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A NOTE ON HAMILTONIAN CIRCUITS*

V. CHVÁTAL

Department of Computing Science, Stanford University, Stanford, Calif. 94305, U.S.A.

P. ERDÖS

Hungarian Academy of Sciences, Mathematical Institute, Budapest XII, Hungary

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The purpose of this note is to prove the following

Theorem 1. *Let G be a graph with at least three vertices. If, for some s, G is s-connected and contains no independent set of more than s vertices, then G has a Hamiltonian circuit.*

This theorem is sharp as the complete bipartite graph $K(s, s+1)$ is s-connected, contains no independent set of more than $s+1$ vertices and has no Hamiltonian circuit. Similarly, the Petersen graph is 3-connected, contains no independent set of more than four vertices and has no Hamiltonian circuit.

Proof. Let G satisfy the hypothesis of Theorem 1. Clearly, G contains a circuit; let C be the longest one. If G has no Hamiltonian circuit, there is a vertex x with $x \notin C$. Since G is s-connected, there are s paths starting at x and terminating in C which are pairwise disjoint apart from x and share with C just their terminal vertices $x_1, x_2, ..., x_s$ (see [1], Theorem 1). For each $i = 1, 2, ..., s$, let y_i be the successor of x_i in a

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fixed cyclic ordering of C. No y_i is adjacent to x - otherwise we would replace the edge $x_i y_i$ in C by the path going from x_i to y_i outside C (via x) and obtain a longer circuit. However, G contains no independent set of $s+1$ vertices and so there is an edge $y_i y_j$. Delete the edges $x_i y_i$, $x_j y_j$ from C and add the edge $y_i y_j$ together with the path going from x_i to x_j outside C. In this way we obtain a circuit longer than C, which is a contradiction.

For s relatively large with respect to the number of vertices of G, our Theorem 1 follows from a stronger statement due to Nash-Williams and Bondy ([2], Lemma 4):

Let G be a graph with n vertices, $n \geqslant 3$. Let G contain no vertex of degree smaller than k where k is an integer such that $k \geqslant \frac{1}{3}(n+2)$. Then G either has a Hamiltonian circuit, or is separable, or has $k+1$ independent vertices.

As an easy consequence of Theorem 1 we obtain

Theorem 2. *Let G be an s-connected graph with no independent set of $s+2$ vertices. Then G has a Hamiltonian path.*

Proof. Indeed, if G satisfies the hypothesis of Theorem 2, then $G+x$ (the graph obtained from G by adding a new vertex x and joining it to all the vertices of G) satisfies the hypothesis of Theorem 1 with $s+1$ in place of s. Therefore $G+x$ has a Hamiltonian circuit and G has a Hamiltonian path. The complete bipartite graph $K(s, s+2)$ shows that Theorem 2 is sharp.

The technique used in the proof of Theorem 1 yields also

Theorem 3. *Let G be an s-connected graph containing no independent set of s vertices. Then G is Hamiltonian-connected (i.e. every pair of vertices is joined by a Hamiltonian path).*

Proof. Let there be a counterexample G. Then G contains three vertices x, y, z such that $x \notin P$ for a longest path P joining y to z. Again, we find s paths from x to P, their terminal vertices being $x_1, ..., x_s$. We may assume $x_i \neq z$ for $i < s$ and denote the successor (in the direction from y to z) of each x_i $(i < s)$ by y_i. Since G has no s independent vertices, there is an edge xy_i or y_iy_j. In both cases we find a path joining y to z and longer than P which is a contradiction. The graph $K(s, s)$ shows that Theorem 3 is sharp.

References

[1] G.A. Dirac, Généralisation du théorème de Menger, C.R. Acad. Sci. Paris 250 (26) (1960) 4252–4253.
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