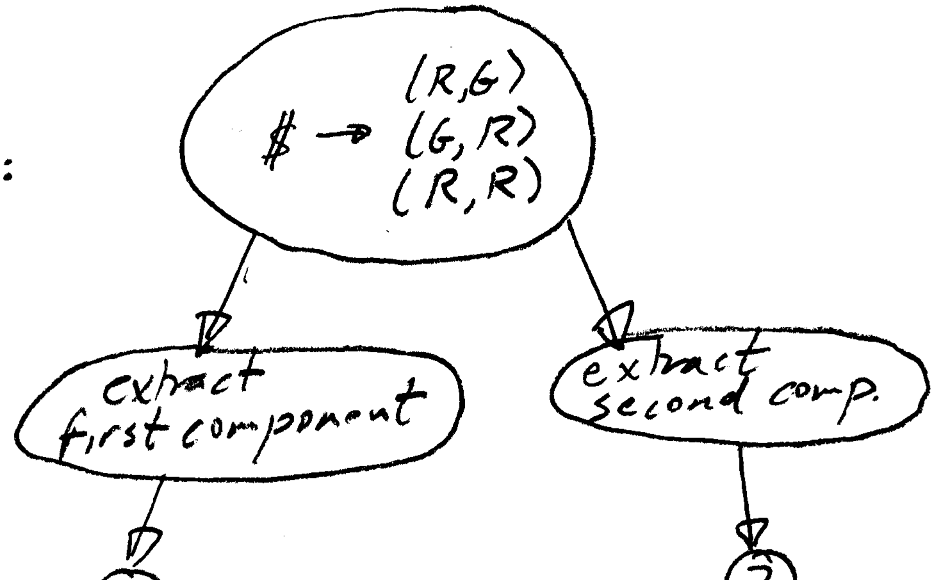
See green: know it is safe to go
See red: probably not safe to go
So go on green, stop on red still a CE

Informal representation:

$\varepsilon = 0$:

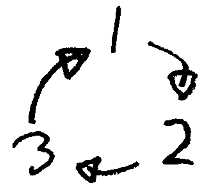


$\varepsilon > 0$:



Another example: Shapley Game

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} (1,0) & (0,1) & (0,0) \\ (0,0) & (1,0) & (0,1) \\ (0,1) & (0,0) & (1,0) \end{pmatrix} \end{array}$$



Row wants to <u>match</u>
Col wants to <u>lead</u>

Only NE: both play $(1/3, 1/3, 1/3)$

A CE: play non-$(0,0)$ entries each with prob. $1/6$

Note if row is told to play 1, all he knows is that col will play 1 <u>or</u> 2, so row 1 is a best response

Advantages/Facts of CE

- Some CE payoffs not achievable by NE
- Any <u>mixture</u> of NE is a CE (so CE always <u>exist</u>!)
- Some CE <u>not</u> a mixture of NE
- "Cooperation" via shared randomization
- Advantages for <u>learning</u>
- More realistic?

Computing CE

Rewrite CE condition:

$$\forall i\ \forall a,a' \quad \sum_{\bar{x}_{-i}} P(\bar{x}_{-i} \mid x_i = a)\, M_i(\bar{x}_{-i}, a) \geq \sum_{\bar{x}_{-i}} P(\bar{x}_{-i} \mid x_i = a)\, M_i(\bar{x}_{-i}, a')$$

$$\sum_{\bar{x}_{-i}} \frac{P(\bar{x}_{-i}, x_i = a)}{P(x_i = a)} M_i(\bar{x}_{-i}, a) \geq \sum_{\bar{x}_{-i}} \frac{P(\bar{x}_{-i}, x_i = a)}{P(x_i = a)} M_i(\bar{x}_{-i}, a')$$

$$\cancel{\frac{1}{P(x_i = a)}} \sum_{\bar{x}_{-i}} P(\bar{x}_{-i}, x_i = a)\, M_i(\bar{x}_{-i}, a) \geq \cancel{\frac{1}{P(x_i = a)}} \sum_{\bar{x}_{-i}} P(\bar{x}_{-i}, x_i = a)\, M_i(\bar{x}_{-i}, a')$$

Thus, each i, a, a' yields a <u>linear</u> inequality in variables $P(\bar{x}_{-i}, x_i = a) = P(\bar{x})$ & coefficients given by M_i

Total # of constraints $\sim n m^2$.

So for (say) 2 players, can <u>efficiently</u> compute a CE $P(i,j)$.

General (large) n:

How will we even <u>represent</u> $P(\bar{x})$???

<u>Bayes</u> & <u>Markov</u> <u>Networks</u>