Analyzing Yield, Duration and Convexity of Mortgage Loans under Prepayment and Default Risks

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Abstract:

In this article, we construct a general model, which considers the borrower's financial and non-financial termination behavior, to derive the closed-form formulae of the mortgage value for analyzing the yield, duration and convexity of the risky mortgage. Since the risks of prepayment and default are reasonably expounded in our model, our formulae are more appropriate than traditional mortgage formulae. We also analyze the influence the prepayment penalty and partial prepayment have on the yield, duration and convexity of a mortgage, and provide lenders with an upper-bound for the mortgage default insurance rate. Our model provides portfolio managers a useful framework to more effectively hedge their mortgage holdings. From the results of sensitivity analyses, we find that higher interest-rate, prepayment and default risks will increase the mortgage yield and reduce the duration and convexity of the mortgage.

Keywords: yield, duration, convexity, default insurance, prepayment penalty, partial prepaymen

1. Introduction

Mortgage-related securities are prevalent in the financial market as they satisfy investors' demands for high yields and secure credit quality. Due to uncertain cash flows resulting from borrowers' default and prepayment behaviors, investors require a premium to compensate for potential losses. Determining an appropriate premium is important for portfolio managers and financial intermediaries. Furthermore, hedging the interest-rate risk of mortgages is an extremely difficult assignment. Because borrower's prepayment behavior influences the duration, the price sensitivity of mortgages to changes in interest rates becomes highly uncertain. Targeting duration and convexity can be one of the most momentous approaches to managing portfolios of mortgage-related securities.

The intention of this article is to utilize the intensity-form approach to price a mortgage and then investigate the influence of interest-rate, prepayment and default risks on the yield, duration and convexity of a mortgage. Moreover, we also discuss how the yield, duration and convexity of a mortgage change under various situations—mortgages with a prepayment penalty, partial prepayment and default insurance.Mortgage yield spreads reflect premiums that compensate investors for exposure to prepayment and default risks. Most mortgage market practitioners and academic researchers are concerned with the impact of prepayment and default risks on mortgage yield, and how to measure the mortgage termination risk in deciding hedging strategies for their portfolios. Literature shows the premium and the termination risk of mortgages have typically been analyzed using the contingent-claim approach, intensity-form approach and empirical analysis.

Under the contingent-claim approach, researchers use the option pricing theory to investigate the premiums of prepayment and default. They argue that borrowers are endowed with the option to prepay (call) or default (put) the mortgage contract. The values of prepayment and default options are calculated through specifying relevant variable processes such as interest rates, house prices and so forth (see e.g., Kau et al., 1993; Yang, Buist and Megbolugbe, 1998; Ambrose and Buttimer, 2000; Azevedo-Pereira, Newton and Paxson, 2003). Some studies use the option pricing theory to determine prepayment and default premiums. Furthermore, they investigate the effects of relevant variables (such as interest rate volatility, yield curve slope, etc.) on the values of prepayment and default options (see, Childs, Ott and Riddiough, 1997). The intensity-form approach evaluates the probabilities of prepayment and default based on hazard rates information (see e.g., Schwartz and Torous, 1989, 1993; Quigley and Van Order, 1990; Lambrecht, Perraudin and Satchell, 2003; Ambrose and Sanders, 2003). Some researchers investigate mortgage risk premiums using the intensity-form approach. They insert the termination probability into the model and derive the equilibrium mortgage rate by calculating the risky mortgage yield. Comparing the mortgage rate of the risk-free mortgage and the risky mortgage can determine the risk premium required to compensate for expected losses (see e.g., Gong and Gyourko, 1998). Recent literature using mortgage market data demonstrates that individual characteristics are related to prepayment and default risks. Some studies use empirical analyses to express relationships between the mortgage risk premium and various observable variables specific to the borrower, such as loan-to-value ratio, income, pay-to-income ratio and so on (see, Berger and Udell, 1990; Chiang, Chow and Liu, 2002).

The duration, which simply reflects the change in price for a given change in yield, is widely applied in interest rate risk management. When prepayment or default occurs, cash flows of the mortgage contract are altered. Thus, the duration will be different at various price levels as the prepayment and default expectations change. Measuring mortgage duration is more complicated and increases the difficulty in hedging mortgage-related securities. Valuation of mortgage duration can be classified into two methods: theoretical and empirical.

As for the theoretical aspect, Ott (1986) provided a foundation by deriving the duration of an adjustable-rate mortgage (ARM) under a discrete time framework. He revealed that the index used to adjust the mortgage rate tends to be more important than the adjustment frequency in determining the duration of an ARM. Haensly, Springer and Waller (1993) used a continuous payment formula to derive the fixed-rate mortgage duration. They found that duration monotonically increases with maturity when the market rate of interest is at or below the coupon rate. On the other hand, duration increases with maturity, peaks and subsequently declines as the market interest rate exceeds the coupon rate.

Empirical measure is another way to derive duration. It describes the relationship between changes in mortgage prices and changes in market yields as measured by Treasury securities. This method argues there is a market consensus on the impact of yield changes reflected in the behavior of market prices (see e.g., Derosa, Goodman and Zazzarino, 1993).

When considering hedging methods for mortgages, duration-matching is the most

commonly used strategy. However, this strategy does not properly reflect the interest rate risk if there is a great change in the interest rate. Thus, managers should make a hedging analysis by measuring the convexity of a mortgage. Previous literature seldom investigated the issue concerning the convexity of the mortgage, but is essential for the hedging analysis of the mortgage. We intend to construct a general model to derive the formulae for the duration and convexity of the risky mortgage, and to discuss the influence of interest rates, prepayments and default risks on them.

Most studies examine the termination risk of the mortgage and evaluate the mortgage by the contingent-claim approach. Moreover, when investigating the mortgage under contingent-claim models, terminations frequently occur when the options are not in the money (i.e., options are exercised under suboptimal conditions) (see e.g., Dunn and McConnell, 1981a, b; Kau, Keenan and Kim, 1993). Suboptimal termination occurs as a result of trigger events. Deng, Quigley and Van Order (1996) found the importance of trigger events, such as unemployment and divorce, in affecting mortgage borrower's termination behavior. Dunn and McConnell (1981a, b) model suboptimal prepayment as a jump process and embody it into the process of the mortgage value to overcome the suboptimal termination under contingent-claim models. However, it is hard to deal with the prepayment risk as well as the default and interest-rate risks under this model. With this approach, it is also difficult to identify the critical region of early exercise and embody the relevant variables into the models. By applying the intensity-form approach, we not only avoid these problems, but also can consider optimal and suboptimal terminations about borrower's prepayment and default behavior to more accurately measure the value, the duration and the convexity of the risky

mortgage.

In this study, we use the backward recursion method to express an implicit mortgage value formula, and then derive the closed-form solution of the mortgage value, yield, duration and convexity under the continuous-time intensity-form model. The key point for accurately pricing the mortgage value and measuring the yield, duration and convexity of the mortgage is appropriately modeling the prepayment and default risks. In our model, the hazard rates of prepayment and default are assumed to be linear functions of influential variables such as interest rates. Furthermore, because trigger events, such as job loss or divorce, influence a borrower's ability to fulfill monthly payment obligations and the mortgage termination incentive by prepayment or default, the likelihood of a borrower's prepayment and default will change under these situations. To reasonably expound mortgage prepayment and default risks, we model the occurrence of non-financial events as jump processes into the specification of hazard rates of prepayment and default. We derive a general closed-form formula for risky mortgages, which considers the borrower's non-financial termination (i.e., suboptimal termination) behavior and can integrate relevant economic variables under this framework..

Our yield formulae, duration and convexity are more appropriate than traditional formulae as our formulae are more sensitive to changes in prepayments and default risks. This point is very important for risk management strategies. Additionally, lenders can use loss avoidance strategies to protect themselves from prepayment and default risks. Prepayment penalties are widely used to eliminate the prepayment risk in a vast majority of

mortgages. Mortgage insurance is also usually required to reduce default losses (see, Riddiough and Thompson, 1993; Ambrose and Capone, 1998; Kelly and Slawson, 2001). Failure to consider the effects of these two factors could lead to an inefficient immunization strategy. We take into account the effects of the prepayment penalty, partial prepayment and default insurance on the risk-adjusted yield, duration and convexity of a risky mortgage.

This article is organized as follows: Part two presents the valuation model, which identifies the mortgage contract components; defines the probabilities and recovery rates of prepayment and default; and derives the closed-form solution of the mortgage value. In part three, we develop the yield, duration and convexity of a mortgage; conduct a sensitivity analysis to investigate the impact of interest rates, intensity rates and loss rates of prepayment and default, and the intensity rate of non-financial termination on the yield, duration and convexity of the mortgage. Part four analyzes the influence of partial prepayment, prepayment penalty and insurance on the yield, duration and convexity of a mortgage. The final section is the conclusion.

2. The Model

2.1 A General Pricing Framework

The focus of our investigation is on a fixed-rate mortgage (FRM)— the mortgage market's basic building block. We consider a fully amortized mortgage, having an initial mortgage principal M_0, with a fixed coupon rate c and time to maturity of T years. The payment Y_t in each period can be written as follows:

$$Y_t \equiv Y = M_0 \times \frac{c}{1 - e^{-cT}}.$$ (1)

The outstanding principal at time t, M_t, is given by

$$M_t = M_0 \times \frac{1 - e^{-c(T-t)}}{1 - e^{-cT}}.$$ (2)

We let A_t and V_t represent the value of the riskless mortgage and the value of a mortgage with prepayment and default risks at time t, respectively, $0 \leq t \leq T$. We have $A_t = \int_t^T Y \exp(-\int_t^u r_s ds) du$. From the risk premium point of view, the value of a risky mortgage is less than the riskless mortgage for the investor because the risky yield must be higher than the riskless yield. Thus, $1 - \frac{V_t}{A_t}$ represents a discounted proportion for the termination risk.

We assume that the borrower is endowed with the options to prepay, default or maintain the mortgage. The optimal strategy can be decided by the option that provides the greatest benefit. If $M_t > Y_t + V_t$, a rational borrower will not prepay because there is no profit. However, under the condition of $M_t < Y_t + V_t$ (such as when the interest rate declines), borrowers will pay M_t to redeem their loan, as the benefit is greater to prepay the mortgage. The lender has a loss from prepayment because the present value of the balance of the mortgage is less than the present value of the mortgage loan.[1] We define the prepayment loss rate at time t to be δ_t, $0 < \delta_t \leq 1$, which is a random variable representing the fractional loss of the mortgage market value at prepayment.

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[1] The loss of prepayment ($Y_t + V_t - M_t$) is the opportunity cost of the lenders.

The bank requires that the value of the collateral must be greater than the value of the mortgage at the initial time of the mortgage in order to avoid a huge loss in the case of default. If $H_t > Y_t + V_t$, where H_t is the market value of collateral at time t, the rational borrower will not default since there is no profit. If the collateral value is less than the mortgage value ($H_t < Y_t + V_t$, such as in a depressed market), the borrower will profit by defaulting on the mortgage. It also means that the borrower pays H_t to buy back the mortgage contract. Under this circumstance, the bank has a loss from default. We denote the loss rate of default at time t as η_t, $0 < \eta_t \leq 1$, which is a random variable representing the fractional loss of the mortgage market value at default.

In our framework, the probabilities of prepayment and default exist at each time point prior to the maturity date. Let us denote random variables τ_P and τ_D as the time of prepayment and default during the period from t to T, respectively. The conditional probabilities of prepayment, P_t^P, and default, P_t^D, can be expressed as follows:

$$P_t^D \equiv P(t - \Delta t < \tau^D < t \mid \tau^D > t - \Delta t) \text{ with initial probability } P_0^D \equiv 0. \tag{3}$$

$$P_t^P \equiv P(t - \Delta t < \tau^P < t \mid \tau^P > t - \Delta t) \text{ with initial probability } P_0^P \equiv 0. \tag{4}$$

We use a discrete time approximation (see, Broadie and Glasserman, 1997) to calculate the value of the mortgage, and then derive the limiting form of the corresponding formula. Let $n = \dfrac{T}{\Delta t}$, the valuation is at the point of i, $i = 1, \cdots, n$, where i represents the payment date. The risk neutral pricing method is used to evaluate the mortgage including the stochastic processes of interest rate, and default and prepayment risks. The mortgage value at

point i under the risk-neutral measure is

$$V_i = E_i[PV(Y_{i+1} + V_{i+1})], \tag{5}$$

where $PV(\cdot)$ and $E_i[\cdot]$ represent the present value and expected value conditional on the information of point i under the risk-neutral measure, respectively.

Figure 1 shows the composition of the mortgage value in each period. If the mortgage contract is not terminated before the payment, the lender will receive the promised amount, Y_i, and the mortgage value is $Y_i + V_i$. The probability is $1 - P_i^P - P_i^D$. Otherwise, in the event of prepayment or default, the cash flow is assumed to be $\alpha_i(Y_i + V_i)$ or $\beta_i(Y_i + V_i)$, where α_i and β_i represent the recovery rates of prepayment and default at point i, respectively. Let $\alpha_i = 1 - \delta_i$ and $\beta_i = 1 - \eta_i$. Considering the termination probability, and the losses of prepayment and default in the model, the mortgage value can be written as

$$V_i = E_i[(1 - P_{i+1}^P - P_{i+1}^D)\exp(-r_{i+1}\Delta t)(Y_{i+1} + V_{i+1}) + P_{i+1}^P \exp(-r_{i+1}\Delta t)\alpha_{i+1}(Y_{i+1} + V_{i+1})$$

$$+ P_{i+1}^D \exp(-r_{i+1}\Delta t)\beta_{i+1}(Y_{i+1} + V_{i+1})], \tag{6}$$

where r_{i+1} is the annualized riskless interest rate between i to $i+1$.

In Equation (6), the first term is the expected value of a mortgage contract, which does not terminate until point $i+1$. The second term is the expected value of a mortgage that is prepaid between points i and $i+1$. The third term is the expected value of a mortgage that defaults between points i and $i+1$. Equation (5) can also be rewritten as

$$V_i = E_i[(Y_{i+1} + V_{i+1})[(1 - P_{i+1}^P - P_{i+1}^D)\exp(-r_{i+1}\Delta t) + P_{i+1}^P \alpha_{i+1}\exp(-r_{i+1}\Delta t)$$

$$+ P_{i+1}^D \beta_{i+1}\exp(-r_{i+1}\Delta t)]]. \tag{7}$$

t=i   **t=i+1**

$$Y_{i+1} + V_{i+1}$$

$$1 - P_{i+1}^{P} - P_{i+1}^{D}$$

$$P_{i+1}^{P}$$

$$V_i \qquad\qquad\qquad\qquad\qquad\qquad \alpha_{i+1}(Y_{i+1} + V_{i+1})$$
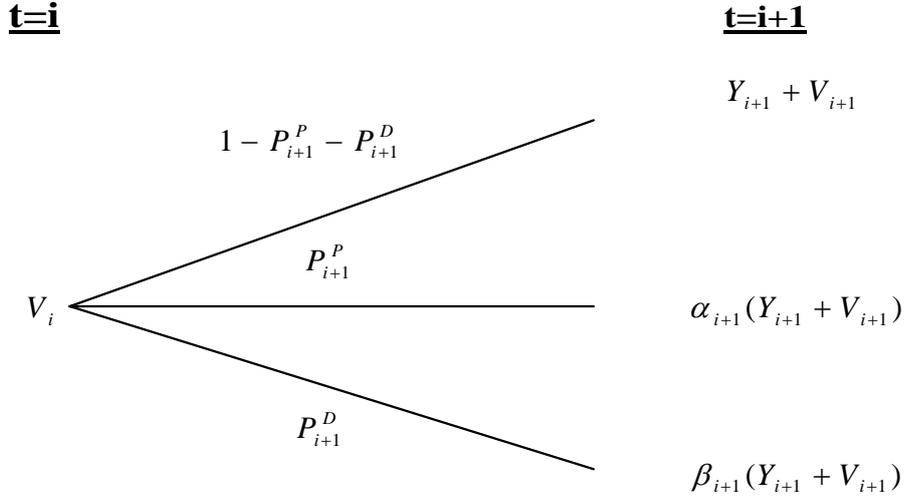
$$P_{i+1}^{D}$$

$$\beta_{i+1}(Y_{i+1} + V_{i+1})$$

Figure 1: Evolution of the mortgage value

Replacing α_{i+1} and β_{i+1} with $1 - \delta_{i+1}$ and $1 - \eta_{i+1}$, we have

$$V_i = E_i[(Y_{i+1} + V_{i+1})Q_{i+1}], \tag{8}$$

where $Q_{i+1} = (1 - P_{i+1}^{P} - P_{i+1}^{D})\exp(-r_{i+1}\Delta t) + P_{i+1}^{P}(1 - \delta_{i+1})\exp(-r_{i+1}\Delta t)$

$$+ P_{i+1}^{D}(1 - \eta_{i+1})\exp(-r_{i+1}\Delta t).$$

For small time periods, using the approximation of $\exp(-c)$ with c, given by $1 - c$, we can

rewrite the expression of Q_{i+1} as

$$Q_{i+1} \cong 1 - [r_{i+1}\Delta t + P_{i+1}^{P}\delta_{i+1}(1 + r_{i+1}\Delta t) + P_{i+1}^{D}\eta_{i+1}(1 + r_{i+1}\Delta t)]$$

$$\cong \exp(-[r_{i+1}\Delta t + P_{i+1}^{P}\delta_{i+1}(1 + r_{i+1}\Delta t) + P_{i+1}^{D}\eta_{i+1}(1 + r_{i+1}\Delta t)]). \tag{9}$$

The mortgage value at any time, $T - (i-1)\Delta t$, is equal to its discounted expected value at

time $T - i\Delta t$, so that

$$V_{T-(i-1)\Delta t} = E_{T-(i-1)\Delta t}[Q_{T-i\Delta t}(Y_{T-i\Delta t} + V_{T-i\Delta t})]. \tag{10}$$

According to Equation (10), the mortgage value at time $T - \Delta t$ can be expressed as

$$V_{T-\Delta t} = E_{T-\Delta t}[Q_T(Y_T + V_T)].$$

At time $T - 2\Delta t$, the mortgage value is

$$V_{T-2\Delta t} = E_{T-2\Delta t}[Q_{T-\Delta t}(Y_{T-\Delta t} + V_{T-\Delta t})]. \tag{11}$$

Since the FRM is a fully amortized and fixed-rate, we have $V_T = 0$ and $Y_i = Y$. Thus, we substitute $V_{T-\Delta t}$ into Equation (11) and obtain

$$V_{T-2\Delta t} = E_{T-2\Delta t}[Q_{T-\Delta t}Y + \prod_{j=T-\Delta t}^{T} Q_j Y].$$

Iterated backwards until the initial time and using an iterated condition, i.e. $E_t[E_{t+i}[\cdot]] = E_t[\cdot]$, we obtain the initial value of mortgage as follows:

$$V_0 = YE_0[\sum_{i=1}^{n}(\prod_{j=1}^{i} Q_j)]. \tag{12}$$

Therefore, substituting Equation (9) into Equation (12), we obtain the mortgage pricing formula as follows:

$$V_0 = YE_0[\sum_{i=1}^{n} \exp(-(\sum_{j=1}^{i} r_j \Delta t + \sum_{j=1}^{i} P_j^P \delta_j(1 + r_j \Delta t) + \sum_{j=1}^{i} P_j^D \eta_j(1 + r_j \Delta t)))]. \tag{13}$$

When the time interval Δt approaches 0, the discrete time series is transformed into a continuous time process, thus allowing us to appraise the mortgage in a continuous time framework. We define the intensity rate of prepayment and default as λ_t^P and λ_t^D. The conditional probabilities of prepayment and default are then represented as $P_t^P = \lambda_t^P dt$ and $P_t^D = \lambda_t^D dt$, respectively. We can obtain the mortgage pricing formula as follows:

$$\lim_{\Delta t \to 0} V_0 = Y \int_0^T E_0 [\exp(-(\int_0^t r_u du + \int_0^t \lambda_u^P \delta_u (1 + r_u) du + \int_0^t \lambda_u^D \eta_u (1 + r_u) du))] dt . \qquad (14)$$

According to our foregoing definition for the loss rates of prepayment and default, the loss rates are random variables. Since the loss rate can be estimated by using market data, it is usually assumed to be an exogenous variable and treated as a constant or a deterministic variable in reduced-form models within studies (see e.g., Jarrow and Turnbull, 1995; Duffie and Singleton, 1999). Moreover, some empirical evidence shows it causes no significant difference on pricing mortgages no matter if the loss rate is assumed to be a stochastic variable or a constant (e.g., Jokivuolle and Peura, 2003). Therefore, the loss rates have been simplified and treated as a constant in our model to derive a closed-form solution of the mortgage. Under the assumptions of $\delta_t = \delta$, $\eta_t = \eta$, letting $\lambda_u^P \delta_u r_u du \cong 0$ and $\lambda_u^D \eta_u r_u du \cong 0$ (since they are quite small in the real world), the mortgage value can be expressed as follows:

$$V_0 = Y \int_0^T E_0 [\exp(-(\int_0^t (r_u + \delta \lambda_u^P + \eta \lambda_u^D) du)] dt . \qquad (15)$$

According to this formula, the value of the mortgage can be expressed as the present value of the promised payoff Y discounted by the risk-adjusted rate $r_u + \delta \lambda_u^P + \eta \lambda_u^D$. This is similar to the formula obtained in Duffie and Singleton (1999).

2.2 A Closed-Form Formula for Mortgage Values under Some Specific Assumptions

To obtain the closed-form solution of the mortgage valuation presented in Equation (15), we need to specify the stochastic processes of the interest rate and hazard rates in our model. We

use the extended Vasicek (1977) model for the term structure.[2] The extended Vasicek interest rate model is a single-factor model with deterministic volatility and can match an arbitrary initial forward-rate curve through the specification of the long-run spot interest rate $\bar{r}_t$ (Vasicek (1977), and Heath, Jarrow and Morton (1992)). Under the risk-neutral measure, the term-structure evolution is described by the dynamics of the short interest rates

$$dr_t = a(\bar{r}_t - r_t)dt + \sigma_r dZ_r(t), \tag{16}$$

where

a is the speed of adjustment, a positive constant;

σ_r is the volatility of the spot rate, a positive constant;

$\bar{r}_t$ is the long-run spot interest rate, a deterministic function of t; and

$Z_r(t)$ is a standard Brownian motion under a risk neutral measure.

The spot interest rates follow a mean-reverting process under a risk-neutral measure based on Equation (16). As shown in Heath et al. (1992), to match an arbitrary initial forward-rate curve, one can set

$$\bar{r}(u) = f(t,u) + \frac{1}{a}\left(\frac{\partial f(t,u)}{\partial u} + \frac{\sigma_r^2(1-e^{-2a(u-t)})}{2a}\right).$$

Combining the above two equations, the evolution of the short interest rate can be shown as

$$r_u = f(t,u) + \frac{\sigma_r^2(e^{-a(u-t)}-1)^2}{2a^2} + \int_t^u \sigma_r e^{-a(u-v)}dZ_r(v), \tag{17}$$

where $f(t,u)$ is the instantaneous forward rate. Let $\Theta_t^r \equiv \int_0^t r_u du$, the evolution of the Θ_t^r can be written as

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[2] Some research has shown that the Vasicek model (and hence the extended Vasicek model) performs well in the pricing of mortgage-backed securities (Chen and Yang (1995)).

$$\Theta_t^r = \int_0^t f(0,u)du + \int_0^t \frac{b(0,u)^2}{2} du + \int_0^t \int_0^u \rho(v,u)dZ_r(v)du \,. \tag{18}$$

The following expressions show the expected value and variance of Θ_t^r (see, Appendix A):

$$\mu_t^r \equiv E[\Theta_t^r] = f(0,t)t + \frac{\sigma_r^2}{2a^2}(t - \frac{2}{a}(1 - e^{-at}) + \frac{1}{2a}(1 - e^{-2at})) \,, \tag{19}$$

and

$$\Sigma_t^r = Var(\Theta_t^r) = \frac{\sigma_r^2}{a^2}(t - \frac{2}{a}(1 - e^{-at}) + \frac{1}{2a}(1 - e^{-2at})) \,. \tag{20}$$

Most empirical models show that the prepayment and default intensity rates are highly significant to the change in interest rates (see, Schwartz and Torous, 1989; Collin-Dufresne and Harding, 1999). According to the specification in Collin-Dufresne and Harding (1999), intensity rates of prepayment and default are designated to depend on the particular variable in the model that is related to the termination risk, such as the interest rate. Moreover, prepayment and default events occur for both financial reasons (such as a change in the interest rate) and non-financial reasons (such as job change, divorce and seasoning). Previous studies demonstrate that trigger events (non-financial states), such as employment and divorce, can affect the probabilities of a borrower's termination decision (e.g., Deng, Quigley and Van Order, 1996). In order to make this model more reasonable without loss of generality, we not only assume that the hazard rates of prepayment and default are linear functions of short interest rates, but also use the Poisson processes to model the random arrivals of non-financial prepayment and default events. Our specification of hazard rates for prepayment and default reasonably depicts the termination risk because it is readily recognized that prepayment and default occur in both financial and non-financial circumstances. When a

non-financial termination event occurs, there is a jump for the hazard rate of prepayment or default. We allow the jump size to be a random variable because trigger events result in an uncertain change in the termination intensity rate. The hazard rates of prepayment and default are set as follows:

$$\lambda_u^P = \lambda_0^P + \lambda_1^P r_u + \xi^P dN(u), \tag{21}$$

and

$$\lambda_u^D = \lambda_0^D + \lambda_1^D r_u + \xi^D dN(u), \tag{22}$$

where $N(u)$ represents the random arrival of non-financial prepayment and default, which is a Poisson process with intensity rate of ϑ; ξ^P and ξ^D respectively, representing the random jump magnitudes of prepayment and default that are assumed independent of the Poisson process $N(u)$.

As mentioned in Collin-Dufresne and Harding (1999), the restriction of the right hand side of Equations (21) and (22) to only a single financial variable simplifies the finding of a closed-form solution. However, other time dependent variables can be added to the model. The Poisson process $N(u)$ counts the number of jumps that occur at or before time u. If there is one jump during the period $[u, u+du]$ then $dN(u)=1$, and $dN(u)=0$ represents no jump during this period. We assume the diffusion component, $dZ_r(u)$, and the jump component, $dN(u)$, are independent. In addition, the random variable ξ_i^l, $l=P,D$, represents the size of the ith jump, $i=1,2,\cdots$ that is a sequence of identical distributions assumed to be independent of each other.

According to the above specifications of λ_u^P and λ_u^D, we obtain

$$\int_0^t \lambda_u^P \, du = \Theta_t^P + J_t^P \text{, and} \tag{23}$$

$$\int_0^t \lambda_u^D \, du = \Theta_t^D + J_t^D \text{,} \tag{24}$$

where $\Theta_t^l = \int_0^t \lambda_0^l + \lambda_1^l r_u \, du$ and $J_t^l = \sum_{i=1}^{N(t)} \xi_i^l$, $l = P, D$. The J_t^l is defined as a compound

Poisson process. As mentioned in Shreve (2004), the jumps in J_t^l occur at the same time as

the jumps in $N(t)$. However, the jump sizes in $N(t)$ are always 1 in each jump; the jump

sizes in J_t^l are of random size.

Then, the mortgage value that contains relevant variable and non-financial factors can

be described as follows:

$$V_0 = Y \int_0^T (\exp(-(\delta\lambda_0^P + \eta\lambda_0^D)t) \times E_0[\exp(-((1 + \delta\lambda_1^P + \eta\lambda_1^D)\int_0^t r_u \, du)]$$

$$\times E_0[\exp(-(\delta J^P + \eta J^D))]) dt \text{.} \tag{25}$$

Since Θ_t^r is normally distributed with mean μ_t^r and variance Σ_t^r, we can use the moment

generating function method to obtain the value of $E_0[\exp(-((1 + \delta\lambda_1^P + \eta\lambda_1^D)\int_0^t r_u \, du)]$. That is

$$E_0[\exp(-((1 + \delta\lambda_1^P + \eta\lambda_1^D)\int_0^t r_u \, du)]$$

$$= \exp(-(1 + \delta\lambda_1^P + \eta\lambda_1^D)\mu_t^r + \frac{1}{2}(1 + \delta\lambda_1^P + \eta\lambda_1^D)^2 \Sigma_t^r)) dt \text{.} \tag{26}$$

The closed-form solution of a mortgage contract can be obtained when given the

distribution of ξ^l, such as a normal distribution (see, e.g., Merton, 1976) and a double

exponential distribution (see, e.g., Kou and Wang, 2001). In our model, we assume that ξ^P and ξ^D follow a normal distribution with mean μ_ξ^P and μ_ξ^D, variances Σ_ξ^P and Σ_ξ^D and covariance $\Sigma_\xi^{P,D}$. Thus, we can obtain the following results (see, Shreve, 2004):

$$E[\exp(-\delta J^P - \eta J^D)] = \exp(\vartheta t(\exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})) - 1)). \quad (27)$$

Substituting Equations (26) and (27) into Equation (25), we have

$$V_0 = Y\int_0^T [\exp(-((\delta\lambda_0^P + \eta\lambda_0^D)t + (1 + \delta\lambda_1^P + \eta\lambda_1^D)\mu_t^r - \frac{1}{2}(1 + \delta\lambda_1^P + \eta\lambda_1^D)^2\Sigma_t^r$$

$$- \vartheta t(\exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})) - 1))]dt. \quad (28)$$

The above equation is the closed-form solution of the mortgage. According to this result, we calculate the yield, duration and convexity of the mortgage, and discuss the influence of interest rates, prepayments and default risks on them.

3. Yield, Duration and Convexity Analyses of Risky Mortgages

3.1 Yield Measure for Risky Mortgages

The yield of a fixed-income security is the discount factor that equates the present value of a security's cash flows to its initial price. Thus, in order to calculate the yield, one needs to know the amount and the timing of the cash flow of the mortgage and the probabilities of prepayment and default. These variables lead to different yield spreads and random changes of the yield over time, since a mortgage has different degrees of risk over time.

Defining R as the risk-adjusted yield of a risky mortgage required by the mortgage holders at time 0, the mortgage value at time 0 can be expressed as

$$V_0 = Y \int_0^T \exp(-Rt)dt \,. \tag{29}$$

According to Equations (28) and (29), under the assumption of no arbitrage opportunities (see, Jacoby, 2003), the yield of a risky mortgage is

$$R = (\delta\lambda_0^P + \eta\lambda_0^D) + (1 + \delta\lambda_1^P + \eta\lambda_1^D)f(0,t) - (1 + \delta\lambda_1^P + \eta\lambda_1^D)(\delta\lambda_1^P + \eta\lambda_1^D)\sigma_r^2 A$$

$$- \vartheta(\exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})) - 1) \,, \tag{30}$$

where $A = \dfrac{1}{2a^2}(1 - \dfrac{2}{at}(1 - e^{-at}) + \dfrac{1}{2at}(1 - e^{-2at}))$, $A > 0$ and $\dfrac{\partial A}{\partial a} < 0$.[3]

The above formula gives the lender a better understanding of mortgage analysis. Note that when the probabilities of prepayment and default are zero (i.e., $\lambda_u^P = \lambda_u^D = \vartheta = 0$), there is no risk premium to the lender. Otherwise, if positive termination probabilities exist but recovery rates are full (i.e. $\delta = \eta = 0$), the risk premium is also zero.

We conduct the sensitivity analyses to investigate how different variables (such as the interest rate, the probabilities of prepayment and default including financial and non-financial states) influence the yield of a mortgage. The partial derivative of the instantaneous yield with respect to different variables can be shown as

$$\frac{\partial R}{\partial f(0,t)} = 1 + \delta\lambda_1^P + \eta\lambda_1^D \,, \tag{31}$$

$$\frac{\partial R}{\partial \sigma_r^2} = -(1 + \delta\lambda_1^P + \eta\lambda_1^D)(\delta\lambda_1^P + \eta\lambda_1^D)A \,, \tag{32}$$

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[3] According to Equation (20), we have $A > 0$. With regard to the result of $\dfrac{\partial A}{\partial a} < 0$, we checked these results by conducting numerical analyses.

$$\frac{\partial R}{\partial a} = -(1 + \delta\lambda_1^P + \eta\lambda_1^D)(\delta\lambda_1^P + \eta\lambda_1^D)\sigma_r^2 \frac{\partial A}{\partial a}, \tag{33}$$

$$\frac{\partial R}{\partial \lambda_0^P} = \delta, \tag{34}$$

$$\frac{\partial R}{\partial \lambda_0^D} = \eta, \tag{35}$$

$$\frac{\partial R}{\partial \lambda_1^P} = \delta(f(0,t) - (1 + 2\delta\lambda_1^P + 2\eta\lambda_1^D)\sigma_r^2 A), \tag{36}$$

$$\frac{\partial R}{\partial \lambda_1^D} = \eta(f(0,t) - (1 + 2\delta\lambda_1^P + 2\eta\lambda_1^D)\sigma_r^2 A), \tag{37}$$

$$\frac{\partial R}{\partial \delta} = \lambda_0^P + \lambda_1^P(f(0,t) + \sigma_r^2 A) - 2\lambda_1^P(1 + \delta\lambda_1^P + \eta\lambda_1^D)\sigma_r^2 A$$

$$+ \vartheta(\mu_\xi^P - \delta\Sigma_\xi^P - \eta\Sigma_\xi^{P,D})\exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})), \tag{38}$$

$$\frac{\partial R}{\partial \eta} = \lambda_0^D + \lambda_1^D(f(0,t) + \sigma_r^2 A) - 2\lambda_1^D(1 + \delta\lambda_1^P + \eta\lambda_1^D)\sigma_r^2 A$$

$$+ \vartheta(\mu_\xi^D - \eta\Sigma_\xi^D - \delta\Sigma_\xi^{P,D})\exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})), \tag{39}$$

and

$$\frac{\partial R}{\partial \vartheta} = 1 - \exp(-(\delta\mu_\xi^P + \eta\mu_\xi^D) + \frac{1}{2}(\delta^2\Sigma_\xi^P + \eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D})). \tag{40}$$

By observing the above partial derivative, one cannot entirely judge whether the impact of the parameter on the mortgage yield is positive or negative. Thus, we discuss the direction regarding the influence of the parameter on the mortgage yield based on some conditions. We analyze the influence of the interest rate on the mortgage yield based on Equations (31) to (33). Common knowledge suggests there is negative relation between the mortgage value

and the forward rate. Thus, the forward rate positively influences the mortgage yield resulting

in $\frac{\partial R}{\partial f(0,t)} = 1 + \delta\lambda_1^P + \eta\lambda_1^D > 0$. Empirical results from some previous literature demonstrate

that the influences of interest rates on prepayment and default probabilities are negative (see,

Schwartz and Torous, 1993). Thus, we can infer that $\lambda_1^P < 0$ and $\lambda_1^D < 0$, obtaining

$0 < \frac{\partial R}{\partial f(0,t)} < 1$. This result shows that if we consider the prepayment and default risks in

pricing a mortgage value, the influential magnitude of a forward rate on yield will decrease.

According to the results of $1 + \delta\lambda_1^P + \eta\lambda_1^D > 0$ and $\delta\lambda_1^P + \eta\lambda_1^D < 0$, we can also infer that

$\frac{\partial R}{\partial \sigma_r^2} > 0$ and $\frac{\partial R}{\partial a} < 0$. The variance (the speed of adjustment) of a short interest rate has a

positive (negative) effect on the mortgage yield.

As for the influence of the termination risk on mortgage yield, we find that changes in

λ_0^P and λ_0^D affect the mortgage yield in the positive direction based on Equations (34) and

(35). The impact of λ_1^P and λ_1^D on the mortgage yield depends on whether the value of

$f(0,t)$ is large or less than the value of $(1 + 2\delta\lambda_1^P + 2\eta\lambda_1^D)\sigma_r^2 A$. Under the condition of

$\mu_t^r > \Sigma_t^r$, we have $f(0,t) > (1 + 2\delta\lambda_1^P + 2\eta\lambda_1^D)\sigma_r^2 A$, resulting in $\frac{\partial R}{\partial \lambda_1^P} > 0$ and $\frac{\partial R}{\partial \lambda_1^D} > 0$.

Observing Equations (38) and (39) one can note the affects loss rates of termination risk have

on yield. Because the hazard rates are a positive value in practice, we have

$E[\Theta_t^P] = (\lambda_0^P + \lambda_1^P(f(0,t) + \sigma_r^2 A))t > 0$. Therefore, we infer $\lambda_0^P + \lambda_1^P(f(0,t) + \sigma_r^2 A) > 0$. If we

reasonably assume that $\mu_\xi^P > \delta\Sigma_\xi^P + \eta\delta\Sigma_\xi^{P,D}$, then we have $\frac{\partial R}{\partial \delta} > 0$. By the same way, we

also have $\dfrac{\partial R}{\partial \eta} > 0$. Thus, both loss rates of prepayment and default positively influence

mortgage yield. Moreover, the result of $\dfrac{\partial R}{\partial \vartheta} > 0$ can be obtained under the condition of

$$\delta \mu_\xi^P + \eta \mu_\xi^D > \frac{1}{2}(\delta^2 \Sigma_\xi^P + \eta^2 \Sigma_\xi^D + 2\eta \delta \Sigma_\xi^{P,D}) \,.$$

The above discussions show that when the forward rate goes up, the expected present value of the amount received will decrease requiring a higher yield. As long as there is a great degree of change in the short interest rate (i.e. σ_r^2 is large), the mortgage becomes risky. Lenders will require a higher yield. Furthermore, since an increase in the value of *a* will lead to stability in the short interest rate lenders will require a lower yield. No matter when the financial or non-financial intensity rates of prepayment and default go up, the mortgage becomes riskier. Lenders will then require a higher yield to compensate for the higher termination risk. If the loss rates of prepayment and default increase, the amount received when prepayment or default occurs will be lower and the lender will require a higher yield. Therefore, we conclude that there are positive relationships between the risk premium of a risky mortgage and the factors including the forward rate, the variance of short interest rate, the financial and non-financial intensity rates of prepayment and default, and prepayment and default loss rates. Moreover, a negative relationship exists between the risk premium of a risky mortgage and the speed of adjustment for the short interest rate. These results also conform to our economic institutions.

3.2 Duration and Convexity Measure for Risky Mortgages

The main challenge for portfolio managers is determining the duration and convexity of their mortgage holdings. Because the hedge ratios in mortgages have to be adjusted for changes in relevant variables, portfolio managers also need to investigate how the sensitivity of duration and convexity. Since the influence of a borrower's prepayment and default decisions has to be considered, quantifying durations and convexities in mortgage securities cannot be straightforwardly determined as in non-callable Treasury or corporate securities. To begin with we define the risk-adjusted duration for a risky mortgage as

$$D = -\frac{1}{V_0}\frac{\partial V_0}{\partial R}. \tag{41}$$

The partial derivative of mortgage value with respect to the yield is:

$$\frac{\partial V_0}{\partial R} = Y\int_0^T -t\exp(-Rt)dt. \tag{42}$$

The duration can be obtained as follows:

$$D = \int_0^T tW_t dt, \tag{43}$$

where $W_t = \dfrac{F_t}{V_0} = \dfrac{\exp(-Rt)}{\int_0^T \exp(-Rt)dt}$, which represents the weight of cash flows at time t.

$F_t = Y\exp(-Rt)$.

The termination intensity rate, loss rates of prepayment and default, and recovered amount should all be considered in order to measure the risk-adjusted duration of a risky mortgage. Equation (43) shows that any decrease in F_t will result in a reduced duration of the mortgage because the value of W_t becomes smaller as t approaches the maturity date.

There is a positive relationship between F_t and duration. Apparently, the increase in the interest rate causes F_t to decrease; a decrease in the duration of the mortgage contract then follows. The influence of relevant variables on the duration of the mortgage is more complicated as the change in a factor incurs a trade-off between positive and negative effects. We therefore use sensitivity analysis to show how the change in duration will be affected by different variables. The partial derivatives of the duration with respect to variables ϕ, which represents $f(0,t)$, a, σ_r^2, λ_0^P, λ_0^D, λ_1^P, λ_1^D, δ, η and ϑ, can be developed as follows (see Appendix B):

$$\frac{\partial D}{\partial \phi} = \frac{\partial R}{\partial \phi} G_1, \tag{44}$$

where $G_1 = \int_0^T tW(t)(D-t)dt < 0$.

The above partial derivative expressions show that the influences of variables ϕ on the mortgage duration are contrast to the influences of ϕ on the mortgage yield. Under the foregoing conditions, which are used for the mortgage yield discussion, we have the results as follows: When the forward rate ($f(0,t)$), the variance of short interest rate (σ_r^2), and intensity rates of financial or non-financial prepayment and default (λ_0^P, λ_0^D, λ_1^P, λ_1^D and ϑ) go up, the duration of the mortgage becomes smaller. Similarly, as the loss rates of prepayment and default (δ and η) increase, the duration of the mortgage will also decrease. Furthermore, an increase in the speed of adjustment of short interest rate (a) will cause the mortgage duration to become larger. These results infer that a higher risk of interest rate, prepayment or default will lead to smaller mortgage duration. This inference is similar to the

arguments in Chance (1990), and Derosa et al. (1993).

The definition of the convexity for a risky mortgage is:

$$C = \frac{1}{V_0} \frac{\partial^2 V_0}{\partial R^2}. \tag{45}$$

Since $\dfrac{\partial^2 V_0}{\partial R^2} = Y \int_0^T t^2 \exp(-Rt)dt$, we can obtain the convexity of the mortgage as

$C = \int_0^T t^2 W_t dt$. Then, we have the following:

$$\frac{\partial C}{\partial \phi} = \frac{\partial R}{\partial \phi} G_2, \tag{46}$$

where $G_2 \equiv \int_0^T t^2 W_t (D - t)dt < 0$.

According to Equation (46), one can find that the influences of variables ϕ on the mortgage convexity are similar to the influences of variables ϕ on the mortgage duration. Nevertheless, the affects of variables ϕ on the mortgage convexity are less than the affects of variables ϕ on the mortgage duration because $G_2 < G_1$.

4. Application of the Model

In the mortgage market, partial prepayment and mortgages subject to prepayment penalty clauses are common phenomena. When borrowers have surplus money, but not enough to prepay the whole amount of the mortgage loan, they often decide to make a partial prepayment in order to reduce interest expense. However, most literature often only focuses

on full mortgage prepayment. In this section, we analyze how partial prepayment risk influences the yield, duration and convexity of the mortgage. Furthermore, since the borrower owns the call option to prepay the mortgage, the lender always uses self-protection strategies to hedge the exercise of the option. Prepayment penalty clauses are appended to a vast majority of mortgages in order to reduce prepayment risk. Partial prepayment and the prepayment penalty alter both the termination time and the amount of cash flow promised to mortgage holders. Thus, accurately calculating the yield, the duration and the convexity of a mortgage is more difficult in these situations.

In addition, mortgage insurance is usually required to reduce losses in case of default. Realistically, before issuing mortgage securities to investors and in order to enhance the credit of the contract, the originator of a mortgage may use insurance to reduce the default risk. What is a fair fee to pay insurance companies? How do relevant variables influence insurance rate levels? In this section, we investigate the influence of all these factors on the measure of mortgage yield, duration and convexity.

4.1 Partial Prepayment

We assume the borrower's partial prepayment amount is $(1-\varphi_{t-1})M_{t-1}$ at the time $t-1$, where $\varphi_{t-1}<1$. Thus, the payment at every point and the outstanding mortgage principal at the next point will be $\varphi_t Y_t$ and $\varphi_t M_t$, respectively. If the lender reduces the value of the collateral to the same percentage as the mortgage principal, the collateral value H_t changes to $\varphi_t H_t$ simultaneously. The mortgage value changes from V_t to $\varphi_t V_t$ under the assumption of the same probabilities of prepayment and default.

Moreover, since the borrower's incentive for deciding to make a partial prepayment is different from the incentive to make a full prepayment, there will be a change in the prepayment probability. Thus, we assume that the intensity rate of prepayment changes from λ_t^P to $\tilde{\lambda}_t^P$, and then λ_0^P, λ_1^P and ϑ become $\tilde{\lambda}_0^P$, $\tilde{\lambda}_1^P$ and $\tilde{\vartheta}$. Because the same proportional changes in Y_t, H_t and V_t leads to the identical incentive for the borrower's default decision, (for the purpose of simplicity) we assume there is no change in the default intensity rate. Replacing $\tilde{\lambda}_0^P$, $\tilde{\lambda}_1^P$ and $\tilde{\vartheta}$ into Equations (30), (43) and (46), we obtain the yield, the duration and the convexity of a mortgage that includes partial prepayment, $\tilde{R}$, $\tilde{D}$ and $\tilde{C}$.

According to the previous discussion, we know there are positive relations between the mortgage yield and the intensity rates of financial and non-financial prepayment, and also negative relations between the duration and the convexity of a mortgage and the intensity rates of financial and non-financial prepayment. If $\tilde{\lambda}_t^P > \lambda_t^P$, the yield of a mortgage subject to a partial prepayment risk ($\tilde{R}$) is higher than the yield of a mortgage without partial prepayment risk (R). The duration and the convexity of a mortgage that includes a partial prepayment ($\tilde{D}$ and $\tilde{C}$) must be smaller than the duration and the convexity of a mortgage with no partial prepayment (D and C). Alternatively, if $\tilde{\lambda}_t^P < \lambda_t^P$, the yield of a mortgage including partial prepayment ($\tilde{R}$) is lower than the yield of a mortgage without partial prepayment (R). The duration and the convexity of a mortgage that includes a partial prepayment ($\tilde{D}$ and $\tilde{C}$) become larger than the duration and the convexity of a mortgage with no partial prepayment (D and C).

It is worth noting that if the lender does not alter the collateral value to the same percentage as the mortgage principal (i.e., there is no change in the collateral H_t), the mortgage value under the condition of partial prepayment will be greater than $\varphi_t V_t$ because of the increasing default recovery rate. Additionally, the default probability will decrease because the ratio of the value of collateral to the loan will increase. According to the analyzed results in the previous section, we found that a positive relationship exists between the yield of a mortgage and the default loss rate, and the intensity rates of financial and non-financial defaults. There are negative relationships between the duration and the convexity of a mortgage and the default loss rate, and the intensity rates of financial and non-financial defaults. Thus, if $\tilde{\lambda}_u^P < \lambda_u^P$, we can infer $\tilde{R} < R$, $\tilde{D} > D$ and $\tilde{C} > C$. However, if $\tilde{\lambda}_u^P > \lambda_u^P$, the influence of the borrower's partial prepayment on the yield, the duration and the convexity of a mortgage are difficult to estimate because the increase in prepayment risk and the decrease in default risk lead to two opposite effects on the yield, duration and convexity of a mortgage.

4.2 Prepayment Penalty

In general, prepayment penalty clauses will deter mortgage prepayments. Thus, prepayment penalties have the effect of reducing prepayment risk (see, Kelly and Slawson, 2000). This implies that the yield, duration and convexity must change due to the changes in prepayment probability and prepayment recovery rate.

To analyze this problem, we introduce the fixed penalty into our model. Assume a constant fractional penalty, μ_1, for the preceding k periods of the mortgage contract, then μ_2

fractional penalty for the remaining periods of mortgage contract, where $\mu_1 > \mu_2$. If k is equal to the maturity date, then the fixed penalty becomes the permanent penalty which has a constant percentage penalty for the entire life of mortgage. If the mortgage is prepaid at point j, when $j \leq k$, $\mu_1 M_j$ amount is charged. Alternatively, $\mu_2 M_j$ amount is paid if the borrower prepays at point j, when $k < j < T$. Then one can get the prepayment recovery rate $\hat{\alpha}_1$ and $\hat{\alpha}_2$, which represent the recovery rate at the preceding k periods and the remaining period respectively, where $\hat{\alpha}_1 = \alpha(1 + \mu_1)$ and $\hat{\alpha}_2 = \alpha(1 + \mu_2)$.

The prepayment penalty increases the borrower's cost of eliminating the mortgage liability and should affect the incentive for the borrower's prepayment decision. We assume that prepayment probability changes from λ_u^P to $\hat{\lambda}_u^P$, and then λ_0^P, λ_1^P and ϑ become $\hat{\lambda}_0^P$, $\hat{\lambda}_1^P$ and $\hat{\vartheta}$. Because the prepayment penalty reduces the prepayment risk, it implies that $\hat{\lambda}_u^P < \lambda_u^P$. Replacing $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\lambda}_0^P$, $\hat{\lambda}_1^P$ and $\hat{\vartheta}$ into Equations (28), (44) and (47), we obtain the yield, the duration and the convexity of a mortgage that includes prepayment penalty, $\hat{R}_1$, $\hat{D}_1$ and $\hat{C}_1$ for $j \leq k < T$, and $\hat{R}_2$, $\hat{D}_2$ and $\hat{C}_2$ for $k < j < T$.

According to the discussion in the previous section, we know that there is a positive relation between the mortgage yield and the intensity rates of financial and non-financial prepayments and the loss rate of prepayment. These conditions imply that either $\hat{R}_1$ or $\hat{R}_2$ is lower than the original yield (R). $\hat{R}_1$ is less than $\hat{R}_2$ because $\mu_1 > \mu_2$ will lead to the greater decrease in the loss rate and the probability of prepayment for $j \leq k < T$ than for $k < j < T$.

Since the decrease in the intensity rates of financial and non-financial prepayments and the loss rate of prepayment will cause the increases in the duration and the convexity of a mortgage, the duration and the convexity of a mortgage with penalty ($\hat{D}$ and $\hat{C}$) must be larger than the duration of a mortgage with no penalty (D and C). $\hat{D}_2$ and $\hat{C}_2$ are less than $\hat{D}_1$ and $\hat{C}_1$ because $\mu_1 > \mu_2$. These results show that the risk premium decreases and the duration and the convexity of a mortgage are larger due to the prepayment penalty decreasing the loss rate and the probability of prepayment.

4.3 Insurance Rate

Since the mid-1990s, a growing number of countries have been interested in mortgage default insurance (see, Blood, 2001). Mortgage default insurance can protect lenders and investors against losses when borrowers default and their collaterals are insufficient to fully pay off the mortgage obligation. Numerous market practitioners and academic researchers focus their studies on how to decide an appropriate insurance rate.

In our model, the insurance premium is calculated as an up-front fee defined as the difference between the value of the mortgage (V_j) and the recovered amount ($\beta_j M_j$) in the event a default occurs at point j. Because the default risk is avoidable through insurance, the upper-bound insurance rate is the spread between the yield of the defaultable mortgage and the yield of the mortgage with no default risk. According to the derived mortgage yield (Equation (30)), the upper-bound insurance rate, I, is described as follows: [4]

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[4] Equation (47) can be obtained by calculating the value of the yield without default risk minus the value of the yield with default risk.

$$I = \eta \lambda_0^D + \eta \lambda_1^D f(0,t) - (1 + 2\delta\lambda_1^P + \eta\lambda_1^D)\eta\lambda_1^D \sigma_r^2 A$$

$$+ \vartheta \exp(-\delta\mu_\xi^P + \frac{1}{2}\delta^2\Sigma_\xi^P)\,(1 - \exp(-\eta\mu_\xi^D + \frac{1}{2}(\eta^2\Sigma_\xi^D + 2\eta\delta\Sigma_\xi^{P,D}))). \qquad (47)$$

Since the default risk is not entirely eliminated, it implies the fair insurance rate for partial mortgage default insurance is less than I. Alternatively, the fair insurance rate for full mortgage default insurance is I as the default-free mortgage should have a zero default risk premium. Thus, the level of insurance rate paid depends on the degree of default risk eliminated by the mortgage default insurance. Moreover, in addition to the default risk, prepayment risk also influences the insurance rate, I. This is because we consider the correlation between the risks of prepayment and default in our model. Previous literature does not take into account the prepayment risk to investigate the mortgage default insurance. It implies that the mortgage default insurance that we provide should be more appropriate due to the consideration for the relationship between prepayment and default risks. Insurance institutions can decide an appropriate insurance rate through the rate that we provide.

5. Conclusion

Measuring yields, durations and convexities of mortgage contracts is quite complex due to the borrowers' prepayment and default behaviors that cause uncertainty for both the termination time and promised cash flows. This causes uncertainty in the changes in the yield, duration and convexity of a mortgage. Therefore, the measure of a mortgage yield, duration and convexity should appropriately reflect prepayment and default risks in addition to the interest-rate risk. This paper provides a framework for investigating the effects of various factors on the yield, duration and convexity of a mortgage. These factors do not only include

the term structure of the interest rate, financial termination probability, and prepayment and default loss rates, but also include non-financial termination probability, prepayment penalty and partial prepayment risk. Thus, our formulae for yield, duration and convexity, which precisely accounts for prepayment and default risks, may help regulators and financial institutions reduce their solvency risk.

Hedging a mortgage with termination risk is an extremely difficult endeavor. Therefore, the measure of a mortgage yield, duration and convexity should appropriately reflect prepayment and default risks, in addition to the interest-rate risk. Since our formulae for duration and the convexity more sensitively reflect the impact of prepayment and default risks, it is more appropriate in the management of interest rate risk than traditional duration and convexity. Therefore, our model can provide an appropriate framework for portfolio managers to more effectively hedge their mortgage holdings.

According to the sensitivity analyses, we find that there are positive relationships between the yield and the intensity rates of financial and non-financial termination, the loss rates of prepayment and default, the forward rate and the variance of the short interest rate. Additionally, there is a negative relationship between the yield and the speed of adjustment of the short interest rate. These results confirm that securities with higher termination and interest-rate risks have a higher risk premium. Moreover, the influence of all these factors on the duration contrasts the influence they have on yield. We can infer that higher interest-rate, prepayment and default risks will reduce the mortgage duration. This assertion is also consistent with Chance (1990) and Derosa et al. (1993). Furthermore, to compare the

analyzed results for mortgage duration, the influence of these factors on the mortgage convexity have the same results.

The facts that prepayment penalties and borrowers' partial prepayment behavior significantly affect the yield, duration and convexity measures of a mortgage are well known in the mortgage market. However, few studies have investigated mortgage yield and duration with these phenomena. In this paper, we analyze the impact of prepayment penalty and partial prepayment on the yield and duration. Our model shows that yield decreases and duration increases when a mortgage has a prepayment penalty. Furthermore, the influence of a borrower's partial prepayment behavior on the yield and duration of the mortgage are ambiguous due to the positive and negative effects of partial prepayment on risks of prepayment and default.

Finally, we analyze the problem concerning mortgage default insurance and provide a reference upper-bound for lenders and investors. They can measure an appropriate insurance rate by the mortgage default insurance we provide because it considers the relationship between prepayment and default risks, lenders and investors.

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Appendix A

In this appendix, we provide the derivation of Equation (19). From Equation (17), one can obtain the following results (see, Heath et al., 1992):

$$r_u = f(0,u) + \frac{\sigma_r^2 (e^{-au} - 1)^2}{2a^2} + \int_0^u \sigma_r e^{-a(u-v)} dZ_r(v).$$

and

$$\int_0^t r_u du = \int_0^t f(0,u) du + \int_0^t \frac{b(v,t)^2}{2} dv + \int_0^t b(v,t) dZ_r(v),$$

where $b(s,u) = \sigma_r \dfrac{1 - e^{-a(u-s)}}{a}$. To assume the initial yield curve is flat, a direct computation gives:

$$\mu_t^r = E[\int_0^t r(u) du] = \int_0^t f(0,u) du + \int_0^t \frac{b(0,u)^2}{2} du$$

$$= f(0,t)t + \frac{\sigma_r^2}{2a^2}(t - \frac{2}{a}(1 - e^{-at}) + \frac{1}{2a}(1 - e^{-2at})).$$

By Ito's Lemma, we obtain $E_t[dZ_r(t)dZ_r(t)] = dt$ and $E_t[dZ_r(t)dZ_r(u)] = 0$.

Then,

$$\Sigma_t^r = Var(\int_0^t r(u) du) = E[\int_0^t b(v,t) dZ_r(v) \times \int_0^t b(v,t) dZ_r(v)]$$

$$= \int_0^t b(v,t)^2 dv = \frac{\sigma_r^2}{a^2}(t - \frac{2}{a}(1 - e^{-at}) + \frac{1}{2a}(1 - e^{-2at})).$$

Appendix B

From Equation (44), we have

$$\frac{\partial D}{\partial \phi} = \int_0^T t \frac{\partial W_t}{\partial \phi} dt , \tag{B1}$$

where ϕ represents $f(0,t)$, a, σ_r^2, λ_0^P, λ_0^D, λ_1^P, λ_1^D, δ, η and ϑ. The partial derivative of W_t with respect to variance ϕ can be get as follows:

$$\frac{\partial W_t}{\partial \phi} = \frac{\partial F_t}{\partial \phi} \frac{1}{V_0} - \frac{F_t}{V_0^2} \frac{\partial V_0}{\partial \phi}$$

$$= -(t F_t \frac{\partial R}{\partial \phi} \frac{1}{V_0} - \frac{F_t}{V_0^2} \int_0^T s F_s \frac{\partial R}{\partial \phi} ds)$$

$$= -(t W_t \frac{\partial R}{\partial \phi} - W_t \int_0^T s W_s \frac{\partial R}{\partial \phi} ds).$$

Since $\dfrac{\partial R}{\partial \phi}$ are not varying with time (see Equations (31) to (40)), we have:

$$\frac{\partial W_t}{\partial \phi} = W_t \frac{\partial R}{\partial \phi} (\int_0^T s W_s ds - t)$$

$$= W_t \frac{\partial R}{\partial \phi} (D - t). \tag{B2}$$

Substituting Equation (B2) into Equation (B1), we have

$$\frac{\partial D}{\partial \phi} = \frac{\partial R}{\partial \phi} \int_0^T t W_t (D - t) dt = \frac{\partial R}{\partial \phi} G_1 , \tag{B3}$$

where $G_1 \equiv \int_0^T t W_t (D - t) dt$.

Moreover, we have:

$$G_1 = \int_0^T t W_t (D - t) dt$$

$$= \int_0^D t W_t (D - t) dt + \int_D^T t W_t (D - t) dt$$

$$< \int_0^D D W_t (D - t) dt + \int_D^T D W_t (D - t) dt$$

$$= \int_0^T D W_t (D - t) dt$$

$$= D^2 \int_0^T W_t dt - D \int_0^T t W_t dt = 0 . \tag{B4}$$

The inequality $\int_0^D t W_t (D-t)dt < \int_0^D D W_t (D-t)dt$ holds because $\int_0^D t W_t (D-t)dt$ and

$\int_0^D D W_t (D-t)dt$ are positive and $t < D$ in this region. Moreover, another inequality

$\int_D^T t W_t (D-t)dt < \int_D^T D W_t (D-t)dt$ holds since because $\int_0^D t W_t (D-t)dt$ and

$\int_0^D D W_t (D-t)dt$ are negative and $t > D$ in this region. Therefore, the first inequality in

Equation (B4) holds. Additionally, the last equality holds because $\int_0^T W_t dt = 1$ and

$\int_0^T t W_t dt = D$. Therefore, we have the result of $G_1 < 0$.

With regard to the analyses of convexity, by the same way, we have

$$\frac{\partial C}{\partial \phi} = \int_0^T t^2 \frac{\partial W_t}{\partial \phi} dt = \frac{\partial R}{\partial \phi} \int_0^T t^2 W_t (D-t)dt \equiv \frac{\partial R}{\partial \phi} G_2 . \tag{B5}$$

Moreover, we have

$$G_2 \equiv \int_0^T t^2 W_t (D-t)dt < \int_0^T D^2 W_t (D-t)dt = 0 .$$

Therefore, we have the results of Equations (44) and (46).