Applications of Non-Linear Analysis in Topology*

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If there is any single characterizing feature of the mathematics of the last few years, it is the interactions among subdisciplines. This activity is well-documented by the talks at ICM90. This lecture contains a small part of the background for applications of non-linear analysis in the field of topology. My discussion covers roughly the last twelve years, with emphasis on the earlier period of this time. There are many other articles in this volume which describe in more detail current areas of research. We refer particularly to Floer's Plenary address, and section lectures by McDuff, Simpson, Tian, Kronheimer, and many talks related to topological quantum field theory.

The specific mathematical tools I am considering in this paper are those of "hard" analysis, by which we mean two things. Hard analysis refers in graduate student slang to the use of estimates. But here it refers as well to what Gromov [Gr] calls "hard" — the realization of "soft" or "flabby" topological concepts via the solution of specific rigid partial differential equations. These techniques are naturally not to every mathematician's taste. Topologists have spent a great deal of effort reproving theorems such as the Bott periodicity theorem in ways more related to the general constructive methods of algebraic topology. For mathematicians like myself these geometric methods provide a concrete geometric realization of what is otherwise very much algebraic abstraction.

1. Background Discussion

Our first step is to illustrate the content of this talk with an example which uses only advanced calculus. We define a real-valued function

$$f : \mathbb{C}^{n+1} - \{0\} \to \mathbb{R}$$

by the formula

$$f(\mathbf{v}) = \frac{(A\mathbf{v} \cdot \bar{\mathbf{v}})}{(\mathbf{v} \cdot \bar{\mathbf{v}})}$$

for $\mathbf{v} = (z^1, \dots, z^n) \in \mathbb{C}^{n+1} - \{0\}$. Here A is any Hermitian complex $(n+1) \times (n+1)$ matrix. It is easy to see that if $\alpha \in \mathbb{C} - \{0\}$, $f(\alpha\mathbf{v}) = f(\mathbf{v})$. So f induces a map

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$$[f] = (\mathbb{C}^{n+1} - \{0\})/(\mathbb{C} - \{0\}) = \mathbb{C}P^n \to \mathbb{R}$$

by $[f]([\mathbf{v}]) = f(\mathbf{v})$. The critical points of f (and $[f]$) are points where the derivative vanishes. It is a standard exercise in advanced calculus to show that the equations for critical points of $[f]$ are the lines $[\mathbf{v}]$ for

$$A\mathbf{v} - \lambda\mathbf{v} = 0$$

where $\lambda = f(\mathbf{v})$.

Now the connection with topology comes from the relation between the critical points of $[f]$ and the topology of $\mathbb{C}P^n$. If the eigenvalues of A are distinct, one can compute that the smallest is the minimum, the largest the maximum, and the $n-1$ other critical points occur with index (number of negative directions in the second derivative) $2, 4, \ldots, 2(n-1)$. Standard Morse theory tells us that $\mathbb{C}P^n$ is built from one handle in each even dimension. The Betti numbers $b_{2i} = 1$, $0 \leq i \leq n$ and $b_{2i+1} = 0$. Of course, this is not the usual way of computing the Betti numbers of $\mathbb{C}P^n$.

However, much closer in spirit to many of our examples is the case when A is a Hermitian projection operator of rank $n-k$. Then the minimum of $[f]$ on $\mathbb{C}P^n$ is zero and it occurs on a $\mathbb{C}P^k$ sitting in $\mathbb{C}P^n$. The other critical points consist of a $\mathbb{C}P^{n-k-1}$ on which the maximum occurs. We obtain a very simple topological result by using the gradient flow for $[f]$ to retract $\mathbb{C}P^n - \mathbb{C}P^{n-k-1}$ into $\mathbb{C}P^k$. The result is that the embedding $\mathbb{C}P^k \subseteq \mathbb{C}P^n$ is a homotopy equivalence up to dimension k.

The prototype theorem for this talk is the Bott Periodicity theorem for the unitary group, as originally proved by Bott in 1959 [B]. Here the theory of ordinary differential equations replaces the advanced calculus of our first example. The space which replaces $\mathbb{C}P^n$ is $C^{\infty}_{-I,I}([0,1], SU(2m))$, or the parametrized smooth curves between $-I$ and $+I$ in the special unitary group $SU(2m)$. The function $[f]$ is replaced by energy

$$E(s) = \frac{1}{2}\int_0^1 |\dot{s}(t)|^2\, dt.$$

The equation for critical curves is the equation for geodesics

$$D_t\dot{s}(t) = 0.$$

The minimum for this functional occurs on the set of great circles between $(-I, +I)$. This can be checked to be the complex Grassmannian $G(m, 2m)$ of m planes in $2m$ space. All the other critical points have index at least $2m+2$. The gradient flow provides a homotopy equivalence between the loop space and the Grassmannian up to dimension $2m$. Namely for $0 \leq i \leq 2m$, we have the result that

$$\pi_{i+1} SU(2m) = \pi_i\big(\Omega SU(2)\big) = \pi_i G(m, 2m).$$

There are methods from algebraic topology which show that

$$\pi_{i-1} SU(m) = \pi_{i-1} SU(k)$$

for $i \leq 2m \leq 2k$ and

$$\pi_i G(m, 2m) = \pi_{i-1} SU(m) \quad \text{for } i \leq 2m.$$

Bott Periodicity Theorem [B].

$$\pi_{i-1}SU(k) = \pi_{i+1}SU(k), \qquad i \leq k.$$

Other older results which use the theory of geodesics have more to do with differential geometry.

Theorem (Hadamard). *Suppose M is a compact, connected manifold with negative sectional curvature, then $\pi_{i+1}(M) = 0$ for $i \geq 1$.*

One version of this proof is obtained by showing that all geodesics are of shortest length. As a consequence, every connected component of the loop space is topologically trivial. Hence $\pi_i(\Omega(M)) = \pi_{i+1}(M) = 0$ for $i \geq 1$.

Theorem (Myers). *If M is compact with positive Ricci curvature, then $\pi_1(M)$ is finite.*

This can be proved by showing all the minimizing geodesics are short. A general reference is Milnor's book on Morse theory [Mi].

The infinite dimensional loop spaces $\Omega(M)$ were originally handled by retraction onto finite dimensional spaces using piecewise solutions as approximations. This does not work in more than one variable because of the difficulties involved in gluing small solution pieces together. It may have been that mathematicians hoped that the multivariable problems could be easily handled once the proper tools for treating global problems were developed. This turned out to be not quite true.

The modern developments *do* rest entirely on the foundations of functional analysis and elliptic operator theory. The analytic tools are Hölder spaces, Sobolev spaces, embedding theorems, interpolation theorems and the fundamental estimates for elliptic and parabolic systems.

In the 1960s, an ambitious subject called "global analysis" developed with the explicit goal of solving non-linear problems via methods from infinite dimensional differential topology. During this period, a different set of tools was developed. A short list of these tools includes: the notion of Fredholm operator and the the Atiyah-Singer index theorems (1963) [A-S]; the definition of infinite dimensional manifolds [L]; metric structures and refinements such as layer structures and Fredholm structures; the definition of a non-linear Fredholm operator and Smale's extension of the Sard theorem [Sm] (1965); the Palais-Smale conditions and applications in the calculus of variations (1964) [P-S]; in addition, several variants of infinite dimensional degree theorems, K-theory and transversality. A good sense of the spirit of this development can be obtained by browsing through the three volumes of the proceedings of the Berkeley 1968 AMS conference organized by S.S. Chern and S. Smale [C-S].

The optimism of the era of global analysis has ultimately been justified, but this did not happen immediately. The problem is essentially as follows: In order to discover properties of solutions of ordinary or partial differential equations which have global significance, it is essential to make estimates. Now, it usually happens that certain estimates are natural to the problem. Sometimes it may be an estimate on a maximum of a norm, or more usually an integral estimate on solutions is available. Typically the estimate is on the L^2 norm of the first derivative of a solution. In other words, an estimate in the Banach space L_1^2 is natural. However, in order to obtain the results which have topological meaning, the estimates have to imply something about the space of continuous solutions, or at the very least, some information about continuity. This occurs when the Banach space L_1^2 lies in C^0. However, $L_1^2 \subset C^0$ is a dimensionally dependent inclusion. It holds for $n = 1$, or for the case of ordinary differential equations, but not for $n \geq 2$, the dimensions of partial differential equations.

Hence the explanation for the success of the cited examples on loop space is not exactly what it was expected to be. The problems work at least partly because the Sobolev embedding theorem $L_1^2 \subset C^0$ is true in dimension 1, and naive attempts to apply the theories to partial differential equations do not work except in restrictive cases. Many of the applications which were ultimately found are extremely deep. Especially significant are the dimensional differences. In retrospect, one could not expect one single aspect of non-linear analysis to magically provide for a variety of deep applications.

2. Results

S.T. Yau's proof of the Calabi conjecture, published in 1978, showed for the first time the effect which modern methods of solving partial differential equations could have on other fields. The analytic theorem Yau proved was for an arbitrary complex Kähler manifold with non-positive first Chern class. Yau proved that there is a Kähler metric which solves Einstein's equation.

Theorem [Y-1]. *Let M be a complex Kähler manifold with $c_1(M) \leq 0$. Then there is a Kähler metric g in the same Kähler class as the given one with*

$$\text{Ricci}(g) - Rg = 0\,.$$

The topological conditions imply that $R \leq 0$, where the constant R is a multiple the constant scalar curvature of the Einstein metric.

This result had been conjectured by Calabi and partially proved by Aubin. However, its importance in algebraic geometry lies in the following application.

Corollary. *If M is a complex Kähler manifold with $c_1(M) \leq 0$, then*

$$(-1)^n c_2(M) c_1(M)^{n-2}[M] \geq \frac{(-1)^n n}{2(n+1)} c_1(M)^n [M]\,.$$

This theorem is cited in every theorem on the classification of higher dimensional algebraic varieties. It is essentially the only topological restriction known for algebraic manifolds. The theorem is not true for positive Chern class. One of the satisfying results of Yau's proof is that the importance of the topological condition $c_1(M) \leq 0$ is apparent. Yau's original theorem (in the case of complex 3-folds with $c_1 = 0$) is fundamental in the current model for fundamental physics (string theory).

In the ensuing years, the applications of partial differential equations have been extensive, and we give a very brief survey of the initial results in each field. It would not be possible to list all the latest results and their fine points in a general survey article.

Minimal Surfaces

Yau and his coworkers obtained a number of interesting results. Many but not all of these results have alternate proofs. The first theorem is in the spirit of Myer's theorem, and was published in 1980 by Schoen and Yau. I call it the topological positive mass theorem, as it is the topological version of the well-known positive mass theorem of general relativity. (Schoen and Yau used the same techniques to prove the theorem in general relativity.) Scalar curvature is local mass. An alternate proof using linear analysis (Dirac operators) now exists. The analytical basis for the Schoen-Yau proof is a theorem on minimal surfaces.

Theorem [S-Y, S-U]. *If Σ_g is a surface of genus $g \geq 0$, and M any compact manifold with $\pi_1(\Sigma) \subseteq \pi_1(M)$, then there exists an area minimizing branched immersion of Σ in M.*

Application (Topological Positive-Mass) [S-Y]. *If $M = M^3$ has non-negative scalar curvature, then the group $\pi_1(M)$ does not contain $\pi_1(\Sigma_g)$ as a subgroup unless $\Sigma_g = S^1 \times S^1$ and $M^3 = S^1 \times S^1 \times S^1/\Gamma$ is a quotient of the flat torus.*

To prove this, look at the second variation of the minimal surface. The positive scalar curvature forces it to have a negative direction.

Meeks and Yau proved a series of results which show that minimal surfaces in 3-manifolds are embedded. In many cases, this provides alternate more rigid proofs of basic theorems in 3-manifold topology, such as Dehn's lemma, the loop theorem, and the sphere theorem. As part of this program they gave the first proof of the equivariant loop theorem.

Equivariant Loop Theorem (Meeks-Yau). *Let M^3 be a handle-body with boundary Σ_g, and $K \subseteq \mathrm{Diff}(M^3, \Sigma_g)$ be a finite group. Assume M^3 is given a metric in which K acts as isometries, and in which Σ_g has positive scalar curvature (outward like $S^2 = \partial D^3$). Then there exists an embedded minimal disk $(D^2, S^1) \subseteq (M^3, \Sigma_g)$ such that the elements $k \in K$ either leave D^2 invariant, or map D^2 to a disk $k(D^2)$ with $k(D^2) \cap D^2 = \emptyset$.*

The first step in the analysis consists in showing that there is indeed a smallest disc in M^3 with boundary on Σ_g (this leads to a boundary value problem which is a combination of Dirichlet and Neumann conditions). If the solution is not embedded, or if it intersects an iterate, it turns out that it cannot really be of smallest area. Meeks and Yau pioneered the use of 3-manifold techniques to show that minimal surfaces in 3-manifolds are often embedded rather than immersed [M-Y].

The proof due to Siu and Yau of the Frankel conjecture dates from the same year as the proof due to Mori.

Frankel Conjecture. *If M^n is a complex Kähler manifold with positive bisectional curvature, then M^n is biholomorphically equivalent to $\mathbb{C}P^n$.*

Siu and Yau use the minimal 2-spheres shown to exist by Sacks and Uhlenbeck [S-U]. Positive curvature tends to place restrictions on what can be minimal (as in Myer's theorem and in Schoen and Yau's proof of the positive mass conjecture). In this case, Siu and Yau show that the minimal sphere is actually a holomorphic curve [Si-Y].

Finally, we mention a much more recent result. The method due to Sacks and myself for finding minimal 2-spheres in manifolds had been used by Meeks and Yau to handle embedding problems for spheres in 3-manifolds, as well as by Siu and Yau in the Frankel conjecture. However, these proofs use the area minimizing spheres, whereas Micallef and Moore [M-M] later found a use for the non-minimizing critical points. Their isotropic curvature condition is satisfied if the Riemannian curvature is pinched between K and $4K$.

Theorem [S-U]. *If M is a compact manifold and $\pi_i(M) \neq 0$ for some $i \neq 1$, then there is a 2-sphere which is a stationary point of the area functional.*

Sphere Theorem of Micallef and Moore. *If M^n is simply connected and has positive curvature on isotropic 2-planes, then M^n has the homotopy type of a n-sphere.*

Proof. Show that $\pi_i(M) = 0$ for $i \leq [n/2]$. If this is true, the Hurewicz homomorphism and Poincaré duality complete the proof. If $\pi_i(M)$ is the first non-zero homotopy group, a difficult minimax argument leads to the construction of a minimal 2-sphere of index at most $(i - 2)$. However, the curvature condition forces the existence of at least $[n/2] - 1$ directions in which the second variation is negative. This leads to a contradiction.

Gauge Theory and 4-Manifolds

Donaldson's announcement of the restrictions on the topology of 4-manifolds with differentiable structures is a more recent mathematical event. Donaldson's startling use of gauge theory in four dimensions followed almost immediately the successful use of minimal surfaces, and development of these gauge theory techniques is still an active field.

The program instigated and to a great extent carried out by Donaldson consists of encoding the properties of differential structures on 4-manifolds by studying the self-dual Yang-Mills equations on the manifold. The difficulty is that the mathematician must introduce a metric onto the smooth manifold. The first step is to understand the analysis, then the dependence on the metric must still be analyzed.

The non-linear analysis part of Donaldson's theory consists in construction of the space of self-dual solutions of Yang-Mills equations over a conformal manifold M in a bundle with structure group $SU(2)$ and second Chern class $-k$. (In the more recent literature, the orientation is reversed to study the anti-self-dual equations in a bundle of second Chern class $+k$. This fits in better with complex analysis). The basic ingredients is the list of theorems developed by global analysts. The Atiyah-Singer theorem determines the dimension of the moduli space and the Sard-Smale theorem can be used to show it is generically a manifold. Taubes' implicit function theorem developed to construct solutions by gluing instantons on a manifold was later modified by Donaldson to include the construction of solutions on the connected sum $M_1 \# M_2$ from solutions on M_1 and M_2. The solution spaces are not compact. However, the boundary is well-understood via exactly the arguments developed to understand the convergence of minimal surfaces.

Theorem (Donaldson) [D-2]. *If M^4 is a simply connected 4-manifold with positive definite self-intersection form, then the moduli space of solutions to the self-dual Yang-Mills equations with $k = -1$ and group $SU(2)$ is generically an oriented manifold with isolated singularities whose boundary can be identified with M.*

The isolated singularities correspond to solutions where $E = L \oplus L^{-1}$ splits into line bundles. This is a theorem an analyst might well have proved, although it would certainly not be obvious to include the orientability. However, the topological use of this theorem appears a a corollary.

Corollary. *M^4 is topologically the connected sum of $\mathbb{C}P^2$'s.*

Each singularity looks like a $\mathbb{C}P^2$ with a positive orientation. There isn't anything else with definite form oriented cobordant to this sum of $\mathbb{C}P^2$'s.

The theory has been developed further, and more elaborate properties of the solution space of the Yang-Mills equation are used in later results. We refer the readers to a forthcoming survey by Freed [Fr] and the article by Donaldson [D-1].

Some of the properties of the moduli spaces in gauge theory are quite similar to properties of Gromov's pseudoholomorphic curves. McDuff has used these to distinguish different symplectic forms [Gr, McD].

Complex Moduli Spaces

The results about 4-manifolds obtained from the Yang-Mills equation are obtained by looking at the topology of the "moduli spaces of solutions." Here by moduli space we refer to the actual solutions to Yang-Mills divided out by the natural geometric equivalence. This is in many ways similar to older examples of moduli spaces, such as the Riemann moduli space of conformal structures on surfaces. In a development which is related to the moduli space of self-dual Yang-Mills equations, a whole class of problems in algebraic geometry can be put in a general framework which we might call infinite dimensional geometric invariant theory.

The foundational paper is the paper by Atiyah and Bott on Yang-Mills equations over Riemann surfaces [A-B]. An older result of Narasimhan and Sheshadri [N-S] proves that the moduli space of stable holomorphic bundles M_g over a complex curve Σ_g of genus $g \geq 2$ can be identified with the moduli space of projectively flat connections. These connections can be identified with the minima of the Yang-Mills functional on connections in Σ_g. Atiyah and Bott conjectured that there is a very beautiful analytic picture which fits this functional. The symmetry group of Yang-Mills is the real group of gauge transformations $\mathcal{G}$. However, its complexification $\mathcal{G}_{\mathbb{C}}$ acts and stratifies the sets of connections $\mathfrak{A}$ into classes of holomorphic structures $\mathfrak{A}/\mathcal{G}_{\mathbb{C}}$ on the bundle. The Yang-Mills functional is the L^2 norm of moment map for this action. Stable orbits are (essentially) the orbits on which the action is free and the quotient Hausdorff. By a very general principle, the equivariant topology of the stable moduli space can be computed from the topology of $\mathfrak{A}/\mathcal{G}$ by examining the Morse theory.

Atiyah and Bott were not able to carry through the analysis, but obtained their results from algebraic geometry. Frances Kirwan [Ki] carried out this very general program in a finite dimensional setting. This pattern of development is strikingly similar to the original construction of the Morse theory of geodesics, which inspired the very useful finite dimensional development. Donaldson [Do-3,4], Hitchin [Hi], Daskalapoulos [Da], Bradlow [Br], Corlette [Co] and Simpson [Si] have carried out the analysis and extended the picture to cover coupled equations and complex manifolds of higher dimension. Naturally, the topological results are much better in complex dimension 1. This is because of the Sobolev inequalities. We discuss this in more detail in the next section.

We finish this section by stating one of the basic results of these computations. It corresponds to the bundle version of Yau's solution of the Calabi conjecture. The topological consequences are similar, although as with the Calabi and Frankel conjectures, there is also an algebraic proof.

Theorem (Donaldson, Uhlenbeck-Yau) [D-5, U-Y]. *The moduli space of stable bundles on a complex Kähler manifold is isomorphic to the space of irreducible solutions to the holomorphic Yang-Mills equations (an extension of the anti-self dual equations to arbitrary dimension).*

Corollary. *If E is a stable holomorphic bundle of rank r on a complex Kähler manifold of dimension n, then*

$$c_1(E)^2 \wedge \omega^{n-2}[M] \leq \frac{2r}{r+1} c_2(E) \wedge \omega^{n-2}[M].$$

This is a purely topological statement about the cohomology class of the Kähler form and the characteristic classes of the complex tangent bundle. This inequality is easily seen to be true by applying Chern-Weil theory to the holomorphic Yang-Mills connections.

The Poincaré Conjecture

The Poincaré conjecture refers to one of the best known problems in topology. If a manifold M^n has the homotopy type of S^n, is it S^n (differentiably or continuously)? I don't think that analysts have gotten very close to proving or disproving the Poincaré conjecture. It is perhaps not so well-known that this conjecture has been the inspiration for a number of fundamental developments in analysis.

Most of the ideas have focussed on Einstein's equation. This equation is the Euler-Lagrange equation for a critical metric for the variational integral $\int_M K_g \, d\mu_g$. Here $\operatorname{vol} M = \int_M d\mu_g$ is kept fixed. Here K_g is the scalar curvature of the metric g. Einstein's equation reads

$$\operatorname{Ricci}(g) - \lambda_n K g = 0.$$

It goes without saying that we know more about a manifold if it has a solution of Einstein's equation on it. In particular, for a 3-manifold Einstein's equation is equivalent to the metric having constant Riemannian curvature. Clearly not every 3-manifold supports a solution to Einstein's equation because there is no metric of constant Riemannian curvature on most 3-manifolds. Nevertheless, this fact may not have been widely recognized by analysts in the past.

Solving Einstein's equation is very difficult. Yamabe [Yam] proposed as the first step to fix the conformal structure and vary only the function describing lengths. This leads to a much studied conformally invariant problem usually called the Yamabe Problem [L-P].

Palais was motivated to construct a very general theory of the calculus of variations [P]. He claims that he hoped to apply it to the Einstein functional. However, he very quickly realized that the critical points of the Einstein functional have infinite index and coindex. Hence they cannot be detected by the topological methods he developed. There is still some hope of using a minimax argument and the known solution to the Yamabe problem. I am not sure how much faith any of us have in this project, though.

Hamilton's results on the Einstein equation (1982) were very surprising and promising. He showed via a heat flow argument that a 3-manifold with positive Ricci curvature supports a solution to Einstein's equation. This makes it the quotient of S^3 by a finite group. This result has been very influential on analysis in general without providing a solution to the Poincaré conjecture.

Finally, I would like to comment that the inspiration of the Poincaré conjecture is still very much with analysts. Thurston's results on 3-manifolds go a long way towards describing the geometry of 3-manifolds. Present thinking is that it may be possible to use some other variant of a curvature integral. An example might be

$$\int_M |\mathrm{Riem}(g)|^p \, d\mu(g)$$

for $p \geq 3/2$ in 3 dimensions. The hope is that, by transposing gauge theoretic techniques over to manifolds, the obstructions to minimizing such integrals can be better understood.

Conjecture (Due to Deane Yang). *If M^3 is aspherical and atoroidal, then the minimum of the integral*

$$\int_M |\mathrm{Riem}(g)|^{3/2} \, d\mu(g)$$

under the constraint $\operatorname{vol} M = 1$ *is either zero, or is taken on by a metric of constant negative curvature.*

This result is in fact nearly implied by Thurston's conjectures on 3-manifolds. Yang's conjecture should lead to interesting analysis, even if it doesn't touch the Poincaré conjecture.

3. Analytical Technique

The variety of topological results cited in the previous section is matched by the variety of different analytical methods which were used. There is no neat classification which matches a set of results with a set of techniques. I roughly classify the methods under the subheadings:

Continuity method,
Borderline dimension,
Gauge theory,
Heat equation methods.

This is not an exhaustive list of methods. A theory such as that constructed by Andreas Floer uses nearly all the ideas mentioned in this section and more. Many of the latest developments involve topological index theory constructions on the solution spaces as are needed for Donaldson's invariants of 4-manifolds and applications of Gromov's pseudo-holomorphic curves.

Continuity Method

The continuity method is on the surface naive. The goal is to solve an equation

$$F(g) = 0$$

where F is a non-linear elliptic system. To use the continuity method, start with a trial g_0, and compute $F(g_0) = F_0$. Put in a parameter $\varepsilon \in [0,1]$ and solve

$$F(g_\varepsilon) = (1-\varepsilon)F_0 .$$

This is done by showing that $dF(g_\varepsilon)$ is invertible as a map between appropriately introduced Banach spaces, and that the solution g_t stays bounded in the tangent norm. The invertibility of dF implies via the implicit function theorem that the set of $\varepsilon \in [0,1]$ for which we have a solution to our equation is open. The estimates (and some weak convergence) show this set is closed. The sophistication comes from the usual necessity of dealing with a number of Banach spaces, and from the further necessity of estimating the inverse of dF.

This is the method used by S.T. Yau to solve the Calabi conjecture. While the equation is Einstein's equation

$$\mathrm{Ricci}(g) - \lambda(g) = 0 ,$$

in the Kähler case an equation can be written for a Kähler potential $\varphi(\varepsilon)$ where $g(t) = g_0 + \partial\bar{\partial}\varphi(\varepsilon)$ is a new metric. The equation becomes one for the potential function φ. The first stage of the estimates follow easily from the maximum principle. However, essentially all derivatives of φ have to be estimated, although the estimates become iterative after the third derivatives. No method of solution has been found which avoids estimates (nor would we expect this to happen). However the estimates themselves have been given a more geometric foundation and have found a wider application in the general study of Monge-Ampere equations.

Uhlenbeck and Yau used the same method to solve the holomorphic Yang-Mills equations on a Kähler manifold. Here, the metric is in a bundle, and there is no potential, but there is a maximum principle. By adding an ε, we obtain an invertible equation

$$F_g + \varepsilon \ln g = 0 .$$

For $\varepsilon > 0$, it is easy to solve this equation. As $\varepsilon \to 0$, the solution does blow up unless an extra geometric condition of stability is satisfied. We show that, if blow-up does occur, that the normalized solutions $\frac{g(\varepsilon)}{\mu(\varepsilon)}$ converge to a degenerate metric π which violates the stability condition. The sophisticated analysis occurs in higher order estimates, not in the initial outline.

Finally, Taubes' construction of instantons on four manifolds is via an implicit function theorem proved in the same style [T-1]. His original iterative proof can be rewritten using the continuity method. He glues in the standard instantons on $\mathbb{R}^4$, localized to lie in a ball of radius λ into a ball centered at p on an arbitrary compact four manifold. The resulting approximate solution can be modified by a small amount to give a solution to the instanton equation. Donaldson further needs to obtain a moduli space with the parameters (λ, p). Again, it is the estimates and the invertibility of an operator very close to the derivative operator which are essential [D-2].

The continuity method is best suited to cases where invertibility of the derivative is built into the situation. In all the cases just cited, there is a moduli space of solutions whose dimension and structure can be understood by using the appropriate choice of coordinates (or gauge) when looking for the solution. A maximum principle is available for the Kähler-Einstein and Holomorphic Yang-Mills examples. Invertibility in Taubes' construction comes from topological constraints and knowledge of S^4. It remains a question when Taubes' gluing of point localized solutions applies to other partial differential equations. Schoen has applied a somewhat similar idea in constructing solutions of the Yamabe problem with point singularities [S] and Kapouleas [Ka] has shown the existence of many complicated surfaces of constant curvature in $\mathbb{R}^3$ by a not unrelated technique. These problems all exhibit conformal invariance, which allows the scaling of solutions. The resulting approximate solutions have a number of parameters. In the simple cases there is a projection onto a moduli space; in complicated examples the parameters must be chosen carefully.

Borderline Dimension

The critical ingredient in geometric problems is the Sobolev embedding theorems. For the usual mapping examples, $L_1^2 \subset C^0$ is true in dimension 1, false in higher dimensions. Dimension 2 is the borderline or scale invariant dimension, because the integral $\iint |ds|^2 (dx)^2$ scales the same way as $\max_{x \in M} |s(x)|$. Both are scale invariant. A similar phenomena is observed for the Yamabe problem. In dimension n, $L_1^2 \subset L^p$ for $p < \frac{2n}{n-2}$. The relevant power of p in the Yamabe problem is $p = \frac{2n}{n-2}$ [L-P].

For borderline problems, solutions can easily be found using minimization and weak convergence methods. The difficulty is that the limiting functions may not satisfy the constraint satisfied by the approximating functions. A typical example would be to fix a domain bounded by a simple closed curve Γ in the plane. Consider maps $s : D^2 \to \mathbb{R}^2$ such that $s|S^1 : S^1 \to \Gamma$ has degree one. Now minimize

$$\iint_{D^2} \left(\left| \frac{\partial s}{\partial x} \right|^2 + \left| \frac{\partial s}{\partial y} \right|^2 \right) dx\, dy .$$

The minimum occurs on a conformal map, and will provide a solution of the Riemann mapping problem. The difficulty is to keep the weak limit of a minimizing sequence from being trivial (i.e., to preserve the condition of $s|S^1 : S^1 \to \Gamma$ is of degree one). One solution is to use the conformal invariance to fix the map on three points of S^1. C.B. Morrey [Mo] used these ideas (which originate with Douglas' solution of the Plateau problem) to find minimal surfaces in arbitrary Riemannian manifolds.

The fundamental observation is that in conformally invariant problems energy estimates plus scale invariance imply that the estimates and convergence are valid except possibly at an isolated set of points. At these points, small neighborhoods may conformally expand to cover large pieces of the geometric solution. The mathematician can recover the geometric solution by blow-up. For $\varepsilon_i \to 0$, set

$$s_i \left(\frac{x - x_0}{\varepsilon_i} \right) = \hat{s}_i(x) .$$

Then $\hat{s}_i(x) \to s : \mathbb{R}^n = S^n - \{p\} \to M$ will be a solution on S^n [S-U1].

This principle allows the construction of a large number of two-dimensional minimal surfaces, and underlines our present understanding of the Yamabe problem. Later it came to be fundamental in the analysis of solutions of the self-dual Yang-Mills equation on 4-manifolds. The borderline dimension for Yang-Mills turns out to be 4 instead of 2, and many results for harmonic maps and minimal surfaces have counterparts in theorems on Yang-Mills in dimension four [Ma, Se]. However, the topological results have come from Donaldson's study of the solution spaces of the self-dual Yang-Mills equations. Here existence theorems are proved by the implicit function theorem mentioned in the previous section. Whatever compactness of the solution space has is shown by the same techniques used in the construction of minimal surfaces [Do-2, F-U]. Gromov later applied these same ideas back in two dimensions to study pseudo-holomorphic 2-spheres in symplectic manifolds [Gr, McD].

It is an interesting question whether mathematicians will discover new and useful scale-invariant geometric partial differential equations. These techniques are just waiting to be used again!

Gauge Theory (New Sobolev-Inequalities)

Until gauge field theory appeared in mathematics, well-posed, natural, topological calculus of variations problems seemed confined to one dimension. It is possible to artificially construct variational problems of all sorts. However, the geometrically natural variational problems seem to be the only ones which are useful in topology. These are (a) first order in derivatives and (b) quadratic. Conditions (a) and (b) imply that the natural estimates are L_1^2. Topological results come from C^0, so we are stuck in the classical case of dimension one with geodesics for good problems and in the case of dimension 2 with minimal surfaces for the borderline case.

However, in gauge field theory on a manifold M, the unknown is the connection A which is a one-form. The natural function is the L^2 norm of curvature

$F_A = dA + [A, A]$, which is the first derivative of A. However, topology comes from the overlap functions, which relate A's via their derivatives. Intuitively speaking, curvature is the second derivative of the structure functions of a bundle. The relevant Sobolev embedding is $L_2^2 \subset C^0$. This embedding is true in dimensions 2 and 3, and borderline in dimension 4.

Some basic estimates need to be obtained before analysis can be done. Gauge theory problems have an infinite dimensional symmetry group, and the curvature $F_A = dA + [A, A]$ contains only part of the full derivative of A. These problems are related and are taken care of in the same analytical lemma. Because of the nonlinearity in the problem, the estimates are stated globally in terms of convergence.

Theorem [U-1]. *Let D_i be a sequence of connections on a compact manifold M of dimension n. If F_i is the curvature of D_i, and $F_i \in L^p$ forms a bounded sequence for $p > n/2$, then there exists a subsequence $D_{i'}$ and a sequence of gauge transformations $s_{i'}$ such that $s_{i'}^* D_{i'} \rightharpoonup D$ in L_1^p.*

A weaker version of this lemma applies in the borderline case $p = n/2$.

Yang-Mills in dimensions 2 and 3 gives us new examples of variational problems in which the topology and analysis match. The 2-dimensional example has been extensively studied [A-B] and gives us the topological results on the moduli space of stable bundles over Riemann surfaces. Recently the analytical details were completed by Daskalapoulos [Da]. The 3-dimensional problem is not very well understood geometrically. We do not as yet know how to use the many solutions to Yang-Mills which exist on a 3-manifold.

Heat Equation Methods

One of the earliest global non-linear results was that of Eells and Sampson [E-S]. Recall that the energy functional is defined on maps between two compact Riemannian manifolds $s : M \to N$

$$E(s) = \int_M |ds|^2 \, d\mu .$$

Critical maps are called harmonic maps. Eells and Sampson constructed a harmonic map in every homotopy class of maps $s \in [M, N]$ when N has non-positive sectional curvature. They did this by following L^2 gradient curves. Thus L^2 flow is a non-linear system of parabolic equations

$$\frac{\partial s}{\partial t} = \nabla_s^* \, ds .$$

In fact, early in his career S.T. Yau wrote a number of papers which apply this existence theory [Y-2].

A potential method of solving a non-linear elliptic equation is to solve the associated non-linear parabolic equations and follow the solutions as $t \to \infty$. This at first seems unduly cumbersome, and at least one mathematician expended some effort in finding "better" ways to find the Eells-Sampson harmonic maps [U-2]. In 1983 Richard Hamilton [H] was able to solve Einstein's equation on some 3-manifolds by solving the associated parabolic equation and following the time dependent solution to a solution of the elliptic equation. This result has already been noted in the section on the Poincaré conjecture in section 2. Hamilton's result has had a remarkable effect on geometric PDE, since this result surely cannot be obtained by variational methods. In the wake of Hamilton's work, both the Einstein and the Kähler-Einstein equation have been reexamined and solved in certain contexts using the heat equation method.

The non-linear heat flow was the geometric gradient used by Atiyah and Bott to study the Yang-Mills equations. Their analysis was shown to be rigorous by Donaldson, and was used by Donaldson to solve the holomorphic version of Yang-Mills in arbitrary complex dimension [Do-3,Do-4]. Carlos Simpson [Si] has shown that this approach fits in very nicely with geometric invariant theory. The method can be used to obtain the same results as obtained by Yau and myself [U-Y] and is more philosophically satisfying than the perturbation method.

Strüwe has gone a long way towards showing that many of the useful properties of harmonic maps (or minimal surfaces) can be obtained via the parabolic equation [St]. In many cases, the rigid method of looking for solutions of a geometric elliptic equation via the parabolic equation will give the most delicate results. This fits in best with the topology when there are moduli spaces of solutions, the equation is geometrically natural, and fewer choices are involved in following a parabolic equation compared to picking a minimizing sequence.

4. Failures

There are a number of theorems and conjectures which seem to be suitable for attack by the methods discussed in the previous section. Some mathematicians might regard these as "open problems." This is a matter of perspective. Perhaps, since I confess to having spent considerable time on them myself, the failure is personal.

Mostow-Rigidity Theorem. *If $s : M^n \to N^n$, $n > 2$, is a diffeomorphism between hyperbolic manifolds, then s is homotopic to an isometry.*

The proof of this theorem ought to go somewhat as follows. Let $\hat{s}$ be the harmonic map homotopic to s. The existence of this map dates back to [E-S] in 1963. Because M^n and N^n have constant curvature -1, the map $\hat{s}$ is an isometry.

Unfortunately, the only way we know that $\hat{s}$ is an isometry is by applying Mostow rigidity in a circular argument. This is, of course, only the simplest version of the Mostow rigidity theorem. There are some extensions due to Gromov which ought to be accessible by these means. Ultimately we would generally like to know better the relationship between curvature and volume. So far partial differential equations have not been helpful.

A related theorem by Siu uses harmonic maps to establish rigidity of complex manifolds with negative bisectional curvature [Si]. Other work has been done by Corlette, Gromov and Schoen, but a partial differential equation's proof of Mostow's theorem remains elusive.

Theorem [Se]. *Let $C_d^\infty(S^2, S^2)$ be the space of maps of degree d between S^2 and S^2. Let $M_d^\infty(S^2, S^2) \subseteq C_d^\infty(S^2, S^2)$ be the meromorphic maps of degree d. Then the inclusion of M_d in C_d^∞ is a homotopy equivalence up to dimension d.*

The proof ought to proceed as follows: Consider the energy integral

$$E(s) = \frac{1}{2} \iint_{S^2} |ds|^2 \, d\mu$$

on maps $s : S^2 \to S^2$. The meromorphic functions are the set on which E takes on its minimum. After removing a set of codimension $d + 2$, the gradient flow retracts the remainder of the space of C^∞ functions onto the minimum.

This is a conformally invariant problem and we have not learned to handle the finer points of the topology of the flow! Segal's theorem is actually very extensive, applying to surfaces of arbitrary genus as the domain and a whole class of (positively curved) complex manifolds as the image [Se]. It is related to the Atiyah-Jones conjecture on the topology of instantons embedded in the space of connections [A-J].

There is actually an analytic proof of Segal's theorem. It can pieced together by applying a theorem of Donaldson [Do-6] on monopoles and Taubes' proof of a Morse theory for the monopole equation [T-2]. This seems too unwieldy to be a model proof, however.

Thurston's Techniques for 3-Manifolds. As part of his theory of 3-manifold topology, Thurston develops techniques for analyzing hyperbolic 3-manifolds which depend heavily on rigidly embedded surfaces in the manifolds. These surfaces are geodesically embedded with constant Gaussian curvature, and are broken along geodesics. It is more natural from the point of view of analysis to examine the properties of constant mean curvature surfaces in 3-manifolds. (Zero mean curvature characterizes locally minimizing surfaces.) Up until this date, attempts to replace Thurston's broken surfaces with smooth surfaces have been strikingly unsuccessful. While some results in 3-manifolds are obtained by partial differential equations techniques, Thurston's program has been unaffected by these methods.

Jones-Witten Invariants. Two of the four fields medals at this congress were given out at least in part for a theory designed to produce 3-manifold invariants. There are many approaches to this theory, but the unifying approach of Witten is to start with a classical geometrical integral similar to those which have been used already in the applications of analysis to topology. Witten in fact takes the integral used by Floer. The classical Chern-Simons integral is defined for an $su(N)$ valued one-form A on a 3-manifold.

$$CS(A) = \frac{1}{4\pi} \int_{M^3} \operatorname{tr} \left(dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right) .$$

The Jones-Witten invariant for the manifold has the form

$$W(k, N) = ⨎_{\text{all } A} e^{ikCS(A)} \, dA .$$

The symbol $⨎$ indicates the Feynman path integral used heavily in quantum field theory. Unfortunately, this is not a situation which has been made completely rigorous. However, many tools exist as input into calculations in quantum field theory. We list a few, in the hopes of giving some flavor of the subject.

Perturbation Theory. Calculations in quantum field theory which relate to physical experiment are meant to obtain asymptotic formulas as $1/k \to 0$. They are done via Feynman diagrams, which represent power series expansion around a vacuum (usually $A = 0$). In all physical cases I know of, the domain manifold is $\mathbb{R}^n$, $n \leq 4$. These methods would fail here, due to the lack of a proper Green's function.

Stationary Phaze (and Ghosts). There are two separate problems which can be formally dealt with in more geometric problems. One is that of several important classical solutions and the other is the lack of ellipticity. Witten has carried out these asymptotic expansions and obtained the first order asymptotic in k [W-2]. There are clearly severe problems in the theory since further calculations cannot even be guessed at. No one has yet obtained lower order terms for manifolds, although some calculations exist for knots [B-N].

Finite Approximation. The approximation of the infinite dimensional integral $\not\int$ by finite integrals is known as lattice gauge theory. A large amount of supercomputer time is spent on more down-to-physics calculations than those of Chern-Simons theory. The value of these type of calculations is not clear. They certainly shed no light on arbitrary 3-manifolds.

Axiomatic Approach The quantum field theory involved in the Chern-Simons theory is particularly simple, and the formulation of the correct axioms seems to have been the greatest success of the theory so far. Atiyah has formuated the axioms for topological quamtum field theory (TQFT) [A-1].

Geometric Quantization. This formulation of quantum theory is probably the best understood by mathematicians. Witten and his students have a successful formulation of the Chern-Simons theory in terms of calculations on the moduli space of flat bundles [A-PD-W].

Canonical Quantization. It is usually necessary to use both the path integral and the Hamiltonian approach in computing the ingredients of quantum field theories. In many contexts, the Hamiltonian approach makes contact with group representations. We know that the input from 2-*d* conformal field theory is very useful in setting up the building blocks for the 3-*d* Chern-Simons theory. We can hence include on our list of useful mathematics:

Group representation theory,
Integrable lattice models of statistical mechanics,
Quantization of completely integrable systems.

Quantum Groups. Finally, some topologists have found that the direct quantum group (Hopf algebra) approach leads to the most direct construction of the Jones-Witten invariants [Q,Wa].

In classical mechanics, Lagrangian and Hamiltonian formulations are equivalent. In quantum field theory, both are necessary and complementary. What remains in question is the consistency (not equivalence) of the facts gained from the different approaches. To someone like myself, who has worked in topological applications of partial differential equations, the situation is analogous to the plurality of approaches which can be made to understanding the Laplace operator on a Lie group. It is possible to get input from a wide variety of mathematical directions: separation of variables, special functions, finite-element approximation schemes, group representations, asymptotic heat kernel methods and a variety of different types of geometric constructions. However, we know what the Laplace operator itself is. This holds the different ideas together. The study of Jones-Witten invariants is similar, but we are missing the central ingredient which corresponds to the Laplace operator — a proper definition of $\oint$.

Recent computer calculations of D. Freed and R. Gompf indicate that two approaches (both due to Witten), from conformal field theory and from stationary phase approximation actually agree for lens spaces and some homology spheres [F-G]. Any mathematical demonstrations of this agreement would of necessity contain the proof tht analytic torsion agrees with combinatorial (Reidemeister) torsion. Perhaps we should look forward to a construction of $\oint$ which will pull this theory together.

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