Computing the radius of positive semidefiniteness of a multivariate real polynomial via a dual of Seidenberg's method

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Joint Work With: Erich Kaltofen, and Lihong Zhi

Motivation

$$\forall x \in \mathbb{R}\colon\ 0.33\,x^2 - 0.66\,x + 0.33 \geq 0 \quad ???$$

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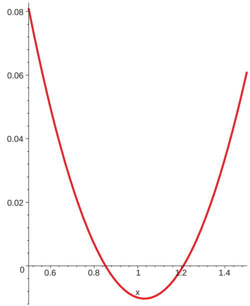

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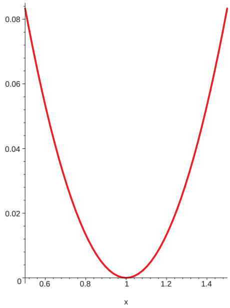

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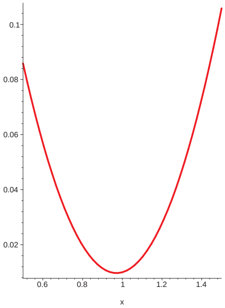

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Main Theorem [Stetter; Corless, Gianni, Hitz, Hutton, Kaltofen, Karmarkar, Lakshman, Sciabica, Ruatta, Szanto, Trager, Watt, Zhi]

Given $\alpha \in \mathbb{R}^n$:
$$\mathcal{N}_2^{[f]}(\alpha) = \inf_{\tilde{f} \in \mathbb{R}[x_1,\ldots,x_n]} \|f - \tilde{f}\|_2^2 \quad \text{s. t. } \tilde{f}(\alpha) = 0, \ \deg(\tilde{f}) \le \deg(f)$$
$$\boxed{= \frac{f(\alpha)^2}{\|\tau\|_2^2}},$$

where $\tau = [1, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_n^{i_n}, \ldots]_{(i_1,\ldots,i_n)\le\deg(f)}$.

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Note: Generalizes to **complex** roots and/or **complex** coefficients!

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  - $\rho_2(\, x^2 + 1 \,) = 1, \quad \tilde{f} = x^2$
  - $f = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 + 1 \geq 1$
    Nearest polynomial with a real root has distance $\rho_2(\, f \,) = 0$
    because $(\frac{1}{\epsilon}, \frac{1}{\epsilon}, \frac{1}{\epsilon})$ is a root of $f - \epsilon^6 x^2y^2z^2$

Example

- $f = x^2 + y^2 + 1$,
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- $\mathcal{N}_2^{[f]}(0,0) = \inf\limits_{\tilde{f} \in \mathbb{R}[x_1,\ldots,x_n]} (1 - a_{20})^2 + (0 - a_{11})^2 +$
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  - $\tilde{f} = x^2 + y^2$
- $\rho_2 = \inf_{(\alpha,\beta)} \mathcal{N}_2^{[f]}(\alpha, \beta) = 1$

Our contributions:

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  - customized formulas for equality constraints on coeff's
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- New proof by Lagrangian multipliers
  - customized formulas for equality constraints on coeff's
  - can have inequality constraints via Karush-Kuhn-Tucker conditions for fixed root α get linear program
  - nearest consistent system with infinity coefficient norm
- SOS certificates for rational lower bound $\tilde{\rho}_2(f) < \rho_2(f)$
  - degree bounded SOS certificates for Motzkin like polynomials
  - Seidenberg's problem with imprecise coefficients

Weighted Norms

Weighted norms $\|\vec{f}\|_{2,w}^2 = \sum_j w_j(\vec{f})_j^2$ for weights $w_i > 0$

$$\mathcal{N}_{2,w}^{[f]}(\alpha) = \frac{f(\alpha_1, \ldots, \alpha_n)^2}{\sum_{(i_1,\ldots,i_n)\leq \deg(f)} \frac{1}{w_{i_1,\ldots,i_n}} \alpha_1^{2i_1} \cdots \alpha_n^{2i_n}}$$

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$w_j \to \infty$: coefficient remains fixed, e.g., 0

Note: for $\alpha = 0$ cannot fix non-zero constant coefficient $\mathcal{N} = \dfrac{1}{0}$

$w_i \to 0$: coefficient is a "don't care" case

$$\tilde{f}(x) = f(x) - \frac{f(\alpha)}{\alpha^i} x^i, \quad \alpha \neq 0$$

Stetter's Results

- The dual norm $\|\ldots\|^*$ for $v \in \mathbb{C}^n$ is defined by $\|v^T\|^* = sup_{u \neq 0} \frac{|v^T u|}{\|u\|} = sup_{\|u\|=1} |v^T u|$.

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- $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$,

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- Theorem:

$$\|\vec{f} - \vec{\tilde{f}}\|^* \geq \frac{|f(\alpha)|}{\|\tau\|}$$

Related Results

- Hölder's Inequality: $u, v \in \mathbb{C}^n$, weights w_i, and $1/w = (\ldots, 1/w_i, \ldots)$.
  - $|v^T u| \leq \|u\|_{\infty,w} \|v\|_{1,1/w}$
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  - $|v^T u| \leq \|u\|_{2,w} \|v\|_{2,1/w}$

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where $v = [1, \mathrm{sgn}(\tau_i), \ldots]$.

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- Theorem: $\mathcal{N}_{\infty,w}^{[f]}(\alpha) = \frac{|f(\alpha)|}{\|\tau\|_{1,1/w}}$ and

$$\vec{\tilde{f}}_i = \begin{cases} \vec{f}_i & \text{for } i \neq i_{max} \\ \vec{f}_i - \operatorname{sgn}(\tau_i)\frac{f(\alpha)}{\|\tau\|_{\infty,1/w}}\frac{1}{w_i} & \text{for } i = i_{max} \end{cases}$$

  where $i_{max} = argmax_i\{\frac{|\tau_i|}{w_i}\}$

Motzkin Example

- $Mot(x,y) = x^4y^2 + x^2y^4 + 2 - 3x^2y^2$, which is $\geq 1 \ \forall x, y \in \mathbb{R}$ and which is **not** a SOS
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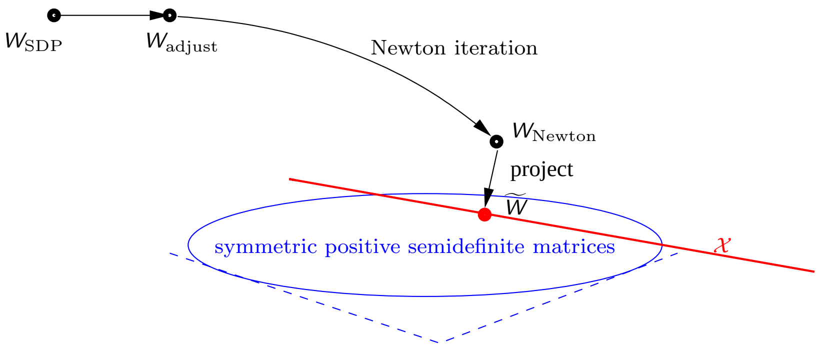
- Can we certify a lower bound?
  - If yes, proves that the polynomial has no real root (Seidenberg's Problem)

Rationalizing a Sum-Of-Squares: "Easy Case" [Peyrl, Parrilo, '07, '08; Kaltofen, Li, Yang, Zhi, '08]



where the affine linear hyperplane, $\mathcal{X}$, is tangent to the cone boundary

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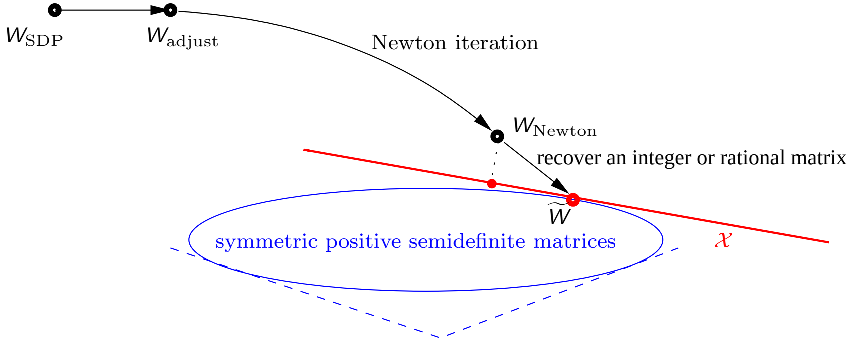
Rationalizing a Sum-Of-Squares: "Hard Case" [Kaltofen, Li, Yang, Zhi, '09]



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Singular W: real optimizers, fewer squares, missing terms

Sum of Squares Certificate for Motzkin Example

- W = matrix obtained from SOS solver in Matlab
  $m = [1, x^2y^2, x^2y^4, x^4y^2, xy^2, x^3y^2, x^2y, x^2y^3, xy, x^3y^3]^T$
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  $Mot(x, y)^2 - 12854802625942833/100000000000000000$
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- This means that the non-zero coefficients of *Mot* need to be perturbed (by at least 0.128 in ℓ^2 -norm squared) for *Mot* to have a real root.
- $Mot(x, y) > 0\ \forall x,y \in \mathbb{R}$ via a polynomial SOS certificate!

Linear Constraints

$$\mathcal{N}_{2,w}^{[f;H]}(\alpha) = \inf_{\tilde{f}\in\mathbb{R}[x_1,\ldots,x_n]} \|f-\tilde{f}\|_{2,w}^2$$
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- Jacobian of Lagrange function constitutes a linear system in coefficients of $\tilde{f}$ & multipliers!

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- Lagrange function

$$L = (\vec{f} - \vec{\tilde{f}})^T D_w (\vec{f} - \vec{\tilde{f}}) + \lambda_0 \tau^T \vec{\tilde{f}} + \lambda_1^T (H\vec{\tilde{f}} - p) + \mu^T (G\vec{\tilde{f}} - q).$$

KKT Conditions

The KKT conditions:

$$
\left.
\begin{array}{l}
\dfrac{\partial L}{\partial \vec{\tilde{f}}_i} = 0, \quad i = 1, \ldots, s, \\
\tau^T \vec{\tilde{f}} = 0 \\
H\vec{\tilde{f}} = p, \\
G\vec{\tilde{f}} \leq q, \\
\mu_i \geq 0, \quad i = 1, \ldots, m, \\
\mu^T (G\vec{\tilde{f}} - q) = 0.
\end{array}
\right\} \tag{1}
$$

The last orthogonality conditions constitute branching: $\mu_i = 0$ or $(G\vec{\tilde{f}} - q)_i = 0$, and (1) form linear programs.

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- Can be generalized to include complex coefficients and/or complex roots

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- 20 digits: $x = 2.5645,\ y = -0.2751,\ \mathcal{N}_2^{f_1,f_2} = 0.9180.$

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- Why? Truncate coefficients of fsolve

Conjecture

- If there exists a weight vector such that the radius of positive semidefiniteness is > 0 then there is a SOS certificate for that polynomial.
- If $f \in \mathbb{R}[x_1, \ldots, x_n]$ then $\exists$ w, a vector of positive and infinite weights, s.t. $\rho_{2,w}(f) > 0$ then $r^* > 0$ in

$$\left.\begin{array}{lll} r^* := & \sup\limits_{r \in \mathbb{R}, W} & r \\ & \text{s. t.} & f(\mathbf{X})^2 - r\tau^T D_w^{-1}\tau = m(\mathbf{X})^T \, W \, m(\mathbf{X}) \\ & & W \succeq 0, \; W^T = W \end{array}\right\} \tag{2}$$

Conjecture

- If there exists a weight vector such that the radius of positive semidefiniteness is > 0 then there is a SOS certificate for that polynomial.
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- Note that we have seen that $\rho_{2,w}(f) \leq 0$, if f has a projective root at infinity, and $\rho_{2,w}(f) > 0$ makes f and w quite special.

Danke schön!

(Thank You)

"End Key" Wrong!