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The Geometry of 4-Manifolds

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I. Introduction. The title of this lecture is appropriate because, while the results we describe lie in the field of differential topology, the methods used are geometrical, exploiting the "instantons" or "Yang-Mills fields" introduced by physicists. Before going on to a detailed survey of results and techniques we will first contrast these developments with the general pattern of manifold topology.

A topological n-manifold is constructed from domains in n-dimensional Euclidean space, pieced together by homeomorphisms. The manifold is provided with a differentiable or smooth structure if these homeomorphisms are differentiable. The basic equivalence relation among topological manifolds is that of *homeomorphism* and among smooth manifolds is *diffeomorphism* (homeomorphism defined by smooth functions). In the 1960s and 1970s topologists developed a comprehensive theory of manifolds in dimension 5 or more. This theory explained the relationship between the smooth and topological categories [**28**]; and for many classes of manifolds it gave a complete classification in terms of invariants from algebraic topology [**3**, **26**, **33**]. The coarsest of these are *homotopy invariants*, for example, the homology groups. Next most important are the *Pontrayagin classes* $p_i(X)$ in $H^{4i}(X;\mathbf{Z})$ of a smooth manifold X—characteristic classes of the tangent bundle. Let us focus on four facts from this high-dimensional theory:

(1) Simply connected smooth manifolds of dimension 5 or more are diffeomorphic if they are h-cobordant.

(2) The homotopy type and Pontrayagin classes of a compact simply connected manifold of dimension 5 or more determine the smooth structure up to a finite number of possibilities.

(3) The reductions of the Pontrayagin classes to $H^*(X;\mathbf{Q})$ are topological invariants.

(4) A contractible topological manifold of dimension 5 or more has a unique smooth structure.

In high dimensions the h-cobordism technique gives an effective method for constructing equivalences between manifolds, and the classification of smooth and topological manifolds differ by only a "finite amount."

In 1982 Freedman [16] showed that the basic constructions used in high dimensions could be carried out with *topological* 4-manifolds. The sole classical invariant of a compact simply connected 4-manifold is the intersection form on 2-dimensional homology. By Hirzebruch's theorem the first Pontrayagin number $p_1(X^4)[X^4]$ is 3 times the signature $b_2^+ - b_2^-$, where b_2^+ and b_2^- are the dimensions of the positive and negative parts of this quadratic form. Freedman's theory asserts that, up to a finite ambiguity, the topological classification of 4-manifolds mimics the algebraic classification of forms.

Now among recent results for smooth 4-manifolds we have:

(1)′ There are simply connected, smooth 4-manifolds which are h-cobordant but nondiffeomorphic.

(2)′ There is a countably infinite family of smooth, simply connected 4-manifolds, all mutually homeomorphic but with distinct smooth structures.

(3)′ There are rational cohomology invariants of smooth 4-manifolds which (unlike the Pontrayagin classes) depend essentially on the smooth structure.

(4)′ There is an uncountable family of smooth 4-manifolds, each homeomorphic to $\mathbf{R}^4$ but with mutually distinct smooth structures.

(See §III below for more precise statements and references.) These facts, all coming from Yang-Mills theory, emphasize the very different picture we are beginning to see in four dimensions.

II. Techniques.

(i) *The first order Yang-Mills equations.* These equations in 4-dimensional geometry are in some ways analogous to the Cauchy-Riemann equations in dimension 2. In place of the functions on a Riemann surface the basic geometric objects we take are the *connections* on a bundle E over an oriented Riemannian 4-manifold X. The structure group of E is some compact Lie group G, for example SU(2) or SO(3). In place of the splitting of the derivative of a function into holomorphic and antiholomorphic parts we have the splitting of the curvature F_A of a connection A into self-dual and anti-self-dual parts: $F_A = F_A^+ + F_A^-$. These are the components in the eigenspaces of the Hodge * operator, acting on bundle-valued 2-forms. In place of the holomorphic functions we have the *anti-self-dual connections* (or *instantons*), solutions of the equation $F_A^+ = 0$. Geometrically this condition means that the curvature F_A takes opposite values on any pair of orthogonal 2-planes in the tangent space of X.

This anti-self-dual equation is a first-order partial differential equation for the connection A. It has a large group of symmetries: the group of automorphisms or "gauge transformations" of the bundle E. When this is taken into account (by identifying solutions which differ by a gauge transformation) the equation becomes elliptic. It depends only on the conformal class of the Riemannian metric on X. Many of its special features spring from a fundamental identity linking the "energy" of a solution with the topology of the bundle E. If X is compact and A is anti-self-dual, then

$$\int_X |F_A|^2 \, d\mu = c(G) \cdot k(E), \tag{1}$$

where $c(G)$ is a normalizing constant and $k(E)$ is an integer—a characteristic number of E. For example, if G is SO(3), then k is minus the first Pontrayagin class of E, evaluated on X. Again, analogous identities hold for the energy of holomorphic maps.

(ii) *Nonlinear Fredholm theory.* Differential topology in infinite-dimensional manifolds provides a convenient language to describe many properties of the Yang-Mills instantons, primarily those stemming from the implicit function theorem in Banach spaces. This is applied to nonlinear differential operators between suitable Sobolev spaces. For a fixed bundle $E \to X$ one defines a space $\mathcal{B}_E$ of all gauge equivalence classes of connections, orbits under the group of gauge transformations of E. An open dense subset $\mathcal{B}_E^*$ of $\mathcal{B}_E$ is an infinite-dimensional manifold; its complement, the singular set of $\mathcal{B}_E$, represents *reducible connections* with holonomy group a subgroup of G whose centralizer properly contains the center of G.

The second stock of ideas which can be applied are those based on Sard's theorem and transversality. As Smale observed [**30**] these basic constructions of differential topology carry over to infinite-dimensional problems involving *Fredholm mappings*: smooth maps whose derivatives have finite-dimensional kernels and cokernels. As an illustration (particularly relevant to the definition of the invariants in §III(iv) below) consider a Fredholm map $\varphi\colon E \to F$ between Banach spaces whose *index* (the integer $\dim(\ker d\varphi)_x - \dim(\operatorname{coker} d\varphi)_x$, calculated for any x in E) is zero. Generic points y in F are regular values of φ and for these $\varphi^{-1}(y)$ is a discrete subset of X. If φ is a proper map, then this set is finite. We can attach a sign to each point, in such a way that the algebraic sum over the fibers yields an integer invariant, independent of y. The proof in the general case is not significantly different from that in finite dimensions. In the same way this integer—the "degree" of φ—is a deformation invariant unchanged by proper Fredholm homotopies. More generally, if E is replaced by a Banach manifold B we can associate homology classes in $H_d(B; \mathbf{Z})$ to suitable Fredholm maps φ, where d is the index of φ.

The anti-self-dual equations fit into this framework. The *moduli space* M_E (space of equivalence classes of the equation $F_A^+ = 0$) is defined by a Fredholm mapping whose index was calculated by Atiyah, Hitchin, and Singer [**2**] (applying the Atiyah-Singer index theorem to the linearized operator). They gave a general formula:

$$\dim M_E = 2a_G \cdot k(E) - \dim G(1 - b_1(X) + b_2^+(X)), \tag{2}$$

where a_G is an integer depending only on G (equal to 1 when $G = \mathrm{SO}(3)$, for example). This is the "virtual dimension" of M_E, and typically one expects the part of the moduli space in $\mathcal{B}_E^*$ to be a smooth manifold of this dimension. More precisely, Freed and Uhlenbeck prove in [**15**] that for nontrivial SU(2) and SO(3) bundles E this is the case for typical metrics on X. In general one can achieve the same end by making small perturbation of the anti-self-duality equations. Note here that the parity of $\dim M_E$ is *independent of the bundle E.*

The same kinds of ideas can be applied to the reducible connections, the nonmanifold points in $\mathcal{B}$. That is, we can describe the behavior of the moduli spaces there in typical situations. The important reductions are those with abelian structure group S^1, and these can be studied by Hodge theory. As the Riemannian metric on X varies, these reducible solutions to the anti-self-dual equations appear on subsets of codimension b_2^+. So we can avoid them in families of metrics of dimension less than b_2^+. In families of dimension b_2^+ we encounter a fundamental singularity in the associated moduli spaces. For example, if X has a negative definite intersection form, so $b_2^+ = 0$, the singularities are always present and, for typical metrics, are cones on complex projective spaces.

(iii) *Compactification.* The Yang-Mills moduli spaces are not, in general, compact, but a theorem of Uhlenbeck singles out a natural compactification. This control "at infinity" in the space $\mathcal{B}_E$ of connections stands in for the properness of the Fredholm map defining the moduli spaces, which holds only in special cases.

Uhlenbeck's theorem [**37**] supplies information on connections given bounds on their energy. For anti-self-dual connections these come from the identity (1). Let us restrict for simplicity to SO(3) bundles E, which are determined topologically by characteristic classes $p_1(E)$ in $H^4(X;\mathbf{Z}) \cong \mathbf{Z}$ and $w_2(E)$ in $H^2(X;\mathbf{Z}/2)$ with $w_2^2 = p_1 \bmod 4$. So if we fix $w_2 = U$ there is a family of moduli spaces, $M_j = M_{j,U}$ say, with $j \geq 0$. Then one can define a topology on

$$M_j \cup X \times M_{j-4} \cup S^2(X) \times M_{j-8} \cup \cdots$$

such that the closure $\overline{M}_j$ of M_j is compact. Here the $S^i(X)$ denote the *symmetric products* of i points in X. The points in the lower "strata" $S^i(X) \times M_{j-4i}$ represent "ideal connections" whose energy density $|F_A|^2$ is augmented by δ-functions at i points in X.

Thanks to work of Taubes [**34**], extended in [**7**], we have a good hold on the structure of neighborhoods of the lower strata in $\overline{M}_k$, that is, of the "ends" of the moduli spaces. By making a detailed analysis of the relevant implicit function theorem one describes neighborhoods of $S^i(X) \times M_{j-4i}$ in M_j in terms of a connection in M_{j-4i}, i copies of the "fundamental instanton" at points of X, and "glueing data" which identifies these component parts. For example, the link of $X \times M_{j-4}$ in $\overline{M}_j$ is typically a copy of the structure group SO(3).

Ideas of this kind, describing the behavior of differential operators "at infinity" in a function space, have appeared recently in a number of different geometric problems. In gauge theory, Taubes has used them to construct a calculus of variations—see Taubes's lecture at this Congress.

(iv) *The anti-self-dual equations and holomorphic geometry.* Suppose the base space X is a 2-dimensional complex surface with a Hermitian metric. If E is a complex vector bundle over X (with structure group a subgroup of the unitary group), any connection defines an almost complex structure on E. If the connection is anti-self-dual its curvature has type $(1,1)$ and this implies that the structure is integrable. We get, in this way, a map from the anti-self-dual connections over X to the holomorphic bundles, the latter depending only on the complex geometry of X.

A holomorphic bundle is, by definition, locally trivial whereas there are many local solutions to the anti-self-dual equations. However globally we can reconstruct the connection from its holomorphic bundle. There is nothing special here about the dimension of the base space. If (Y, ω) is any compact Kahler manifold and $E \to Y$ a holomorphic bundle with structure group $\mathrm{SL}(r, \mathbf{C})$ (say), any metric on E determines a reduction of the structure group to $\mathrm{SU}(r)$ and also a preferred $\mathrm{SU}(r)$ connection. We look for metrics such that the curvature F of this connection satisfies

$$F \cdot \omega = 0 \tag{3}$$

at every point of Y. This is a second-order elliptic equation for the metric on E. On the other hand, in algebraic (or holomorphic) geometry there is a notion of a *stable* vector bundle, introduced by algebraic geometers in moduli problems. We have:

PROPOSITION. *The holomorphic bundle E is stable if and only if it carries an irreducible solution of the differential equation* (3). *The solution is then unique.*

This was proved recently by Uhlenbeck and Yau [**38**]. The result had been conjectured (independently) by Hitchin and Kobayashi; in the simplest case when Y is a complex curve, it is equivalent to a theorem of Narasimhan and Seshadri, and this was developed from the point of view of Yang-Mills theory by Atiyah and Bott [**1**]. For an algebraic surface Y the result was proved in [**6**].

So on compact Kahler manifolds of any dimension, this theorem of Uhlenbeck and Yau gives a holomorphic description of the unitary connections whose curvature is of type $(1,1)$ and perpendicular to the Kahler form. The special feature of complex surfaces is that these are *precisely* the anti-self-dual connections. Thus for algebraic surfaces the moduli spaces M_E can be described using algebraic geometry. They are quasi-projective complex varieties. From this point of view the best algebraic construction is that of Gieseker [**20**].

III. Results and applications.

(i) *Realizing intersection forms.* Here we discuss results forbidding the construction of smooth 4-manifolds with given intersection forms. They can be seen alternatively as obstructions to smoothing the topological manifolds constructed by Freedman. Equally, they imply that it is impossible to do smooth *surgery* on homology classes in many existing manifolds.

The first theorem of this kind asserted that nonstandard (nondiagonalizable) definite forms cannot be realized by smooth, simply connected 4-manifolds [**5**]. The proof used a 5-dimensional moduli space of $\mathrm{SU}(2)$ connections. The result has since been extended in two different ways.

On the one hand, Fintushel and Stern found a rather simple proof for negative definite forms which represent -2 or -3. They considered SO(3) connections, choosing w_2 and p_1 to get moduli spaces of low dimensions [**12**]. Their proof dealt with manifolds with no 2-torsion in H_1. In both proofs the moduli spaces are truncated to give manifolds with a known boundary. Boundary contributions come from the links of singularities at reducible connections and from the lower strata in the compactified moduli space. Then one asserts that the moduli manifold gives a cobordism, and a fortiori homology, in $\mathcal{B}_E^*$ between these boundaries.

On the other hand, the proof of [**5**] was extended by Furuta [**18**] and the author [**8**] to take account of fundamental group. This required a more extensive use of transversality and also a detailed study of the *orientation* of the moduli spaces. The upshot is the optimal result for definite forms:

THEOREM 1 [**8**]. *If a smooth, compact, oriented 4-manifold has a definite intersection form, then the form can be diagonalized over the integers.*

There are also results for some indefinite forms [**7**]. These are proved in a similar way, using more complicated analysis and topology. One can define a map

$$\mu\colon H_2(X;\mathbf{Z}) \to H^2(\mathcal{B}_E^*;\mathbf{Z}) \tag{4}$$

by decomposing the 4-dimensional characteristic class of the "universal" bundle over $\mathcal{B}_E^* \times X$. For indefinite manifolds the moduli spaces typically avoid the reductions, so, by restricting μ, we construct cohomology classes over the moduli spaces. For the same reason the only boundary contributions are now those from the lower strata. There are further, mod 2, cohomology classes which detect the links of the lower strata in the homology of $\mathcal{B}_E^*$. The best result so far is

THEOREM 2 [**7**]. *If a smooth, compact, oriented 4-manifold has no 2-torsion in H_1 and an even intersection form with a positive part of rank 2, then the form is*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The method appears to run out of steam as b_2^+ grows because the relative size of the contributions to the ends from different lower strata changes.

(ii) *Orbifolds and the representation of homology classes.* Fintushel and Stern began the study of Yang-Mills equations on 4-dimensional orbifolds: spaces with a discrete set of singularities modelled on finite quotients of $\mathbf{R}^4$. These occur naturally as the quotients of smooth 4-manifolds by finite groups or of 5-manifolds by circle actions. They are rational homology manifolds, and analysis on them is quite similar to that on smooth manifolds—the chief modification is the appearance of extra terms in the index formula (2) owing to the singularities. Using variants of their argument for manifolds, Fintushel and Stern obtained restrictions on the existence of orbifolds with certain intersection forms and singularities. Their results have many applications, notably to the group θ_H^3 of homology 3-spheres modulo homology cobordism. Before Fintushel and Stern's work it had seemed possible that this group was rather small, perhaps of order 2. In fact we have

THEOREM 3 [**13**]. *The Poincaré homology sphere P has infinite order in* θ^3_H *(i.e., no connected sum* $P\#\cdots\#P$ *bounds an acyclic smooth 4-manifold). Moreover* $\theta^3_H/\langle P\rangle$ *is nonzero.*

(By contrast, the Poincaré sphere itself bounds an acyclic topological 4-manifold [**16**].)

Another application was to give an alternative proof of a theorem of Kuga. Any 2-dimensional homology class in a 4-manifold can be represented by a smoothly embedded surface. It is an interesting general problem to find lower bounds on the genus of such a representative. Kuga's theorem considers classes in $H_2(S^2 \times S^2)$, written in the standard basis as pairs (p, q) of integers.

THEOREM 4 [**23**]. *The class* (p,q) *in* $H_2(S^2 \times S^2)$ *can be represented by a smoothly embedded 2-sphere if and only if either p or q is* 0, $+1$, *or* -1.

(By contrast, if p and q are co-prime the class can be represented by a topologically flat embedded sphere.)

Kuga's original proof was indirect, applying the results of §III(i).

(Similar arguments for other manifolds have been made by Lawson [**25**] and Suciu [**31**].) Fintushel and Stern gave a simpler proof using orbifolds. If a 2-sphere, embedded in a 4-manifold with nonzero self-intersection number, is collapsed to a point, the resulting space is an orbifold (whose singularity is a cone on a lens space). Later Furuta [**19**] gave an even more direct proof using other moduli spaces on these orbifolds. In another direction Lawson [**24**] used these techniques to study embedded projective planes.

The techniques in Fintushel and Stern's first paper have recently been extended by Fintushel, Lawson, and Stern. One application yields results on the exceptional orbits of circle actions on S^5, partially proving a conjecture of Montgomery and Yang [**14**].

(iii) *Exotic structures on* $\mathbf{R}^4$. Freedman's theory asserts that direct sum decompositions of the intersection form of a 4-manifold can be realized, by surgery, as topological decompositions of the manifold. The results of §III(i) prevent these being made smoothly. This conflict implies that there exist "exotic $\mathbf{R}^4$'s": smooth manifolds homeomorphic but not diffeomorphic to Euclidean space [**21**]. The first examples were open subsets of $S^2 \times S^2$ or $\mathbf{CP}^2$. The proofs of their exotic nature were indirect. Later Gompf found a countably infinite family of exotic $\mathbf{R}^4$'s in this way.

Dramatic further progress was made by Taubes [**35**], carrying out a program suggested by Freedman. Taubes extended the fundamentals of Yang-Mills theory to "end-periodic" 4-manifolds. These are noncompact manifolds whose end has a periodic configuration $W_1 \cup W_2 \cup \cdots \cup W_n \cup \cdots$, where the W_i are overlapping copies of an open manifold W. The simplest examples are manifolds whose end is a tube $Y^3 \times (0,\infty)$ and W is $Y \times (0,1)$. Taubes showed that given conditions

(1) on the homology of W and

(2) on the *representations* $\pi_1(W) \to \mathrm{SU}(2)$,

these behave like compact manifolds from the point of view of the anti-self-dual equations. By substituting the resulting theorems on the intersection forms of end-periodic manifolds into Freedman's analysis of the failure of smooth surgery Taubes proved:

THEOREM 5 [**22**, **35**]. *There exists a family $R_{s,t}$ of smooth 4-manifolds, parametrized by $(s,t) \in \mathbf{R}^2$, each homeomorphic to $\mathbf{R}^4$ but no two diffeomorphic.*

Thus there are "moduli" of smooth structures on the topological manifold $\mathbf{R}^4$. Moreover Taubes's family does not contain all exotic $\mathbf{R}^4$'s. None of the $R_{s,t}$ can be embedded in the standard $\mathbf{R}^4$, but the failure of the h-cobordism theorem (§III(iv)) implies that examples with this property do exist.

In a similar spirit to this work of Taubes, the author and Sullivan have extended the fundamentals of Yang-Mills theory to *quasi-conformal* 4-manifolds, whose local co-ordinates compare by quasi-conformal maps of domains in $\mathbf{R}^4$ [**11**]. The work is allied to that of Teleman on Lipschitz manifolds [**36**]. Essentially all the results proved for smooth 4-manifolds and diffeomorphisms, using the anti-self-duality equations, extend to quasi-conformal manifolds and quasi-conformal maps. (In particular there are exotic quasi-conformal structures on $\mathbf{R}^4$.) A fortiori the results extend to Lipschitz 4-manifolds. This is in sharp contrast with the theorem of Sullivan [**32**]: in high dimensions every topological 4-manifold has a unique Lipschitz structure.

(iv) *New invariants.* The results here are the other side of the coin displayed in §III(i). Many compact topological 4-manifolds cannot be smoothed: those that can may carry many different smooth structures. This is established by constructing differential topological invariants from the Yang-Mills moduli spaces, along the lines indicated in §II(i).

Let us restrict attention to simply connected 4-manifolds. For any bundle E over X the rational cohomology of $\mathcal{B}^*_E$ is generated as a ring by classes c/α, where c is a rational characteristic class of the universal bundle on the product $\mathcal{B}^*_E \times X$ and α is a homology class in X. In particular, all the rational cohomology of $\mathcal{B}^*_E$ lies in *even dimensions*. For example, if G is SO(3), then $H^*(\mathcal{B}^*_E;\mathbf{Q})$ is a polynomial algebra, generated by the image of the map μ in (4) and a further class in H^4. Since the invariants we expect to see with the ideas of §II(ii) lie, roughly speaking, in the *homology* of $\mathcal{B}^*_E$ we anticipate good results in cases when the moduli spaces are *even-dimensional*, and by (2) this happens exactly when $b_2^+(X)$ is *odd.*

Manifolds with $b_2^+ = 1$ form a rather special class here, since reducible solutions appear in the moduli space for a codimension 1 family of metrics. We consider the two-dimensional moduli space of SU(2) connections with Chern class 1 (SO(3) connections with Pontrayagin class -4) over such a manifold X. This will not always be compact, but there is a way to introduce a correction term for the boundary. Then one can associate to generic metrics on X a homology class in $\mathcal{B}_E^*$ which changes only through the appearance of reducible solutions. In the end one obtains a differential topological invariant of X having the form of a map:

$$\Gamma_X \colon \mathcal{C}_X \to H^2(X; \mathbf{Z}),$$

where $\mathcal{C}_X$ is a set of "chambers" in $H^2(X; \mathbf{R})$ [**9**].

For algebraic surfaces X one can hope to calculate this invariant using the holomorphic description of anti-self-dual connections via stable bundles. Any rational surface has $b_2^+ = 1$, but there are also irrational examples, in particular, a family $D_{p,q}$ (p, q coprime integers) constructed by Dolgachev. The difference in the complex geometry of the rational and irrational manifolds is reflected in the stable bundles and so in the moduli spaces and the invariant Γ. This gives

THEOREM 6 [**9**]. *A Dolgachev surface $D_{p,q}$ is homotopy equivalent (hence homeomorphic and h-cobordant) but not diffeomorphic to a connected sum* $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$.

So the h-cobordism theorem does not extend to smooth 4-manifolds. Going further, Friedman and Morgan and Okonek and Van de Ven used the Γ-invariant to prove:

THEOREM 7 [**17**, **27**]. *There are infinitely many diffeomorphism types among the homotopy equivalent manifolds $D_{p,q}$.*

When b_2^+ is odd and bigger then 1, many other invariants can be defined. For any bundle E the moduli space is of even dimension $2d(E)$. The classes $\mu(\alpha)$, for α in $H_2(X)$, can be represented by cochains with "small" support in $\mathcal{B}_E^*$. Our description of the end of the moduli space then allows the construction (in a stable range $k(E) \gg 0$) of a pairing between the powers $\mu(\alpha)^d$ and the fundamental class of M_E. Considering SO(3) connections one gets:

THEOREM 8 [**10**]. *Let X be a simply connected, smooth, oriented 4-manifold with $b_2^+(X) = 2p+1$, $p > 0$. Fix an orientation of a maximal positive subspace for the intersection form on H^2. Then for any u in $H^2(X; \mathbf{Z}/2)$ with $u^2 = \alpha$* mod 4 *and for $j > j_0(p)$ the homology class of the* SO(3) *moduli space $M_{u,j}$ defines a polynomial*

$$q_{u,j,X} \colon S^d(H_2(X)) \to \mathbf{Z}$$

of degree $d = j - 3(1+p)$, independent of the metric on X.

(Here the orientation of the positive subspace orients the moduli space M_E.) So for roughly "half" of the possible simply connected 4-manifolds we can define *infinitely many* new invariants. At present they are very hard to calculate; their main application has come from the tension between two general properties of the moduli spaces. On the one hand, we have for connected sums a "vanishing theorem":

THEOREM 9 [**10**]. *If the 4-manifold X of Theorem 8 is a connected sum $X_1 \# X_2$ with each $b_2^+(X_i) > 0$, then all the invariants $q_{u,j,X}$ are* 0.

On the other hand, if X is a projective algebraic surface there is a preferred "hyperplane" class $[H]$ in $H_2(X)$. For large values of j the moduli spaces are quasi-projective varieties of the "proper" dimension. Gieseker's construction shows that $\mu(H)$ is the first Chern class of an ample line bundle over the moduli space. So $\langle \mu(H)^d, M\rangle$ is positive and we deduce

THEOREM 10 [**10**]. *If a simply connected compact complex algebraic surface can be written as a connected sum, then the intersection form of one of the summands is negative definite.*

This immediately gives many more examples of manifolds, with the same classical invariants, distinguished by the new invariants $q_{u,j,X}$.

IV. Problems. The techniques described here are a long way from becoming a systematic theory. One notable feature is that, while the only known proofs of the ten theorems in §III use Yang-Mills instantons, there are, in most cases, a number of alternative proofs available (arguing with different moduli spaces, etc.). This suggests that there may be some more fundamental principle, relating 4-manifold topology with Yang-Mills theory, of which these arguments are different manifestations. If we could find such a principle, it might point the way to attack problems which seem to lie beyond the methods discussed above. The most obvious general questions are:

(1) Which even indefinite forms are the intersection forms of smooth, simply connected 4-manifolds? (The simplest open case is the rank 38 form $4E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.)

(2) In which homotopy types are there compact, simply connected, 4-manifolds with distinct smooth structures?

For (oriented) manifolds with b_+^2 odd there are many new invariants with which one can hope to distinguish smooth structures, so we are led to ask:

(3) Are there homotopy equivalent, simply connected, 4-manifolds with b_2^+ *even* having distinct smooth structures? The smooth 4-dimensional Poincaré conjecture is an instance of this.

On the other hand many problems to do with our new invariants for manifolds with b_2^+ odd present themselves:

(4) Are there universal relations among the invariants $q_{k,u,X}$?

(5) Can we systematically calculate the invariants given some standard description of a 4-manifold?

Two avenues seem to be promising. First we have the holomorphic description of the moduli spaces when the base manifold is a complex surface. Perhaps there are general relations between our invariants and the usual algebro-geometric invariants of surfaces. A concrete question is:

(6) If X is a minimal algebraic surface, are the invariants $q_{k,u,X}$ of Theorem 8 all polynomials in the canonical class $c_1(K_X)$ and the intersection form of X?

An extreme possibility is that they are given by universal polynomials in these two variables, with coefficients depending on k.

Second, a new slant on the picture in 4 dimensions may come from the work of Casson [**4**]. He defines an integer invariant for homology 3-spheres Y^3 using the representations $\pi_1(Y^3) \to \mathrm{SU}(2)$. The Casson invariant can be calculated from a Dehn surgery description of Y. Now these representations also come to the fore in Taubes's extension of Yang-Mills theory to noncompact (end periodic) 4-manifolds. Recently, Taubes has shown that Casson's invariant can be put into the same framework of Fredholm maps over Banach manifolds described in §II (ii). Moreover Casson and Taubes found different proofs, as corollaries of their work, of

THEOREM 11 [**4**, **35**]. *There exist topological 4-manifolds which are not homeomorphic to a simplicial complex.*

These two proofs are circumstantial evidence for the existence of some link between Casson's invariant and the anti-self-dual equations over 4-manifolds. Perhaps there is a path through Casson's work which will allow our new invariants to be defined using more familiar methods of geometric topology.

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REFERENCES

1. M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1982), 523–615.

2. M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.

3. W. Browder, *Surgery on simply connected manifolds*, Springer-Verlag, Berlin, 1971.

4. A. Casson (to appear).

5. S. K. Donaldson, *An application of gauge theory to 4-dimensional topology*, J. Differential Geom. **18** (1983).

6. ——, *Anti-self-dual Yang-Mills connections on complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3) **50** (1985), 1–26.

7. ——, *Connections, cohomology and the intersection forms of four manifolds*, J. Differential Geom. **24** (1986), 275–341.

8. ——, *The orientation of Yang-Mills moduli spaces and four-manifold topology*, J. Differential Geom. (to appear).

9. ——, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **25** (1987).

10. ——, *Polynomial invariants for smooth 4-manifolds* (in preparation).

11. S. K. Donaldson and D. P. Sullivan (in preparation).

12. R. Fintushel and R. Stern, SO(3) *connections and the topology of 4-manifolds*, J. Differential Geom. **20** (1984), 523–539.

13. ——, *Pseudo-free orbifolds*, Ann. of Math. (2) **122** (1985), 325–364.

14. ——, $O(2)$ *actions on the 5-sphere*, Invent. Math. (to appear).

15. D. S. Freed and K. K. Uhlenbeck, *Instantons and 4-manifolds*, Math. Sci. Res. Inst. Pub., Springer-Verlag, New York, 1984.

16. M. H. Freedman, *The topology of 4-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

17. R. Friedman and J. W. Morgan, *On the diffeomorphism types of certain algebraic surfaces*, submitted to J. Differential Geom.

18. M. Furuta, *Perturbation of moduli spaces of self-dual connections*, submitted to J. Fac. Sci. Univ. Tokyo Sect. I A Math.

19. ——, *On self-dual pseudo-connections on some orbifolds*, Preprint, Univ. of Tokyo.

20. D. Gieseker, *On the moduli of vector bundles over an algebraic surface*, Ann. of Math. (2) **106** (1977), 45–60.

21. R. Gompf, *Three exotic* $\mathbf{R}^4$*'s and other anomalies*, J. Differential Geom. **18** (1983), 317–328.

22. ——, *An infinite set of exotic* $\mathbf{R}^4$*'s*, J. Differential Geom. **21** (1985), 283–300.

23. K. Kuga, *Representing homology classes in* $S^2 \times S^2$, Topology **23** (1984), 133–137.

24. T. Lawson, *Normal bundles for an embedded* $\mathbf{RP}^2$ *in a positive definite 4-manifold*, J. Differential Geom. **22** (1985), 215–231.

25. ——, *Representing homology classes in an almost definite 4-manifold*, Preprint.

26. S. P. Novikov, *Homotopy equivalent smooth manifolds*, Izv. Akad. Nauk. SSSR. Ser. Mat. **28** (1964), 365–474; English transl. in Amer. Math. Soc. Transl. (2) **48** (1965).

27. C. Okonek and A. Van de Ven, *Stable bundles and differentiable structures on certain elliptic surfaces*, Invent. Math. **86** (1986), 357–370.

28. L. C. Siebenmann, *Structures on topological manifolds*, Proc. Internat. Congr. Math. (Nice, 1970), Vol. 2, Gauthier-Villars, Paris, 1971, pp. 133–163.

29. S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1964), 387–399.

30. ——, *An infinite dimensional version of Sard's Theorem*, Amer. J. Math. **87** (1965), 861–866.

31. A. I. Suciu, *Immersed spheres in* $\mathbf{CP}^2$ *and* $S^2 \times S^2$, Preprint, Yale University.

32. D. P. Sullivan, *Hyperbolic geometry and homeomorphisms*, Geometry Topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), J. Cantrell, editor, Academic Press, New York, 1978, pp. 543–555.

33. ——, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–332.

34. C. H. Taubes, *On the existence of self-dual connections on manifolds with indefinite intersection matrix*, J. Differential Geom. **19** (1984), 517–560.

35. ——, *Gauge theory on asymptotically periodic 4-manifolds*, Preprint, Harvard Univ., Cambridge, Mass.

36. N. Teleman, *The index of signature operators on Lipschitz manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 39–79.

37. K. K. Uhlenbeck, *Connections with* L^p*-bounds on curvature*, Comm. Math. Phys. **83** (1982), 11–30.

38. K. K. Uhlenbeck and S. T. Yau, *On the existence of Hermitian Yang-Mills connections on stable bundles*, Preprint, University of Chicago.

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