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Some results on 4^m2^n designs with clear two-factor interaction components

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Abstract Clear effects criterion is one of the important rules for selecting optimal fractional factorial designs, and it has become an active research issue in recent years. Tang et al. derived upper and lower bounds on the maximum number of clear two-factor interactions (2fi's) in $2^{n-(n-k)}$ fractional factorial designs of resolutions III and IV by constructing a $2^{n-(n-k)}$ design for given k, which are only restricted for the symmetrical case. This paper proposes and studies the clear effects problem for the asymmetrical case. It improves the construction method of Tang et al. for $2^{n-(n-k)}$ designs with resolution III and derives the upper and lower bounds on the maximum number of clear two-factor interaction components (2fic's) in 4^m2^n designs with resolutions III and IV. The lower bounds are achieved by constructing specific designs. Comparisons show that the number of clear 2fic's in the resulting design attains its maximum number in many cases, which reveals that the construction methods are satisfactory when they are used to construct 4^m2^n designs under the clear effects criterion.

Keywords: **clear, mixed levels, resolution, two-factor interaction component**

MSC(2000): **62K15**

1 Introduction

Orthogonal arrays with mixed levels have been much widely used in experimental design. When the arrays have m 4-level factors and n 2-level factors, they are said to be 4^m2^n designs. A 4^m2^n design can be constructed by the method of replacement, which was first formally introduced in [1]. This class of designs is useful in practice because in factorial investigations, especially those involving physical experiments, the number of factorial levels seldom exceeds four. [2] improved the construction method in [1] by introducing the method of grouping. [3] extended the grouping scheme in [2] to cover more general $s^m(s^{r_1})^{n_1}\cdots(s^{r_t})^{n_t}$ designs for any prime power s and some integers r_i and n_i.

In this paper, we consider 4^m2^n designs with $N=2^k$ runs and suppose that such designs are constructed by the method of grouping. Let $A_1,\dots,A_m$ and $b_1,\dots,b_n$ denote the 4-level factors and 2-level factors of a 4^m2^n design, respectively. Suppose a 4^m2^n design is obtained

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by replacing three 2-level factors $\{a_{i1}, a_{i2}, a_{i3}\}$ with a 4-level factor A_i, where $a_{i1}a_{i2}a_{i3} = I, i = 1, \ldots, m$ and I is the column with all entries zero. Such a design is determined by $B = \{a_{11}, a_{12}, a_{13}, \ldots, a_{m1}, a_{m2}, a_{m3}, b_1, \ldots, b_n\}$, where $a_{i1}a_{i2}a_{i3} = I$, $i = 1, \ldots, m$, in the following sections. We call $a_{i_1j_1}$ the main-effect component of A_{i_1}, and $a_{i_1j_1}a_{i_2j_2}$ (or $a_{i_1j_1}b_l$) the two-factor interaction component (2fic) of A_{i_1} and A_{i_2} (or A_{i_1} and b_l), where $i_1, i_2 = 1, \ldots, m, i_1 \neq i_2, j_1, j_2 = 1, 2, 3, l = 1, \ldots, n$. For convenience, we call both the main effects of 2-level factors and the main-effect components of 4-level factors the main-effect components. For the same reason, the two-factor interactions (2fi's) of two 2-level factors, the 2fic's of two 4-level factors, and 2fic's of a 2-level factor and a 4-level factor are all called 2fic's.

When the experimenter's knowledge is diffuse, a reasonable assumption people can make is the effect hierarchical assumption. Under such circumstances, resolution in [4] and minimum aberration in [5] are the most often used criteria for selecting good designs. Extending them to the mixed-level case, [6] gave the definitions of resolution and minimum aberration criteria for selecting good 4^m2^n designs. For $m = 1$, suppose that $a_1, a_2, a_3, b_1, \ldots, b_n$ are columns chosen from the $2^k - 1$ columns of a saturated design with 2^k runs such that $a_1a_2a_3 = I$. A 4^12^n design can be obtained by replacing $\{a_1, a_2, a_3\}$ with a 4-level factor. It is easy to see that there are two types of defining contrasts for this design. The first involves only the b_j's, which is called type 0. The second involves one of the a_i's and some of the b_j's, which is called type 1. For a 4^12^n design D, let $W_{i0}(D)$ and $W_{i1}(D)$ be the numbers of type 0 and type 1 words of length i in the defining contrasts of D, respectively. The resolution of D is defined to be the smallest i such that $W_{ij}(D)$ is positive for at least one j. For $m = 2$, the resolution for 4^22^n designs is defined similarly as that of 4^12^n designs. Furthermore, [7] deliberated a method for constructing this class of asymmetric minimum aberration designs through symmetric minimum aberration ones, [8] obtained two types of minimum aberration designs with mixed levels in terms of complementary sets, and [9] improved the results in [8].

Different situations call for different designs. Clear effects criterion[10] is another criterion for selecting good designs. A main-effect component of a factor is said to be clear if it is not aliased with any main-effect component of the other factors or any 2fic. A main effect is said to be clear if all its components are clear. A 2fic is said to be clear if it is not aliased with any main-effect component or any other 2fic. A two-factor interaction (2fi) is said to be clear if all its components are clear. As usual, we assume that interaction components involving three or more factors are negligible. A design of resolution V or higher permits the estimation of all the main effects and 2fi's. In what follows, we look at the case where the experimenter cannot afford a design of resolution V or higher. A resolution IV design with the maximum number of clear 2fic's allows the joint estimation of the whole main effects and the clear 2fic's as many as possible in the presence of other 2fic's. It is a desirable design when we are interested in estimating 2fic's besides the main effects. For a resolution III design, we can assume the magnitude of the main-effect components is much larger than that of the 2fic's. Although the presence of 2fic's which are not clear can bias the estimates of the main-effect components, this bias will not be substantial. Thus, in this paper, we are interested in estimating as many 2fic's as possible. A 4^m2^n design with the maximum number of clear 2fic's will be called a MaxC2cR design if it has resolution R.

Recent results on the clear effects criterion include [11] and [12]–[23]. In particular, [11] derived upper and lower bounds on the maximum number of clear 2fi's in $2^{n-(n-k)}$ fractional factorial designs of resolutions III and IV by constructing a $2^{n-(n-k)}$ design for given k. [17] investigated the structure of 4^m2^n designs with resolution III or IV from a different angle if one's goal is to check the existence of clear 2fic's in the design. It showed that a 4^m2^n design has a clear 2fic if and only if $n \leqslant 2^{k-1} - 3m$ for $m = 1, 2$. For a resolution IV 4^m2^n design to have a clear 2fic, it is proved that the necessary and sufficient condition is $n \leqslant 2^{k-2} + 2 - 3m$ for $m = 1, 2$. When $2^{k-2} + 2 - 3m < n \leqslant 2^k - 1 - 3m$ for $m = 1, 2$, resolution IV design does not exist. This paper tries to modify the method in [11] to improve the lower bound on the maximum number of clear 2fi's in $2^{n-(n-k)}$ designs of resolution III and derive the upper and lower bounds on the maximum number of clear 2fic's in 4^m2^n designs of resolutions III and IV for $m = 1, 2$.

The paper is organized as follows. Section 2 presents the construction method for $2^{n-(n-k)}$ designs containing as many clear 2fi's as possible. Sections 3 and 4 obtain the lower and upper bounds on the maximum numbers of clear 2fic's in 4^m2^n designs for $m = 1, 2$. And Section 5 examines the performance of these bounds for $k = 5$.

2 Bounds on the maximum number of clear two-factor interactions for $2^{n-(n-k)}$ designs with resolution III

This section sketches out a construction method for resolution III $2^{n-(n-k)}$ designs containing as many clear 2fi's as possible when $k \geqslant 5$.

Let $\alpha(k, n)$ be the maximum number of clear 2fi's in a $2^{n-(n-k)}$ design with resolution III. Let $n_j = 2^j + 2^{k-j} - 2, j = 1, \dots, J$, where $J = \lfloor k/2 \rfloor$ and $\lfloor z \rfloor$ denotes the largest integer not exceeding z. Clearly, we have $n_1 > \cdots > n_J$. If $n > n_1$, there does not exist any design containing clear 2fi (see [12]). One needs only to examine values of n in the range of $M(k) < n \leqslant n_1$, where $M(k)$ is the maximum value of n for a $2^{n-(n-k)}$ design to have resolution at least V.

Suppose that $n = n_j$ for some $j = 1, \dots, J$. Let $a_1, \dots, a_k$ be the k independent columns and H_k be the saturated design generated by $a_1, \dots, a_k$. Define D_j as

$$D_j = H_j \cup H_{k-j}, \tag{1}$$

where $H_j = H_j(a_1, \dots, a_j)$ is a subset of H_k, generated by $a_1, \dots, a_j$, $H_{k-j} = H_{k-j}(a_{j+1}, \dots, a_k)$ is the subset of H_k, generated by $a_{j+1}, \dots, a_k$. Note that H_j and H_{k-j} contain $2^j - 1$ and $2^{k-j} - 1$ columns, respectively. Then design D_j contains $n_j = 2^j + 2^{k-j} - 2$ columns. For any $a \in H_j$ and $b \in H_{k-j}$, ab is clear. In addition, there is no clear 2fi within H_j or H_{k-j}. This implies that the number of clear 2fi's in D_j is $(2^j - 1)(2^{k-j} - 1)$.

When $n = n_j + 1$ for some $j = 2, \dots, J$, let $D' = D_j \cup \{a_1a_{j+1}\}$. Then D' has $n = n_j + 1$ columns. Note that any 2fi which is not clear in D_j is still not clear in D', we need only to calculate the number l_1 of 2fi's which are clear in D_j but not in D' any more and the number l_2 of 2fi's which are clear in D' but not in D_j originally. From the discussion in the last paragraph, a_1a_{j+1} is clear in D_j and there are $(2^j - 1)(2^{k-j} - 1)$ clear 2fi's in D_j. Let d denote the column a_1a_{j+1} in D'. For any $p \in H_j$ and $q \in H_{k-j}$, if pq is not clear in D', then $p = a_1, q = a_{j+1}$ or there must exist $c \in D_j$ such that $cpq = d$. There are two cases:

(i) $p \in H_j \backslash \{a_1\}, q = a_{j+1}, c = a_1p \in H_j$ and $cpq = d$;

(ii) $p = a_1, q \in H_{k-j} \backslash \{a_{j+1}\}, c = a_{j+1}q \in H_{k-j}$ and $cpq = d$.

Then there are $l_1 = (2^j - 2) + (2^{k-j} - 2) + 1 = 2^j + 2^{k-j} - 3$ 2fi's which are clear in D_j but not in D'. Note that dp is not clear in D' for any $p \in D_j$, hence $l_2 = 0$. Therefore D' has $(2^j - 1)(2^{k-j} - 1) - l_1 + l_2 = 2^k - 2^{j+1} - 2^{k-j+1} + 4$ clear 2fi's.

Now consider the case $n_j > n > n_{j+1} + 1$ for some $j = 1, \ldots, J$, where n_{J+1} is defined as $n_{J+1} = 2(2^J - 1) + 1$. When k is even, $n_J < n_{J+1}$, and the case $n_J > n > n_{J+1} + 1$ in fact does not exist and can be ignored. The case $n_J > n > n_{J+1} + 1$ is only non-trivial for odd k. Let $D = H_j \cup H^*_{k-j}$, where H_j is the same as that in D_j defined by (1) and H^*_{k-j} is obtained from H_{k-j} in D_j by deleting any $n_j - n$ columns from H_{k-j}. Note that for any $a \in H_j$ and $b \in H^*_{k-j}$, ab is clear in D. In addition, there is no clear 2fi within H_j or H^*_{k-j}. Hence the number of clear 2fi's in D is simply $(2^j - 1)(n - 2^j + 1)$.

When k is odd and $n = n_{J+1} + 1 = 2^{J+1}$, let

$$D_{J+1} = \{a_1\} \cup H'_J \cup H'_{k-J-1}, \tag{2}$$

where $H'_J = H_J(a_2, \ldots, a_{J+1})$ and $H'_{k-J-1} = H_{k-J-1}(a_{J+2}, \ldots, a_k)$ are subsets of H_k generated by $a_2, \ldots, a_{J+1}$ and $a_{J+2}, \ldots, a_k$, respectively. Since for any $p \in H'_J, q \in H'_{k-J-1}$, a_1p, a_1q, pq are all clear in D_{J+1}, the number of clear 2fi's in D_{J+1} is $(2^J - 1) + (2^{k-J-1} - 1) + (2^J - 1)(2^{k-J-1} - 1) = 2^{2J} - 1$. Let $D'_{J+1} = D_{J+1} \cup \{a_1a_2\}$, then D'_{J+1} has $n = 2^{J+1}$ columns. Note that for any $p \in H'_J \backslash \{a_2\}$, $a_1p = (a_1a_2)(a_2p)$ and $a_2p \in D'_{J+1}$, so a_1p is not clear in D'_{J+1}. And for any $q \in H'_{k-J-1}$, $(a_1a_2)q$ is clear in D'_{J+1}. Since a_1a_2 is not clear in D'_{J+1}, there are $2^{2J} - 1 - (2^J - 1) + (2^{k-J-1} - 1) = 2^{2J} - 1$ clear 2fi's in D'_{J+1}.

When k is odd and $n \leqslant n_{J+1}$, let $D' = \{a_1\} \cup H^*_J \cup H^*_{k-J-1}$, where H^*_J is a subset of H'_J with $\lfloor n/2 \rfloor$ columns and H^*_{k-J-1} is a subset of H'_{k-J-1} with $n - \lfloor n/2 \rfloor - 1$ columns, and H'_J, H'_{k-J-1} are the same as those in (2). Since for any $p \in H^*_J, q \in H^*_{k-J-1}$, a_1p, a_1q, pq are all clear in D', the total number of clear 2fi's is at least $\lfloor n/2 \rfloor + (n - \lfloor n/2 \rfloor - 1) + \lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor - 1) = n - 1 + \lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor - 1)$.

When k is even and $n \leqslant n_J$, let $D = \tilde{H}_J \cup \tilde{H}_{k-J}$, where $\tilde{H}_J$ and $\tilde{H}_{k-J}$ are subsets of H_J and H_{k-J} defined by (1) for $j = J$, respectively, and there are $\lfloor n/2 \rfloor$ columns in $\tilde{H}_J$ and $n - \lfloor n/2 \rfloor$ columns in $\tilde{H}_{k-J}$. Since for any $p \in \tilde{H}_J, q \in \tilde{H}_{k-J}$, pq is clear in D, the number of clear 2fi's in D is at least $\lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor)$.

We summarize the above results in the following

Theorem 1. *Suppose $k \geqslant 5$, then a lower bound $\alpha_l(k, n)$ on the maximum number of clear 2fi's of a $2^{n-(n-k)}$ design is given by*

$$\alpha_l(k,n) = \begin{cases} (2^j - 1)(n - 2^j + 1), & \text{if } n_j \geqslant n > n_{j+1} + 1, \text{ for } j = 1, \ldots, J, \\ 2^k - 2^{j+1} - 2^{k-j+1} + 4, & \text{if } n = n_j + 1, \text{ for } j = 2, \ldots, J, \\ 2^{2J} - 1, & \text{if } n = n_{J+1} + 1, \text{ for odd } k, \\ n - 1 + e_1(n - e_1 - 1), & \text{if } n \leqslant n_{J+1}, \text{ for odd } k, \\ e_1(n - e_1), & \text{if } n \leqslant n_J, \text{ for even } k, \end{cases}$$

where $J = \lfloor k/2 \rfloor$, $n_j = 2^j + 2^{k-j} - 2$ for $j = 1, \ldots, J$, $n_{J+1} = 2(2^J - 1) + 1$, $e_1 = \lfloor n/2 \rfloor$.

The following remark is useful for constructing 4^m2^n designs in the subsequent sections.

Remark 1. When $k \geqslant 5$, we can always construct a $2^{n-(n-k)}$ design D for any $M(k) < n \leqslant n_1$ such that there are six factors $\{c_1, c_2, c_3\}$ and $\{c_4, c_5, c_6\}$ in D satisfying $\{c_1, c_2, c_3\} \cap \{c_4, c_5, c_6\} = \emptyset$ and $c_1c_2c_3 = c_4c_5c_6 = I$. When $n_1 \geqslant n > n_2 + 1$, the columns $\{a_2, a_3, a_2a_3\}$ and $\{a_4, a_5, a_4a_5\}$ satisfy the condition from the construction of D_j in (1) for $j = 1$. When $n_j + 1 \geqslant n > n_{j+1} + 1, j = 2, \ldots, J$, the columns $\{a_1, a_2, a_1a_2\}$ and $\{a_{j+1}, a_{j+2}, a_{j+1}a_{j+2}\}$ satisfy the condition from the construction of D_j in (1). When $n = n_{J+1} + 1$ and k is odd, $\{a_2, a_3, a_2a_3\}$ and $\{a_{J+2}, a_{J+3}, a_{J+2}a_{J+3}\}$ satisfy the condition from the construction of D_{J+1} in (2). For $n \leqslant n_{J+1}$ and odd k, H_J^* and H_{k-J-1}^* can be selected such that there are a subset $\{c_1, c_2, c_3\}$ of H_J^* satisfying $c_1c_2c_3 = I$ and a subset $\{c_4, c_5, c_6\}$ of H_{k-J-1}^* satisfying $c_4c_5c_6 = I$. When $n = n_J + 1$ and k is even, $\{a_1, a_2, a_1a_2\}$ and $\{a_{J+1}, a_{J+2}, a_{J+1}a_{J+2}\}$ satisfy the condition from the construction of D_J in (1) for $j = J$. For $n \leqslant n_J$ and even k, $\tilde{H}_J$ and $\tilde{H}_{k-J}$ can be selected such that there are a subset $\{c_1, c_2, c_3\}$ of $\tilde{H}_J$ satisfying $c_1c_2c_3 = I$ and a subset $\{c_4, c_5, c_6\}$ of $\tilde{H}_{k-J}$ satisfying $c_4c_5c_6 = I$.

3 Bounds on the maximum number of clear 2fic's for 4^12^n designs

In this section, construction methods for 4^12^n designs with resolutions III and IV are provided. The results in Section 5 indicate that the construction method performs well for 4^12^n designs with resolution III, but this does not hold for 4^12^n designs with resolution IV. Let $k \geqslant 5$ and $\alpha(k, n, R)$ be the maximum number of clear 2fic's in 4^12^n designs with $N = 2^k$ runs and resolution R.

3.1 Bounds on the maximum number of clear 2fic's for 4^12^n designs with resolution III

This subsection is devoted to establishing upper and lower bounds on $\alpha(k, n, \mathrm{III})$. The fact that the $n+3$ main-effect components and $\alpha(k, n, \mathrm{III})$ clear 2fic's are not mutually aliased with each other implies that $n + 3 + \alpha(k, n, \mathrm{III}) \leqslant 2^k - 1$. Therefore $\alpha(k, n, \mathrm{III}) \leqslant 2^k - n - 4$, and an upper bound on $\alpha(k, n, \mathrm{III})$ is thus established.

Theorem 2. *The maximum number $\alpha(k, n, R)$ of clear 2fic's in a 4^12^n design with $R = \mathrm{III}$ is bounded above by $\alpha_u(k, n, \mathrm{III}) = 2^k - n - 4$.*

We now describe the method for constructing resolution III 4^12^n designs with clear 2fic's. Let $n_j = 2^j + 2^{k-j} - 5, j = 1, \ldots, J$, where $J = \lfloor k/2 \rfloor$. Clearly, we have $n_1 > \cdots > n_J$. If $n > n_1$, there does not exist any 4^12^n design containing clear 2fic's (see [17]). We need only to examine values of n in the range of $M'(k) < n \leqslant n_1$, where $M'(k)$ denotes the maximum value of n for a 4^12^n design to have resolution at least V.

Note that a 4^12^n design with 2^k runs can be constructed from a $2^{(n+3)-(n+3-k)}$ design. When $M'(k) < n \leqslant n_1$, we can construct a $2^{(n+3)-(n+3-k)}$ design such that there are three factors $\{c_1, c_2, c_3\}$ satisfying $c_1c_2c_3 = I$ as discussed in Remark 1. Then replacing $\{c_1, c_2, c_3\}$ with a 4-level factor, we obtain a 4^12^n design with 2^k runs, which has the same number of clear 2fic's as the $2^{(n+3)-(n+3-k)}$ design has. And from Theorem 2, the number of clear 2fic's attains the upper bound when $n = n_j$. The results are summarized in the following

Theorem 3. Suppose $k \geqslant 5$, *then the design constructed above for $n = n_j$ with $j = 1, \ldots, J$ has the maximum number of clear 2fic's, and more generally, a lower bound $\alpha_l(k, n, \mathrm{III})$ on the* *maximum number of clear 2fic's of a* $4^1 2^n$ *design is given by*

$$\alpha_l(k,n,\mathrm{III})=\begin{cases}(2^j-1)(n-2^j+4), & \text{if } n_j \geqslant n > n_{j+1}+1, \text{ for } j=1,\dots,J,\\ 2^k-2^{j+1}-2^{k-j+1}+4, & \text{if } n=n_j+1, \text{ for } j=2,\dots,J,\\ 2^{2J}-1, & \text{if } n=n_{J+1}+1, \text{ for odd } k,\\ n+2+e_2\,(n+2-e_2), & \text{if } n\leqslant n_{J+1}, \text{ for odd } k,\\ e_2\,(n+3-e_2), & \text{if } n\leqslant n_J, \text{ for even } k,\end{cases}$$

where $J=\lfloor k/2\rfloor$, $n_j=2^j+2^{k-j}-5$ *for* $j=1,\dots,J$, $n_{J+1}=2(2^J-1)-2$, $e_2=\lfloor (n+3)/2\rfloor$.

3.2 Bounds on the maximum number of clear 2fic's for $4^1 2^n$ designs with resolution IV

The upper and lower bounds on $\alpha(k,n,\mathrm{IV})$ are established in this subsection. To build the upper bound, an approach similar to one used in [20] for obtaining an upper bound on the maximum number of clear 2fi's for blocked 2-level fractional factorial designs is utilized here. First let us see some notations from [13, 24]. Let E be a $4^m 2^n$ design determined by $B=\{a_{11},a_{12},a_{13},\dots,a_{m1},a_{m2},a_{m3},b_1,\dots,b_n\}$, where $a_{i1}a_{i2}a_{i3}=I, i=1,\dots,m$. And let $m_j(E)$ be the number of 2fic's in the j-th alias set not containing main-effect components, where $j=1,\dots,f_m$, $f_m=2^k-1-n-3m$. Also let $I(E)$ denote the number of 2fic's of E, and $N_i=\#\{1\leqslant j\leqslant f_m : m_j(E)=i\}$ for $i\geqslant 0$ be the number of alias sets that contain i 2fic's. Let $C(E)$ and $U(E)$ denote the numbers of clear 2fic's and unclear 2fic's of E, respectively. Then $C(E)=N_1$, $I(E)=(n+3m)(n+3m-1)/2-3m$ and $U(E)=I(E)-C(E)=(n+3m)(n+3m-1)/2-3m-C(E)$.

Note that any two 2fic's in the same alias set do not share a common letter. Thus $m_j(E)\leqslant r$ and $N_i=0$ for $i>r$, where $r=\lfloor (n+3m)/2\rfloor$. If $N_i>0$, there exists an alias set with i 2fic's. These 2fic's contain $2i$ letters, and any two of which form an unclear 2fic, thus $U(E)\geqslant i(2i-1)$. Note that there is at most one of a_{j1},a_{j2} and a_{j3} for any $j=1,\dots,m$ among these $2i$ letters, so $N_{r-m+1}=\cdots=N_r=0$. Then we have the following Lemma 1.

Lemma 1. *If* $N_i>0$ *for some* i*, where* $2\leqslant i\leqslant r-m$*, then* $U(E)\geqslant i(2i-1)$.

Lemma 2. (i) *If* $N_i=0$ *for* $i=j+1,\dots,r$*, where* $2<j<r$*, then* $C(E)\leqslant\{(j-1)f_m+N_j-I(E)\}/(j-2)$.

(ii) *If* $N_i=0$ *for* $i=j,\dots,r$*, where* $2<j\leqslant r$*, then* $C(E)\leqslant\{(j-1)f_m-I(E)\}/(j-2)$.

The proof of Lemma 2 is similar to that of Lemma 4.6 in [13], we omit it here.

For $m=1$, it follows that $N_r=0$, then from Lemma 2 (ii), we can obtain

$$C(E)\leqslant\begin{cases}C_{1o}=\{(n+1)f_1-n^2-5n\}/(n-1), & \text{if } n>1 \text{ is odd},\\ C_{1e}=\{nf_1-n^2-5n\}/(n-2), & \text{if } n>2 \text{ is even}.\end{cases}\tag{3}$$

Next, let us consider the two cases (i) $N_r=0$, $N_{r-1}>0$, and (ii) $N_r=0$, $N_{r-1}=0$ for $m=1$. For Case (i), from Lemma 1, we have

$$C(E)\leqslant\begin{cases}C_{2o}=2n, & \text{if } n \text{ is odd},\\ C_{2e}=3n, & \text{if } n \text{ is even}.\end{cases}\tag{4}$$

On the other hand for Case (ii), from Lemma 2 (ii), we have

$$C(E) \leqslant \begin{cases} C_{3o} = \{(n-1)f_1 - n^2 - 5n\}/(n-3), & \text{if } n > 3 \text{ is odd}, \\ C_{3e} = \{(n-2)f_1 - n^2 - 5n\}/(n-4), & \text{if } n > 4 \text{ is even}. \end{cases} \tag{5}$$

Then based on (3), (4) and (5), we obtain

Theorem 4. *If $n > 4$, the maximum number $\alpha(k,n,R)$ of clear 2fic's in a $4^1 2^n$ design with $R = \mathrm{IV}$ is bounded above by*

$$\alpha_u(k,n,\mathrm{IV}) = \begin{cases} \min\{\lfloor C_{1o} \rfloor, \max\{C_{2o}, \lfloor C_{3o} \rfloor\}\}, & \textit{if } n \textit{ is odd}, \\ \min\{\lfloor C_{1e} \rfloor, \max\{C_{2e}, \lfloor C_{3e} \rfloor\}\}, & \textit{if } n \textit{ is even}. \end{cases}$$

A lower bound $\alpha_l(k,n,\mathrm{IV})$ on $\alpha(k,n,\mathrm{IV})$ is derived through constructing $4^1 2^n$ designs with resolution IV. Theorem 5 summarizes the results and the detailed construction is given in Appendix for simplicity.

Theorem 5. Suppose $k \geqslant 5$, *then a lower bound $\alpha_l(k,n,\mathrm{IV})$ on the maximum number of clear 2fic's of a $4^1 2^n$ design is given by*

$$\alpha_l(k,n,\mathrm{IV}) = \begin{cases} 2n, & \textit{if } n_2 \geqslant n > n_3, \\ 2^k - 3 \times 2^{k-j} - 3 \times 2^j + 10, & \textit{if } n = n_j,\ \textit{for } j = 3, \ldots, J, \\ 2^k - 3 \times 2^{k-j} - 3 \times 2^j + 11, & \textit{if } n = n_j - 1,\ \textit{for } j = 3, \ldots, J, \\ 2^k - 3 \times 2^{k-j} - 2^{j+1} + 5, & \textit{if } n = n_j - 2,\ \textit{for } j = 3, \ldots, J, \\ (2^j - 3)(n - 2^j + 5) + n - 1, & \textit{if } n_j - 3 \geqslant n > n_{j+1},\ \textit{for } j = 3, \ldots, J, \\ 2^{2J} - 2^{J+1} - 4, & \textit{if } n = n_{J+1},\ \textit{for odd } k, \\ e_3\,(n + 2 - e_3) + n - 1, & \textit{if } n < n_{J+1}, \end{cases}$$

where $J = \lfloor k/2 \rfloor$, $n_j = 2^j + 2^{k-j} - 5$ for $j = 2, \ldots, J$, $n_{J+1} = 2(2^J - 2) - 2$, $e_3 = \lfloor (n+2)/2 \rfloor$.

Remark 2. When k is even, $n_{J+1} + 1 = n_J$, and the case $n = n_J - 2$ is included in both items $n = n_J - 2$ and $n < n_{J+1}$ in Theorem 5, thus we can select

$$\alpha_l(k, n_J - 2, \mathrm{IV}) = \max\left\{2^k - 3 \times 2^{k-J} - 2^{J+1} + 5,\ e_3\,(n_J - e_3) + n_J - 3\right\},$$

where $e_3 = \lfloor (n+2)/2 \rfloor$.

4 Bounds on the maximum number of clear 2fic's for $4^2 2^n$ designs

This section provides the construction methods for $4^2 2^n$ designs with resolutions III and IV. The results in Section 5 indicate that the construction methods perform well for both cases of resolutions III and IV. Suppose $k \geqslant 5$ and let $\beta(k,n,R)$ denote the maximum number of clear 2fic's in $4^2 2^n$ designs with $N = 2^k$ runs and resolution R.

4.1 Bounds on the maximum number of clear 2fic's for $4^2 2^n$ designs with resolution III

The upper and lower bounds on $\beta(k,n,\mathrm{III})$ are established in this subsection. The fact that the $n+6$ main-effect components and $\beta(k,n,\mathrm{III})$ clear 2fic's are not mutually aliased with each other implies that $n + 6 + \beta(k, n, \mathrm{III}) \leqslant 2^k - 1$. Therefore $\beta(k, n, \mathrm{III}) \leqslant 2^k - n - 7$. Thus an upper bound on $\beta(k, n, \mathrm{III})$ is established. This result is summarized in Theorem 6.

Theorem 6. *The maximum number $\beta(k, n, R)$ of clear 2fic's in a $4^2 2^n$ design with $R = \mathrm{III}$ is bounded above by $\beta_u(k, n, \mathrm{III}) = 2^k - n - 7$.*

Let $n_j = 2^j + 2^{k-j} - 8, j = 1, \ldots, J$, where $J = \lfloor k/2 \rfloor$. Clearly, we have $n_1 > \cdots > n_J$. If $n > n_1$, there does not exist any $4^2 2^n$ design containing clear 2fic's (see [17]). We need only to examine values of n in the range of $M''(k) < n \leqslant n_1$, where $M''(k)$ denotes the maximum value of n for a $4^2 2^n$ design to have resolution at least V.

Note that a $4^2 2^n$ design with 2^k runs can be constructed from a $2^{(n+6)-(n+6-k)}$ design. Then similarly to the case of $4^1 2^n$ design, when $M''(k) < n \leqslant n_1$, we can construct a $2^{(n+6)-(n+6-k)}$ design such that there are six columns $\{c_1, c_2, c_3\}$ and $\{c_4, c_5, c_6\}$ satisfying $\{c_1, c_2, c_3\} \cap \{c_4, c_5, c_6\} = \emptyset$ and $c_1c_2c_3 = c_4c_5c_6 = I$ as discussed in Remark 1. Then replacing $\{c_1, c_2, c_3\}$ and $\{c_4, c_5, c_6\}$ with two 4-level columns, respectively, we obtain a $4^2 2^n$ design with 2^k runs, which has the same number of clear 2fic's as the $2^{(n+6)-(n+6-k)}$ design has. And from Theorem 6, the number of clear 2fic's attains the upper bound when $n = n_j$. The results are summarized in the following

Theorem 7. *Suppose $k \geqslant 5$, the design constructed above for $n = n_j$ with $j = 1, \ldots, J$ has the maximum number of clear 2fic's, and more generally, a lower bound $\beta_l(k, n, \mathrm{III})$ on the maximum number of clear 2fic's of a $4^2 2^n$ design is given by*

$$\beta_l(k,n,\mathrm{III}) = \begin{cases} (2^j - 1)(n - 2^j + 7), & \textit{if } n_j \geqslant n > n_{j+1} + 1, \ \textit{for } j = 1, \ldots, J, \\ 2^k - 2^{j+1} - 2^{k-j+1} + 4, & \textit{if } n = n_j + 1, \ \textit{for } j = 2, \ldots, J, \\ 2^{2J} - 1, & \textit{if } n = n_{J+1} + 1, \ \textit{for odd } k, \\ n + 5 + e_4\,(n + 5 - e_4)\,, & \textit{if } n \leqslant n_{J+1}, \ \textit{for odd } k, \\ e_4\,(n + 6 - e_4)\,, & \textit{if } n \leqslant n_J, \ \textit{for even } k, \end{cases}$$

where $J = \lfloor k/2 \rfloor$, $n_j = 2^j + 2^{k-j} - 8$ for $j = 1, \ldots, J$, $n_{J+1} = 2(2^J - 1) - 5$, $e_4 = \lfloor (n+6)/2 \rfloor$.

4.2 Bounds on the maximum number of clear 2fic's for $4^2 2^n$ designs with resolution IV

In this subsection, the upper and lower bounds on $\alpha(k, n, \mathrm{IV})$ are established.

For $m = 2$, it follows from the discussions in Subsection 3.2 that $N_{r-1} = N_r = 0$, then from Lemma 2 (ii), we can obtain

$$C(E) \leqslant \begin{cases} C'_{1o} = \{(n+1)f_2 - (n+2)(n+9)\}/(n-1), & \text{if } n > 1 \text{ is odd}, \\ C'_{1e} = \{(n+2)f_2 - (n+2)(n+9)\}/n, & \text{if } n \text{ is even}. \end{cases} \tag{6}$$

Next, let us consider the two cases (i) $N_{r-1} = N_r = 0$, $N_{r-2} > 0$, and (ii) $N_{r-1} = N_r = 0$, $N_{r-2} = 0$ for $m = 2$. For Case (i), from Lemma 1, we have

$$C(E) \leqslant \begin{cases} C'_{2o} = 5n + 9, & \text{if } n \text{ is odd}, \\ C'_{2e} = 4n + 8, & \text{if } n \text{ is even}. \end{cases} \tag{7}$$

On the other hand for Case (ii), from Lemma 2 (ii), we have

$$C(E) \leqslant \begin{cases} C'_{3o} = \{(n-1)f_2 - (n+2)(n+9)\}/(n-3), & \text{if } n > 3 \text{ is odd}, \\ C'_{3e} = \{nf_2 - (n+2)(n+9)\}/(n-2), & \text{if } n > 2 \text{ is even}. \end{cases} \tag{8}$$

Then based on (6), (7) and (8), the following theorem can be obtained.

Theorem 8. *If $n > 3$, the maximum number $\beta(k,n,R)$ of clear 2fic's in a $4^2 2^n$ design with $R = \text{IV}$ is bounded above by*

$$\beta_u(k,n,\text{IV}) = \begin{cases} \min\{\lfloor C'_{1o} \rfloor, \max\{C'_{2o}, \lfloor C'_{3o} \rfloor\}\}, & \text{if } n \text{ is odd}, \\ \min\{\lfloor C'_{1e} \rfloor, \max\{C'_{2e}, \lfloor C'_{3e} \rfloor\}\}, & \text{if } n \text{ is even}. \end{cases}$$

Remark 3. When $k = 5, m = 2, n = 2$ or 3, $r = 4$, $N_3 = N_4 = 0$, clearly we have $N_2 > 0$, hence $\beta_u(5,2,\text{IV}) = \min\{\lfloor C'_{1e} \rfloor, C'_{2e}\} = 16$ and $\beta_u(5,3,\text{IV}) = \min\{\lfloor C'_{1o} \rfloor, C'_{2o}\} = 14$.

By constructing $4^2 2^n$ designs with resolution IV, a lower bound $\beta_l(k,n,\text{IV})$ on $\beta(k,n,\text{IV})$ is obtained, which is shown in Theorem 9. For simplicity, the detailed construction is given in Appendix.

Theorem 9. *Suppose $k \geqslant 5$, then a lower bound $\beta_l(k,n,\text{IV})$ on the maximum number of clear 2fic's of a $4^2 2^n$ design is given by*

$$\beta_l(k,n,\text{IV}) = \begin{cases} 4(n_2 - n) + 6, & \text{if } n_2 \geqslant n > n_3, \\ 2^k - (2^j - 8)(n_j - n) - 4n_j - 13, & \text{if } n_j \geqslant n > n_{j+1}, \text{ for } j = 3, \ldots, J, \\ e_5(n - e_5 - 4) + 2^{J+2} + 2n + 1, & \text{if } 2^J - 4 < n \leqslant n_{J+1}, \text{ for odd } k, \\ e_5(n - e_5 + 4) + 2^{J+3} - 6n - 33, & \text{if } 2^J - 4 < n \leqslant n_{J+1}, \text{ for even } k, \\ e_5(n + 4 - e_5) + 2n + 3, & \text{if } n \leqslant 2^J - 4, \end{cases}$$

where $J = \lfloor k/2 \rfloor$, $n_j = 2^j + 2^{k-j} - 8$ for $j = 2, \ldots, J$, $n_{J+1} = 2(2^J - 2) - 5$, $e_5 = \lfloor (n+4)/2 \rfloor$.

5 Performance of the construction methods

This section examines the performance of the lower and upper bounds on the maximum number of clear 2fic's obtained in Sections 3 and 4 for $k = 5$. All the MaxC2cR $4^m 2^n$ designs with 2^k runs in the following tables are constructed from $2^{(3m+n)-(3m+n-k)}$ designs which are obtained through computer searches. Here the details are omitted for simplicity.

For $k = 5$, let a_1, a_2, a_3, a_4 and a_5 denote the five independent columns $(10000)'$, $(01000)'$, $(00100)'$, $(00010)'$ and $(00001)'$, respectively. Then any product of a_1, a_2, a_3, a_4 and a_5 also corresponds to a binary sequence, for example $a_1 a_3 a_5$ corresponds to $(10101)'$. After converting these binary sequences into base-ten system in Table 1, a $2^{m-(m-k)}$ design D' can be obtained by selecting a subset of m columns of $C = \{1, \ldots, 31\}$, consisting of k independent columns and $m - k$ additional columns. Then we can get a $4^1 2^n$ design D by replacing three columns, say $\{b_1, b_2, b_3\}$ of D' satisfying $b_1 b_2 b_3 = I$, with a 4-level column. $4^2 2^n$ designs can also be obtained similarly.

For simplicity, we omit the independent columns in the following tables. The 4-level column of each design in Table 2 is obtained from $\{1, 2, 3\}$ in Table 1. And the two 4-level columns of each design in Table 3 are obtained from $\{1, 2, 3\}$ and $\{8, 16, 24\}$ in Table 1, respectively.

Table 1 Design matrices for 16- and 32-run designs

1	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2 MaxC2cR $4^1 2^n$ designs with 32 runs

R	n	Additional columns	$\alpha_l(5,n,R)$	$\alpha(5,n,R)$	$\alpha_u(5,n,R)$
III	5	6 29	15	18	23
	6	5 6 31	18	21	22
	7	12 20 24 28	21	21	21
	8	10 12 22 24 30	12	12	20
		6 12 20 24 30	12	12	20
		6 12 20 24 28	12	12	20
	9	9 10 17 18 24 26	11	11	19
		9 10 11 18 24 26	11	11	19
		9 10 18 19 24 25	11	11	19
	10	9 10 18 19 24 25 27	12	12	18
		9 10 17 18 24 25 26	12	12	18
	11	9 10 11 17 18 24 25 26	13	13	17
	12	9 10 11 17 18 19 24 25 26	14	14	16
	13	9 10 11 17 18 19 24 25 26 27	15	15	15
IV	5	13 26	5	13	21
	6	14 26 28	7	12	16
	7	14 22 26 28	6	14	14

We can easily get that $M'(5) = 4$. When $n \leqslant 13$ there exist $4^1 2^n$ designs containing clear 2fic's, and when $4 < n \leqslant 7$ there exist $4^1 2^n$ designs with resolution IV containing clear 2fic's (see [17]). Table 2 tabulates MaxC2cR $4^1 2^n$ designs with resolutions III and IV and gives the values of $\alpha(k,n,R)$ along with $\alpha_l(k,n,R)$ and $\alpha_u(k,n,R)$ for $k = 5$, where $\alpha(k,n,R)$ is defined in Section 3, $\alpha_l(k,n,R)$ and $\alpha_u(k,n,R)$ are the lower and upper bounds on $\alpha(k,n,R)$, respectively. From Table 2, we can find that the lower bound $\alpha_l(k,n,\mathrm{III})$ behaves better than the upper bound $\alpha_u(k,n,\mathrm{III})$. The lower bound $\alpha_l(k,n,\mathrm{III})$ equals $\alpha(k,n,\mathrm{III})$ in many cases but the upper bound $\alpha_u(k,n,\mathrm{III})$ equals $\alpha(k,n,\mathrm{III})$ only in a few cases. Table 2 also shows that the lower bound $\alpha_l(k,n,\mathrm{IV})$ and the upper bound $\alpha_u(k,n,\mathrm{IV})$ behave well only in a few cases.

Also, we can get $M''(5) = 1$. When $n \leqslant 10$ there exist $4^2 2^n$ designs containing clear 2fic's and when $1 < n \leqslant 4$ there exist $4^2 2^n$ designs with resolution IV containing clear 2fic's (see [17]). Table 3 tabulates the MaxC2cR $4^2 2^n$ designs with resolutions III and IV and gives the values of $\beta(k,n,R)$ along with $\beta_l(k,n,R)$ and $\beta_u(k,n,R)$ for $k = 5$, where $\beta(k,n,R)$ is defined in Section 4, $\beta_l(k,n,R)$ and $\beta_u(k,n,R)$ are the lower and upper bounds on $\beta(k,n,R)$, respectively. From Table 3 we can find that the lower bound $\beta_l(k,n,\mathrm{III})$ equals $\beta(k,n,\mathrm{III})$ for all the cases but

Table 3 MaxC2cR 4^22^n designs with 32 runs

R	n	Additional columns									$\beta_l(5,n,R)$	$\beta(5,n,R)$	$\beta_u(5,n,R)$
	2	12									15	15	23
	3	12	20								18	18	22
	4	12	20	28							21	21	21
	5	10	12	22	30						12	12	20
		6	12	20	30						12	12	20
		6	12	20	28						12	12	20
	6	9	10	17	18	26					11	11	19
III		9	10	11	18	26					11	11	19
		9	10	18	19	25					11	11	19
	7	9	10	18	19	25	27				12	12	18
		9	10	17	18	25	26				12	12	18
	8	9	10	11	17	18	25	26			13	13	17
	9	9	10	11	17	18	19	25	26		14	14	16
	10	9	10	11	17	18	19	25	26	27	15	15	15
	2	14									14	16	16
IV	3	14	21								10	12	14
	4	14	21	31							6	6	12

the upper bound $\beta_u(k,n,\mathrm{III})$ does not do so. Table 3 also shows that both the lower bound $\beta_l(k,n,\mathrm{IV})$ and the upper bound $\beta_u(k,n,\mathrm{IV})$ behave well.

Those comparisons reveal that our construction methods are satisfactory for constructing 4^m2^n designs with resolution III, and are okey for constructing 4^m2^n designs with resolution IV under the consideration of maximizing the number of clear 2fic's in the designs.

6 Summary and concluding remarks

In this paper, we first sketch out a construction method for $2^{n-(n-k)}$ designs containing as many clear 2fi's as possible, and then derive the upper and lower bounds on the maximum number of clear 2fic's in 4^m2^n designs with resolutions III and IV. The lower bounds are achieved by constructing 4^m2^n designs with 2^k runs for given k. Finally, we examine the performance of the bounds obtained above for $k=5$. The construction methods are satisfactory when they are used to construct 4^m2^n designs with resolution III. The number of clear 2fic's in the constructed design attains the maximum number in many cases. And such designs can be easily obtained following the construction methods.

The maximum estimation capacity (see [24, 25]) is another optimality criterion for evaluating factorial designs. It aims at selecting a design that retains full information on the main effects and as much information as possible on the 2fi's in the sense of entertaining the maximum possible model diversity, under the assumption of absence of interactions involving three or more factors. The aim of clear effects criterion is to find a design containing as many clear main effects and 2fi's as possible that can be estimated without being aliased in a single model. For the former, the estimability of effects requires all the 2fi's not in the model to be absent, while for the latter, the estimability of effects does not require this. This is also the advantage of the clear effects criterion over the others. These two criteria can behave quite differently because of this major difference.

Although the clear effects criterion has the advantage above, there are situations under which no design containing any clear 2fi exists. For example, a $2^{n-(n-k)}$ design has no clear 2fi when $n > 2^{k-1}$ (see [12]). The clear effects criterion does not apply to these situations. It can be considered as a disadvantage of the clear effects criterion.

When there exists a design with clear 2fic's and the experimenter hopes to estimate the 2fic's of some factors, he or she can arrange the important factors on those main effect columns corresponding to the clear 2fic's. Thus he or she can estimate the clear 2fic's without assuming the absence of the other 2fic's not in the model in doing the data analysis.

As explained in [6], different types of 2fic's have not the same importance, so they should be treated differently. Obtaining the related bounds for different types of 2fic's will be welcome. And the results here can be further extended to the case of general $(s^r)^m s^n$ designs, where s is a prime or prime power. But how to obtain the bounds and how to extend the results are the open problems and need to be further investigated.

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Appendix. Proofs of two theorems

A general construction method used in the proof of Theorem 5 is introduced firstly. First, we construct a $2^{(n+2)-(n+2-k)}$ design D' of resolution IV with as many clear 2fi's as possible. Then by choosing a suitable two-factor interaction column, say d_1 which is clear in D' and adding it into D', a $2^{(n+3)-(n+3-k)}$ design D'' of resolution III with only one length three word is obtained. Thus a 4^12^n design D of resolution IV can be obtained by replacing the three columns which form one length three word with a 4-level factor. Note that the number of clear 2fic's in D equals the number of clear 2fi's in D'', we need only to calculate the number of clear 2fi's in D''. Clearly, the number of clear 2fi's in D'' is $z_0 - z_1 + z_2$, where z_0 is the number of clear 2fi's in D', z_1 is the number of 2fi's which are clear in D' but not in D'' anymore, and z_2 is the number of 2fi's which are clear in D'' but not in D' originally. Note that a 2fi which is clear in D'' but not in D' originally must contain d_1 which is added into D'.

Proof of Theorem 5. Suppose that $n = 2^{k-2} - 1$. Let $a_1, a_2, b_1, \ldots, b_{k-2}$ be the k independent columns. Let

$$O_b = \{b_{i_1} \cdots b_{i_p} | \text{ where } p \geqslant 1 \text{ is odd and } 1 \leqslant i_1 < \cdots < i_p \leqslant k-2\}, \tag{9}$$

$$E_b = \{b_{i_1} \cdots b_{i_p} | \text{ where } p \geqslant 2 \text{ is even and } 1 \leqslant i_1 < \cdots < i_p \leqslant k-2\}. \tag{10}$$

It is obvious that $|O_b| = 2^{k-3}$ and $|E_b| = 2^{k-3} - 1$, where for example $|O_b|$ denotes the number of elements in O_b. Let

$$a_1a_2E_b = \{a_1a_2c | c \in E_b\}. \tag{11}$$

Obviously, we have $|a_1a_2E_b| = 2^{k-3} - 1$. Consider the following design:

$$D_2' = S \cup C, \text{ where } S = \{a_1, a_2\} \text{ and } C = O_b \cup (a_1a_2E_b). \tag{12}$$

From the discussion of [11], a_1a_2 is clear in D_2', and D_2' is of resolution IV with $|D_2'| = 2^{k-2} + 1$ columns, and for any $a \in S$ and $c \in C$, ac is clear in D_2'. Let d_{12} denote the interaction column a_1a_2. Now we can obtain a design D_2'' by adding the column d_{12} to D_2'. Since a_1a_2 is clear in design D_2', the new design D_2'' is a design with only one word $a_1a_2d_{12}$ of length three. Let

$$D_2 = \{A_1\} \cup C, \tag{13}$$

where A_1 is a 4-level factor obtained from $\{a_1, a_2, d_{12}\}$. Then D_2 has one 4-level factor and $n = 2^{k-2} - 1$ 2-level factors. Clearly, for any $c \in C$, a_1c and a_2c are clear and cd_{12} is not clear in D_2. And for any $c_1, c_2 \in C$, c_1c_2 is not clear in D_2, too. Therefore, the number of clear 2fic's in design D_2 is $\beta_{1l}(k, n) = 2n$.

Now suppose that $n = 2^j + 2^{k-j} - 5$, where $3 \leqslant j \leqslant \lfloor k/2 \rfloor$. Let $a_1, \dots, a_j$, $b_1, \dots, b_{k-j}$ be k independent columns. Define $O_a, E_a, O_b, E_b, a_1a_2E_b$ and $b_1b_2E_a$ the same way as in (9), (10) and (11). For example, $O_a = \{a_{i_1} \cdots a_{i_p} |$ where $p \geqslant 1$ is odd and $1 \leqslant i_1 < \cdots < i_p \leqslant j\}$. Consider design D'_j given by

$$D'_j = P \cup Q, \text{ where } P = O_a \cup (b_1b_2E_a \backslash \{a_1a_2b_1b_2\}) \text{ and } Q = O_b \cup (a_1a_2E_b). \tag{14}$$

Clearly, $|P| = 2^j - 2, |Q| = 2^{k-j} - 1$. Following the discussion of [11], we have $|D'_j| = (2^j - 2) + (2^{k-j} - 1) = n + 2$, D'_j has resolution IV, pq is clear for any $p \in P$ and $q \in Q\backslash\{a_1a_2b_1b_2\}$, and D'_j have $(2^j - 2)(2^{k-j} - 2)$ clear 2fi's. By adding a_1b_1 to D'_j, we can obtain a design $D''_j = D'_j \cup \{a_1b_1\}$ with resolution III. Let d_1 denote the column a_1b_1 in this and the following paragraphs. Since a_1b_1 is clear in D'_j, $a_1b_1d_1$ is the only word of length three in D''_j. Now we need only to calculate the number of 2fi's which are clear in D'_j but not in D''_j anymore and the number of 2fi's which are clear in D''_j but not in D'_j originally. For any $p \in P$ and $q \in Q\backslash\{a_1a_2b_1b_2\}$, pq is clear in D'_j, if pq is not clear in D''_j, then $p = a_1, q = b_1$ or there exists a factor $c \in P \cup Q$ such that $cpq = d_1$. There are two cases:

(i) $p \in P\backslash\{a_1\}, q = b_2, c = a_1b_1b_2p$ and $cpq = d_1$;

(ii) $p = a_2, q \in Q\backslash\{b_1, a_1a_2b_1b_2\}, c = a_1a_2b_1q$ and $cpq = d_1$.

Note that $p = a_2, q = b_2, c = a_1a_2b_1b_2$ are included in both cases (i) and (ii), and a_1b_1 is not clear in D''_j, thus the number of 2fi's which are clear in D'_j but not in D''_j anymore is $(2^j - 3) + (2^{k-j} - 3) - 1 + 1 = 2^j + 2^{k-j} - 6$. Note that d_1p is not clear in D''_j for any $p \in P \cup Q$, the number of 2fi's which are clear in D''_j is $(2^j - 2)(2^{k-j} - 2) - (2^j + 2^{k-j} - 6) = 2^k - 3 \times 2^{k-j} - 3 \times 2^j + 10$. Let

$$D_j = \{A_1\} \cup D''_j \backslash \{a_1, b_1\}, \tag{15}$$

where $\{A_1\}$ is the 4-level factor obtained from $\{a_1, b_1, d_1\}$. The resulting 4^12^n design D_j has resolution IV and $2^k - 3 \times 2^{k-j} - 3 \times 2^j + 10$ clear 2fic's.

Let $n_j = 2^j + 2^{k-j} - 5$ for $j = 2, \dots, J$, where $J = \lfloor k/2 \rfloor$. For $n_2 > n > n_3$, let $D = \{A_1\} \cup C^*$, where C^* is a subset of C obtained by deleting any $n_2 - n$ elements from C given in (13). For any $c \in C^*$, a_1c and a_2c are still clear in D, hence D has at least $2n$ clear 2fic's. For $n = n_j - 1$ with $j = 3, \dots, J$, the design D is constructed by deleting the column $a_1a_2b_1b_2$ in D_j given in (15). Then $p \neq a_2$ in case (i) and $q \neq b_2$ in case (ii) above. Note that $p = a_2$ in case (i) and $q = b_2$ in case (ii) are the same in fact, there are $2^k - 3 \times 2^{k-j} - 3 \times 2^j + 11$ clear 2fic's in the resulting design. For $n = n_j - 2$ with $j = 3, \dots, J$, the design D is constructed by deleting the columns $a_1a_2b_1b_2$ and b_2 in D_j given in (15). Then case (i) cannot occur anymore and $q \neq b_2$ in case (ii). Note that a_1b_1 is not clear in the design D and the 2fic d_1p is clear for any $p \in P\backslash\{a_1, a_2\}$. There are $(2^j - 2)(2^{k-j} - 3) - (2^{k-j} - 4) - 1 + (2^j - 4) = 2^k - 3 \times 2^{k-j} - 2^{j+1} + 5$ clear 2fic's in the resulting design. For $n_{j+1} < n \leqslant n_j - 3$ with $j = 3, \dots, J$, where $n_{J+1} = 2(2^J - 2) - 2$, the design

D is constructed by deleting the column $a_1a_2b_1b_2, a_2, b_2$ and any additional $n_j - n - 3$ columns from $Q\backslash\{b_1\}$. Then both the cases (i) and (ii) cannot occur anymore. The 2fic d_1p is clear for any $p \in (P \cup Q)\backslash\{a_1, b_1\}$. There are $(2^j - 3)(n + 2 - 2^j + 3) - 1 + n = (2^j - 3)(n - 2^j + 5) + n - 1$ clear 2fic's in the resulting design.

Note that the case $n_J - 3 \geqslant n > n_{J+1}$ is non-trivial only for odd k. When k is odd and $n = n_{J+1}$, let $D'_{J+1} = P^* \cup Q^*$, where $P^* = P\backslash\{a_2\}, Q^* \subset Q\backslash\{b_2, a_1a_2b_1b_2\}$ such that $b_1 \in Q^*, |Q^*| = 2^J - 1$, and P, Q are the same as those given in (14) for $j = J$. Note that pq is clear in D'_{J+1} for any $p \in P^*$ and $q \in Q^*$, there are $(2^J - 3)(2^J - 1)$ clear 2fi's in D'_{J+1}. By adding a_1b_1 to D'_{J+1} we can get a design D''_{J+1} with $n + 3$ columns. For any $p \in P^*$ and $q \in Q^*$ if pq is not clear in D''_{J+1}, then $p = a_1, q = b_1$. Note that d_1p is clear in D''_{J+1} for any $p \in (P^* \cup Q^*)\backslash\{a_1, b_1\}$, there are $(2^J - 3)(2^J - 1) - 1 + (2^{J+1} - 6) = 2^{2J} - 2^{J+1} - 4$ clear 2fi's in D''_{J+1}. Replacing $\{a_1, b_1, d_1\}$ with a 4-level factor we get a $4^1 2^n$ design D. And D has $2^{2J} - 2^{J+1} - 4$ clear 2fic's.

When $n < n_{J+1}$, let $D' = P^* \cup Q^*$, where P^* is a subset of $P\backslash\{a_2\}$ with $\lfloor (n + 2)/2 \rfloor$ columns and Q^* is a subset of $Q\backslash\{b_2, a_1a_2b_1b_2\}$ with $(n + 2) - \lfloor (n + 2)/2 \rfloor$ columns such that $a_1 \in P^*, b_1 \in Q^*$, and P, Q are given in (14) for $j = J$. Note that for any $p \in P^*, q \in Q^*$, pq is clear in D'. Then by adding a_1b_1 to D' and replacing $\{a_1, b_1, d_1\}$ with a 4-level factor we get a $4^1 2^n$ design D. For any $p \in P^*, q \in Q^*$, pq, d_1p, d_1q are all clear in D except for $p = a_1$ and $q = b_1$. Thus D is a design with at least $\lfloor (n+2)/2 \rfloor((n + 2) - \lfloor (n + 2)/2 \rfloor) - 1 + n$ clear 2fic's.

A construction method which is similar to that described in the proof of Theorem 5 is used in Theorem 9. First, a $2^{(n+4)-(n+4-k)}$ design E' of resolution IV is constructed. Then by choosing two suitable two-factor interaction columns, say d_1, d_2 which are clear in E' and adding them into E', a $2^{(n+6)-(n+6-k)}$ design E'' of resolution III with only two length three words is obtained. Thus a $4^2 2^n$ design E of resolution IV can be obtained by replacing the six columns which form the two length three words with two 4-level factors. Then the number of clear 2fic's in E equals the number of clear 2fi's in E''. And the number of clear 2fi's in E'' is calculated just like that in D'' in Theorem 5.

Proof of Theorem 9. Suppose that $n = 2^{k-2} - 4$. Let

$$E'_2 = D'_2\backslash\{a_1a_2b_1b_2\}, \quad E''_2 = E'_2 \cup \{a_1b_1\} \cup \{a_2b_2\},$$

where $D'_2 = S \cup C$, $S = \{a_1, a_2\}$ and $C = O_b \cup (a_1a_2E_b)$ are given in (12), and $|O_b| = 2^{k-3}, |a_1a_2E_b| = 2^{k-3} - 1$. Then E''_2 has $n + 6$ columns. For any $p \in S$ and $q \in C\backslash\{a_1a_2b_1b_2\}$, pq is clear in E'_2. Note that a_1a_2 is clear in E'_2, E'_2 has $2(2^{k-2} - 2) + 1$ clear 2fi's. Let d_1 and d_2 denote a_1b_1 and a_2b_2 in this and the following paragraphs, respectively. Since a_1b_1 and a_2b_2 are clear in design E'_2, E''_2 is a design with only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. Clearly, for any $p \in C\backslash\{b_1, b_2, a_1a_2b_1b_2\}$, there exists $c = a_1a_2b_2p \in C\backslash\{a_1a_2b_1b_2\}$ such that $a_1cp = d_2$. A similar argument is valid for a_2 and d_1. Therefore, for any $p \in C\backslash\{b_1, b_2, a_1a_2b_1b_2\}$ the 2fi's a_ip, d_ip are not clear in E''_2 for $i = 1, 2$. Note that $a_1b_1, a_2b_2, a_1d_1, a_1d_2, a_2d_1, a_2d_2, b_1d_1, b_2d_2$ are not clear and d_1d_2, b_1d_2, b_2d_1 are clear in E''_2, the design E''_2 has $2(2^{k-2} - 2) + 1 - 2(2^{k-2} - 4) - 2 + 3 = 6$ clear 2fi's. Thus we can obtain a $4^2 2^n$ design E_2 by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ with two 4-level factors in E''_2. And the number of clear 2fic's of E_2 is 6.

Now suppose that $n = 2^j + 2^{k-j} - 8$ for some $j = 3, \dots, J$, where $J = \lfloor k/2 \rfloor$. Let the k independent columns be $a_1, \dots, a_j, b_1, \dots, b_{k-j}$. And let

$$E'_j = D'_j \backslash \{a_1a_2b_1b_2\}, \quad E''_j = E'_j \cup \{a_1b_1\} \cup \{a_2b_2\},$$

where $D'_j = P \cup Q$ is given in (14) and $|P| = 2^j - 2, |Q| = 2^{k-j} - 1$. Thus E''_j has $n+6$ columns. For any $p \in P$ and $q \in Q \backslash \{a_1a_2b_1b_2\}$, pq is clear in E'_j, hence E'_j has $(2^j - 2)(2^{k-j} - 2)$ clear 2fi's. Since a_1b_1 and a_2b_2 are clear in E'_j, E''_j is a design with only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. Now we need to find out the number of 2fi's which are clear in E'_j but not in E''_j anymore and the number of 2fi's which are clear in E''_j but not in E'_j originally. For any $p \in P$ and $q \in Q \backslash \{a_1a_2b_1b_2\}$, if pq is clear in E'_j but not in E''_j, then $p = a_1, q = b_1$ or $p = a_2, q = b_2$, or there exists a factor $c \in E'_j$ such that $cpq = d_i, i = 1$ or 2. There are four cases:

(i) $p \in P \backslash \{a_1, a_2\}, q = b_1, c = a_2b_1b_2p \in P$ and $cpq = d_2$;

(ii) $p \in P \backslash \{a_1, a_2\}, q = b_2, c = a_1b_1b_2p \in P$ and $cpq = d_1$;

(iii) $p = a_1, q \in Q \backslash \{b_1, b_2, a_1a_2b_1b_2\}, c = a_1a_2b_2q \in Q \backslash \{a_1a_2b_1b_2\}$ and $cpq = d_2$;

(iv) $p = a_2, q \in Q \backslash \{b_1, b_2, a_1a_2b_1b_2\}, c = a_1a_2b_1q \in Q \backslash \{a_1a_2b_1b_2\}$ and $cpq = d_1$.

From cases (i) and (ii) we can find that for any $p \in P \backslash \{a_1, a_2\}, b_ip$ and d_ip are not clear in E''_j for $i = 1, 2$. And cases (iii) and (iv) show that for any $q \in Q \backslash \{b_1, b_2, a_1a_2b_1b_2\}$, a_iq and d_iq are not clear in E''_j for $i = 1, 2$. Note that $a_{l_1}d_{l_2}$ and $b_{l_1}d_{l_2}$ are not clear in E''_j for $l_1, l_2 = 1, 2$, and d_1d_2 is clear, thus the number of clear 2fi's in E''_j is $(2^j - 2)(2^{k-j} - 2) - 2(2^j - 4 + 2^{k-j} - 4) - 2 + 1 = 2^k - 2^{j+2} - 2^{k-j+2} + 19$. We can obtain a $4^2 2^n$ design E_j with resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ in E''_j with two 4-level factors and E_j has $2^k - 2^{j+2} - 2^{k-j+2} + 19$ clear 2fic's.

Let $n_j = 2^j + 2^{k-j} - 8, j = 2, \dots, J$. For $n_2 > n > n_3$, let

$$E' = S \cup O_b \cup (a_1a_2E^*_b), \quad E'' = E' \cup \{a_1b_1\} \cup \{a_2b_2\},$$

where $a_1a_2E^*_b$ denotes a subset of $a_1a_2E_b \backslash \{a_1a_2b_1b_2\}$ obtained by deleting $n_2 - n$ columns from it, $S = \{a_1, a_2\}$ and $O_b, a_1a_2E_b$ are the same as those given in (9) and (11), $|a_1a_2E^*_b| = 2^{k-3} - 2 - (n_2 - n)$ and E'' has $n+6$ columns. For any $p \in S, q \in O_b \cup (a_1a_2E^*_b)$, pq is clear in E'. Note that a_1a_2 is also clear in E', the design E' has $2(2^{k-2} - 2 - n_2 + n) + 1$ clear 2fi's. Clearly, for any $p \in a_1a_2E^*_b$, there exists $q = a_1a_2b_2p \in O_b$ such that $a_1pq = d_2$. Thus a_1p, a_1q, a_1d_2, d_2p and d_2q are not clear in E''. Note that for any $q \in O_b \backslash \{b_2\}$ satisfying $a_1a_2b_2q \notin a_1a_2E^*_b, d_2q$ is clear in E''. A similar argument is valid for a_2 and d_1. Note that a_1b_1, a_2b_2 are not clear and d_1d_2 is clear in E'', design E'' has $2(2^{k-2} - 2 - n_2 + n) + 1 - 2 - 4(2^{k-3} - 2 - n_2 + n) + 2(n_2 - n + 1) + 1 = 4(n_2 - n) + 6$ clear 2fi's. Thus we can obtain a $4^2 2^n$ design E with resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ in E'' with two 4-level factors. And E has $4(n_2 - n) + 6$ clear 2fic's.

When $n_j > n > n_{j+1}$ for some $j = 3, \dots, J$, where $n_{J+1} = 2(2^J - 2) - 5$, let

$$E' = P \cup Q^*, \quad E'' = E' \cup \{a_1b_1\} \cup \{a_2b_2\},$$

where $P = O_a \cup (b_1b_2E_a) \backslash \{a_1a_2b_1b_2\}, Q^* = O_b \cup (a_1a_2E^*_b)$, $a_1a_2E^*_b$ denotes a subset of $a_1a_2E_b \backslash \{a_1a_2b_1b_2\}$ obtained by deleting $n_j - n$ columns from it, and $O_a, O_b, b_1b_2E_a$, $a_1a_2E_b$ are the same as those in (14). Clearly, $|O_a| = 2^{j-1}$, $|O_b| = 2^{k-j-1}$, $|b_1b_2E_a| = 2^{j-1} - 1$,

$|a_1a_2E_b^*| = 2^{k-j-1} - 2 - (n_j - n)$ and E'' has $n+6$ columns. For any $p \in P$ and $q \in Q^*$, pq is clear in E'. Thus E' has $(2^j-2)(2^{k-j}-2-n_j+n)$ clear 2fi's. Since a_1b_1 and a_2b_2 are clear in E', E'' is a design with only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. For any $p \in P$ and $q \in Q^*$, if pq is not clear in E'', then $p = a_1, q = b_1$ or $p = a_2, q = b_2$, or there exists a factor $c \in E'$ such that $cpq = d_i, i = 1$ or 2. There are four cases:

(i) $p \in P\backslash\{a_1, a_2\}, q = b_1, c = a_2b_1b_2p \in P$ and $cpq = d_2$;

(ii) $p \in P\backslash\{a_1, a_2\}, q = b_2, c = a_1b_1b_2p \in P$ and $cpq = d_1$;

(iii) $p = a_1, q \in Q^*\backslash\{b_1, b_2\}, c = a_1a_2b_2q \in Q^*\backslash\{b_1, b_2\}$ and $cpq = d_2$;

(iv) $p = a_2, q \in Q^*\backslash\{b_1, b_2\}, c = a_1a_2b_1q \in Q^*\backslash\{b_1, b_2\}$ and $cpq = d_1$.

Case (i) shows that for any $p \in P\backslash\{a_1, a_2\}$, b_1p and d_2p are not clear in E''. And case (iii) shows that for any $q \in a_1a_2E_b^*$ and $c = a_1a_2b_2q \in O_b$, a_1q, a_1c, d_2q and cd_2 are not clear in E''. For any $c \in O_b\backslash\{b_1, b_2\}$ satisfying $a_1a_2b_2c \notin a_1a_2E_b^*, cd_2$ is clear in E''. A similar argument is valid for cases (ii) and (iv). Note that $a_{l_1}d_{l_2}$, and $b_{l_1}d_{l_2}$ are not clear for $l_1, l_2 = 1, 2$ and d_1d_2 is clear in E'', the number of clear 2fi's in E'' is $(2^j-2)(2^{k-j}-2-n_j+n) - 2(2^j-4) - 4(2^{k-j-1}-2-n_j+n) - 2 + 2(n_j-n) + 1 = 2^k - (2^j-8)(n_j-n) - 2^{j+2} - 2^{k-j+2} + 19$. Thus we can obtain a $4^2 2^n$ design E of resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ in E'' with two 4-level factors and E has $2^k - (2^j-8)(n_j-n) - 2^{j+2} - 2^{k-j+2} + 19$ clear 2fic's.

For $2^J - 4 < n \leqslant n_{J+1}$ and odd k, let $e_5 = \lfloor (n+4)/2 \rfloor$ and $E' = P^* \cup O_b^*$, where $P^* = O_a \cup (b_1b_2E_a^*)$, $b_1b_2E_a^*$ is a subset of $b_1b_2E_a\backslash\{a_1a_2b_1b_2\}$ with $e_5 - 2^{J-1}$ elements, O_b^* is a subset of O_b with $(n+4) - e_5$ elements such that $b_1, b_2 \in O_b^*$, $O_a, O_b, b_1b_2E_a$ are defined in (14) for $j = J$. Adding a_1b_1 and a_2b_2 to E', we get a $2^{(n+6)-(n+6-k)}$ design E''. Since a_1b_1 and a_2b_2 are clear in E', E'' has only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. For any $p \in P^*$ and $q \in O_b^*$, pq is clear in E'. Hence E' has $e_5((n+4)-e_5)$ clear 2fi's. For any $p \in P^*$ and $q \in O_b^*$, if pq is not clear in E'', then $p = a_1, q = b_1$ or $p = a_2, q = b_2$, or there exists a factor $c \in E'$ such that $cpq = d_i, i = 1$ or 2. There are two cases:

(i) $p \in P^*\backslash\{a_1, a_2\}, q = b_1, c = a_2b_1b_2p \in P^*\backslash\{a_1, a_2\}$ and $cpq = d_2$;

(ii) $p \in P^*\backslash\{a_1, a_2\}, q = b_2, c = a_1b_1b_2p \in P^*\backslash\{a_1, a_2\}$ and $cpq = d_1$.

Let us consider case (i) firstly. For any $p \in b_1b_2E_a^*$ and $c = a_2b_1b_2p \in O_a, b_1p, b_1c, d_2p$ and cd_2 are not clear in E''. And for any $q \in O_a\backslash\{a_2\}$ satisfying $a_2b_1b_2q \notin b_1b_2E_a^*, d_2q$ is clear in E''. A similar argument is valid for b_2 and d_1 in case (ii). For any $p \in O_b^*\backslash\{b_1, b_2\}$, d_ip is clear in E'' for $i = 1, 2$. Since d_1d_2 is clear in E'', the number of clear 2fi's in E'' is $e_5((n+4)-e_5) - 4(e_5 - 2^{J-1}) - 2 + 2(2^J - 1 - e_5) + 2(n+2-e_5) + 1 = e_5(n-e_5) + 2^{J+2} - 4e_5 + 2n + 1$. Then we can obtain a $4^2 2^n$ design E of resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ with two 4-level factors in E'' and E has $e_5(n-e_5) + 2^{J+2} - 4e_5 + 2n + 1$ clear 2fic's.

For $2^J - 4 < n \leqslant n_{J+1} (= n_J - 1)$ and even k, let $E' = P^* \cup Q^*$, where $P^* = O_a \cup (b_1b_2E_a^*), Q^* = O_b \cup (a_1a_2E_b^*)$, $b_1b_2E_a^*$ is a subset of $b_1b_2E_a\backslash\{a_1a_2b_1b_2\}$ with $e_5 - 2^{J-1}$ elements, $a_1a_2E_b^*$ is a subset of $a_1a_2E_b\backslash\{a_1a_2b_1b_2\}$ with $(n+4) - e_5 - 2^{J-1}$ columns, and $O_a, O_b, b_1b_2E_a, a_1a_2E_b$ are defined in (14) for $j = J$. Adding a_1b_1 and a_2b_2 to E', we get a $2^{(n+6)-(n+6-k)}$ design E''. Since a_1b_1 and a_2b_2 are clear in E', E'' has only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. For any $p \in O_a \cup (b_1b_2E_a^*)$ and $q \in O_b \cup (a_1a_2E_b^*)$, pq is clear in E'. Hence E' has $e_5((n+4)-e_5)$ clear 2fi's. For any $p \in O_a \cup (b_1b_2E_a^*)$ and $q \in O_b \cup (a_1a_2E_b^*)$, if

pq is not clear in E'', then $p = a_1, q = b_1$ or $p = a_2, q = b_2$, or there exists a factor $c \in E'$ such that $cpq = d_i, i = 1$ or 2. There are four cases:

(i) $p \in P^*\backslash\{a_1, a_2\}, q = b_1, c = a_2b_1b_2p \in P^*\backslash\{a_1, a_2\}$ and $cpq = d_2$;

(ii) $p \in P^*\backslash\{a_1, a_2\}, q = b_2, c = a_1b_1b_2p \in P^*\backslash\{a_1, a_2\}$ and $cpq = d_1$;

(iii) $p = a_1, q \in Q^*\backslash\{b_1, b_2\}, c = a_1a_2b_2q \in Q^*\backslash\{b_1, b_2\}$ and $cpq = d_2$;

(iv) $p = a_2, q \in Q^*\backslash\{b_1, b_2\}, c = a_1a_2b_1q \in Q^*\backslash\{b_1, b_2\}$ and $cpq = d_1$.

Consider case (i) firstly. For any $p \in b_1b_2E_a^*, c = a_2b_1b_2p \in O_a, b_1c, b_1p, cd_2$ and d_2p are not clear in E''. For any $q \in O_a\backslash\{a_1, a_2\}$ satisfying $a_2b_1b_2q \notin b_1b_2E_a^*, qd_2$ is clear in E''. A similar argument is valid for the pairs $(b_2, d_1), (a_1, d_2)$ and (a_2, d_1) in the other three cases. Since d_1d_2 is clear in E'', the number of clear 2fi's in E'' is $e_5((n+4)-e_5) - 4(e_5 - 2^{J-1} + (n+4) - e_5 - 2^{J-1}) - 2 + 2(2^J - 2 - e_5) + 2(2^J + e_5 - n - 6) + 1 = e_5(n - e_5) + 4e_5 + 2^{J+3} - 6n - 33$. We can thus obtain a $4^2 2^n$ design E with resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ with two 4-level factors in E'' and E has $e_5(n - e_5) + 4e_5 + 2^{J+3} - 6n - 33$ clear 2fic's.

For $n \leqslant 2^J - 4$, let $E' = O_a^* \cup O_b^*$, where O_a^* is a subset of O_a with e_5 elements and O_b^* is a subset of O_b with $(n+4) - e_5$ elements such that $a_1, a_2 \in O_a^*, b_1, b_2 \in O_b^*$. For any $p \in O_a^*$ and $q \in O_b^*$, pq is clear in E'. Hence E' has $e_5((n+4) - e_5)$ clear 2fi's. Adding a_1b_1 and a_2b_2 to E', we get a $2^{(n+6)-(n+6-k)}$ design E''. Since a_1b_1 and a_2b_2 are clear in E', the design E'' has only two words $a_1b_1d_1$ and $a_2b_2d_2$ of length three. For any $p \in E'\backslash\{a_1, b_1\}, d_1p$ is clear in E''. A similar argument is valid for d_2. Since a_1b_1 and a_2b_2 are not clear and d_1d_2 is clear in E'', the number of clear 2fi's in E'' is $e_5((n+4) - e_5) - 2 + 2(n+2) + 1 = e_5(n + 4 - e_5) + 2n + 3$. We can thus obtain a $4^2 2^n$ design E with resolution IV by replacing $\{a_1, b_1, d_1\}$ and $\{a_2, b_2, d_2\}$ with two 4-level factors in E'' and E has $e_5(n + 4 - e_5) + 2n + 3$ clear 2fic's.