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Perpetual Reductions in λ-Calculus*

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This paper surveys a part of the theory of β-reduction in λ-calculus which might aptly be called *perpetual reductions*. The theory is concerned with *perpetual reduction strategies*, i.e., reduction strategies that compute infinite reduction paths from λ-terms (when possible), and with *perpetual redexes*, i.e., redexes whose contraction in λ-terms preserves the possibility (when present) of infinite reduction paths. The survey not only recasts classical theorems in a unified setting, but also offers new results, proofs, and techniques, as well as a number of applications to problems in λ-calculus and type theory. © 1999 Academic Press

* This paper draws on several sources. In late 1994, van Raamsdonk and Severi [59] and Sørensen [66, 67] independently developed several results in the theory of perpetual reductions. Later, Xi [82] independently discovered related techniques. First Xi submitted the paper [82]. Then van Raamsdonk, Severi, and Sørensen suggested improvements and were adopted as coauthors. Finally, Sørensen rewrote the paper based on [68, Chap. 1], thereby covering the papers mentioned above as well as previously unpublished material.

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1. INTRODUCTION

Considerable attention has been devoted to the classification of *reduction strategies* in type-free λ-calculus [4, 6, 7, 15, 38, 44, 81]—see also [2, Chap. 13]. We are concerned with strategies differing in the *length* of reduction paths.

(i) A *maximal* strategy computes for a term a *longest* reduction path to normal form, if one exists, otherwise some infinite reduction path.

(ii) A *minimal* strategy computes for a term a *shortest* reduction path to normal form, if one exists, otherwise some infinite reduction path.

(iii) A *perpetual* strategy computes for a term an *infinite* reduction path, if one exists, otherwise some finite reduction path to normal form.

(iv) A *normalizing* strategy computes for a term a *finite* reduction path to normal form, if one exists, otherwise some infinite reduction path.[1]

Perpetual and normalizing strategies are opposite, in some sense, as are maximal and minimal strategies.

Another classification is concerned with *redexes* rather than strategies. For instance, a redex Δ with contractum Δ' is *perpetual* if, for any context C such that $C[\Delta]$ has an infinite reduction path, $C[\Delta']$ also has an infinite reduction path.

This paper presents a theory of perpetual and maximal β-reduction strategies and β-redexes. The paper not only recasts in a unified setting classical theorems due to Barendregt, Bergstra, Klop, and Volken, to Church and Rosser, to Curry and Feys, and to de Vrijer, but also presents new results, proofs, and techniques, as well as a number of applications to problems in λ-calculus and type theory, demonstrating the elegance and relevance of the theory.

The paper is organized as follows. Section 2 classifies reduction strategies and redexes in λ-calculus and proves equivalence between different formulations from the literature of perpetual and maximal strategies and redexes.

Section 3 is about perpetual and maximal β-reduction strategies. This is a central theme in work of de Vrijer [78, 79, 81], who uses the technique of counting steps to establish several strong normalization results. The counting functions in fact define reduction strategies.

We first prove a result which we call the *fundamental lemma of perpetuality*. The lemma is used—often implicitly—in many strong normalization proofs in the literature. An attempt is then made to show that the core of the recent techniques by van Raamsdonk and Severi and by Xi for proving strong normalization results is captured by this lemma. The section presents several perpetual reduction strategies; perpetuality is in each case an immediate consequence of the fundamental lemma of perpetuality.

The section then proves a stronger form of the fundamental lemma of perpetuality which we call the *fundamental lemma of maximality*. This result is often used implicitly in strong normalization proofs which establish upper bounds for the lengths of reduction paths. We use the lemma to show maximality of a certain reduction

[1] In this paper, attention is restricted to the usual λ-calculus. In the so-called *infinite λ-calculus* one also studies infinite reductions ending in infinite normal forms.

strategy and to give a certain, trivial technique for computing upper bounds for the lengths of reduction paths from λ-terms without infinite reduction paths. We also prove that, in a certain sense, the trivial technique cannot be improved.

Sections 4–6 present applications of perpetual and maximal β-reduction strategies. Section 4 presents the recent *Ω-theorem*, due to Sørensen, stating that every λ-term in every infinite reduction path contains the λ-term Ω as a substring. The proof uses a certain perpetual reduction strategy. Section 5 studies approaches to proving strong normalization of *simply typed λ-calculus* based on the fundamental lemma of perpetuality and based on the related techniques by van Raamsdonk and Severi and by Xi. In particular, a new perspicuous proof is presented. Section 6 similarly studies approaches to proving *finiteness of developments* and in particular gives a new, perspicuous proof of this theorem.

Section 7 is about perpetual β-redexes (as we shall see, maximal β-redexes turn out to be trivial). A well-known proof technique is refined and used to give smooth proofs of the conservation theorem for Λ_I, of the conservation theorem for Λ_K, and of a related theorem due to Bergstra and Klop; these results together give characterizations of perpetual redexes in Λ_I and Λ_K. The technique is also demonstrated to yield the normalization theorem with little effort. The section ends with a very short proof of the conservation theorem for Λ_I using the normalization theorem.

Klop [39] surveys some results about reduction strategies in first-order term rewriting systems. Due to the absence of abstractions and the presence of patterns in the term language, some parts of that theory are rather different from what is presented in this paper; therefore, we shall not consider such systems any further. Several notions of higher-order term rewriting system exist, some of which contain as special cases λ-calculus with β-reduction. We will not consider such systems, although we do try to give references to results that generalize those for λ-calculus presented in this paper.

2. CLASSIFICATION OF STRATEGIES AND REDEXES

In this section we classify strategies and redexes as outlined in the Introduction. The first subsection reviews preliminary notions. The second subsection introduces some notation and properties pertaining to reductions. The third and fourth subsections then classify strategies and redexes and prove equivalence between different classifications from the literature.

2.1. *Preliminaries*

Most notation, terminology, and conventions are adopted from [2]; in this subsection we merely fix the notation for some well-known concepts.

Λ_K is the set of type-free λ-terms. Some example terms are $\mathbf{K} \equiv \lambda x.\lambda y.x$, $\mathbf{I} \equiv \lambda x.x$, $\omega \equiv \lambda x.x\,x$, and $\Omega \equiv \omega\omega$. We use $x, y, z, \ldots$ to range over the set V of variables. Familiarity is assumed with conventions for omitting parentheses in λ-terms. Familiarity is also assumed with the notions of free and bound variables, the variable convention, substitution, and the subterm relation, which is denoted by $\subseteq$. Syntactic equality up to renaming of bound variables is denoted by $\equiv$. $\mathrm{FV}(M)$

denotes the set of free variables in M. $\|M\|_x$ denotes the number of free occurrences of x in M. $\|M\|$ denotes the size of M, i.e., the number of occurrences of abstractions, applications, and variables in M. Λ_I is the set of all λ-terms where for every subterm $\lambda x.M$, $x \in \mathrm{FV}(M)$. Thus, $\mathbf{I}, \omega, \Omega \in \Lambda_I$, whereas $\mathbf{K} \notin \Lambda_I$. A λ-context C is a term with a single occurrence of the symbol [ ]; the result of replacing [ ] by the term M in C is denoted by $C[M]$. Occasionally the name of bound variables matters, e.g., when dealing with contexts. In such cases, $\mathrm{BV}(M)$ denotes the set of variables bound in M.

We occasionally use vector notation $\mathbf{P}$ for a sequence of terms $P_1 \cdots P_n$ (where $n \geqslant 0$), e.g., $Q\mathbf{P}$ for $QP_1 \cdots P_n$, and $\mathbf{P} \in S$ for $P_1, ..., P_n \in S$.

A notion of reduction on a set S is a binary relation $R \subseteq S \times S$. If MRN, then M is an R-redex and N its R-contractum. By $R_1 R_2$ we denote the union of two notions of reduction R_1 and R_2. For a notion of reduction R, the corresponding reduction relation $\rightarrow_R$ is the compatible closure (relative to some set of contexts). For a reduction relation $\rightarrow_R$, $\twoheadrightarrow_R$ is the reflexive, transitive closure, $\twoheadrightarrow_R^+$ is the transitive closure, and $=_R$ is the transitive, reflexive, symmetric closure. We assume that the reader is familiar with the notion of reduction β on Λ_K. Several elementary properties about substitution and β-reduction will be used implicitly.

Let $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$. The following calculation rules are convenient: $\min \varnothing = \max \varnothing = \infty$. Moreover, $\max U = \infty$ if $U \subseteq \mathbb{N}$ is *unbounded*, i.e., if, for all $m \in U$, there is an $n \in U$ with $n > m$. Also, $\infty - k = \infty + k = \infty + \infty = k \cdot \infty = \infty$, for any $k \in \mathbb{N}$. Finally, for $m^*, n^* \in \mathbb{N}^+$ we write $m^* < n^*$ iff either $m^* \neq \infty$ and $n^* = \infty$, or $m^*, n^* \in \mathbb{N}$ and $m^* < n^*$ by the usual ordering on $\mathbb{N}$. We write $m^* \leqslant n^*$ iff $m^* < n^*$ or $m^* = n^*$.

We use $\Rightarrow$, $\Leftrightarrow$, &, $\forall$, $\exists$ as connectives and quantifiers in the informal meta-language. For a map $F\colon S \rightarrow S$ on some set S, we define $F^0(M) = M$ and $F^{n+1}(M) = F(F^n(M))$.

2.2. *Some Notation Concerning Normalization*

In this subsection R denotes a notion of reduction on some set S, and $\rightarrow_R$ denotes the corresponding reduction relation.

2.1. Definition. A finite or infinite sequence

$$M_0 \rightarrow_R M_1 \rightarrow_R \cdots$$

is called an *R-reduction path* from M_0. We say that M_0 *has* this R-reduction path. If the sequence is finite it *ends* in the last term M_n and has *length* n, and then we write $M_0 \twoheadrightarrow_R^n M_n$. If the sequence is infinite, it has length ∞.

2.2. Definition.

$$\infty_R = \{M \mid M \text{ has an infinite } R\text{-reduction path}\}$$

$$n_R = \{M \mid M \text{ has an } R\text{-reduction path of length } n\}$$

$$\mathrm{NF}_R = \{M \mid M \text{ has no } R\text{-reduction path of length 1 or more}\}$$

$$\mathrm{SN}_R = \{M \mid M \text{ has no infinite } R\text{-reduction path}\}$$

$$\mathrm{WN}_R = \{M \mid M \text{ has a finite } R\text{-reduction path ending in an } N \in \mathrm{NF}_R\},$$

where $n \in \mathbb{N}$ in the notation n_R.

2.3. DEFINITION.

$$\mathrm{CR}_R = \{M \mid M \text{ for all } L, N, \text{ if } L\ {}_R\!\twoheadleftarrow M \twoheadrightarrow_R N \text{ then } L \twoheadrightarrow_R K\ {}_R\!\twoheadleftarrow N \text{ for a } K\}$$

$$\mathrm{FB}_R = \{M \mid M \to_R N \text{ for only finitely many different } N\}.$$

2.4. TERMINOLOGY.

(i) $M \in \mathrm{NF}_R \Leftrightarrow M$ is an *R-normal form*;

(ii) $M \in \mathrm{SN}_R \Leftrightarrow M$ is *R-strongly normalizing*;

(iii) $M \in \mathrm{WN}_R \Leftrightarrow M$ is *R-weakly normalizing*;

(iv) $M \in \mathrm{CR}_R \Leftrightarrow M$ is *R-Church–Rosser*;

(v) $M \in \mathrm{FB}_R \Leftrightarrow M$ is *R-finitely branching*.

We often omit R, relying on the context to resolve the ambiguity. When R is a notion of reduction on a set S, and $M \in \mathrm{FB}_R$ for all $M \in S$, we simply write FB_R. Similarly with the other sets introduced above.

2.5. LEMMA. *Assume* FB_R. *Then* $M \in \infty_R \Leftrightarrow \forall n \in \mathbb{N}$: $M \in n_R$.

Proof. "⇒" is obvious; "⇐" is by König's lemma. ∎

We shall denote by $s_R(M) \in \mathbb{N}^*$ the length of a shortest finite reduction path from M to normal form, if a finite reduction path to normal form exists; otherwise $s_R(M) = \infty$. Also, $l_R(M) \in \mathbb{N}^*$ denotes the length of a longest finite reduction path from M to normal form, if there is an upper bound on the length of these reduction paths; otherwise $l_R(M) = \infty$. In symbols:

2.6. DEFINITION.

(i) $s_R(M) = \min\{n \mid \exists N \in \mathrm{NF}_R \colon M \twoheadrightarrow^n_R N\}$.

(ii) $l_R(M) = \max\{n \mid \exists N \in \mathrm{NF}_R \colon M \twoheadrightarrow^n_R N\}$.

2.7. LEMMA. *Assume* CR_R, FB_R. *Then*

(i) $M \in \mathrm{WN}_R \Leftrightarrow s_R(M) < \infty$;

(ii) $M \in \infty_R \Leftrightarrow l_R(M) = \infty$.

Proof. (i) "⇒": If $M \in \mathrm{WN}_R$ then $M \twoheadrightarrow^n_R N \in \mathrm{NF}_R$ for some $n \in \mathbb{N}$, so $s_R(M) < \infty$.

"⇐": If $s_R(M) < \infty$ then $M \twoheadrightarrow^n_R N \in \mathrm{NF}_R$ for some $n \in \mathbb{N}$, so $M \in \mathrm{WN}_R$.

(ii) "⇒": Assume $M \in \infty_R$.

1. $M \notin \mathrm{WN}_R$. Then $l_R(M) = \infty$.

2. $M \in \mathrm{WN}_R$. Then $M \twoheadrightarrow_R N \in \mathrm{NF}_R$ for some N. Since $M \in \infty_R$, for any $n \in \mathbb{N}$ there is K such that $M \twoheadrightarrow^n_R K$. By CR_R, $K \twoheadrightarrow_R N$. Thus, for any $m \in \mathbb{N}$ there is $n > m$ such that $M \twoheadrightarrow^n_R N \in \mathrm{NF}_R$. Then $l_R(M) = \infty$.

"$\Leftarrow$": Assume $l_R(M) = \infty$. There are two ways this can happen.

1. $M \notin \mathrm{WN}_R$. Then $M \in \infty_R$.

2. For arbitrarily large $n \in \mathbb{N}$ there is $N \in \mathrm{NF}_\beta$ with $M \twoheadrightarrow^n_R N$. Then $M \in \infty_R$ by Lemma 2.5. ∎

2.8. *Remark.* Although seemingly trivial, the above proof uses the rules $\min \varnothing = \max \varnothing = \max U = \infty$ (U unbounded) in subtle ways. For instance, as shown in (ii) "$\Rightarrow$", if $m \in \infty_R$, then $\{n \mid \exists N \in \mathrm{NF}_R \colon M \twoheadrightarrow^n_R N\}$ is either empty (if $M \notin \mathrm{WN}_R$) or unbounded (if $M \in \mathrm{WN}_R$). In either event, the two latter conventions imply $l_R(M) = \infty$.

2.9. *Remark.* The statement formulated in Lemma 2.7(ii) will be used at various places later; an equivalent statement is $M \in \mathrm{SN}_R \Leftrightarrow l_R(M) < \infty$.

2.3. *Classification of Strategies*

In this subsection we introduce rigorously the classification of reduction strategies that was mentioned informally in the Introduction. Throughout the subsection, R denotes a notion of reduction on some set S, and $\rightarrow_R$ denotes the corresponding reduction relation.

2.10. Definition. (Barendregt *et al.* [2, 4]). (i) An *R-reduction strategy* is a map $F \colon S \rightarrow S$ such that $M \rightarrow_R F(M)$ if $M \notin \mathrm{NF}_R$, and $F(M) = M$ otherwise.

(ii) Let F be an R-reduction strategy. Define

$$L_F(M) = \min\{n \mid F^n(M) \in \mathrm{NF}_R\}.$$

The *F-reduction path* from M is the reduction path

$$M \rightarrow_R F(M) \rightarrow_R F^2(N) \rightarrow_R \cdots$$

of length $L_F(M)$.

2.11. *Remark.* Reduction strategies are *history insensitive*; that is, given some $M \in \Lambda_K$, the act of a reduction strategy on M is independent on how we might have arrived at M. For instance, "for any $M \in \Lambda_K$, reduce alternately the left-most and right-most β-redex, beginning with the left-most one" does not specify a reduction strategy; a reduction strategy receives a term as input and must return as output another term that arises from the former by one reduction step.

Barendregt *et al.* [2, 4] use the terminology *one-step reduction strategy* for what we call *reduction strategy*. In the following definition, (ii)–(iv) are also taken from [2, 4], but what we call *minimal* is there called *L-1-optimal.*

2.12. Definition. Let F be an R-reduction strategy.

(i) F is *R-maximal* iff $L_F(M) = l_R(M)$;

(ii) F is *R-minimal* iff $L_F(M) = s_R(M)$;

(iii) F is *R-perpetual* iff $M \in \infty_R \Rightarrow L_F(M) = \infty$;

(iv) F is *R-normalizing* iff $M \in \mathrm{WN}_R \Rightarrow L_F(M) < \infty$.

This classification of strategies is "global" in that it is formulated in terms of the whole reduction path of the strategy. The following formulations of minimality and maximality are "local" in that they are formulated in terms of one step of the strategy. The local classifications have the advantage that they give rise to analogous classifications of redexes.

2.13. LEMMA. *Assume* CR_R, FB_R. *Let F be an R-reduction strategy.*

(i) *F is R-minimal iff for all $M \notin \mathrm{NF}_R$: $s_R(M) = s_R(F(M)) + 1$.*

(ii) *F is R-maximal iff for all $M \notin \mathrm{NF}_R$: $l_R(M) = l_R(F(M)) + 1$.*

Proof. (i) "$\Rightarrow$": Assume that F is R-minimal. Then, for any $M \notin \mathrm{NF}_R$,

$$\begin{aligned} s_R(M) &= L_F(M) \\ &= \min\{n \mid F^n(M) \in \mathrm{NF}_R\} \\ &= \min\{n \mid F^n(F(M)) \in \mathrm{NF}_R\} + 1 \\ &= L_F(F(M)) + 1 \\ &= s_R(F(M)) + 1. \end{aligned}$$

"$\Leftarrow$": Assume for all $M \notin \mathrm{NF}_R$ that $s_R(M) = s_R(F(M)) + 1$. If $s_R(M) = \infty$, then also $L_F(M) = \infty$. Now assume that $s_R(M) < \infty$. We show by induction on $s_R(M)$ that $s_R(M) = L_F(M)$.

1. $s_R(M) = 0$. Then $M \in \mathrm{NF}_R$, so $L_F(M) = 0$.
2. $0 < s_R(M) < \infty$. Then $M \notin \mathrm{NF}_R$. By the induction hypothesis,

$$\begin{aligned} s_R(M) &= s_R(F(M)) + 1 \\ &= L_F(F(M)) + 1 \\ &= L_F(M). \end{aligned}$$

(ii) "$\Rightarrow$": Assume that F is maximal. Then, for any $M \notin \mathrm{NF}_R$,

$$\begin{aligned} l_R(M) &= L_F(M) \\ &= L_F(F(M)) + 1 \\ &= l_R(F(M)) + 1 \end{aligned}$$

"$\Leftarrow$": Assume for all $M \notin \mathrm{NF}_R$ that $l_R(M) = l_R(F(M)) + 1$. If $L_F(M) = \infty$, then, by Lemma 2.7, also $l_R(M) = \infty$. Now assume that $L_F(M) < \infty$. We show $l_R(M) = L_F(M)$ by induction on $L_F(M)$.

1. $L_F(M)=0$. Then $M \in \mathrm{NF}_R$, so $l_R(M)=0$.
2. $0<L_F(M)<\infty$. Then $N \notin \mathrm{NF}_R$. By the induction hypothesis,

$$\begin{aligned} L_F(M) &= L_F(F(M))+1 \\ &= l_R(F(M))+1 \\ &= l_R(M) \end{aligned}$$

Note that we need Lemma 2.7 in (ii), but not in (i). ▮

The following gives another local formulation of perpetuality and maximality, due to Bergstra and Klop [7] and Regnier [60], respectively.

2.14. Lemma. *Assume* CR_R, FB_R. *Let F be an R-reduction strategy.*

(i) *F is R-perpetual iff for all M: $M \in \infty_R \Rightarrow F(M) \in \infty_R$;*

(ii) *F is R-maximal iff for all M and $n \geqslant 1$: $M \in n_R \Rightarrow F(M) \in (n-1)_R$.*

Proof. (i) "$\Rightarrow$": Assume $M \in \infty_R$. By assumption, $L_F(M)=\infty$; i.e., the path $M \to_R F(M) \to_R F^2(M) \to_R \cdots$ is infinite, so $F(M) \in \infty_R$.

"$\Leftarrow$": Assume $M \in \infty_R$. By induction on n show that $F^n(M) \in \infty_R$, in particular $F^n(M) \notin \mathrm{NF}_R$, so $L_F(M)=\infty$.

(ii) "$\Rightarrow$": Assume that $M \in n_R$. By CR_R, $n \leqslant l_R(M)=L_F(M)$, i.e., $F^{n-1}(M) \notin \mathrm{NF}_R$, so $F(M) \in (n-1)_R$.

"$\Leftarrow$": If $L_F(M)=\infty$, then, by Lemma 2.7, $l_R(M)=\infty$. Assume $L_F(M)<\infty$. We show $L_F(M)=l_R(M)$ by induction on $L_F(M)$.

1. $L_F(M)=0$. Then $M \in \mathrm{NF}_R$, so $l_R(M)=0$.
2. $0<L_F(M)<\infty$. Then $M \notin \mathrm{NF}_R$. By the induction hypothesis and Lemma 2.13,

$$\begin{aligned} L_F(M) &= L_F(F(M))+1 \\ &= l_R(F(M))+1 \\ &= l_R(M). \quad ▮ \end{aligned}$$

2.15. Proposition. *Assume* CR_R, FB_R. *Let F be an R-reduction strategy.*

(i) *If F is R-maximal then F is R-perpetual.*

(ii) *If F is R-minimal then F is R-normalizing.*

Proof. (i) If $M \in \infty_R$ then, by Lemma 2.7, $L_F(M)=l_R(M)=\infty$.

(ii) If $M \in \mathrm{WN}_R$ then, by Lemma 2.7, $L_F(M)=s_R(M)<\infty$. ▮

2.16. *Remark.* No other general containment exists between our four types of strategies than the two mentioned above.

Perpetual reduction strategies are often useful to prove properties about infinite reduction paths. In these cases we are usually not interested in how the strategy behaves on strongly normalizing terms. This motivates the following.

2.17. DEFINITION. A *partial, perpetual R-reduction strategy* is a mapping $F: \infty_R \rightarrow \infty_R$ such that for all $M \in \infty_R$: $M \rightarrow_R F(M)$.

2.4. Classification of Redexes

In this subsection we introduce rigorously the classification of redexes from the Introduction. Throughout the subsection, R denotes a notion of reduction on Λ_K, and $\rightarrow_R$ denotes the corresponding reduction relation.

In the following definition, (i) is taken from [7].

2.18. DEFINITION. Let Δ be an R-redex with contractum Δ'.

(i) Δ is *R-perpetual* iff, for all C: $C[\Delta] \in \infty_R \Rightarrow C[\Delta'] \in \infty_R$;

(ii) Δ is *R-maximal* iff, for all $n \geqslant 1$ and C: $C[\Delta] \in n_R \Rightarrow C[\Delta'] \in (n-1)_R$.

2.19. *Remark.* As was the case for strategies, one can vary the formulation of perpetual and maximal redexes; we shall not study such equivalent formulations.

2.20. DEFINITION. Let Δ be an R-redex with contractum Δ'. Then Δ is *R-minimal* iff for all C: $s_R(C[\Delta]) = s_R(C[\Delta']) + 1$.

2.21. *Discussion.* A strategy that always contracts perpetual redexes is perpetual. Similarly, strategies that always contract maximal and minimal redexes are maximal and minimal, respectively. This is easy to verify simply by noting the analogy between on the one hand the local formulations of perpetual, maximal, and minimal strategies in Lemmas 2.13 and 2.14, and on the other hand the formulations of perpetual, maximal, and minimal redexes in Definitions 2.18 and 2.20.

Perpetual strategies may also contract non-perpetual redexes. The reason is that a *strategy* is confronted with a redex in a given context, and needs only to make sure that contracting the redex in *this particular* context preserves the possibility, if present, of an infinite reduction. A perpetual *redex*, on the other hand, must preserve the existence of infinite reduction paths in *all* contexts. Similar remarks apply to maximal and minimal strategies.

We do not know how to give a formulation of the notion of a *normalizing redex* which satisfies the property that a strategy contracting only normalizing redexes is itself normalizing. This problem stems from the fact that the above classifications of redexes were derived from *local* formulations of the notions of a perpetual, maximal, and minimal strategy, whereas we have no local formulation of the notion of a normalizing strategy.

2.22. PROPOSITION. *Assume* FB_R. *A redex which is R-maximal is also R-perpetual.*

Proof. Given R-maximal redex Δ with contractum Δ' and a context C, assume $C[\Delta] \in \infty_R$. To prove $C[\Delta'] \in \infty_R$ it suffices by Lemma 2.5 to show that $C[\Delta'] \in n_R$ for all $n \in \mathbb{N}$. Since $C[\Delta] \in \infty_R$ we have by Lemma 2.5 for all $n \in \mathbb{N}$, $C[\Delta] \in n_R$ and thereby $C[\Delta] \in (n+1)_R$. Thus $C[\Delta'] \in n_R$ for all $n \in \mathbb{N}$ by maximality. ∎

2.23. *Remark.* The converse of the preceding proposition does not hold.

3. PERPETUAL AND MAXIMAL STRATEGIES

In this section we study perpetual and maximal β-reduction strategies. The first subsection presents the *fundamental lemma of perpetuality*. The second subsection presents two recent characterizations of strongly normalizing terms due to van Raamsdonk and Severi and to Xi, respectively, and shows that the core of these characterizations is made up of the fundamental lemma of perpetuality and a certain lexicographic induction principle. The third subsection presents two (partial) perpetual β-reduction strategies; the proof of perpetuality in each case uses the fundamental lemma of perpetuality.

The fourth subsection presents the *fundamental lemma of maximality*, analogous to the fundamental lemma of perpetuality. The fifth subsection presents an effective, maximal β-reduction strategy; the proof of maximality uses the fundamental lemma of maximality. The sixth subsection shows that to compute an upper bound on the length of a longest β-reduction path for some term, one cannot do better, in a certain sense, than try to reduce the term to normal form and count the number of steps along the way.

The property CR_β is used freely in this and the following sections.

3.1. The Fundamental Lemma of Perpetuality

The following lemma is used in many strong normalization proofs in the literature—see Section 5. As will be seen below, the lemma is also useful to show that reduction strategies are perpetual.

3.1. LEMMA (Fundamental Lemma of Perpetuality). *Assume that $M_1 \in \mathrm{SN}_\beta$ if $x \notin \mathrm{FV}(M_0)$. For all $n \geqslant 1$,*

$$M_0\{x := M_1\}\ M_2 \cdots M_n \in \mathrm{SN}_\beta \Rightarrow (\lambda x.M_0)\ M_1 \cdots M_n \in \mathrm{SN}_\beta.$$

Proof. Let $M_0\{x := M_1\}\ M_2 \cdots M_n \in \mathrm{SN}_\beta$. Then $M_0, M_2, ..., M_n \in \mathrm{SN}_\beta$. If $x \notin \mathrm{FV}(M_0)$, then, by assumption, $M_1 \in \mathrm{SN}_\beta$. If $x \in \mathrm{FV}(M_0)$, then also $M_1 \subseteq M_0\{x := M_1\}\ M_2 \cdots M_n$, so $M_1 \in \mathrm{SN}_\beta$. If $(\lambda x.M_0)\ M_1 \cdots M_n \in \infty_\beta$, then any infinite reduction must therefore have the form

$$\begin{aligned}(\lambda x.M_0)\ M_1 \cdots M_n &\twoheadrightarrow_\beta (\lambda x.M_0')\ M_1' \cdots M_n' \\ &\to_\beta M_0'\{x := M_1'\}\ M_2' \cdots M_n' \\ &\to_\beta \cdots.\end{aligned}$$

Since

$$M \twoheadrightarrow_\beta M' \;\&\; N \twoheadrightarrow_\beta N' \Rightarrow M\{x := N\} \twoheadrightarrow_\beta M'\{x := N'\}$$

there is an infinite reduction sequence

$$M_0\{x := M_1\}\ M_2 \cdots M_n \twoheadrightarrow_\beta M'_0\{x := M'_1\}\ M'_2 \cdots M'_n \\ \rightarrow_\beta \cdots$$

contradicting $M_0\{x := M_1\}\ M_2 \cdots M_n \in \mathrm{SN}_\beta$. ▮

3.2. COROLLARY. *If $M_1 \in \mathrm{SN}_\beta$, then for all $n \geqslant 1$,*

$$M_0\{x := M_1\}\ M_2 \cdots M_n \in \mathrm{SN}_\beta \Rightarrow (\lambda x.M_0)\ M_1 \cdots M_n \in \mathrm{SN}_\beta.$$

Proof. By the fundamental lemma of perpetuality. ▮

3.3. *Remark.* The fundamental lemma of perpetuality gives a condition ensuring that a contraction $(\lambda x.M_0)\ M_1 \cdots M_n \rightarrow_\beta M_0\{x := M_1\}\ M_2 \cdots M_n$ preserves the possibility, if present, of an infinite reduction. The corollary requires a slightly simpler condition.

3.2. Two Characterizations of Strongly Normalizing Terms

Next we introduce two characterizations of SN_β due to van Raamsdonk and Severi [59] (also [58, 65]) and to Xi [82], respectively.

3.4. DEFINITION. Let $X \subseteq \Lambda_K$ be the smallest set closed under:

(i) $M_1, ..., M_n \in X \Rightarrow xM_1 \cdots M_n \in X$;

(ii) $M \in X \Rightarrow \lambda x.M \in X$;

(iii) $M_1 \in X\ \&\ M_0\{x := M_1\}\ M_2 \cdots M_n \in X \Rightarrow (\lambda x.M_0)\ M_1 \cdots M_n \in X$.

3.5. PROPOSITION. $\mathrm{SN}_\beta = X$.

Proof. We first prove $M \in \mathrm{SN}_\beta \Rightarrow M \in X$ by induction on lexicographically ordered pairs $\langle l_\beta(M), \|M\| \rangle$.

1. $M \equiv xP_1 \cdots P_n$. Then $P_1, ..., P_n \in \mathrm{SN}_\beta$. By the induction hypothesis $P_1, ..., P_n \in X$, so $M \in X$.

2. $M \equiv \lambda x.P$. Similar to Case 1.

3. $M \equiv (\lambda x.P_0)\ P_1 \cdots P_n$. Then $P_1 \in \mathrm{SN}_\beta$, $P_0\{x := P_1\}\ P_2 \cdots P_n \in \mathrm{SN}_\beta$, so by the induction hypothesis, $P_1 \in X$, $P_0\{x := P_1\}\ P_2 \cdots P_n \in X$, so $M \in X$.

It remains to prove $M \in X \Rightarrow M \in \mathrm{SN}_\beta$. We proceed by induction on the derivation of $M \in X$.

1. $M \equiv xP_1 \cdots P_n$, where $P_1, ..., P_n \in X$. By the induction hypothesis $P_1, ..., P_n \in \mathrm{SN}_\beta$, so $M \in \mathrm{SN}_\beta$.

2. $M \equiv \lambda x.P$. Similar to Case 1.

3. $M \equiv (\lambda x.P_0)\ P_1 \cdots P_n$, where $P_1 \in X$, $P_0\{x := P_1\}\ P_2 \cdots P_n \in X$. By the induction hypothesis, $P_1 \in \mathrm{SN}_\beta$, $P_0\{x := P_1\}\ P_2 \cdots P_n \in \mathrm{SN}_\beta$, so by the fundamental lemma of perpetuality, $M \in \mathrm{SN}_\beta$. ▮

3.6. *Remark.* Given an assertion of form $M \in \mathrm{SN}_\beta \Rightarrow P(M)$, we may prove instead $M \in X \Rightarrow P(M)$ by induction on the derivation of $M \in X$; this is very similar to proving the original assertion by induction on lexicographically ordered pairs $\langle l_\beta(M), \|M\| \rangle$. Given an assertion of form $P(M) \Rightarrow M \in \mathrm{SN}_\beta$, we may prove instead $P(M) \Rightarrow M \in X$; this is very similar to proving the original assertion and using the fundamental lemma of perpetuality in the case $M \equiv (\lambda x.P_0)\, P_1 \cdots P_n$. Thus, the two main ingredients in the proof of Proposition 3.5—lexicographic induction on $\langle l_\beta(M), \|M\| \rangle$ and the fundamental lemma of perpetuality—are used implicitly when one uses X to reason about SN_β. Van Raamsdonk and Severi [59] prove strong normalization results in λ-calculus using this characterization—see Sections 5 and 6.

3.7. Definition. Define $F_l\colon \Lambda_K \to \Lambda_K$ as follows. If $M \in \mathrm{NF}_\beta$ then $F_l(M) = M$; otherwise,

$$F_l(x\mathbf{P}Q\mathbf{R}) = x\mathbf{P}F_l(Q)\,\mathbf{R} \qquad \text{if} \quad \mathbf{P} \in \mathrm{NF}_\beta, \quad Q \notin \mathrm{NF}_\beta$$

$$F_l(\lambda x.P) = \lambda x.F_l(P)$$

$$F_l((\lambda x.P)\, Q\mathbf{R}) = P\{x := Q\}\,\mathbf{R}.$$

Write $M \to_l N$ if $M \notin \mathrm{NF}_\beta$ and $F_l(M) \equiv N$, and $M \in \infty_l$ if $L_{F_l}(M) = \infty$.

3.8. Definition. Define the relation $\rhd$ by:

$$\rhd \; = \; \sqsupset \; \cup \; \to_l,$$

where $\sqsupset$ denotes the smallest relation closed under the rules

$$\lambda x.M \sqsupset M \qquad M_1 M_2 \sqsupset M_1 \qquad M_1 M_2 \sqsupset M_2.$$

Define

$$\mathscr{H}(M_0) = \max\{n \mid M_0 \rhd M_1 \rhd \cdots \rhd M_n\} \in \mathbb{N}^*.$$

3.9. Proposition. $\mathrm{SN}_\beta = \{M \in \Lambda_K \mid \mathscr{H}(M) < \infty\}$.

Proof. We first prove $M \in \mathrm{SN}_\beta \Rightarrow \mathscr{H}(M) < \infty$ by induction on lexicographically ordered pairs $\langle l_\beta(M), \|M\| \rangle$. First note that if $\mathscr{H}(M_0) = \infty$ then by König's lemma there is an infinite sequence $M_0 \rhd M_1 \rhd \cdots$, and so there is an M_1 with $M_0 \rhd M_1$ and $\mathscr{H}(M_1) = \infty$.

1. $M \equiv x$. Then $\mathscr{H}(M) = 0 < \infty$.

2. $M \equiv PQ$. Then $P, Q \in \mathrm{SN}_\beta$. Moreover, if $M \to_l M'$ then $M' \in \mathrm{SN}_\beta$. By the induction hypothesis $\mathscr{H}(P) < \infty$, $\mathscr{H}(Q) < \infty$, and $\mathscr{H}(M') < \infty$. Thus, for all N with $M \rhd N$, $\mathscr{H}(N) < \infty$. Thus, $\mathscr{H}(M) < \infty$.

3. $M \equiv \lambda x.P$. Similar to Case 2.

Next we prove $\mathscr{H}(M) < \infty \Rightarrow M \in \mathrm{SN}_\beta$ by induction on $\mathscr{H}(M)$.

1. $M \equiv xP_1 \cdots P_n$. Then $\mathscr{H}(P_1) < \infty$, ..., $\mathscr{H}(P_n) < \infty$. By the induction hypothesis $P_1, ..., P_n \in \mathrm{SN}_\beta$, so $M \in \mathrm{SN}_\beta$.

2. $M \equiv \lambda x.P$. Similar to Case 1.

3. $M \equiv (\lambda x.P_0)\, P_1 \cdots P_n$. Then $\mathscr{H}(P_0\{x := P_1\}\, P_2 \cdots P_n) < \infty$ and $\mathscr{H}(P_1) < \infty$. By the induction hypothesis, $P_0\{x := P_1\}\, P_2 \cdots P_n \in \mathrm{SN}_\beta$ and $P_1 \in \mathrm{SN}_\beta$. By the fundamental lemma of perpetuality it then follows that $M \in \mathrm{SN}_\beta$. ▮

3.10. *Remark.* The point in Remark 3.6 may be repeated with "$M \in X$" replaced by "$\mathscr{H}(M) < \infty$." Xi [82] proves strong normalization results in λ-calculus using this characterization—see Sections 5 and 6.

3.11. *Remark.* The above characterizations of SN_β, especially the second one, are similar to the *successor relation*, defined by Terlouw [75], who proves this relation to be well-founded and uses it to show a connection between higher type levels and transfinite recursion (see also [77]).

Whether one should prove results in λ-calculus using the fundamental lemma of perpetuality and lexicographic induction, or one should use one of the characterizations by van Raamsdonk and Severi and by Xi, seems to be a matter of taste.

3.3. Some Perpetual β-Reduction Strategies

The following strategy is due to Bergstra and Klop [7].

3.12. Definition. Define $F_1 \colon \infty_\beta \to \Lambda_K$ by:

$$
\begin{array}{lcll}
F_1(x\mathbf{P}Q\mathbf{R}) & = & x\mathbf{P}\, F_1(Q)\, \mathbf{R} & \text{if} \quad \mathbf{P} \in \mathrm{SN}_\beta, \quad Q \notin \mathrm{SN}_\beta \\
F_1(\lambda x.P) & = & \lambda x.F_1(P) & \\
F_1((\lambda x.P)\, Q\mathbf{R}) & = & P\{x := Q\}\, \mathbf{R} & \text{if} \quad Q \in \mathrm{SN}_\beta \\
F_1((\lambda x.P)\, Q\mathbf{R}) & = & (\lambda x.P)\, F_1(Q)\mathbf{R} & \text{if} \quad Q \notin \mathrm{SN}_\beta.
\end{array}
$$

3.13. *Remark.* For every $M \in \infty_\beta$ either $M \equiv xP_1 \cdots P_n$, where $n \geqslant 1$ and $P_i \in \infty_\beta$ for some i, or $M \equiv \lambda x.P$, or $M \equiv (\lambda x.P_0)\, P_1 \cdots P_n$, where $n \geqslant 1$. It follows that F_1 is defined on all elements of ∞_β.

3.14. Proposition. *F_1 is a partial, perpetual β-reduction strategy.*

Proof. By induction on the size of M prove that $M \in \infty_\beta \Rightarrow F_1(M) \in \infty_\beta$; the only non-trivial case is when $M \equiv (\lambda x.P)\, Q\mathbf{R}$ and $Q \in \mathrm{SN}_\beta$, in which case use Corollary 3.2. ▮

The following strategy is a variant of a strategy in [67].

3.15. Definition. Define $F_2\colon \infty_\beta \to \Lambda_K$ by:

$$
\begin{array}{llll}
F_2(x\mathbf{P}Q\mathbf{R}) & = x\mathbf{P}F_2(Q)\,\mathbf{R} & \text{if} & \mathbf{P}\in \mathrm{SN}_\beta,\quad Q\notin \mathrm{SN}_\beta \\
F_2(\lambda x.P) & = \lambda x.F_2(P) & & \\
F_2((\lambda x.P)\,Q\mathbf{R}) & = P\{x:=Q\}\,\mathbf{R} & \text{if} & P\in \mathrm{SN}_\beta,\quad Q\in \mathrm{SN}_\beta \\
F_2((\lambda x.P)\,Q\mathbf{R}) & = (\lambda x.F_2(P))\,Q\mathbf{R} & \text{if} & P\notin \mathrm{SN}_\beta \\
F_2((\lambda x.P)\,Q\mathbf{R}) & = (\lambda x.P)\,F_2(Q)\,\mathbf{R} & \text{if} & P\in \mathrm{SN}_\beta,\quad Q\notin \mathrm{SN}_\beta
\end{array}
$$

3.16. Proposition. *F_2 is a partial, perpetual β-reduction strategy.*

Proof. By induction on the size of M prove that $M\in\infty_\beta \Rightarrow F_2(M)\in\infty_\beta$; the only non-trivial case is when $M\equiv(\lambda x.P)\,Q\mathbf{R}$ and $P, Q\in \mathrm{SN}_\beta$, in which case use Corollary 3.2. ∎

3.4. The Fundamental Lemma of Maximality

The following lemma is used in some of the strong normalization proofs in the literature which, in addition to proving strong normalization, establish upper bounds for the length of reduction paths—see Section 5.

3.17. Definition. Define for any variable x the map $\notin_x\colon \Lambda_K \to \{0, 1\}$ by

$$
\notin_x(M) = \begin{cases} 1 & \text{if} \quad x\notin \mathrm{FV}(M) \\ 0 & \text{if} \quad x\in \mathrm{FV}(M). \end{cases}
$$

3.18. Lemma (Fundamental Lemma of Maximality). *For all $n\geqslant 1$,*

$$
l_\beta((\lambda x.M_0)\,M_1\cdots M_n) = l_\beta(M_0\{x:=M_1\}\,M_2\cdots M_n) + \notin_x(M_0)\cdot l_\beta(M_1) + 1.
$$

Proof. If $l_\beta((\lambda x.M_0)\,M_1\cdots M_n)=\infty$, then by Lemma 2.7 and the fundamental lemma of perpetuality, also $l_\beta(M_0\{x:=M_1\}\,M_2\cdots M_n)=\infty$ or $\notin_x(M_1)\cdot l_\beta(M_1)=\infty$. Thus, in this case the equality holds.

If $l_\beta((\lambda x.M_0)\,M_1M_2\cdots M_n)<\infty$, then $M_0, ..., M_n\in \mathrm{SN}_\beta$ by Lemma 2.7. We consider two cases.

1. $x\notin \mathrm{FV}(M_0)$. A longest reduction from $(\lambda x.M_0)\,M_1\cdots M_n$ has the form

$$
\begin{aligned}
(\lambda x.M_0)\,M_1\cdots M_n &\twoheadrightarrow^m_\beta (\lambda x.M'_0)\,M'_1\cdots M'_n \\
&\to_\beta M'_0M'_2\cdots M'_n \\
&\twoheadrightarrow^k_\beta K\in \mathrm{NF}_\beta,
\end{aligned}
$$

where $M_0 \twoheadrightarrow^{m_0}_\beta M'_0, ..., M_n \twoheadrightarrow^{m_n}_\beta M'_n$, and where $m_0+\cdots+m_n=m$, $l_\beta(M_1)=m_1$, and $l_\beta((\lambda x.M_0)\,M_1\cdots M_n)=m+k+1$. Then

$$\begin{array}{rll} (\lambda x.M_0)\, M_1 \cdots M_n & \twoheadrightarrow_\beta^{m_1} & (\lambda x.M_0)\, M_1'\, M_2 \cdots M_n \\ & \rightarrow_\beta & M_0 M_2 \cdots M_n \\ & \twoheadrightarrow_\beta^{m-m_1} & M_0' M_2' \cdots M_n' \\ & \twoheadrightarrow_\beta^{k} & K \in \mathrm{NF}_\beta \end{array}$$

is another longest reduction path from $(\lambda x.M_0)\, M_1 \cdots M_n$. Thus, $M_0 M_2 \cdots M_n \twoheadrightarrow_\beta^{m-m_1+k} K$ is also a longest reduction path from $M_0 M_2 \cdots M_n$, i.e., $l_\beta(M_0 M_2 \cdots M_n) = m - m_1 + k$. Thus,

$$\begin{array}{rcl} l_\beta((\lambda x.M_0)\, M_1 \cdots M_n) & = & m+k+1 \\ & = & (m-m_1+k)+m_1+1 \\ & = & l_\beta(M_0 M_2 \cdots M_n) + l_\beta(M_1) + 1. \end{array}$$

2. $x \in \mathrm{FV}(M_0)$. A longest reduction from $(\lambda x.M_0)\, M_1 \cdots M_n$ has the form

$$\begin{array}{rl} (\lambda x.M_0)\, M_1 \cdots M_n & \twoheadrightarrow_\beta^{m} (\lambda x.M_0')\, M_1' \cdots M_n' \\ & \rightarrow_\beta M_0'\{x := M_1'\}\; M_2' \cdots M_n' \\ & \twoheadrightarrow_\beta^{k} K \in \mathrm{NF}_\beta, \end{array}$$

where $M_0 \twoheadrightarrow_\beta^{m_0} M_0', ..., M_n \twoheadrightarrow_\beta^{m_n} M_n'$, and where $m_0 + \cdots + m_n = m$ and $l_\beta((\lambda x.M_0)\, M_1 \cdots M_n) = m+k+1$. Since

$$M \twoheadrightarrow_\beta^{m} M' \;\&\; N \twoheadrightarrow_\beta^{n} N' \Rightarrow M\{x := N\} \twoheadrightarrow_\beta^{m+n \cdot \|M\|_x} M'\{x := N'\}$$

also

$$\begin{array}{rll} (\lambda x.M_0)\, M_1 \cdots M_n & \rightarrow_\beta & M_0\{x := M_1\}\; M_2 \cdots M_n \\ & \twoheadrightarrow_\beta^{m_0+m_1 \cdot \|M_0\|_x} & M_0'\{x := M_1'\}\; M_2 \cdots M_n \\ & \twoheadrightarrow_\beta^{m_2+\cdots+m_n} & M_0'\{x := M_1'\}\; M_2' \cdots M_n' \\ & \twoheadrightarrow_\beta^{k} & K \in \mathrm{NF}_\beta. \end{array}$$

Since $\|M_0\|_x \geqslant 1$, $m_0 + m_1 \cdot \|M_0\|_x + m_2 \cdots + m_n + k + 1 \geqslant m+k+1$, so this is, in fact, another longest reduction from $(\lambda x.M_0)\, M_1 \cdots M_n$, so $l_\beta(M_0\{x := M_1\}\; M_2 \cdots M_n) = m_0 + m_1 \cdot \|M_0\|_x + m_2 \cdots + m_n + k$. Thus,

$$\begin{array}{rcl} l_\beta((\lambda x.M_0)\, M_1 \cdots M_n) & = & m_0 + m_1 + \cdots + m_n + k + 1 \\ & \leqslant & m_0 + m_1 \cdot \|M_0\|_x + m_2 + \cdots + m_n + k + 1 \\ & = & l_\beta(M_0\{x := M_1\}\; M_2 \cdots M_n) + 1 \end{array}$$

The converse inequality is trivial. ∎

3.19. COROLLARY. *For all $n \geqslant 1$,*

$$l_\beta((\lambda x.M_0)\, M_1 \cdots M_n) \leqslant l_\beta(M_0\{x := M_1\}\, M_2 \cdots M_n) + l_\beta(M_1) + 1.$$

Proof. By the fundamental lemma of maximality. ▮

3.20. *Remark.* The fundamental lemma of perpetuality and its corollary are special cases of the fundamental lemma of maximality and its corollary, respectively.

3.5. *An Effective Maximal Strategy*

The following strategy is due to Barendregt *et al.* [2, 4].

3.21. DEFINITION. Define $F_\infty : \Lambda_K \to \Lambda_K$ as follows. If $M \in \mathrm{NF}_\beta$ then $F_\infty(M) = M$; otherwise

$$\begin{array}{lcll}
F_\infty(x\mathbf{P}Q\mathbf{R}) & = & x\mathbf{P}F_\infty(Q)\,\mathbf{R} & \text{if}\quad \mathbf{P} \in \mathrm{NF}_\beta,\quad Q \notin \mathrm{NF}_\beta \\
F_\infty(\lambda x.P) & = & \lambda x.F_\infty(P) & \\
F_\infty((\lambda x.P)\, Q\mathbf{R}) & = & P\{x := Q\}\,\mathbf{R} & \text{if}\quad x \in \mathrm{FV}(P) \quad \text{or}\quad Q \in \mathrm{NF}_\beta \\
F_\infty((\lambda x.P)\, Q\mathbf{R}) & = & (\lambda x.P)\, F_\infty(Q)\mathbf{R} & \text{if}\quad x \notin \mathrm{FV}(P) \quad \text{and}\quad Q \notin \mathrm{NF}_\beta.
\end{array}$$

The following theorem has been folklore for some time. De Vrijer [79, 81] uses F_∞ to calculate the maximal length of a reduction path of a simply typed λ-term. In fact, the proof of [79, Theorem 4.9] shows that F_∞ is maximal—see also [79, 2.3.3 and 4.9.2], and the discussion of related work in Section 5. Later, the theorem was proved independently by Regnier [60], Khasidashvili [34], van Raamsdonk and Severi [59], and Sørensen [66]. The proof below is a simplification of the two latter proofs.

3.22. THEOREM. *F_∞ is an effective, maximal β-reduction strategy.*

Proof. It is clear that F_∞ is an effective β-reduction strategy. To prove maximality we use the formulation from Lemma 2.14. Given $M \in \Lambda_K$ and $m \geqslant 1$, we must show that $M \in m_\beta \Rightarrow F_\infty(M) \in (m-1)_\beta$. We proceed by induction on M.

1. $M \equiv x\mathbf{P}Q\mathbf{R}$, where $\mathbf{P} \in \mathrm{NF}_\beta$, $Q \notin \mathrm{NF}_\beta$. Let $\mathbf{R} = R_1, ..., R_n$. Then $Q \in m^0_\beta$, $R_1 \in m^1_\beta, ..., R_n \in m^n_\beta$, where $m = m^0 + m^1 + \cdots + m^n$, and $m^0 \geqslant 1$. By the induction hypothesis, $F_\infty(Q) \in (m^0 - 1)_\beta$. Then $F_\infty(M) = x\mathbf{P}F_\infty(Q)\,\mathbf{R} \in (m-1)_\beta$.

2. $M \equiv \lambda x.P$. Similar to Case 1.

3. $M \equiv (\lambda x.P)\, Q\mathbf{R}$, where $x \in \mathrm{FV}(P)$ or $Q \in \mathrm{NF}_\beta$. By the fundamental lemma of maximality,

$$l_\beta(P\{x := Q\}\,\mathbf{R}) + 1 = l_\beta(M) \geqslant m.$$

Therefore, $l_\beta(P\{x := Q\}\,\mathbf{R}) \geqslant m - 1$, i.e., $F_\infty(M) \in (m-1)_\beta$.

4. $M \equiv (\lambda x.P)\, Q\mathbf{R}$, where $x \notin \mathrm{FV}(P)$ and $Q \notin \mathrm{NF}_\beta$. By the fundamental lemma of maximality,

$$l_\beta(P\mathbf{R}) + l_\beta(Q) + 1 = l_\beta(M) \geqslant m.$$

We consider two cases.

4.1. $Q \in \infty_\beta$. Then, for any $n \geqslant 1$, $Q \in n_\beta$. By the induction hypothesis, for any $n \geqslant 1$, $F_\infty(Q) \in (n-1)_\beta$. In particular, $F_\infty(Q) \in (m-1)_\beta$, and then $F_\infty(M) \in (m-1)_\beta$.

4.2. $Q \notin \infty_\beta$. Then $l_\beta(Q) < \infty$ by Lemma 2.7. By the induction hypothesis, $l_\beta(F_\infty(Q)) \geqslant l_\beta(Q) - 1$. Then

$$\begin{aligned} l_\beta(F_\infty(M)) &= l_\beta((\lambda x.P)\, F_\infty(Q)\mathbf{R}) \\ &= l_\beta(P\mathbf{R}) + l_\beta(F_\infty(Q)) + 1 \\ &\geqslant l_\beta(P\mathbf{R}) + l_\beta(Q) \\ &= l_\beta(M) - 1 \\ &\geqslant m - 1. \end{aligned}$$

Thus, $F_\infty(M) \in (m-1)_\beta$. ∎

3.23. COROLLARY (Barendregt *et al.* [2, 4]). *F_∞ is perpetual.*

3.24. *Remark.* As pointed out by van Raamsdonk and Severi [59], the proof in [2, 4] of this corollary can be simplified by using the fundamental lemma of perpetuality or one of the related characterizations.

Khasidashvili [34] studies so-called *limit* reduction strategies in orthogonal expression reduction systems (of which β-reduction on Λ_K is a special case), and shows that any limit reduction strategy is maximal and that F_∞ is a limit reduction strategy in λ-calculus. Sørensen [66] presents a $\beta\eta$-reduction strategy H_∞ and shows that it is $\beta\eta$-maximal and thereby $\beta\eta$-perpetual.

3.6. On Upper Bounds for Lengths of Reductions

One can effectively compute upper bounds for the length of longest developments and longest reduction paths in several typed λ-calculi (see Sections 5 and 6). This raises the question whether there is some formula for upper bounds for lengths of reduction paths in type-free λ-calculus. In this subsection we give a positive and a negative answer to this question.

The following definition gives the most obvious way of counting the number of steps in a longest reduction to normal form.

3.25. DEFINITION. Define $h\colon \mathrm{SN}_\beta \to \mathbb{N}$ by

$$\begin{array}{lcll} h(xP_1 \cdots P_n) & = & h(P_1) + \cdots + h(P_n) & \\ h(\lambda x.P) & = & h(P) & \\ h((\lambda x.P)\, Q\mathbf{R}) & = & h(P\{x := Q\}\, \mathbf{R}) + 1 & \text{if} \quad x \in \mathrm{FV}(P) \quad \text{or} \quad Q \in \mathrm{NF}_\beta \\ h((\lambda x.P)\, Q\mathbf{R}) & = & h(P\mathbf{R}) + h(Q) + 1 & \text{if} \quad x \notin \mathrm{FV}(P) \quad \text{and} \quad Q \notin \mathrm{NF}_\beta. \end{array}$$

3.26. PROPOSITION. *For any $M \in \mathrm{SN}_\beta$: $h(M) = l_\beta(M)$.*

Proof. By induction on $l_\beta(M)$ using the fundamental lemma of maximality. ∎

The map h is defined only for elements in SN_β. It is natural to ask whether there is a "simple formula" f such that $f(M)$ is the length of a longest β-reduction from M when $M \in \mathrm{SN}_\beta$, and $f(M)$ is some unpredictable number when $M \in \infty_\beta$. One could hope that the freedom to return arbitrary values on terms with infinite reductions could give a simple formula on strongly normalizing terms. A reasonable formalization of "simple formula" is the notion of a primitive recursive function. The following proposition, which answers a more general question, shows that our hopes are in vain.

3.27. PROPOSITION. *There is no total effective $l\colon \Lambda_K \to \mathbb{N}$ such that, for all $M \in \mathrm{SN}_\beta$,*

$$l(M) \geqslant l_\beta(M).$$

Proof. Suppose that such an l existed and consider $c\colon \Lambda_K \to \mathbb{N}$:

$$c(M) = \begin{cases} 0 & \text{if} \quad F_\infty^{l(M)}(M) \in \mathrm{NF}_\beta \\ 1 & \text{if} \quad F_\infty^{l(M)}(M) \notin \mathrm{NF}_\beta. \end{cases}$$

Here c is total and effective. Consider the following two cases.

1. $c(M) = 0$. Then $F_\infty^{l(M)}(M) \in \mathrm{NF}_\beta$, i.e., $L_{F_\infty}(M) \leqslant l(M) < \infty$, so $M \in \mathrm{SN}_\beta$ by perpetuality of F_∞.

2. $c(M) = 1$. Then $F_\infty^{l(M)}(M) \notin \mathrm{NF}_\beta$. By maximality of F_∞ it follows that $l_\beta(M) = L_{F_\infty}(M) > l(M)$. By definition of l, $M \notin \mathrm{SN}_\beta$.

Thus, c gives a procedure to decide for any M whether $M \in \mathrm{SN}_\beta$, which is known to be impossible, a contradiction. ∎

4. THE Ω-THEOREM

In the type-free λ-calculus some terms have an infinite reduction path. The simplest example is the term $\Omega \equiv \omega\, \omega$, where $\omega \equiv \lambda x.x\, x$. It has an infinite reduction path where the term reduces to itself in every step:

$$\Omega \to_\beta \Omega \to_\beta \cdots.$$

There are terms that do have an infinite reduction path, but where the path does not have this simple form.[2] For instance, the term $\Psi \equiv \psi\ \psi$, where $\psi \equiv \lambda x.x\,x\,y$, has the infinite reduction path

$$\Psi \to_\beta \Psi\, y \to_\beta \Psi\, y\, y \to_\beta \cdots.$$

In every step the redex Ψ appears as a subterm, and the context of the redex is extended with an application $\bullet y$. As a more complicated example consider the term $\upsilon\, y\, \upsilon$, where $\upsilon \equiv \lambda a.\lambda x.x(a\, y)x$. It has the infinite reduction path

$$\upsilon y \upsilon \to_\beta (\lambda x.x(y\,y)x)\upsilon \to_\beta \upsilon(y\,y)\upsilon \to_\beta (\lambda x.x(yyy)x)\upsilon \to_\beta \upsilon(yyy)\upsilon \to_\beta \cdots.$$

This path is similar to the preceding one, but the extra application $\bullet y$ is added *inside* the redex.

Although these three reduction paths have their differences they have a common property: in all three paths every term has Ω as a substring. It is natural to ask whether this property is shared by all infinite reduction paths. In this section we present the *Ω-theorem*, due to Sørensen [67], which states that this is indeed the case. The proof exploits perpetuality of the strategy F_2 from Section 3.3.

The first subsection introduces the set of all terms that do not have Ω as substring, and the second subsection shows that the elements of this set are strongly normalizing. The third subsection studies applications.

4.1. *The Set* Λ_Ω

We first formalize what it means that one term is a substring of another.

4.1. Definition. Define the relation $\trianglelefteq$ ("substring") on Λ_K by

$$\begin{array}{lll}
x \trianglelefteq x & & \\
P \trianglelefteq Q & \Rightarrow & P \trianglelefteq \lambda x.Q \quad \text{if} \quad x \notin \mathrm{FV}(P) \\
P \trianglelefteq Q & \Rightarrow & P \trianglelefteq Q\,Z \\
P \trianglelefteq Q & \Rightarrow & P \trianglelefteq Z\,Q \\
P \trianglelefteq Q & \Rightarrow & \lambda x.P \trianglelefteq \lambda x.Q \\
P_1 \trianglelefteq Q_1\ \&\ P_2 \trianglelefteq Q_2 & \Rightarrow & P_1 P_2 \trianglelefteq Q_1 Q_2.
\end{array}$$

Example 4.2. (i) $\omega \trianglelefteq \lambda x.x\,x\,Z$;

(ii) $\Omega \trianglelefteq (\lambda x.x\,x\,Z)(\lambda.x\,x\,Z)$;

(iii) $\omega \trianglelefteq \lambda a.\lambda x.x\,Z\,x$;

(iv) $\Omega \trianglelefteq (\lambda a.\lambda x.x\,Z\,x)\,Z(\lambda a.\lambda x.x\,Z\,x)$;

(v) $\omega \trianglelefteq \lambda x.x(\lambda y.x)$;

(vi) $\lambda x.x\,y \not\trianglelefteq (\lambda x.x)\ y$;

[2] Lercher [42] shows that $M \to_\beta M$ iff $M \equiv C[\Omega]$ for some context C.

(vii) $\lambda x.y \not\trianglelefteq (\lambda x.x)\, y$;

(viii) $\lambda x.x\, y \not\trianglelefteq (\lambda x.x)\, y$;

(ix) $\omega \not\trianglelefteq \lambda x.x(\lambda x.x)$;

(x) $\Omega \not\trianglelefteq \lambda x.(x\, x)\omega$.

It is convenient to introduce an inductively defined set Λ_Ω of all terms that do not contain Ω as a substring, and show that all elements of this set are strongly normalizing. The following auxiliary set Λ_ω, studied by Komori [40], Hindley [22], and Jacobs [28], is the set of all terms that do not contain ω as a substring.

4.3. Definition. (i) Define Λ_ω by

$$\begin{array}{lll} x \in \Lambda_\omega & & \\ P \in \Lambda_\omega, \quad \|P\|_x \leqslant 1 & \Rightarrow & \lambda x.P \in \Lambda_\omega \\ P, Q \in \Lambda_\omega & \Rightarrow & P\, Q \in \Lambda_\omega. \end{array}$$

(ii) Define, for $M \in \Lambda_K$, $\|M\|_\omega \in \mathbb{N}$ by

$$\begin{array}{lll} \|x\|_\omega & = & 0 \\ \|\lambda x.P\|_\omega & = & \begin{cases} \|P\|_\omega & \text{if} \quad \|P\|_x \leqslant 1 \\ 1 + \|P\|_\omega & \text{if} \quad \|P\|_x > 1 \end{cases} \\ \|P\, Q\|_\omega & = & \|P\|_\omega + \|Q\|_\omega. \end{array}$$

(iii) An abstraction $\lambda x.P$ is *duplicating* if $\|P\|_x > 1$.

4.4. *Remark.* The following equivalences are easily established:

$$\|M\|_\omega = 0 \Leftrightarrow M \in \Lambda_\omega \Leftrightarrow \omega \not\trianglelefteq M.$$

Each of these equivalent conditions state that M does not contain a subterm which is a duplicating abstraction.

One easily shows that Λ_ω is closed under reduction. The intuition is that if $M \in \Lambda_\omega$ and $N \notin \Lambda_\omega$, then M has no duplicating abstractions while N has at least one. Thus, the reduction $M \to_\beta N$ must duplicate a variable in the body of some abstraction, but this would require a duplicating abstraction in M. It is also easy to prove that reduction in Λ_ω decreases term size, since every step removes an application and an abstraction. With the preceding property this implies that every term in Λ_ω is strongly normalizing.

4.5. Definition. Define the set Λ_Ω as

$$\begin{array}{llll} (1) & x \in \Lambda_\Omega & & \\ (2) & M \in \Lambda_\Omega & \Rightarrow & \lambda x.M \in \Lambda_\Omega \\ (3) & M \in \Lambda_\Omega, \quad N \in \Lambda_\omega & \Rightarrow & M\, N \in \Lambda_\Omega \\ (4) & M \in \Lambda_\omega, \quad N \in \Lambda_\Omega & \Rightarrow & M\, N \in \Lambda_\Omega. \end{array}$$

4.6. *Remark.* It is easy to show $\Lambda_\omega \subseteq \Lambda_\Omega$ and the following equivalence:

$$M \in \Lambda_\Omega \Leftrightarrow \Omega \ntrianglelefteq M.$$

Informally, these two equivalent conditions state that M does not contain two disjoint subterms that are both duplicating abstractions.

Next we show that Λ_Ω is closed under reduction. The intuition is as follows. If $M \in \Lambda_\Omega$ and $N \notin \Lambda_\Omega$, then M has no disjoint duplicating abstractions, while N has at least two. If $M \to_\beta N$ then non-disjoint duplicating abstractions in M are also non-disjoint in N. Therefore, the two disjoint duplicating abstractions in N must arise from M either by duplication into disjoint positions of a single duplicating abstraction or by duplication of a variable in the body of a non-duplicating abstraction which is disjoint with a duplicating abstraction. Both cases are impossible because they entail that M has two disjoint duplicating abstractions.

4.7. Lemma. $M \in \Lambda_\Omega \;\&\; M \to_\beta N \Rightarrow N \in \Lambda_\Omega$.

Proof. First prove by induction on the derivation of $M \in \Lambda_\omega$ that

$$M \in \Lambda_\omega \;\&\; N \in \Lambda_\omega \Rightarrow M\{x := N\} \in \Lambda_\omega \tag{1}$$

$$M \in \Lambda_\omega \;\&\; \|M\|_x \leqslant 1 \;\&\; N \in \Lambda_\Omega \Rightarrow M\{x := N\} \in \Lambda_\Omega. \tag{2}$$

Show by induction on the derivation of $M \in \Lambda_\Omega$, using (1) and $\Lambda_\omega \subseteq \Lambda_\Omega$,

$$M \in \Lambda_\Omega, \quad N \in \Lambda_\omega \Rightarrow M\{x := N\} \in \Lambda_\Omega. \tag{4.3}$$

Now proceed by induction on the derivation of $M \to_\beta N$ using (2) and (3). ∎

4.2. *Strong Normalization of Terms in Λ_Ω*

As for Λ_ω, the idea for proving that all terms in Λ_Ω are strongly normalizing is to find a decreasing measure, but term size $\|\bullet\|$ does not work. Instead we consider the lexicographically ordered measure $\langle \|\bullet\|_\omega, \|\bullet\| \rangle$.

Suppose $M \to_\beta N$ by contraction of the redex $\Delta \equiv (\lambda x.P)Q$. If $\lambda x.P$ is non-duplicating, contraction of Δ creates no new duplicating abstractions. Moreover, the size of N is strictly smaller than the size of M, so the reduction step decreases the measure.

If $\lambda x.P$ is duplicating, the reduction step removes one duplicating abstraction, and any new duplicating abstractions have to come either from proliferation of duplicating abstractions in Q or from duplication of variables in the body of some abstraction. The first case is impossible, since it implies that M has two disjoint duplicating abstractions. In the second case, new duplicating abstractions *may* be created, but they must have their λ to the left of Δ.

Recall that a standard reduction path $M_0 \to_\beta M_1 \to_\beta \cdots$ is such that whenever a redex Δ is contracted in M_i all abstractions to the left of Δ are marked, and a redex with marked abstraction is not allowed to be contracted in M_j for any $j > i$. If a term has an infinite reduction path, then it has a standard infinite reduction path [7].

The idea then is as follows. Suppose that some $M \in \Lambda_\Omega$ has an infinite reduction path and hence a standard infinite reduction path. Then the measure $\langle \| \bullet \|, \| \bullet \| \rangle$ is decreasing on this reduction path if we insist that $\| \bullet \|_\omega$ count only non-marked abstractions, and thus we arrive at a contradiction.

To formalize this reasoning we use the strategy F_2 from Section 3.3, which computes standard infinite reductions. The following map V isolates the part of a term in which F_2 contracts a redex. This part of the term contains all the abstractions to be counted by our measure.

4.8. Definition. Define $V\colon \infty_\beta \to \Lambda_K$ by

$$\begin{array}{lcll}
V(x\,\mathbf{P}\,Q\,\mathbf{R}) & = & V(Q) & \text{if} \quad \mathbf{P} \in \mathrm{SN}_\beta, \quad Q \notin \mathrm{SN}_\beta \\
V(\lambda x.P) & = & V(P) & \\
V((\lambda x.P)\,Q\,\mathbf{R}) & = & (\lambda x.P)\,Q\mathbf{R} & \text{if} \quad P \in \mathrm{SN}_\beta, \quad Q \in \mathrm{SN}_\beta \\
V((\lambda x.P)\,Q\,\mathbf{R}) & = & V(P) & \text{if} \quad P \notin \mathrm{SN}_\beta \\
V((\lambda x.P)\,Q\,\mathbf{R}) & = & V(Q) & \text{if} \quad P \in \mathrm{SN}_\beta, \quad Q \notin \mathrm{SN}_\beta.
\end{array}$$

4.9. Lemma. *For all $M \in \infty_\beta$: $V(M) \subseteq M$.*

Proof. By induction on M. ∎

4.10. Lemma. *For all $M \in \infty_\beta$,*

$$V(M) = (\lambda y.K)\,L\,\mathbf{N}$$

for some K, L, $\mathbf{N} \in \Lambda_K$ with

$$V(F_2(M)) \subseteq K\{x := L\}\,\mathbf{N}.$$

Proof. Induction on M using perpetuality of F_2.

1. $M \equiv x\,\mathbf{P}\,Q\,\mathbf{R}$, where $\mathbf{P} \in \mathrm{SN}_\beta$, $Q \notin \mathrm{SN}_\beta$. By the induction hypothesis,

$$V(M) = V(Q) = (\lambda y.K)\,L\,\mathbf{N}$$

for some K, L, $\mathbf{N}$. By the induction hypothesis and perpetuality of F_2,

$$V(F_2(M)) = V(x\,\mathbf{P}\,F_2(Q)\,\mathbf{R}) = V(F_2(Q)) \subseteq K\{y := L\}\,\mathbf{N}.$$

2. $M \equiv \lambda x.P$. Similar to Case 1.
3. $M \equiv (\lambda y.P)\,Q\,\mathbf{R}$, where $P \in \mathrm{SN}_\beta$ and $Q \in \mathrm{SN}_\beta$. Then

$$V(M) = (\lambda x.P)\,Q\,\mathbf{R}$$

and by Lemma 4.9,

$$V(F_2(M)) = V(P\{x := Q\}\ \mathbf{R}) \subseteq P\{x := Q\}\ \mathbf{R}.$$

The remaining two cases are similar to Case 1. ∎

4.11. LEMMA. $\|P\{x := Q\}\|_\omega = \|P\|_\omega + \|P\|_x \cdot \|Q\|_\omega$.

Proof. By induction on P. ∎

4.12. PROPOSITION. $M \in \Lambda_{\Omega \Rightarrow M} \in \mathrm{SN}_\beta$.

Proof. Suppose that $M \in \Lambda_\Omega$ and $M \in \infty_\beta$. By perpetuality of F_2, there is an infinite reduction path

$$M_0 \to_\beta M_1 \to_\beta \cdots$$

such that for all i, $F_2(M_i) = M_{i+1}$ and, by Lemma 4.7, $M_i \in \Lambda_\Omega$. We now claim that for all i

$$\langle \|V(M_i)\|_\omega, \|V(M_i)\| \rangle > \langle \|V(M_{i+1})\|_\omega, \|V(M_{i+1})\| \rangle. \tag{4}$$

This implies that we have an infinite sequence

$$\langle \|V(M_0)\|_\omega, \|V(M_0)\| \rangle > \langle \|V(M_1)\|_\omega, \|V(M_1)\| \rangle > \cdots$$

which is clearly a contradiction. Thus $M \in \mathrm{SN}_\beta$, provided that we can prove (4).

To prove this, first note that by Lemmas 4.4 and 4.9,

$$V(M_i) = (\lambda y.K)\ L\ \mathbf{N} \subseteq M_i \tag{5}$$

$$V(M_{i+1}) \subseteq K\{y := L\}\ \mathbf{N} \tag{6}$$

for some K, L, $\mathbf{N}$. Since $(\lambda y.K)\ L \subseteq M_i \in \Lambda_\Omega$, also $(\lambda y.K)\ L \in \Lambda_\Omega$. We now prove (4) splitting into the following two cases. Let $\mathbf{N} = N_1, ..., N_n$.

1. $\|K\|_y > 1$. Then $\lambda y.K \in \Lambda_\Omega \backslash \Lambda_\omega$, so $L \in \Lambda_\omega$, and hence $\|L\|_\omega = 0$. By (6), Lemma 4.11, and (5),

$$\begin{aligned}
\|V(M_{i+1})\|_\omega &\leqslant \|K\{y := L\}\ \mathbf{N}\|_\omega \\
&= \|K\| + \|K\|_y \cdot \|L\|_\omega + \|N_1\|_\omega + \cdots + \|N_n\|_\omega \\
&= \|K\| + \|N_1\|_\omega + \cdots + \|N_n\|_\omega \\
&< \|\lambda y.K\|_\omega + \|N_1\|_\omega + \cdots + \|N_n\|_\omega \\
&= \|(\lambda y.K)\ L\ \mathbf{N}\|_\omega \\
&= \|V(M_i)\|_\omega.
\end{aligned}$$

2. $\|K\|_y \leqslant 1$. Then by (6), Lemma 4.11, and (5),

$$\begin{aligned}
\|V(M_{i+1})\|_\omega &\leqslant \|K\{y:=L\}\,\mathbf{N}\|_\omega \\
&= \|K\|_\omega + \|K\|_y \cdot \|L\|_\omega + \|N_1\|_\omega + \cdots + \|N_n\|_\omega \\
&\leqslant \|K\|_\omega + \|L\|_\omega + \|N_1\|_\omega + \cdots + \|N_n\|_\omega \\
&= \|(\lambda y.K)\,L\,\mathbf{N}\|_\omega \\
&= \|V(M_i)\|_\omega.
\end{aligned}$$

Moreover, by (6) and (5),

$$\begin{aligned}
\|V(M_{i+1})\| &\leqslant \|K\{y:=L\}\,\mathbf{N}\| \\
&= \|K\| + \|K\|_y \cdot (\|L\| - 1) + \|N_1\| + \cdots + \|N_n\| + n \\
&< \|K\| + \|L\| + 2 + \|N_1\| + \cdots + \|N_n\| + n \\
&= \|(\lambda y.K)\,L\,\mathbf{N}\| \\
&= \|V(M_i)\|
\end{aligned}$$

as required. ▮

We finally have the Ω-theorem, due to Sørensen [67]:

4.13. THEOREM. *If* $M \in \infty_\beta$ *then* $\Omega \trianglelefteq M$.

Proof. By Remark 4.6 and Proposition 4.12. ▮

4.14. *Remark.* The term $M \equiv (\lambda x.y\,x\,x)(\lambda x.y\,x\,x)$ shows that $\Omega \trianglelefteq M$ does not generally imply $M \in \infty_\beta$. This should come as no surprise: if $\Omega \trianglelefteq M$ had been equivalent to $M \in \infty_\beta$, we would have had a simple syntactic (in particular effective) algorithm for deciding whether $M \in \mathrm{SN}_\beta$, which is an undecidable problem.

Following Gramlich [20] (see also Plaisted [54]) we call an infinite reduction path *constricting* if it has the form

$$C_1[M_1] \to_\beta C_1[C_2[M_2]] \to_\beta C_1[C_2[C_3[M_3]]] \cdots,$$

where M_i is the minimal superterm with an infinite reduction path of the redex contracted in the step $C_1[\cdots C_i[M_i]\cdots] \to_\beta C_1[\cdots C_i[C_{i+1}[M_{i+1}]]\cdots]$.

Van Oostrom [50] sketches a variant of the above proof which, instead of using the perpetual strategy F_2 to obtain standard infinite reductions, uses a so-called *zoom-in* strategy (see Melliès [45]). This is a constricting strategy which in each term contracts the left-most redex of a minimal subterm with an infinite reduction path. The proof presented above is very similar, since F_2 is also constricting—indeed, Lemma 4.10 expresses a very similar property. However, in $(\lambda x.x)\,z\,\Omega$, F_2 contracts the left-most redex, so F_2 is not a zoom-in strategy in the above sense. The following variation F_3, studied by Sørensen [67], is a zoom-in strategy:

4.15. DEFINITION. Define $F_3\colon \infty_\beta \to \Lambda_K$ by

$$
\begin{array}{lcll}
F_3(x\,\mathbf{P}\,Q\,\mathbf{R}) & = & x\,\mathbf{P}\,F_3(Q)\,\mathbf{R} & \text{if } \ \mathbf{P}\in \mathrm{SN}_\beta, \quad Q\notin \mathrm{SN}_\beta \\
F_3(\lambda x.P) & = & \lambda x.F_3(P) & \\
F_3((\lambda x.P)\,Q\,\mathbf{R}) & = & P\{x:=Q\}\,\mathbf{R} & \text{if } \ P, Q, \mathbf{R}\in \mathrm{SN}_\beta \\
F_3((\lambda x.P)\,Q\,\mathbf{R}) & = & (\lambda x.F_3(P))\,Q\,\mathbf{R} & \text{if } \ P\notin \mathrm{SN}_\beta \\
F_3((\lambda x.P)\,\mathbf{R}\,Q\,\mathbf{S}) & = & (\lambda x.P)\,\mathbf{R}\,F_3(Q)\,\mathbf{S} & \text{if } \ P, \mathbf{R}\in \mathrm{SN}_\beta, \quad Q\notin \mathrm{SN}_\beta
\end{array}
$$

Khasidashvili and Ogawa [37] study strategies which in a term contract a so-called *external* redex of a minimal subterm of M with an infinite reduction path; in particular, in λ-terms the left-most redex of a minimal subterm with an infinite reduction is external. They show that any such strategy is perpetual. They also show that the strategy which in each step contracts the left-most among all such redexes is constricting.

Xi [84] calls a reduction path $M_0 \to_\beta M_1 \to_\beta \cdots$ *canonical* if whenever a redex Δ is contracted in M_i all redexes containing Δ as a subterm have their abstractions marked, and a redex with marked abstraction is not allowed to be contracted in M_j for any $j > i$. Any standard reduction is also canonical, but the converse is not true, since a canonical path may contract disjoint redexes from right to left. However, whenever a term M has a canonical reduction which is infinite (or ends in N) then M also has a standard reduction which is infinite (or ends in N). Xi uses canonical reductions to give proofs of the finite developments theorem, the standardization theorem, the conservation theorem for Λ_I, and the normalization theorem.

Böhm *et al.* [9] and Böhm and Dezani–Ciancaglini [10] give, for any β-normal form M, a constructive definition of a set of β-normal forms N for which $M\,N$ has a β-normal form. Since any λ-term can be transformed to an equivalent term which is an applicative combination of β-normal forms, this can be used to generally approximate whether a term has a β-normal form or not. On the other hand, Λ_Ω directly characterizes a class of terms with arbitrary nesting of λ's and application which are all β-strongly normalizing.

4.3. *Applications*

An *S-term* in combinatory logic is a term built of only the S-combinator and application, e.g., $S(SS)\,SSSS$ and $SSS(SS)\,SS$. Barendregt *et al.* [4] show that these two S-terms have infinite reduction paths. Duboué has verified by computer that the remaining 130 other S-terms with 7 or fewer occurrences of S are strongly normalizing. The following shows that only one among the 2622 closed λ-terms of size 9 or less has an infinite reduction path.[3]

[3] T. Mogensen gives a formula $f(n, m)$ for the number of λ-terms of size $n \geqslant 1$ with at most $m \geqslant 0$ free variables:

$$
\begin{array}{lcl}
f(1, m) & = & m \\
f(n+1, m) & = & f(n, m+1) + \sum\limits_{i=1}^{n-1} f(i, m)\cdot f(n-i, m)
\end{array}
$$

4.16. COROLLARY. *Let $M \in \infty_\beta$. Then*

(i) $\|\Omega\| \leqslant \|M\|$.

(ii) $\|M\| \leqslant \|\Omega\| \Rightarrow M \equiv \Omega$.

Proof. (i) By the Ω-theorem, since $O \trianglelefteq M$ clearly implies $\|O\| \leqslant \|M\|$.

(ii) By the Ω-theorem and (i) using the fact that $O \trianglelefteq M$ and $\|M\| \leqslant \|O\|$ implies $M \equiv O$. ∎

The next application gives a technique to reduce proofs that some term is strongly normalizing to proofs that terms are weakly normalizing. The latter is usually easier.

4.17. COROLLARY. *If $N \in \mathrm{WN}_\beta$ for all $N \trianglelefteq M$, then $M \in \mathrm{SN}_\beta$.*

Proof. If $M \in \infty_\beta$ then, by the Ω-theorem, $\Omega \trianglelefteq M$, and $\Omega \notin \mathrm{WN}_\beta$. ∎

The following shows how this corollary may be used to prove strong normalization of a set of terms.

4.18. PROPOSITION. *Let $S \subseteq \Lambda_K$ and let $\preccurlyeq$ be a relation on Λ_K with*

(i) *if $N \in \mathrm{WN}_\beta$ for all $N \preccurlyeq M$, then $M \in \mathrm{SN}_\beta$;*

(ii) *if $M \in S$ and $N \preccurlyeq M$ then $N \in S$.*

Then $S \subseteq \mathrm{WN}_\beta \Rightarrow S \subseteq \mathrm{SN}_\beta$.

Proof. Assume that $\preccurlyeq$ satisfies (i)–(ii) and assume $S \subseteq \mathrm{WN}_\beta$. Given an $M \in S$. By (ii), $N \in S$ for all $N \preccurlyeq M$. Then, by assumption, $N \in \mathrm{WN}_\beta$ for all $N \preccurlyeq M$. Then by (i), $M \in \mathrm{SN}_\beta$, as required. ∎

4.19. *Remark.* The previous result has motivated the search for relations satisfying (i)–(ii) for various sets S, notably the set $\Lambda^{\rightarrow}$ of terms typable in simply typed λ-calculus à la Curry (see Section 5). With such a relation at hand, one can show that all elements of $\Lambda^{\rightarrow}$ are strongly normalizing by demonstrating that they are all weakly normalizing.

As Corollary 4.17 shows, $\trianglelefteq$ satisfies (i). In fact, the proof of the corollary shows that any relation $\preccurlyeq$ satisfying $M \in \infty_\beta \Rightarrow \Omega \preccurlyeq M$ also satisfies (i). However, $\trianglelefteq$ does not satisfy (ii) for $\Lambda^{\rightarrow}$. For instance, $\lambda x.x(x\,\lambda y.y)$ has type $((\alpha \to \alpha) \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ in simply typed λ-calculus, but $\lambda x.\ x \trianglelefteq \lambda x.x(x\,\lambda y.y)$ has no type.

Sørensen [69] and Xi [83, 86] study relations $\preccurlyeq$ satisfying (i) and (ii) for $\Lambda^{\rightarrow}$ which are defined by translations, i.e., $M \preccurlyeq N$ iff $t(N) = M$ for certain translations $t\colon \Lambda_K \to \Lambda_I$.

4.20. *Problem.* Hindley [22] shows that $M \in \Lambda_\omega \Rightarrow M \in \Lambda^{\rightarrow}$; i.e., every $M \in \Lambda_\omega$ can be typed in simply typed λ-calculus à la Curry. Can every M in Λ_Ω be typed in second-order typed λ-calculus à la Curry?

5. STRONG NORMALIZATION IN TYPE THEORY

As mentioned in Section 3, many strong normalization proofs in the literature make use of the fundamental lemma of perpetuality or the fundamental lemma of maximality (see also Remarks 3.6 and 3.10). In this section we study such proofs in more detail in the context of the simply typed λ-calculus.

The first subsection presents the version of simply typed λ-calculus with which we shall be concerned. The second subsection presents a new proof of strong normalization of simply typed λ-calculus due to van Raamsdonk and Severi [59]. While their original proof uses their characterization of SN_β, the present version uses the fundamental lemma of perpetuality. Other proofs are reviewed in less detail.

5.1. Simply Typed λ-Calculus

5.1. Definition. Let T_0 be a set of constants, called *base types*. The set T of *simple types* is the smallest set such that

(i) $T_0 \subseteq T$;

(ii) $A, B \in T \Rightarrow A \to B \in T$.

For $A \in T$, $\|A\|$ denotes the number of arrows in A.

We use association to the right, so $A \to B \to C$ means $A \to (B \to C)$.

5.2. Convention. It is convenient to assume that the set V (the set of variables of Λ_K) is divided into mutually exclusive and together exhaustive non-empty classes V_A, where $A \in T$, i.e.,

$$V = \bigcup_{A \in T} V_A \;\&\; A \neq B \Rightarrow V_A \neq V_B \;\&\; V_A \neq \emptyset.$$

5.3. Definition. For every $A \in T$, the set of *simply typed λ-terms of type A*, written $\Lambda_A^{\to}$, is the smallest set such that

(i) $x \in V_A \Rightarrow x \in \Lambda_A^{\to}$;

(ii) $x \in V_A \;\&\; M \in \Lambda_B^{\to} \Rightarrow \lambda x.M \in \Lambda_{A \to B}^{\to}$;

(iii) $M \in \Lambda_{B \to A}^{\to} \;\&\; N \in \Lambda_B^{\to} \Rightarrow M\,N \in \Lambda_A^{\to}$.

The set of *simply typed λ-terms*, written $\Lambda^{\to}$, is defined by

$$\Lambda^{\to} = \bigcup_{A \in T} \Lambda_A^{\to}.$$

The following two properties, known as the *substitution lemma* and the *uniqueness of types property*, will be used in the next subsection.

5.4. Lemma. (i) $P \in \Lambda_B^{\to} \;\&\; x \in \Lambda_A^{\to} \;\&\; N \in \Lambda_A^{\to} \Rightarrow P\{x := N\} \in \Lambda_B^{\to}$;

(ii) $P \in \Lambda_A^{\to} \;\&\; P \in \Lambda_B^{\to} \Rightarrow A = B$.

Proof. (i)–(ii): by induction on the derivation of $P \in \Lambda_B^{\to}$. ∎

5.2. *Strong Normalization of Simply Typed λ-Calculus*

An attempt to prove directly, by induction on the derivation of $M \in \Lambda^{\rightarrow}_A$, that $M \in \mathrm{SN}_\beta$ breaks down in the application case, where $P \in \mathrm{SN}_\beta$ and $Q \in \mathrm{SN}_\beta$ does not imply $P\,Q \in \mathrm{SN}_\beta$. One way of overcoming this difficulty is to introduce the set $\mathrm{SN}^{\rightarrow}_A$ of strongly normalizing terms of type A and show that $M \in \Lambda^{\rightarrow}_A$ implies $M \in \mathrm{SN}^{\rightarrow}_A$. The crucial step then is to show for any $M \in \mathrm{SN}^{\rightarrow}_{A \rightarrow B}$ and $N \in \mathrm{SN}^{\rightarrow}_A$ that $M\,N \in \mathrm{SN}^{\rightarrow}_B$. This idea is carried out below, following [59].

5.5. Definition. For $A \in T$ define $\mathrm{SN}^{\rightarrow}_A = \mathrm{SN}_\beta \cap \Lambda^{\rightarrow}_A$, and

$$\mathrm{SN}^{\rightarrow} = \bigcup_{A \in T} \mathrm{SN}^{\rightarrow}_A .$$

5.6. *Remark.* For every type A: $\varnothing \subsetneq V_A \subseteq \mathrm{SN}^{\rightarrow}_A \subseteq \Lambda^{\rightarrow}_A$.

5.7. Definition. For $X, Y \subseteq \Lambda_K$ define

$$X \rightarrow Y = \{M \in \Lambda_K \mid \forall N \in X\colon M\,N \in Y\}.$$

5.8. Lemma. $\Lambda^{\rightarrow}_{A \rightarrow B} = \Lambda^{\rightarrow}_A \rightarrow \Lambda^{\rightarrow}_B$.

Proof. Let $M \in \Lambda^{\rightarrow}_{A \rightarrow B}$. For all $N \in \Lambda^{\rightarrow}_A$, $M\,N \in \Lambda^{\rightarrow}_B$, so $M \in \Lambda^{\rightarrow}_A \rightarrow \Lambda^{\rightarrow}_B$. Hence $\Lambda^{\rightarrow}_{A \rightarrow B} \subseteq \Lambda^{\rightarrow}_A \rightarrow \Lambda^{\rightarrow}_B$. Conversely, let $M \in \Lambda^{\rightarrow}_A \rightarrow \Lambda^{\rightarrow}_B$. Pick some $N \in \Lambda^{\rightarrow}_A$. Then $M\,N \in \Lambda^{\rightarrow}_B$. Therefore, $M \in \Lambda^{\rightarrow}_{C \rightarrow B}$ for some $C \in T$ with $N \in \Lambda^{\rightarrow}_C$. By uniqueness of types, $A = C$, so $M \in \Lambda^{\rightarrow}_{A \rightarrow B}$. Hence $\Lambda^{\rightarrow}_A \rightarrow \Lambda^{\rightarrow}_B \subseteq \Lambda^{\rightarrow}_{A \rightarrow B}$. ∎

5.9. Lemma. $\mathrm{SN}^{\rightarrow}_{A \rightarrow B} \supseteq \mathrm{SN}^{\rightarrow}_A \rightarrow \mathrm{SN}^{\rightarrow}_B$.

Proof. Let $M \in \mathrm{SN}^{\rightarrow}_A \rightarrow \mathrm{SN}^{\rightarrow}_B$. Pick some $N \in \mathrm{SN}^{\rightarrow}_A$. Then $M\,N \in \mathrm{SN}^{\rightarrow}_B$. In particular, $M\,N \in \mathrm{SN}_\beta$, and then $M \in \mathrm{SN}_\beta$. Moreover, since $M\,N \in \Lambda^{\rightarrow}_B$ and $N \in \Lambda^{\rightarrow}_A$, also $M \in \Lambda^{\rightarrow}_{A \rightarrow B}$ by uniqueness of types. In conclusion, $M \in \mathrm{SN}^{\rightarrow}_{A \rightarrow B}$. ∎

The converse of the preceding lemma is more difficult to prove. We need the following lemma.

5.10. Lemma. *Let* $P \in \mathrm{SN}^{\rightarrow}_B$, $x \in \Lambda^{\rightarrow}_{A_1 \rightarrow \cdots \rightarrow A_m}$, *and* $N \in \mathrm{SN}^{\rightarrow}_{A_1} \rightarrow \cdots \rightarrow \mathrm{SN}^{\rightarrow}_{A_m}$, *where* A_m *is a base type. Then* $P\{x := N\} \in \mathrm{SN}^{\rightarrow}_B$.

Proof. We use the abbreviation $L^* \equiv L\{x := N\}$ for any $L \in \Lambda^{\rightarrow}$. By Lemma 5.9, $N \in \mathrm{SN}^{\rightarrow}_{A_1 \rightarrow \cdots \rightarrow A_m}$. By the substitution lemma, $P^* \in \Lambda^{\rightarrow}_B$. It remains to show $P^* \in \mathrm{SN}_\beta$. We show this by induction on lexicographically ordered pairs $\langle l_\beta(P), \|P\| \rangle$.

1. $P \equiv y\,P_1 \cdots P_n$. Then $P_1, ..., P_n \in \mathrm{SN}_\beta$. Also, $y \in \Lambda^{\rightarrow}_{B_1 \rightarrow \cdots \rightarrow B_n \rightarrow B}$ and $P_1 \in \Lambda^{\rightarrow}_{B_1}, ..., P_n \in \Lambda^{\rightarrow}_{B_n}$, i.e., $P_1 \in \mathrm{SN}^{\rightarrow}_{B_1}, ..., P_n \in \mathrm{SN}^{\rightarrow}_{B_n}$. By the induction hypothesis, $P_1^*, ..., P_n^* \in \mathrm{SN}_\beta$. Consider two subcases.

1.1. $y \not\equiv x$. Then $P^* \equiv y\,P_1^* \cdots P_n^* \in \mathrm{SN}_\beta$.

1.2. $y \equiv x$. Then $B_1 = A_1, ..., B_n = A_n$ and $B = A_{n+1} \rightarrow \cdots \rightarrow A_m$. By Lemma 5.9, $\mathrm{SN}^{\rightarrow}_{A_{n+1}} \rightarrow \cdots \rightarrow \mathrm{SN}^{\rightarrow}_{A_m} \subseteq \mathrm{SN}^{\rightarrow}_B$. Therefore, $\mathrm{SN}^{\rightarrow}_{A_1} \rightarrow \cdots \rightarrow \mathrm{SN}^{\rightarrow}_{A_m} \subseteq \mathrm{SN}^{\rightarrow}_{A_1} \rightarrow \cdots \rightarrow \mathrm{SN}^{\rightarrow}_{A_n} \rightarrow \mathrm{SN}^{\rightarrow}_B$. So $N \in \mathrm{SN}^{\rightarrow}_{A_1} \rightarrow \cdots \rightarrow \mathrm{SN}^{\rightarrow}_{A_n} \rightarrow \mathrm{SN}^{\rightarrow}_B$. By the substitution lemma,

$P_1^* \in \Lambda_{B_1}^{\rightarrow}, ..., P_n^* \in \Lambda_{B_n}^{\rightarrow}$, i.e., $P_1^* \in \mathrm{SN}_{A_1}^{\rightarrow}, ..., P_n^* \in \mathrm{SN}_{A_n}^{\rightarrow}$. Therefore, $P^* \equiv NP_1^* \cdots P_n^* \in \mathrm{SN}_B^{\rightarrow}$.

2. $P \equiv \lambda y P_0$. Then $P_0 \in \mathrm{SN}_\beta$. Also, $B = B_1 \rightarrow B_0$ and $P_0 \in \Lambda_{B_0}^{\rightarrow}$, i.e., $P_0 \in \mathrm{SN}_{B_0}^{\rightarrow}$. By the induction hypothesis, $P_0^* \in \mathrm{SN}_\beta$. Therefore also $P^* \equiv \lambda y.P_0^* \in \mathrm{SN}_\beta$.

3. $P \equiv (\lambda y.P_0)\, P_1 P_2 \cdots P_n$. Then $P_0\{y := P_1\}\, P_2 \cdots P_n \in \mathrm{SN}_\beta$, $P_1 \in \mathrm{SN}_\beta$. Also, $P_1 \in \Lambda_{B_1}^{\rightarrow}, ..., P_n \in \Lambda_{B_n}^{\rightarrow}$, $y \in \Lambda_{B_1}^{\rightarrow}$, and $P_0 \in \Lambda_{B_2 \rightarrow \cdots \rightarrow B_n \rightarrow B}^{\rightarrow}$. By the induction hypothesis,

$$(P_0\{y := P_1\}\, P_2 \cdots P_n)^* \equiv P_0^*\{y := P_1^*\}\, P_2^* \cdots P_n^* \in \mathrm{SN}_\beta$$

and $P_1^* \in \mathrm{SN}_\beta$. Then $P^* \equiv (\lambda y.P_0^*)\, P_1^* P_2^* \cdots P_n^* \in \mathrm{SN}_\beta$, by the fundamental lemma of perpetuality. ∎

The following crucial lemma states that $M \in \mathrm{SN}_{A \rightarrow B}^{\rightarrow}$ and $N \in \mathrm{SN}_A^{\rightarrow}$ implies $M\,N \in \mathrm{SN}_B^{\rightarrow}$.

5.11. Lemma. $\mathrm{SN}_{A \rightarrow B}^{\rightarrow} \subseteq \mathrm{SN}_A^{\rightarrow} \rightarrow \mathrm{SN}_B^{\rightarrow}$.

Proof. We prove that $M \in \mathrm{SN}_{A \rightarrow B}^{\rightarrow}$ implies $M \in \mathrm{SN}_A^{\rightarrow} \rightarrow \mathrm{SN}_B^{\rightarrow}$. The proof is by induction on lexicographically ordered pairs $\langle \|A\|, l_\beta(M) \rangle$. For each $N \in \mathrm{SN}_A^{\rightarrow}$ we must prove that $M\,N \in \mathrm{SN}_B^{\rightarrow}$. Since obviously $M\,N \in \Lambda_B^{\rightarrow}$, it suffices to show in each case that $M\,N \in \mathrm{SN}_\beta$.

1. $M \equiv y\, P_1 \cdots P_n$. Then $P_1, ..., P_n \in \mathrm{SN}_\beta$. Since $N \in \mathrm{SN}_\beta$, it follows that $M\,N \equiv y\, P_1 \cdots P_n N \in \mathrm{SN}_\beta$.

2. $M \equiv \lambda x.P$. Then $P \in \mathrm{SN}_\beta$. Since $A = A_1 \rightarrow \cdots \rightarrow A_m$ for some base type A_m, the induction hypothesis yields $N \in \mathrm{SN}_{A_1}^{\rightarrow} \rightarrow \cdots \rightarrow \mathrm{SN}_{A_m}^{\rightarrow}$. Since $P \in \mathrm{SN}_B^{\rightarrow}$, Lemma 5.10 implies that $P\{x := N\} \in \mathrm{SN}_\beta$. Then $M\,N \equiv (\lambda x.P)\, N \in \mathrm{SN}_\beta$ by the fundamental lemma of perpetuality.

3. $M \equiv (\lambda y.P_0)\, P_1 P_2 \cdots P_n$. Then $P_0\{y := P_1 P_2 \cdots P_n \in \mathrm{SN}_\beta$ and also $P_1 \in \mathrm{SN}_\beta$. Since $P_0\{y := P_1\}\, P_2 \cdots P_n \in \Lambda_{A \rightarrow B}^{\rightarrow}$, the induction hypothesis yields $P_0\{y := P_1\}$ $P_2 \cdots P_n N \in \mathrm{SN}_\beta$. Since $P_1 \in \mathrm{SN}_\beta$, also $M\,N \equiv (\lambda y.P_0)\, P_1 P_2 \cdots P_n N \in \mathrm{SN}_\beta$ by the fundamental lemma of perpetuality. ∎

5.12. Theorem. *Let A be a simple type. If $M \in \Lambda_A^{\rightarrow}$ then $M \in \mathrm{SN}_\beta$.*

Proof. By induction on the derivation of $M \in \Lambda_A^{\rightarrow}$.

1. $M \equiv x \in V_A$. Then $x \in \mathrm{SN}_\beta$.

2. $M = \lambda x.P$, where $A = A_0 \rightarrow A_1$ and $P \in \Lambda_{A_1}^{\rightarrow}$. By the induction hypothesis, $P \in \mathrm{SN}_\beta$, and therefore $\lambda x.P \in \mathrm{SN}_\beta$.

3. $M \equiv P\,Q$, where $P \in \Lambda_{B \rightarrow A}^{\rightarrow}$ and $Q \in \Lambda_B^{\rightarrow}$. By the induction hypothesis, $P \in \mathrm{SN}_{B \rightarrow A}^{\rightarrow}$ and $Q \in \mathrm{SN}_B^{\rightarrow}$. By Lemma 5.11, $P \in \mathrm{SN}_B^{\rightarrow} \rightarrow \mathrm{SN}_A^{\rightarrow}$. Then $P\,Q \in \mathrm{SN}_A^{\rightarrow} \subseteq \mathrm{SN}_\beta$. ∎

5.13. *Remark.* A similar technique for handling the difficult application case is due to Xi [82].

There are many other proofs of strong normalization of simply typed λ-calculus. The following is an incomplete list. Tait [73] proves (strong) normalization of Gödel's system T, which extends simply typed λ-calculus with primitive recursion. The proof makes use of the notion of (strong) computability and is quite short but complex. The proof uses the fundamental lemma of perpetuality to show that the set of (strongly) computable terms is closed under certain expansions—see, e.g., [23, Appendix 2, Lemma 2].

Girard [19] introduces the notion of candidate of reducibility. He extends Tait's method in order to prove strong normalization of second- and higher-order λ-calculus. In the version of this proof technique expressed in terms of saturated sets, the fundamental lemma of perpetuality is used to show that SN_β is a saturated set—see, e.g., [3, Lemma 4.3.3].

Terlouw, [76] interprets Tait's proof of strong normalization of simply typed λ-calculus in a general model-theoretic framework. This yields a proof of strong normalization of the Calculus of Constructions and other advanced type systems.

Gandy [17] interprets a term in a typed λ-calculus by a strict monotonic functional whose value is an upper bound for the length of reductions from the term —the form of the upper bound is elaborated by Schwichtenberg [62]. Gandy's technique uses implicitly the weak form of the fundamental lemma of maximality (Corollary 3.19). The technique is generalized to higher-order rewrite systems by van de Pol [55] and applied to a variety of systems by van de Pol and Schwichtenberg [57]. Van de Pol [56] discusses the relationship between the proof by Gandy and the proof by Tait.

De Vrijer [79, 81] proves strong normalization of simply typed λ-calculus by translating terms into functionals computing the exact length of the longest reduction path to normal form, and shows that F_∞ computes this path. De Vrijer's proof uses the fundamental lemma of maximality—see the proof of [79, Theorem 4.9], and also [79, 2.3.3 and 4.9.2].

Another technique for computing upper bounds on lengths of reductions is due to Howard [25] which is used by Schwichtenberg [63] to give upper bounds for the length of reductions in simply typed λ-calculus. Whereas the bound h from Definition 3.25 implicitly reduces the term to normal form, i.e., $h((\lambda x.P)Q)$ is expressed in terms of $h(P\{x:=Q\})$, the bounds for reductions of simply typed terms can be expressed in such a way that the bound for $(\lambda x.P)Q$ is expressed in terms of the bounds for P and Q. This technique uses implicitly a version of the fundamental lemma of maximality—see the proof of the main lemma [63, p. 407]. Springintveld [71] applies the technique to the dependent system λP and to the weak version $\lambda\underline{\omega}$ of higher-order typed λ-calculus.

Xi [85] gives a proof of the standardization theorem which provides an upper bound on the length of the standard reduction path obtained from any given reduction path, and Xi uses this to provide upper bounds for the length of reduction paths in simply typed λ-calculus.

Van Daalen proves strong normalization of simply typed λ-calculus using induction on a certain triple—see [48, p. 507]. Lévy [43] uses the technique to prove strong normalization of a labeled λ-calculus with a bounded predicate. This proof yields also that all developments are finite, and standardization, as reported in [14].

Capretta and Valentini [11] prove strong normalization of simply typed λ-calculus by showing strong normalization of an alternative formulation of simply typed λ-calculus which they prove is equivalent to the usual formulation; this latter part is the difficult part of the proof.

Klop [38] shows strong normalization of a labeled λ-calculus by an interpretation in Λ_I. Several of the above techniques also use translations from Λ_K to Λ_I. The technique by Klop was discovered independently from a similar technique by Nederpelt [47] and has been reinvented and extended by many researchers, e.g., Khasidashvili [32], Karr [29], de Groote [16], Kfoury and Wells [30], Xi [83, 86], and Sørensen [69]; the latter paper gives a survey of some of the variations on the technique.

6. DEVELOPMENTS

The preceding section analyzed approaches based on the fundamental lemma of perpetuality, etc., to proving that all reductions of typed terms terminate. In the present section we give a similar analysis for reduction of *labeled terms*, i.e., for so-called developments.

The first subsection presents the fundamental lemma of perpetuality for developments along with two related characterizations due to van Raamsdonk and Severi and to Xi, respectively. The second subsection presents a new proof, due independently to van Raamsdonk and Severi and to Xi, of the finite developments theorem. Whereas the proof by van Raamsdonk and Severi and by Xi use their respective characterizations, the proof presented here uses the fundamental lemma of perpetuality for developments. Other proofs of the theorem are reviewed in less detail.

6.1. Developments

This subsection introduces developments in terms of labeled terms; we follow Barendregt [2, 11.1–2], with some insignificant deviations.

6.1. Definition. (i) The set $\underline{\Lambda}_K$ (*$\underline{\lambda}$-terms* or *labeled* λ-terms) is defined as

$$
\begin{aligned}
&x \in \underline{\Lambda}_K\\
&P \in \underline{\Lambda}_K \quad\;\;\; \Rightarrow \lambda x.P \in \underline{\Lambda}_K\\
&P, Q \in \underline{\Lambda}_K \Rightarrow P\,Q \in \underline{\Lambda}_K\\
&P, Q \in \underline{\Lambda}_K \Rightarrow (\underline{\lambda} x.P)\,Q \in \underline{\Lambda}_K.
\end{aligned}
$$

In the last clause $(\underline{\lambda} x.P)\,Q$ is a *labeled redex*.

(ii) The notions of reduction $\underline{\beta}$,β on $\underline{\Lambda}_K$ are defined by

$$
\begin{aligned}
(\underline{\lambda} x.P)\,Q \quad &\underline{\beta} \quad P\{x := Q\}\\
(\lambda x.P)\,Q \quad &\beta \quad P\{x := Q\}.
\end{aligned}
$$

(iii) The notion of reduction β^* is defined by

$$\beta^* = \underline{\beta} \cup \beta.$$

6.2. *Remark.* As done for λ-terms in Section 2.1 we briefly fix the terminology and notation for some well-known concepts—see [2]. We assume familiarity with conventions for omitting parentheses, with the notions of free and bound variables, with the variable convention, and with substitution. Also, $\subseteq$ denotes the subterm relation,[4] and $\equiv$ denotes syntactic equality up to renaming of bound variables. $\mathrm{FV}(M)$ denotes the set of variables that occur free in M. A $\underline{\lambda}$-context C is a $\underline{\lambda}$-term with a single occurrence of $[\,]$; $C[M]$ denotes the result of replacing the occurrence of $[\,]$ in C by M. $\|M\|$ denotes the number of occurrences of abstractions (labeled and unlabeled), applications, and variables in M. The set $\underline{\Lambda}_I$ is the subset of $\underline{\Lambda}_K$, where, for every $M \in \underline{\Lambda}_I$ and every $\lambda x.P \subseteq M$ and $(\underline{\lambda} x.P)Q \subseteq M$, $x \in \mathrm{FV}(P)$.[5]

6.3. Lemma. (i) $M, N \in \underline{\Lambda}_K \Rightarrow M\{x := N\} \in \underline{\Lambda}_K$;

(ii) $M \in \underline{\Lambda}_K \,\&\, M \to_{\beta^*} N \Rightarrow N \in \underline{\Lambda}_K$.

Proof. (i) By induction on M.

(ii) By induction on the derivation of $M \to_{\beta^*} N$, using (i). ∎

6.4. Definition. (i) A *development* of $M \in \underline{\Lambda}_K$ is a $\underline{\beta}$-reduction path from M.

(ii) A *complete* development of $M \in \underline{\Lambda}_K$ is one which ends in an $N \in \mathrm{NF}_{\underline{\beta}}$.

The *finiteness of developments theorem* states that all developments eventually terminate, i.e., that $M \in \mathrm{SN}_{\underline{\beta}}$ for all $M \in \underline{\Lambda}_K$. A stronger form asserts in addition that the $\underline{\beta}$-normal form of $M \in \underline{\Lambda}_K$ is unique.

6.2. *Fundamental Lemma of Perpetuality and Developments*

The following is an analog of the fundamental lemma of perpetuality for developments. It is used implicitly in several proofs in the literature of finite developments.

6.5. Lemma. *Assume that $N \in \mathrm{SN}_{\underline{\beta}}$ if $x \notin \mathrm{FV}(M)$. Then*

$$M\{x := N\} \in \mathrm{SN}_{\underline{\beta}} \Rightarrow (\underline{\lambda} x.M)\, N \in \mathrm{SN}_{\underline{\beta}}.$$

Proof. Suppose that $M\{x := N\} \in \mathrm{SN}_{\underline{\beta}}$. If $x \notin \mathrm{FV}(M)$, then, by assumption, $N \in \mathrm{SN}_{\underline{\beta}}$. If $x \in \mathrm{FV}(M)$, then $N \subseteq M\{x := N\}$, so again $N \in \mathrm{SN}_{\underline{\beta}}$. Also $M \in \mathrm{SN}_{\underline{\beta}}$. If $(\underline{\lambda} x.M)\, N \in \infty_{\underline{\beta}}$, then any infinite reduction must have the form

$$\begin{aligned}(\underline{\lambda} x.M) N &\twoheadrightarrow_{\underline{\beta}} (\underline{\lambda} x.M')\, N' \\ &\to_{\underline{\beta}} M'\{x := N'\} \\ &\to_{\underline{\beta}} \cdots\end{aligned}$$

[4] Recall that the subterms of $(\underline{\lambda} x.P)Q$ are the subterms of P and Q and the term $(\underline{\lambda} x.P)Q$ itself; that is, $\underline{\lambda} x.P$ is *not* a subterm.

[5] In other words, $\underline{\Lambda}_I$ is the set of all $M \in \underline{\Lambda}_K$ such that replacing every $\underline{\lambda}$ by λ yields an element of Λ_I.

Since

$$M \twoheadrightarrow_{\underline{\beta}} M' \;\&\; N \twoheadrightarrow_{\underline{\beta}} N' \Rightarrow M\{x := N\} \twoheadrightarrow_{\underline{\beta}} M'\{x := N'\}$$

there is an infinite reduction sequence

$$\begin{aligned} M\{x := N\} &\twoheadrightarrow_{\underline{\beta}} M'\{x := N'\} \\ &\rightarrow_{\underline{\beta}} \cdots \end{aligned}$$

contradicting $M\{x := N\} \in \mathrm{SN}_{\underline{\beta}}$. ∎

6.6. Corollary. *If $N \in \mathrm{SN}_{\underline{\beta}}$, then*

$$M\{x := N\} \in \mathrm{SN}_{\underline{\beta}} \Rightarrow (\underline{\lambda} x.M)\, N \in \mathrm{SN}_{\underline{\beta}}$$

Proof. By Lemma 6.5. ∎

6.7. *Remark.* Following van Raamsdonk and Severi [59] one can show that $\mathrm{SN}_{\underline{\beta}}$ is the smallest set closed under the rules:

(i) $x \in X$;

(ii) $P \in X \Rightarrow \lambda x.P \in X$;

(iii) $P \in X \;\&\; Q \in X \Rightarrow P\, Q \in X$;

(iv) $P\{x := Q\} \in X \;\&\; Q \in X \Rightarrow (\underline{\lambda} x.P)\, Q \in X$.

The proof of this uses two principles: induction on lexicographically ordered pairs $\langle l_{\underline{\beta}}(\bullet), \|\bullet\| \rangle$ and the fundamental lemma of perpetuality for developments. Proofs using the characterization correspond to direct proofs using the two principles, as was the case for β-reduction—see Remark 3.6.

6.8. *Remark.* Another characterization of $\mathrm{SN}_{\underline{\beta}}$ is due to Xi [82], who considers a relation $\trianglerighteq$ on $\underline{\Lambda}_K$ defined by

$$\trianglerighteq \;=\; \sqsupseteq \;\cup\; \rightarrow_{\underline{l}},$$

where $\rightarrow_{\underline{l}}$ denotes left-most $\underline{\beta}$-reduction and where $\sqsupseteq$ is the smallest relation closed under the rules

$$\lambda x.M \sqsupseteq M \quad M\, N \sqsupseteq M \quad M\, N \sqsupseteq N \quad (\underline{\lambda} x.M)\, N \sqsupseteq M \quad (\lambda x.M)\, N \sqsupseteq N.$$

Let $\underline{\mathscr{H}}(M_0) = \max\{n \mid M_0 \trianglerighteq M_1 \trianglerighteq \cdots \trianglerighteq M_n\} \in \mathbb{N}^*$. Then, for all $M \in \underline{\Lambda}_K$,

$$\mathrm{SN}_{\underline{\beta}} = \{M \in \underline{\Lambda}_K \mid \underline{\mathscr{H}}(M) < \infty\}$$

The proof and uses of this characterization are very similar to those of the characterization in [59].

6.3. *A New Proof of the Finite Developments Theorem*

The following proof of the finite developments theorem is due to van Raamsdonk and Severi [59]; their proof uses their characterization of $\mathrm{SN}_{\underline{\beta}}$ whereas the following proof uses lexicographic induction and the fundamental lemma of perpetuality—see Remark 6.7.

6.9. LEMMA. $M, N \in \mathrm{SN}_{\underline{\beta}} \Rightarrow M\{x := N\} \in \mathrm{SN}_{\underline{\beta}}$.

Proof. By induction on $\langle l_{\underline{\beta}}(M), \|M\| \rangle$. Let $L^* \equiv L\{x := N\}$.

1. $M \equiv x$. Then $M^* \equiv N \in \mathrm{SN}_{\underline{\beta}}$.
2. $M \equiv y$. Then $M^* \equiv y \in \mathrm{SN}_{\underline{\beta}}$.
3. $M \equiv \lambda x.P$. By the induction hypothesis, $P^* \in \mathrm{SN}_{\underline{\beta}}$. It follows that $M^* \equiv \lambda x.P^* \in \mathrm{SN}_{\underline{\beta}}$.
4. $M \equiv P\,Q$. Similar to the preceding case.
5. $M \equiv (\underline{\lambda} y.P)\,Q$. Then $P\{y := Q\} \in \mathrm{SN}_{\underline{\beta}}$ and $Q \in \mathrm{SN}_{\underline{\beta}}$. By the induction hypothesis $(P\{y := Q\})^* \equiv P^*\{y := Q^*\} \in \mathrm{SN}_{\underline{\beta}}$ and $Q^* \in \mathrm{SN}_{\underline{\beta}}$. By the fundamental lemma of perpetuality for developments it follows that $((\underline{\lambda} y.P)\,Q)^* \equiv (\underline{\lambda} y.P^*)\,Q^* \in \mathrm{SN}_{\underline{\beta}}$. ∎

6.10. THEOREM (Finite Developments). *For all* $M \in \underline{\Lambda}_K$, $M \in \mathrm{SN}_{\underline{\beta}}$.

Proof. By induction on M.

1. $M \equiv x$. Then $M \in \mathrm{SN}_{\underline{\beta}}$.
2. $M \equiv \lambda x.P$. By the induction hypothesis, $P \in \mathrm{SN}_{\underline{\beta}}$, and therefore $M \in \mathrm{SN}_{\underline{\beta}}$.
3. $M \equiv P\,Q$. Similar to Case 2.
4. $M \equiv (\underline{\lambda} x.P)\,Q$. By the induction hypothesis $P, Q \in \mathrm{SN}_{\underline{\beta}}$. By Lemma 6.9 also $P\{x := Q\} \in \mathrm{SN}_{\underline{\beta}}$. By the fundamental lemma of perpetuality for developments, $M \in \mathrm{SN}_{\underline{\beta}}$. ∎

There are many proofs of the finite developments theorem in the literature; the following is an incomplete list. The theorem was first proved by Church and Rosser [12, 13] for Λ_I; they also sketch a proof for Λ_K.[6] Curry and Feys [15] and Schroer [61] give full proofs of the theorem for Λ_K. Other proofs were later given independently by Hyland [27] and Hindley [21]. Barendregt *et al.* [4] subsequently simplified Hyland's proof—see also [2].

Xi [82] gives a proof similar to the above using instead of the fundamental lemma of perpetuality for developments his characterization of $\mathrm{SN}_{\underline{\beta}}$—see Remark 6.8. Van Oostrom [50, 51] shows that Lemma 6.9 can be eliminated by proving in Theorem 6.10 the stronger assertion: for all substitutions σ with $\sigma(x) \in \mathrm{SN}_{\underline{\beta}}$ for all x, it holds that $M\sigma \in \mathrm{SN}_{\underline{\beta}}$.

Another proof due to van Oostrom [50] uses Klop's [38] technique for reducing strong normalization to weak normalization. Other proofs that work by translation

[6] See the end of [13], or the beginning of Chapter V of [12].

into strongly normalizing typed λ-calculi are due to Parigot [53] (see also [41]), van Oostrom and van Raamsdonk [52], van Raamsdonk and Severi [59], Ghilezan [18], and Statman [72].

The theorem has also been proved in several ways for various notions of higher-order rewrite systems. Klop [38] proves it for orthogonal combinatory reduction systems by means of his technique to reduce weak normalization to strong normalization. Van Oostrom [49, 51] proves finiteness of developments for orthogonal higher-order rewriting systems and for pattern rewriting systems. Each of these two results implies finite developments for orthogonal combinatory reduction systems. Melliès [45] gives an axiomatic formulation of developments and shows finite developments for this formulation, which includes orthogonal combinatory reduction systems, but apparently not pattern rewriting systems—see [51]. Khasidashvili [32, 34] gives algorithms to compute longest developments and length of such developments in orthogonal expression reduction systems; these algorithms are special cases of methods to compute longest reductions and the length of such reductions in certain restricted orthogonal expression reduction systems.

One can formulate a version of the fundamental lemma of maximality for developments and use this to give a corresponding effective strategy $\underline{F}_\infty$ computing longest developments and a map $h\colon \mathrm{SN}_{\underline{\beta}} \to \mathbb{N}$ computing the length of longest developments, similarly to the development in Sections 3.5 and 3.6. However, de Vrijer [78] shows that in the case of developments one can do better; he gives a map $f\colon \underline{\Lambda}_K \to \mathbb{N}$ (called h in [78]) computing the length of longest developments where $f((\underline{\lambda} x.P)Q)$ is expressed in terms of $f(P)$ and $f(Q)$; this of course implies finiteness of developments. He also shows that $\underline{F}_\infty$ computes longest developments. Sørensen [70] applies to de Vrijer's technique a principle of duality thereby arriving at a technique to compute shortest development as well as the length of such developments.

7. MAXIMAL AND PERPETUAL REDEXES

Having applied the techniques related to perpetual and maximal β-reduction strategies from Section 3. to various strong normalization problems in Sections 4–6, we now return to study perpetual and maximal β-redexes. This leads to some *conservation theorems*.

The first subsection reviews some fundamental results relating reduction on terms with and without labels, which will be used in the rest of the section. In particular, a scheme employed in several proofs of conservation theorems in the literature is made explicit. The next three subsections prove the conservation theorem for Λ_I, the conservation theorem for Λ_K, and a related conservation theorem due to Bergstra and Klop, using this proof scheme. These results are used in the fifth subsection to characterize perpetual β-redexes (the notion of maximal β-redex turns out to be trivial). The sixth subsection gives a proof of the normalization theorem similar to the proofs of the conservation theorems, and the last subsection gives a very short proof of the conservation theorem for Λ_I using the normalization theorem.

7.1. Reduction on Terms with and without Labels

There are two important ways to move from a term with labels to one without: one can either *erase* all labels or *reduce* all labeled redexes. This is done by the two maps $|\bullet|$, $\varphi(\bullet)$: $\underline{\Lambda}_K \to \Lambda_K$, respectively, introduced below.

7.1. DEFINITION. For $M \in \underline{\Lambda}_K$ define $|M| \in \Lambda_K$ as follows.

$$\begin{array}{lcl} |x| & = & x \\ |\lambda x.P| & = & \lambda x.|P| \\ |P\,Q| & = & |P|\,|Q| \\ |(\underline{\lambda} x.P)\,Q| & = & (\lambda x.|P|)\,|Q|. \end{array}$$

7.2. LEMMA. *Let* $M, N \in \underline{\Lambda}_K$.

(i) $|M|\,\{x := |N|\} \equiv |M\{x := N\}|$.

(ii) (Projection.) $M \to_{\beta*} N \Rightarrow |M| \to_\beta |N|$;

(iii) (Lifting.) $|M| \to_\beta K \Rightarrow \exists N \in \underline{\Lambda}_K$: $M \to_{\beta*} N$ & $|N| \equiv K$.

Proof. (i) By induction on M. (ii) By induction on the derivation of $M \to_{\beta*} N$. (iii) By induction on the derivation of $|M| \to_\beta K$. ∎

7.3. COROLLARY. *Let* $M \in \underline{\Lambda}_K$.

(i) $M \in \mathrm{SN}_{\beta*} \Leftrightarrow |M| \in \mathrm{SN}_\beta$;

(ii) $M \in \mathrm{NF}_{\beta*} \Leftrightarrow |M| \in \mathrm{NF}_\beta$.

The following map $\varphi(M)$ computes a complete *inside-out* development of $M \in \underline{\Lambda}_K$, whereas $M \twoheadrightarrow_\beta N \in \mathrm{NF}_{\underline{\beta}}$ means that N is the result of an arbitrary complete development of M. In the last clause of the definition it is implicit that no previous clause applies.

7.4. DEFINITION. Define φ: $\underline{\Lambda}_K \to \Lambda_K$ as

$$\begin{array}{lcl} \varphi(x) & = & x \\ \varphi(\lambda x.Q) & = & \lambda x.\varphi(Q) \\ \varphi((\underline{\lambda} x.P)\,Q) & = & \varphi(P)\{x := \varphi(Q)\} \\ \varphi(P\ \ Q) & = & \varphi(P)\,\varphi(Q). \end{array}$$

7.5. LEMMA. *For all* $M, N \in \underline{\Lambda}_K$:

(i) $\varphi(M\{x := N\}) = \varphi(M)\{x := \varphi(N)\}$;

(ii)
$$\begin{tikzcd} M \arrow[rr, "\to_{\underline{\beta}}"'] \arrow[dr, "\varphi"'] & & N \arrow[dl, "\varphi"] \\ & K & \end{tikzcd}$$

(iii)
$$\begin{tikzcd} M \arrow[r, "\to_{\beta}"'] \arrow[d, "\varphi"'] & N \arrow[d, "\varphi"] \\ K \arrow[r, "\twoheadrightarrow_{\beta}"'] & L \end{tikzcd}$$

Proof. (i) By induction on M. (ii) By induction on the derivation of $M \to_{\underline{\beta}} N$ using (i). (iii) By induction on the derivation of $M \to_{\beta} N$ using (i). ∎

The following expresses a relation between $|\bullet|$ and $\varphi(\bullet)$.

7.6. Lemma. *Let* $M \equiv C[(\lambda x.P)Q] \in \underline{\Lambda}_K$, $N \equiv C[P\{x := Q\}] \in \Lambda_K$, *and* $L \equiv C[(\underline{\lambda} x.P)Q] \in \underline{\Lambda}_K$. *Then*

$$\begin{tikzcd} M \arrow[dd, "\to_{\beta}"'] & \\ & L \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \\ N & \end{tikzcd}$$

Proof. By induction on the derivation of $M \to_{\beta} N$. ∎

The following proposition expresses the core idea of several proofs of conservation theorems in the literature.

7.7. Proposition. *Let* $M \in \Lambda_K$ *and* $M \to_{\beta} N$. *Then*

$$M \in \infty_{\beta} \Rightarrow N \in \infty_{\beta}$$

if there is an $S \subseteq \underline{\Lambda}_K$ *and* $F^*\colon \infty_{\beta *} \to \infty_{\beta *}$ *with*

(i) $M \equiv C[(\lambda x.P)Q]$, $C[P\{x := Q\}] \equiv N$, $C[(\underline{\lambda} x.P)Q] \in S$ *for some* C, P, Q;

(ii) $L \in S \Rightarrow F^*(L) \in S$;

(iii) *for all* $L \in S$: $L \to_{\beta} F^*(L) \Rightarrow \varphi(L) \twoheadrightarrow_{\beta}^{+} \varphi(F^*(L))$.

Proof. Let $M \to_{\beta} N$, where $M \in \infty_{\beta}$, and let C, P, Q, S, and F^* be as required in (i)–(iii).

Let $L_0 \equiv C[(\underline{\lambda} x.P)Q]$, $N_0 \equiv N$, and $M_0 \equiv M$. By Corollary 7.3, $L_0 \in \infty_{\beta *}$. Since F^* is perpetual,

$$L_0 \to_{\beta *} L_1 \to_{\beta *} L_2 \cdots$$

with $L_i = F^*(L_{i-1})$ is infinite.

By Lemmas 7.5 and 7.6 and the assumptions we can erect the diagram.[7]

[7] The reduction $M_0 \to_{\beta} M_1 \to_{\beta} \cdots$ is constructed by projection of $L_0, L_1, \ldots$, but the former reduction path is not essential.

$$\begin{tikzcd}
M_0 \arrow[rr, "\to_\beta"'] \arrow[dd, "\to_\beta"'] & & M_1 \arrow[rr, "\to_\beta"'] & & M_2 \arrow[rr, "\to_\beta"'] & & \cdots \\
& L_0 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"'] \arrow[rr, "\to_\beta"'] & & L_1 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"'] \arrow[rr, "\to_{\underline{\beta}}"'] & & L_2 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"'] \arrow[rr, "\to_\beta"'] & & \cdots \\
N_0 \arrow[rr, "\twoheadrightarrow^+_\beta"'] & & N_1 \arrow[rr, "\equiv"'] & & N_2 \arrow[rr, "\twoheadrightarrow^+_\beta"'] & & \cdots
\end{tikzcd}$$

Here

$$L_i \to_\beta L_{i+1} \;\Rightarrow\; N_i \twoheadrightarrow^+_\beta N_{i+1}$$
$$L_i \to_{\underline{\beta}} L_{i+1} \;\Rightarrow\; N_i \equiv N_{i+1}.$$

By finiteness of developments $L_i \to_\beta L_{i+1}$ for infinitely many i, giving an infinite β-reduction path from N_0. ∎

7.8. *Remark.* The diagram used in the above proof is an infinite version of the diagram used by Barendregt [2, 11.1] to prove the *strip lemma*, the main lemma in his proof of the Church–Rosser property.

7.2. *The Conservation Theorem for Λ_I*

We now use Proposition 7.7 to prove the conservation theorem for Λ_I.

7.9. Lemma. *For any $M \in \underline{\Lambda}_I$: $M \to_{\beta*} N \Rightarrow N \in \underline{\Lambda}_I$.*[8]

Proof. Show by induction on M that

$$M, N \in \underline{\Lambda}_I \Rightarrow M\{x := N\} \in \underline{\Lambda}_I \qquad (*)$$

and by induction on the derivation of $M \to_{\beta*} N$ that

$$FV(M) \subseteq \mathrm{FV}(N). \qquad (+)$$

Using $(*)$ and $(+)$ proceed by induction on the derivation of $M \to_{\beta*} N$. ∎

7.10. Lemma. *For any $M \in \underline{\Lambda}_I$: $M \to_\beta N \Rightarrow \varphi(M) \twoheadrightarrow^+_\beta \varphi(N)$.*

Proof. Show by induction on M that for all $M \in \underline{\Lambda}_I$,

$$\mathrm{FV}(M) \subseteq \mathrm{FV}(\varphi(M)).$$

Using this property and Lemma 7.5(i), proceed by induction on the derivation of $M \to_\beta N$. ∎

[8] $\underline{\Lambda}_I$ is defined in Remark 6.2.

7.11. Theorem (Conservation for Λ_I). *If* $M \in \Lambda_I$ *and* $M \to_\beta N$, *then*

$$M \in \infty_\beta \Rightarrow N \in \infty_\beta$$

Proof. By the preceding two lemmas we can use Proposition 7.7 with $S = \underline{\Lambda}_I$ and any partial, perpetual β^*-reduction strategy in the role of F^*. ∎

7.12. *Remark.* Since Λ_I is closed under β-reduction, we can view β as a notion of reduction on Λ_I, and we can view any β-reduction strategy on Λ_K as a β-reduction strategy on Λ_I. The conservation theorem for Λ_I states that in Λ_I, all β-redexes and β-reduction strategies are perpetual.

7.13. Corollary. *Let* $M \in \Lambda_I$.

(i) $M \in \mathrm{WN}_\beta \Rightarrow M \in \mathrm{SN}_\beta$;

(ii) $M \in \mathrm{WN}_\beta \;\&\; N \subseteq M \Rightarrow N \in \mathrm{WN}_\beta$.

Proof. (i) If $M \in \mathrm{WN}_\beta$, then $M \twoheadrightarrow_\beta N \in \mathrm{NF}_\beta$, for some N. If $M \in \infty_\beta$, then by the conservation theorem, $N \in \infty_\beta$, a contradiction.

(ii) If $M \in \mathrm{WN}_\beta$ and $N \subseteq M$, then $M \in \mathrm{SN}_\beta$, and therefore $N \in \mathrm{SN}_\beta$, in particular $N \in \mathrm{WN}_\beta$. ∎

As mentioned in Remark 4.19 and at the end of Section 5, a number of techniques to prove strong normalization of typed λ-calculi use translations from Λ_K to Λ_I. Most of these techniques also use some variant of Corollary 7.13(i). For instance, the techniques by Sørensen [69] and Xi [83, 86] use a translation $t\colon \Lambda_K \to \Lambda_I$ such that $t(M) \in \mathrm{SN}_\beta \Rightarrow M \in \mathrm{SN}_\beta$. By the corollary, it then suffices to show $t(M) \in \mathrm{WN}_\beta$ to infer $M \in \mathrm{SN}_\beta$.

The conservation theorem for Λ_I is due to Church and Rosser [12, 13], and was later proved by Curry and Feys [15]. A proof in the spirit of the former proof is given by Barendregt *et al.* [2, 4]. These proofs are all by syntactic methods; a semantic proof appears in [24]. Klop [38] proves a generalization of the theorem for orthogonal non-erasing combinatory reduction systems.

The above proof is a slight simplification of the proof by Barendregt *et al.*; our proof uses inside-out developments rather than arbitrary developments and avoids the explicit notions of redex occurrence and residual (similarly, Takahashi [74] proves Curry and Feys' *standardization theorem* using parallel reductions, arguing that these are more convenient than the arbitrary developments used in, e.g., Mitschke's proof [46]—see also [2]). A very short proof will be given in the last subsection.

7.3. The Conservation Theorem for Λ_K

We now use Proposition 7.7 to prove the conservation theorem for Λ_K.

7.14. Definition. (i) An *I-redex* is a term $(\lambda x.P)Q \in \Lambda_K$, where $x \in \mathrm{FV}(P)$. A *K-redex* is a term $(\lambda x.P)Q \in \Lambda_K$, where $x \notin \mathrm{FV}(P)$.

(ii) We write $\mathbf{K}\, P\, Q$ for $(\lambda x.P)Q$ and $\underline{\mathbf{K}}\, P\, Q$ for $(\underline{\lambda} x.P)Q$ when $x \notin \mathrm{FV}(P)$ and call P and Q the *body* and *argument*, respectively, of the redex.

(iii) $\underline{\Lambda}^I$ is the subset of $\underline{\Lambda}_K$, where for each $M \in \underline{\Lambda}^I$ and each $(\underline{\lambda} x.P)Q \subseteq M$, it holds that $x \in \mathrm{FV}(P)$.

(iv) We write $M \equiv (\lambda^* x.P)Q$ if $M \equiv (\lambda x.P)Q$ or $M \equiv (\underline{\lambda} x.P)Q$.

7.15. Definition. Define $F_1^*\colon \infty_{\beta*} \to \underline{\Lambda}_K$ by

$$
\begin{array}{llll}
F_1^*(x\,\mathbf{P}\,Q\,\mathbf{R}) & = & x\,\mathbf{P}\,F_1^*(Q)\,\mathbf{R} & \text{if} \quad \mathbf{P} \in \mathrm{SN}_{\beta*}, \quad Q \notin \mathrm{SN}_{\beta*} \\
F_1^*(\lambda x.P) & = & \lambda x.F_1^*(P) & \\
F_1^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & P\{x := Q\}\,\mathbf{R} & \text{if} \quad Q \in \mathrm{SN}_{\beta*} \\
F_1^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & (\lambda^* x.P)\,F_1^*(Q)\,\mathbf{R} & \text{if} \quad Q \notin \mathrm{SN}_{\beta*}.
\end{array}
$$

7.16. Lemma. *For all $M \in \infty_{\beta*}$: $F_1^*(M) \in \infty_{\beta*}$.*

Proof. First show that, for all $M \in \infty_{\beta*}$,

$$|F_1^*(M)| = F_1(|M|) \tag{$*$}$$

by induction on M using Corollary 7.3. Since $M \in \infty_{\beta*}$, $|M| \in \infty_\beta$ by Corollary 7.3. By $(*)$ and perpetuality of F_1, $|F_1^*(M)| = F_1(|M|) \in \infty_\beta$. Then by Corollary 7.3, $F_1^*(M) \in \infty_{\beta*}$. ∎

7.17. Lemma. *For all $M \in \underline{\Lambda}^I$: $F_1^*(M) \in \underline{\Lambda}^I$.*

Proof. First prove by induction on M that

$$M, N \in \underline{\Lambda}^I \Rightarrow M\{x := N\} \in \underline{\Lambda}^I.$$

Using this show $F_1^*(M) \in \underline{\Lambda}^I$ by induction on M. ∎

7.18. Lemma. *For all $M \in \underline{\Lambda}^I$: $M \to_\beta F_1^*(M) \Rightarrow \varphi(M) \twoheadrightarrow_\beta^+ \varphi(F_1^*(M))$.*

Proof. By induction on M show that for all $M \in \underline{\Lambda}^I$: $\mathrm{FV}(M) \subseteq \mathrm{FV}(\varphi(M))$. Using this and Lemma 7.5 proceed by induction on M. ∎

7.19. Theorem (Conservation for Λ_K). *If $M \equiv C[\Delta] \to_\beta C[\Delta'] \equiv N$, where $M \in \Lambda_K$ and Δ is an I-redex, then*

$$M \in \infty_\beta \Rightarrow N \in \infty_\beta.$$

Proof. By the preceding three lemmas we can use Proposition 7.7 with $S = \underline{\Lambda}^I$ and $F^* = F_1^*$. ∎

7.20. COROLLARY. *Any I-redex is perpetual.*

7.21. *Discussion* (Barendregt *et al.* [2, 4]). The proof of the conservation theorem for Λ_I does not carry over to Λ_K; i.e., we cannot use Proposition 7.7 with $S = \underline{\Lambda}_K$ and F^* any partial, perpetual β^*-reduction strategy. For instance, $(\lambda x.\mathbf{K}\,\mathbf{I}\,x)\Omega$ is an I-redex, but the diagram in the proof of Proposition 7.7 is

$$
\begin{tikzcd}
(\lambda x.\mathbf{K}\,\mathbf{I}x)\Omega \arrow[rr, "\to_\beta"'] \arrow[dd, "\to_\beta"'] & & (\lambda x.\mathbf{I})\Omega \arrow[rr, "\to_\beta"'] & & (\lambda x.\mathbf{I})\Omega \arrow[rr, "\to_\beta"'] & & \cdots & \\
& (\underline{\lambda} x.\mathbf{K}\,\mathbf{I}\,x)\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & (\underline{\lambda} x.\mathbf{I})\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & (\underline{\lambda} x.\mathbf{I})\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & \cdots \\
\mathbf{K}\,\mathbf{I}\,\Omega \arrow[rr, "\twoheadrightarrow_\beta"'] & & \mathbf{I} \arrow[rr, "\equiv"'] & & \mathbf{I} \arrow[rr, "\twoheadrightarrow_\beta^+"'] & & \cdots &
\end{tikzcd}
$$

After one step, no reductions occur in the lower sequence. The problem is that property (iii) in Proposition 7.7 fails for $S = \underline{\Lambda}_K$ if F^* is arbitrary. This is because in $M \to_\beta N$ the reduction may take place in the argument Q of a labeled K-redex $\mathbf{K}\,P\,Q$, and then $\varphi(M) \equiv \varphi(N)$.

However, (iii) does hold for $S = \underline{\Lambda}^I$; i.e., when only I-redexes are labeled. The rescue then is that labeling an I-redex yields a term in $\underline{\Lambda}^I$, so (i) also holds. Moreover, to turn a $\underline{\Lambda}^I$ term into a term outside $\underline{\Lambda}^I$ would require a reduction step inside P of $(\underline{\lambda} x.P)Q$ which erased all occurrences of x, but F_1^* never reduces a redex inside P of a redex $(\underline{\lambda} x.P)Q$, so (ii) holds too.

The conservation theorem for Λ_K is due to Barendregt *et al.* [2, 4]. Khasidashvili [34] shows a version for orthogonal expression reduction systems, using perpetuality of his limit strategies mentioned earlier (see the end of Section 3.5). Our proof is a slight simplification of the proof by Barendregt *et al.*; apart from the simplifications mentioned in the preceding subsection, our proof uses a simpler perpetual reduction strategy than the proof by Barendregt *et al.*

7.4. Conservation under K-Reduction

The preceding two subsections characterized perpetual I-redexes in Λ_I and Λ_K. Now we characterize perpetual K-redexes in Λ_K.

7.22. DEFINITION. (i) $\underline{\Lambda}^K$ is the subset of $\underline{\Lambda}_K$ such that for all $M \in \underline{\Lambda}_K$ and all $(\underline{\lambda} x.P)Q \subseteq M$, it holds that $x \notin \mathrm{FV}(P)$.

(ii) For $(L, R) = (\Lambda_K, \beta)$ and $(L, R) = (\underline{\Lambda}_K, \beta^*)$, an SN_R-*substitution* is a substitution σ such that $x\sigma \in \mathrm{SN}_R$ for every variable x. For $P, Q \in L$, we write $P \geqslant_\infty^R Q$ iff for all SN_R-substitutions σ,

$$P\sigma \in \infty_R \Leftarrow Q\sigma \in \infty_R.$$

For $Q \in \mathrm{SN}_R$, $\sigma + \{x := Q\}$ maps x to Q and acts as σ on any other variable. By projection and lifting $P \geqslant_\infty^\beta Q \Leftrightarrow P \geqslant_\infty^{\beta^*} Q$ for any $P, Q \in \Lambda_K$.

7.23. Definition. Define $F_2^*\colon \infty_{\beta*} \to \underline{\Lambda}_K$ by

$$
\begin{array}{llll}
F_2^*(x\,\mathbf{P}\,Q\,\mathbf{R}) & = & x\ \mathbf{P}\,F_2^*(Q)\mathbf{R} & \text{if}\quad \mathbf{P}\in \mathrm{SN}_{\beta*},\quad Q\notin \mathrm{SN}_{\beta*}\\
F_2^*(\lambda x.P) & = & \lambda x.F_2^*(P) & \\
F_2^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & P\{x:=Q\}\ \mathbf{R} & \text{if}\quad P, Q\in \mathrm{SN}_{\beta*}\\
F_2^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & (\lambda^* x.F_2^*(P))\,Q\,\mathbf{R} & \text{if}\quad P\notin \mathrm{SN}_{\beta*}\\
F_2^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & (\lambda^* x.P)\,F_2^*(Q)\mathbf{R} & \text{if}\quad P\in \mathrm{SN}_{\beta*},\quad Q\notin \mathrm{SN}_{\beta*}.
\end{array}
$$

7.24. Lemma. *For all* $M\in \infty_{\beta*}$: $F_2^*(M)\in \infty_{\beta*}$.

Proof. First show that, for all $M\in \infty_{\beta*}$,

$$|F_2^*(M)| = F_2(|M|) \tag{$*$}$$

by induction on M using Corollary 7.3. Since $M\in \infty_{\beta*}$, $|M|\in \infty_\beta$ by Corollary 7.3. By $(*)$ and perpetuality of F_2, $|F_2^*(M)| = F_2(|M|)\in \infty_\beta$. Then by Corollary 7.3, $F_2^*(M)\in \infty_{\beta*}$. ∎

7.25. Lemma. *For all* $M\in \underline{\Lambda}^K$, $F_2^*(M)\in \underline{\Lambda}^K$.

Proof. First prove by induction on M that

$$M, N\in \underline{\Lambda}^K \Rightarrow M\{x:=N\}\in \underline{\Lambda}^K.$$

Using this property proceed by induction on M. ∎

7.26. Definition. Let X be a set of variables.

(i) An $\mathrm{SN}_{\beta*}$-substitution σ is *X-neutral*, if $x\sigma = x$ for all $x\in X$;

(ii) M is *X-good* if, for all $\underline{\mathbf{K}}AB\subseteq M$ and X-neutral σ, $A\sigma\in \infty_{\beta*} \Leftarrow B\sigma\in \infty_{\beta*}$;

(iii) *X respects M* if $\mathrm{FV}(M)\subseteq X$ and $X\cap \mathrm{BV}(M) = \{\}$.

7.27. Definition. For $M\in \infty_{\beta*}$, define the set of variables $V(M)$ by

$$
\begin{array}{llll}
V(x\,\mathbf{P}\,Q\,\mathbf{R}) & = & V(Q) & \text{if}\quad \mathbf{P}\in \mathrm{SN}_{\beta*},\quad Q\notin \mathrm{SN}_{\beta*}\\
V(\lambda x.P) & = & \{x\}\cup V(P) & \\
V((\lambda^* x.P)\,Q\,\mathbf{R}) & = & \{\} & \text{if}\quad P, Q\in \mathrm{SN}_{\beta*}\\
V((\lambda^* x.P)\,Q\,\mathbf{R}) & = & \{x\}\cup V(P) & \text{if}\quad P\notin \mathrm{SN}_{\beta*}\\
V((\lambda^* x.P)\,Q\,\mathbf{R}) & = & V(Q) & \text{if}\quad P\in \mathrm{SN}_{\beta*},\quad Q\notin \mathrm{SN}_{\beta*}.
\end{array}
$$

7.28. Lemma. *For all* $M\in \infty_{\beta*}$: $V(M)\subseteq V(F_2^*(M))$.

Proof. By induction on M using perpetuality of F_2^*. ∎

7.29. LEMMA. *Let $M \in \infty_{\beta*} \cap \underline{\Lambda}^K$, M be $X \cup V(M)$-good, X* respect *M.*

(i) *$F_2^*(M)$ is $X \cup V(F_2^*(M))$-good, and X respects $F_2^*(M)$;*

(ii) *$M \to_\beta F_2^*(M) \Rightarrow \varphi(M) \twoheadrightarrow_\beta^+ \varphi(F_2^*(M))$.*

Proof. Let $M \in \infty_{\beta*} \cap \underline{\Lambda}^K$, M be $X \cup V(M)$-good, X respect M.

(i) Since reduction does not invent new free variables, and new bound variables are chosen fresh, X respects $F_2^*(M)$.

We show that $F_2^*(M)$ is $X \cup V(F_2^*(M))$-good by induction on M. Let $\mathbf{K}\ A\ B \subseteq F_2^*(M)$ and let σ be an $X \cup V(F_2^*(M))$-neutral $\mathrm{SN}_{\beta*}$-substitution. We are to show that $A\sigma \in \infty_{\beta*} \Leftarrow B\sigma \in \infty_{\beta*}$.

1. $M \equiv x\ \mathbf{P}\ Q\ \mathbf{R}$, and $\mathbf{P} \in \mathrm{SN}_{\beta*}$, $Q \notin \mathrm{SN}_{\beta*}$. Then $F_2^*(M) = x\ \mathbf{P}\ F_2^*(Q)\ \mathbf{R}$.

1.1. $\underline{\mathbf{K}}\ A\ B \subseteq S$, where $\mathbf{P} = \mathbf{P}_1, S, \mathbf{P}_2$ or $\mathbf{R} = \mathbf{R}_1, S, \mathbf{R}_2$. Then, by Lemma 7.28, $V(M) \subseteq V(F_2^*(M))$. Therefore, σ is $X \cup V(M)$-neutral. Since M is $X \cup V(M)$-good, $A\sigma \in \infty_{\beta*} \Leftarrow B\sigma \in \infty_{\beta*}$.

1.2. $\underline{\mathbf{K}}\ A\ B \subseteq F_2^*(Q)$. Since $V(M) = V(Q)$, Q is $X \cup V(Q)$-good. By the induction hypothesis, $F_2^*(Q)$ is $X \cup V(F_2^*(Q))$-good. Since F_2^* is perpetual, $V(F_2^*(M)) = V(F_2^*(Q))$, so $F_2^*(Q)$ is $X \cup V(F_2^*(M))$-good. Therefore, $A\sigma \in \infty_{\beta*} \Leftarrow B\sigma \in \infty_{\beta*}$.

2. $M \equiv \lambda x.P$. Then $F_2^*(M) = \lambda x.F_2^*(P)$. Then $\underline{\mathbf{K}}\ A\ B \subseteq F_2^*(P)$. Since $V(M) = \{x\} \cup V(P)$, P is $X \cup \{x\} \cup V(P)$-good. Here $X \cup \{x\}$ respects P, so by the induction hypothesis, $F_2^*(P)$ is $X \cup \{x\} \cup V(F_2^*(P))$-good. Since $V(F_2^*(M)) = \{x\} \cup V(F_2^*(P))$, $F_2^*(P)$ is $X \cup V(F_2^*(M))$-good. Then $A\sigma \in \infty_{\beta*} \Leftarrow B\sigma \in \infty_{\beta*}$.

3. $M \equiv (\lambda^* x.P)\ Q\ \mathbf{R}$. We consider three subcases.

3.1. $P \in \infty_{\beta*}$. Then $F_2^*(M) = (\lambda^* x.F_2^*(P))\ Q\ \mathbf{R}$. There are, in turn, three cases to consider.

3.1.1. $\underline{\mathbf{K}}\ A\ B \subseteq S$, where $S \equiv Q$ or $\mathbf{R} = \mathbf{R}_1, S, \mathbf{R}_2$. Similar to Case 1.1.

3.1.2. $\underline{\mathbf{K}}\ A\ B \subseteq F_2^*(P)$. Similar to Case 2.

3.1.3. $\underline{\mathbf{K}}\ A\ B \equiv (\lambda^* x.F_2^*(P))\,Q$. Since F_2^* is perpetual, $F_2^*(P) \in \infty_{\beta*}$, i.e., $A \in \infty_{\beta*}$. Thus $A\sigma \in \infty_{\beta*}$, so $A\sigma \in \infty_{\beta*} \Leftarrow B\sigma \in \infty_{\beta*}$ trivially.

3.2. $P \in \mathrm{SN}_{\beta*}$, $Q \notin \mathrm{SN}_{\beta*}$. As in Case 3.1, there are three subcases.

3.2.1. $\underline{\mathbf{K}}\ A\ B \subseteq S$, where $S \equiv P$ or $\mathbf{R} = \mathbf{R}_1, S, \mathbf{R}_2$. Similar to Case 1.1.

3.2.2. $\underline{\mathbf{K}}\ A\ B \subseteq F_2^*(Q)$. Similar to Case 1.2.

3.2.3. $\underline{\mathbf{K}}\ A\ B \equiv (\lambda^* x.P)\ F_2^*(Q)$. This case is impossible. Indeed, suppose that $\underline{\mathbf{K}}AB \equiv (\lambda^* x.P)\ F_2^*(Q)$, so $\underline{\mathbf{K}}AB' \equiv (\lambda^* x.P)\,Q \subseteq M$. The identity substitution ι is clearly $X \cup V(M)$-neutral, but according to the above, $A\iota \notin \infty_{\alpha*}$ and $B'\iota \in \infty_{\beta*}$, contradicting the assumption that M is $X \cup V(M)$-good.

3.3. $P, Q \in \mathrm{SN}_{\beta*}$. Then $F_2^*(M) = P\{x := Q\}\ \mathbf{R}$.

3.3.1. $\underline{\mathbf{K}}\ A\ B \subseteq S$, where $S \in \mathbf{R}$. Similar to Case 1.1.

3.3.2. $\underline{\mathbf{K}}\ A\ B \subseteq P\{x := Q\}$. We consider three subcases.

(a) $\underline{\mathbf{K}}\ A\ B \subseteq Q$. Similar to Case 1.1.

(b) $\underline{\mathbf{K}}\ A\ B \subseteq P$. Similar to Case 1.1.

(c) $\underline{\mathbf{K}}\ A\ B \equiv \mathbf{K}(I\{x := Q\})(J\{x := Q\})$, where $\mathbf{K}\ I\ J \subseteq P$. Since $\mathrm{FV}(Q) \subseteq \mathrm{FV}(M) \subseteq X$, $y\sigma = y$, for all $y \in \mathrm{FV}(Q)$. Therefore, $\mathbf{K}A\sigma\ B\sigma \equiv \underline{\mathbf{K}}I\sigma'\ J\sigma'$, where $\sigma' = \sigma + \{x := Q\}$. Since $V(M) \subseteq V(F_2^*(M))$, σ is $X \cup V(M)$-neutral. Now $x \notin V(M)$, and $x \in \mathrm{BV}(M)$ so $x \notin X$. Therefore σ' is $X \cup V(M)$-neutral. Thus, since M is $X \cup V(M)$-good, $A\sigma \equiv I\sigma' \in \infty_{\beta*} \Leftarrow B\sigma \equiv J\sigma' \in \infty_{\beta*}$.

(ii) By induction on M. ▮

7.30. Theorem (Conservation of K-Redexes). *Assume that $P \geqslant^{\beta}_{\infty} Q$ and $M \equiv C[\mathbf{K}\ P\ Q] \to_{\beta} C[P] \equiv N$, where $M \in \Lambda_K$. Then*

$$M \in \infty_{\beta} \Rightarrow N \in \infty_{\beta}.$$

Proof. Suppose that $M \in \infty_{\beta}$ and $M \equiv C[\mathbf{K}\ P\ Q] \to_{\beta} C[P] \equiv N$, where $P \geqslant^{\beta}_{\infty} Q$. Let $F^* = F_2^*$ and

$$S = \{J \in \underline{\Lambda}^K \cap \infty_{\beta*} \mid J \text{ is } \mathrm{FV}(M) \cup V(J)\text{-good \& } \mathrm{FV}(M) \text{ respects } J\}.$$

Then condition (i) of Proposition 7.7 is clearly satisfied, and by Lemmas 7.24, 7.25, and 7.29, conditions (ii) and (iii) are also satisfied. ▮

7.31. Corollary. *A K-redex $\mathbf{K}\ P\ Q$ is perpetual if $P \geqslant^{\beta}_{\infty} Q$.*

7.32. Corollary. *A K-redex $\mathbf{K}\ P\ Q$ is perpetual if one of the following conditions is satisfied:*

(i) $P \in \infty_{\beta}$;

(ii) $Q \in \mathrm{SN}_{\beta}$ *and* $\mathrm{FV}(Q) = \varnothing$.

7.33. Corollary. *A redex $(\lambda x.P)Q$ is perpetual if*

$$P\sigma\{x := Q\sigma\} \in \infty_{\beta} \Leftarrow Q\sigma \in \infty_{\beta}$$

for all SN_{β}-substitutions σ.

Proof. If $x \in \mathrm{FV}(P)$ then the redex is perpetual by the conservation theorem for Λ_K. If $x \notin \mathrm{FV}(P)$, then the condition of the theorem is equivalent to $P \geqslant^{\beta}_{\infty} Q$, so the redex is again perpetual by the preceding corollary. ▮

7.34. *Discussion.* It is not true that $M \in \Lambda_K$ and $M \to N$ by contraction of any K-redex implies

$$M \in \infty_{\beta} \Rightarrow N \in \infty_{\beta}.$$

For instance, for the term $M \equiv \mathbf{K}\ \mathbf{I}\ \Omega$ and the reduction step $\mathbf{K}\ \mathbf{I}\ \Omega \to_{\beta} \mathbf{I}$ the assertion is wrong. The diagram from the proof of Proposition 7.7 is

$$
\begin{tikzcd}
\mathbf{K}\,\mathbf{I}\,\Omega \arrow[rr, "\to_\beta"'] \arrow[dd, "\to_\beta"'] & & \mathbf{K}\,\mathbf{I}\,\Omega \arrow[rr, "\to_\beta"'] & & \mathbf{K}\,\mathbf{I}\,\Omega \arrow[rr, "\to_\beta"'] & & \cdots & \\
& \underline{\mathbf{K}}\,\mathbf{I}\,\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & \underline{\mathbf{K}}\,\mathbf{I}\,\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & \underline{\mathbf{K}}\,\mathbf{I}\,\Omega \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_\beta"'] & & \cdots \\
\mathbf{I} \arrow[rr, "\equiv"'] & & \mathbf{I} \arrow[rr, "\equiv"'] & & \mathbf{I} \arrow[rr, "\equiv"'] & & \cdots &
\end{tikzcd}
$$

In the lower sequence every term is identical to its successor, and the problem evidently is the same as earlier: (ii) of Proposition 7.7 fails for $S = \underline{\Lambda}^K$; that is, in $M \to_\beta N$ the reduction step may occur in the argument of a labeled K-redex, and then $\varphi(M) \equiv \varphi(N)$.

However, (ii) holds if the reduction step is not inside an argument of a labeled K-redex.[9] If the initial K-redex $\mathbf{K}\ P\ Q$ is such that $P \geqslant^\beta_\infty Q$ and we use F_2^* to compute the middle reduction path, then no reduction will be inside the argument of labeled K-redex. Indeed, when F_2^* contracts $(\lambda^* x.K)L$, $L \in \mathrm{SN}_{\beta *}$. Since F_2^* computes standard reduction paths, this means, roughly, that every residual of the initial labeled K-redex $\underline{\mathbf{K}}\ P\ Q$ has form $\underline{\mathbf{K}}\ P\sigma\ Q\sigma$, where σ is an $\mathrm{SN}_{\beta *}$-substitution. Since $P \geqslant^{\beta *}_\infty Q$, also $P\sigma \in \infty_{\beta *} \Leftarrow Q\sigma \in \infty_{\beta *}$. Therefore, F_2^* does not contract a redex inside $Q\sigma$. It may happen that F_2^* contracts a redex inside $P\sigma$. In this case, all the following reductions will also be inside $P\sigma$.

Theorem 7.30 is due to Bergstra and Klop [7]. Our proof above is a simplification of the proof of Bergstra and Klop. Xi [82] proves Corollary 7.33 directly, instead of proving conservation for Λ_K and the Bergstra–Klop theorem separately. Khasidashvili and Ogawa [37] independently prove Corollary 7.1, using a variant of the strategy F_2, and study applications to various restricted λ-calculi. Corollary 7.32(ii) is also taken from Khasidashvili and Ogawa [37].

7.5. Perpetual and Maximal Redexes

The following proposition shows that the converse of Theorem 7.30 also holds. The idea of the proof is that one can simulate the effect of substitutions by means of contexts and reductions.

7.35. Proposition (Bergstra and Klop [7]). *Assume that*

$$C[\mathbf{K}\ P\ Q] \in \infty_\beta \Rightarrow C[P] \in \infty_\beta$$

for all contexts C. Then $P \geqslant^\beta_\infty Q$.

Proof. To show $P \geqslant^\beta_\infty Q$, let $\mathbf{R} \in \mathrm{SN}_\beta$, and suppose

$$Q\{\mathbf{x} := \mathbf{R}\} \in \infty_\beta.$$

[9] This observation generalizes the earlier observation that (ii) holds in $\underline{\Lambda}_K$ if all labeled redexes are I-redexes. In that case no reduction can take place inside the argument of a labeled K-redex.

Put $C \equiv (\lambda \mathbf{x}.[\,])\mathbf{R}$. Since

$$(\lambda \mathbf{x}.(\mathbf{K}\,P\,Q))\,\mathbf{R} \twoheadrightarrow_{\beta} \mathbf{K}(P\{\mathbf{x} := \mathbf{R}\})(Q\{\mathbf{x} := \mathbf{R}\})$$

also

$$C[\mathbf{K}\,P\,Q] \in \infty_{\beta}.$$

By our assumptions, this implies $C[P] \in \infty_{\beta}$, i.e., $(\lambda \mathbf{x}.P)\mathbf{R} \in \infty_{\beta}$. Since $\mathbf{R} \in \mathrm{SN}_{\beta}$, for some n

$$F_1^n((\lambda \mathbf{x}.P)\mathbf{R}) = P\{\mathbf{x} := \mathbf{R}\}$$

and by perpetuality of F_1, $P\{\mathbf{x} := \mathbf{R}\} \in \infty_{\beta}$ as required. ∎

The following corollary, in which (i) is due to Barendregt *et al.* [2, 4] and (ii) is due to Bergstra and Klop [7], sums up the situation.

7.36. Corollary. *A redex $(\lambda x.P)Q$ is perpetual iff*

(i) *$(\lambda x.P)Q$ is an I-redex*; *or*

(ii) *$(\lambda x.P)Q$ is a K-redex with $P \geqslant^{\beta}_{\infty} Q$.*

Proof. By Corollary 7.20, Corollary 7.31, and Proposition 7.35. ∎

We now proceed to characterize maximal redexes. The intuition is as follows. Given a redex Δ with contractum Δ', we can conceive a context C which is such that $C[\Delta]$ can duplicate Δ. Therefore the longest reduction path from $C[\Delta]$ is obtained only if we do not contract Δ until it has been duplicated. But then Δ is not maximal. The only escape is when the contractum of Δ has an infinite reduction path. Then $C[\Delta']$ has arbitrarily long reduction paths, so Δ is maximal.

7.37. Proposition. *Redex Δ with contractum Δ' is maximal iff $\Delta' \in \infty_{\beta}$.*

Proof. $\Leftarrow$: If $\Delta' \in \infty_{\beta}$ then for any $n > 0$ and context C, $C[\Delta'] \in (n-1)_{\beta}$.

$\Rightarrow$: We assume $\Delta' \in \mathrm{SN}_{\beta}$ and prove that Δ is not maximal by finding an n such that $C[\Delta] \in n_{\beta}$ but not $C[\Delta'] \in (n-1)_{\beta}$.

Since $\Delta' \in \mathrm{SN}_{\beta}$ there is by König's lemma an $m \in \mathbb{N}$ such that $\Delta' \in (m-1)_{\beta}$ and $\Delta' \notin m_{\beta}$. Then $\Delta \in m_{\beta}$. So for $C \equiv (\lambda x.\lambda y.y\,x\,x)[\,]$ we have for some Q

$$C[\Delta] \rightarrow_{\beta} \lambda y.y\,\Delta\,\Delta \twoheadrightarrow^{2m}_{\beta} \lambda y.y\,Q\,Q,$$

that is, $C[\Delta] \in (2m+1)_{\beta}$.

On the other hand, any reduction of $C[\Delta']$ has form

$$C[\Delta'] \twoheadrightarrow^{k}_{\beta} C[Q'] \rightarrow_{\beta} \lambda y.y\,Q'\,Q' \twoheadrightarrow^{2l}_{\beta} \lambda y.y\,Q''\,Q''$$

for some Q', Q'', where $k + l \leqslant m - 1$, and therefore $k + 1 + 2l < 2m$. So, $C[\Delta'] \notin (2m)_{\beta}$. ∎

7.6. *The Normalization Theorem*

In this subsection we prove the normalization theorem for Λ_K which states that repeated contraction of the left-most redex in a weakly normalizing term eventually leads to a normal form. We use a technique very similar to that used to prove conservation theorems in the preceding subsections.

7.38. Definition. Define $F_l^*: \underline{\Lambda}_K \to \underline{\Lambda}_K$ as follows. If $M \in \mathrm{NF}_{\beta *}$ then $F_l^*(M) = M$; otherwise,

$$\begin{array}{lcll} F_l^*(x\,\mathbf{P}\,Q\,\mathbf{P}) & = & x\,\mathbf{P}\,F_l^*(Q)\,\mathbf{R} & \text{if} \quad \mathbf{P} \in \mathrm{NF}_{\beta *}, \quad Q \notin \mathrm{NF}_{\beta *} \\ F_l^*(\lambda x.P) & = & \lambda x.F_l^*(P) & \\ F_l^*((\lambda^* x.P)\,Q\,\mathbf{R}) & = & P\{x := Q\}\,\mathbf{R}. & \end{array}$$

We write $M \to_{l*} N$ if $M \notin \mathrm{NF}_{\beta *}$ and $F_l^*(M) = N$. More specifically, if $M \equiv C[(\underline{\lambda} x.P)\,Q]$ and $C[P\{x := Q\}] \equiv N$ we write $M \to_{\underline{l}} N$, and if $M \equiv C[(\lambda x.P)\,Q]$ and $C[P\{x := Q\}] \equiv N$ we write $M \to_l N$.

7.39. Lemma. *For all* $M \in \underline{\Lambda}_K$: $|F_l^*(M)| = F_l(|M|)$.

Proof. By induction on M. ∎

7.40. Lemma. *Let* $M \in \underline{\Lambda}_K$.

$$\begin{tikzcd} M \arrow[r, "\to_{\underline{l}}"'] \arrow[d, "\varphi"'] & N \arrow[d, "\varphi"] \\ K \arrow[r, dotted, "\to_l"'] & L \end{tikzcd}$$

Proof. By induction on M. ∎

We prove the contrapositive of the normalization theorem: if the left-most reduction path from M does not terminate, then no reduction path does. For this it suffices to show the following result, very similar to the conservation theorems seen earlier—this explains why the technique of the previous subsections is useful.

7.41. Theorem. *If* $M \in \Lambda_K$ *and* $M \to_\beta N$, *then*

$$M \in \infty_l \Rightarrow N \in \infty_l.$$

Proof. Let $M \equiv C[(\lambda x.P)\,Q] \to_\beta C[P\{x := Q\}] \equiv N$. Suppose $M \in \infty_l$, i.e.,

$$M \equiv M_0 \to_l M_1 \to_l M_2 \to_l \cdots.$$

Let $L_0 = C[(\underline{\lambda} x.P)\,Q]$, and $N_0 \equiv N$. By Lemmas 7.5, 7.6, 7.39, and 7.40, we can erect the diagram

$$
\begin{tikzcd}
M_0 \arrow[rr, "\to_l"'] \arrow[dd, "\to_\beta"'] & & M_1 \arrow[rr, "\to_l"'] & & M_2 \arrow[rr, "\to_l"'] & & \cdots \\
& L_0 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_l"'] & & L_1 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_l"'] & & L_2 \arrow[ul, "|\bullet|"'] \arrow[dl, "\varphi"] \arrow[rr, "\to_l"'] & & \cdots \\
N_0 \arrow[rr, "\to_l"'] & & N_1 \arrow[rr, "\equiv"'] & & N_2 \arrow[rr, "\to_l"'] & & \cdots
\end{tikzcd}
$$

where

$$L_i \to_l L_{i+1} \;\Rightarrow\; N_i \to_l N_{i+1}$$

$$L_i \to_l L_{i+1} \;\Rightarrow\; N_i \equiv N_{i+1}.$$

By finiteness of developments, $L_i \to_l L_{i+1}$, for infinitely many i, giving an infinite left-most reduction path from N_0. ▮

7.42. Corollary (Normalization of Left-most Reduction). *F_l is normalizing.*

Proof. Suppose $M \in \mathrm{WN}_\beta$, i.e., $M \twoheadrightarrow_\beta N \in \mathrm{NF}_\beta$. If M had an infinite left-most reduction, then by Theorem 7.41, so did N, a contradiction. ▮

7.43. Definition. Let $M \in \Lambda_K$. A finite or infinite reduction path

$$M_0 \to_\beta M_1 \to_\beta M_2 \to_\beta \cdots$$

is *quasi-left-most* if it is finite or for all $i \in \mathbb{N}$ there is $j > i$ with $M_j \to_l M_{j+1}$.

7.44. Corollary (Normalization of Quasi-Left-most Reductions). *If $M \in \mathrm{WN}_\beta$, then any quasi-left-most reduction from M is finite.*

Proof. First show as in Theorem 7.41 that if $M \to_\beta N$ and M has an infinite quasi-left-most reduction, then so does N. Then proceed as in Corollary 7.42. ▮

The normalization theorem is due to Curry and Feys [15]. Barendregt [2] infers the normalization theorem from the standardization theorem, and uses both of these theorems to prove normalization of quasi-left-most reductions.

Barendregt *et al.* [5] define a β-redex Δ to be *needed* in a term M, if Δ (or a residual of Δ) is contracted in every reduction of M to normal form. They then show that every term not in normal form has at least one needed redex, and that a reduction strategy that contracts only needed redexes is normalizing. They also show that it is undecidable, in general, whether a redex is needed in a term; however, the left-most redex is always needed, and this yields another proof of the normalization theorem. Similar results were shown by Huet and Lévy [26] in their early study of neededness in the context of orthogonal term rewriting systems, and much has been done since in various contexts—see [36] for references to some papers. Similar results were discovered independently by Khasidashvili [31] (see also [33, 35]); in particular, the proof of Theorem 7.41 can be viewed as a special case of a proof due to Khasidashvili [31].

For more on normalization, see [38, 58].

7.7. *Conservation from Normalization*

In this last subsection we give a very short proof of the conservation theorem for Λ_I, using the fact that F_∞ is perpetual and F_l is normalizing.

7.45. Lemma (Regnier [60]). *For all* $M \in \Lambda_I$, $F_l(M) = F_\infty(M)$.

Proof. If $\lambda x.P \subseteq M \in \Lambda_I$, then $x \in \mathrm{FV}(P)$. ∎

7.46. Corollary. (i) *For all* $M \in \Lambda_I$, $M \in \mathrm{WN}_\beta \Leftrightarrow M \in \mathrm{SN}_\beta$.

(ii) *For all* $M \in \Lambda_I$, $M \in \infty_\beta \;\&\; M \to_\beta N \Rightarrow N \in \infty_\beta$.

Proof. (i) Since F_∞ is perpetual and F_l is normalizing, Lemma 7.45 implies

$$M \in \mathrm{WN}_\beta \Leftrightarrow \exists n\colon F_l^n(M) \in \mathrm{NF}_\beta \Leftrightarrow \exists n\colon F_\infty^n(M) \in \mathrm{NF}_\beta \Leftrightarrow M \in \mathrm{SN}_\beta.$$

(ii) Suppose $M \to_\beta N$. If $M \in \infty_\beta$, then by (i), $M \notin \mathrm{WN}_\beta$. Hence $N \notin \mathrm{WN}_\beta$, in particular $N \in \infty_\beta$. ∎

7.47. *Remark.* The same technique can be used to prove that in Λ_ω (see Definition 4.3) all reduction paths have the same length: one proves directly that in Λ_ω, F_∞ is minimal. Since F_∞ is also maximal, the longest and shortest reduction path have the same length, and so *all* reduction paths have the same length.

7.48. *Remark.* Not all strategies are maximal in Λ_I; for instance, the strategy which always contracts the right-most redex is not maximal, as the example $(\lambda x.\lambda y.y\,x\,x)(\mathbf{II}) \twoheadrightarrow_l^3 \lambda y.y\,\mathbf{I}\,\mathbf{I}$ shows.

7.49. *Remark.* A simpler proof of the above corollary, which does not use F_∞, can be obtained by proving directly that F_l is perpetual in Λ_I using the fundamental lemma of perpetuality, rather than inferring this from $F_l = F_\infty$ and perpetuality of F_∞. Slight variations of this technique are due to Curry and Feys [15] and to van Raamsdonk [58].

Barendregt *et al.* [5] show that leftmost reduction paths have maximal length among all reduction paths in which only needed redexes are contracted, and that in Λ_I all redexes are needed. This gives another proof that in Λ_I, F_l is maximal and thereby perpetual.

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