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SOME PROBLEMS IN HARMONIC ANALYSIS SUGGESTED BY SYMMETRIC SPACES AND SEMI-SIMPLE GROUPS

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CONTENTS

Our purpose is to survey some recent contributions and also to suggest several avenues of further development in the area of analysis indicated by the title of this talk.

1. Introduction: euclidean background.

We begin by saying a few words about the classical case corresponding to $\mathbb{R}^1$. In order to facilitate the presentation that follows we single out three main concerns of that theory as points of reference. These are

A. The Fourier transform

B. The Hilbert transform, $\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\,dy$

C. Harmonic and holomorphic functions in the upper half-plane,

$$\mathbb{R}^2_+ = \{ (x, y), y > 0, x \in \mathbb{R}^1 \}.$$

By B we mean of course the whole apparatus that goes with the Hilbert transform, including maximal functions, operators of fractional integration (Riesz potentials), etc., and by C such things as Fatou's theorem, Poisson integrals, Hardy spaces, etc.

Now the upper half-plane is the arena of action of the group $SL(2, \mathbb{R})$ of fractional linear transformations; it is the symmetric space of that group. In this setup the harmonic analysis is taking place, in effect, on the space $\mathbb{R}^1$ which is the *boundary* of the symmetric space (1).

There are two points of view we may take about extending these theories, and in particular A, B and C, in the context of symmetric spaces and semi-simple groups. The first point of view, and the one I have already suggested, is to start with a semi-simple group and its corresponding symmetric space (of non-compact type), and consider a " boundary ". One then performs the harmonic analysis on the boundary, relating it of course to the objects on the group or symmetric space, such as harmonic or holomorphic functions on the symmetric space, or the theory of unitary representations of the group, etc. The first point of view will be taken up in Parts I and II below.

The second point of view is that of considering the (semi-simple) group itself as the primary object of the analysis what we have in mind will be described later, but the best known example that one may cite is that of the " Plancherel formula " for the group (2). We shall be dealing with other problems, however.

A few more words about the Euclidean background may be in order. Much of what is indicated by our points of reference A, B, and C can be extended to the context of Euclidean $\mathbb{R}^n$. We shall here comment only on the singular integral operators generalizing B (3). Our concern is then with operators of the form

$$f \to Tf = \int_{\mathbb{R}^n} K(y) f(x-y) dy,$$

where K is a suitable singular kernel. Under appropriate conditions of existence these operators can also be realized as multiplier operators, namely $(Tf)\hat{}(x) = m(x)\hat{f}(x)$, where $\hat{}$ denotes the Fourier transform, and m is in effect the Fourier transform of the kernel K. In the well-known and important case studied by Mihlin and Calderón and Zygmund $K(x)$ is, besides some regularity, homogeneous of degree $-n$, and has mean value zero on the unit sphere. The multiplier m is then homogeneous of degree 0. The Mihlin-Calderón-Zygmund theory and its variants take care of one important class of singularities of the kernel, but there are many other types of singularities and the study of their corresponding operators represents serious difficulties which are still unsurmounted. I cite an example which is both fundamental for the Euclidean theory and has some bearing on our later discussion.

PROBLEM 1 (4). — *Consider the case of T when the multiplier m is the characteristic*

(1) For the theory in the closely related and analogous setting where the unit disc replaces the upper half-plane, see ZYGMUND [36].

(2) See GELFAND and NEUMARK [7], and HARISH-CHANDRA [10], [11].

(3) See e. g. STEIN [29], and the references given there.

(4) For some recent progress in the direction of the solution of this problem, see FEFFERMAN [4]. (Added in proof). A counterxample for $p \neq 2$ has been found by FEFFERMAN.

function of the unit ball in $\mathbb{R}^n$. *It is known that* T *is not bounded on* $L^p(\mathbb{R}^n)$, *when* $1 \leq p \leq 2n/(n+1)$, *or* $2n/(n-1) \leq p \leq \infty$. *Is it bounded when*

$$2n/(n+1) < p < 2n/(n-1)?$$

PART I. — ANALYSIS ON THE BOUNDARY

2. Examples of boundaries.

We shall come more quickly to the main points if instead of giving a systematic discussion of the class of spaces X which arise as " boundaries " of non-compact semi-simple groups or symmetric spaces, we list some typical examples ([5]) ([6]).

One type of boundary (that could properly be called the *maximal distinguished boundary*) arises from an Iwasawa decomposition of G as KAN. Then the boundary in question of the symmetric space has two essentially equivalent realizations; either in its non-compact form, when it is isomorphic to the nilpotent group N, or in its compact form as K/M; M is the centralizer of A in K. One example of this is

$$G = SU(n-1), \tag{2.1}$$

G/K is the complex n-ball, K/M is its boundary $2n-1$ sphere. Here X is isomorphic with N, and is defined below; it is the genuine boundary of the realization of G/K as a Siegal domain of type II, equivalent to the complex ball via a Cayley transform ([7]).

X is $\{(z, \omega), z \in \mathbb{C}^{n-1}, \omega \in \mathbb{R}^1\}$, with the multiplication law

$$(z_1, \omega_1) \circ (z_2, \omega_2) = (z_1 + z_2, \omega_1 + \omega_2 - 2\,\mathrm{Im}\, z_1 . \bar{z}_2).$$

Another example of a maximal distinguished boundary is

$$G = SL(n, \mathbb{R}), \tag{2.2}$$

and X is isomorphic with $N = n \times n$ strictly upper triangular matrices of G.

Notice that when $n = 3$ in (2.2) we get a boundary which is isomorphic with the one that arises in (2.1) for $n = 2$. The problems that will arise however will be quite different since in the context of (2.1) we are dealing with a rank one situation, and in (2.2) we are in the higher rank case.

Other examples, which do not arise from the Iwasawa decomposition, are:

$$G = Sp(n, \mathbb{R}), \tag{2.3}$$

G/K is the Siegel upper half-space $= \{ x + iy$, x, y real symmetric $n \times n$ matrices,

([5]) See however the general theory of SATAKE [24], FURSTENBERG [5] and C. C. MOORE [21].

([6]) We shall consider primarily the realizations of the boundaries in their non-compact form, as nilpotent groups.

([7]) For the realization of bounded Cartan domains as Siegel domains of type II, see PJATECKIĬ-SAPIRO [22].

and y is pos. def.$\}$. Here X = set of real sym. $n \times n$ matrices, with the additive structure.

(2.4) $$G = SL(2n, \mathbb{R}),$$

but if portioned into $n \times n$ blocks, then the appropriate boundary is isomorphic with $\left\{\begin{matrix} I_n & x \\ 0 & I_n \end{matrix}\right\}$ as x ranges over $M_n(\mathbb{R}) = n \times n$ real matrices. Thus X can be taken to be $M_n(\mathbb{R})$, with its additive structure

Notice that X, in both (2.3) and (2.4), is a Euclidean space (of dimensions n^2 and $\frac{n(n+1)}{2}$ respectively); but the problems of interest in these examples will not be the Euclidean ones alluded to in section 1.

3. Singular integrals on nilpotent groups.

In generalizing the Euclidean theory to the nilpotent groups which arise as boundaries two fundamental notions need to be introduced: that of *dilations* (8), and that of a *norm function* (9). The first concept generalizes the standard dilations in $\mathbb{R}$ given by scalar multiplication, i. e. $x \to \delta x$, $\delta > 0$, $x \in \mathbb{R}^n$, and is prompted by the observation that broadly speaking, much of the usual harmonic analysis on $\mathbb{R}^n$ is not only translation invariant, but also dilation invariant. The precise definition of dilations is as follows. We assume that with our group X (which is nilpotent and simply connected) we are given a one-parameter group of automorphisms of X, namely $\{\alpha_\delta\}_{0<\delta<\infty}$, so that $\alpha_{\delta_1} \circ \alpha_{\delta_2} = \alpha_{\delta_1 \delta_2}$, α_1 = identity, which is continuous in δ and also contractive. The idea we want is that $\lim\limits_{\delta \to 0} \alpha_\delta(K)$ reduces to the group identity, for any compact set K. A more useful, and somewhat stronger assumption, and the one we shall adopt here, is that when we consider its effect on the Lie algebra of X, namely α_δ^*, then $\alpha_\delta^* = \delta^A$, where A is diagonable with all positive eigenvalues.

Given such a one-parameter group of dilations we introduce a norm function $x \to |x|$ on X as follows. We have $|x| = |x^{-1}|$, also:

(3.1) $$|x| \geq 0$$

(3.2) $$|x| \quad \text{is } C^\infty \text{ on the set where} \quad |x| > 0$$

the measure

(3.3) $$\frac{dx}{|x|}$$

is invariant under dilations; here dx is Haar measure on X.

For the purposes of Part I we add the important assumption:

(3.4) $$|x| = 0,$$

if and only if x is the group identity.

(8) See Stein [27].

(9) See Knapp and Stein [16].

This is equivalent with the statement that the sets $\{ |x| \leq C \}$ are bounded. We shall see that whether we impose (3.4) or not makes a crucial difference in the theory.

We cite two quick examples. First in $\mathbb{R}^n$ $\alpha_\delta(x) = \delta . x$, and $|x| = ||x||^n$, where $|| . ||$ is the usual Euclidean norm. Secondly for the boundary X corresponding to the unit ball cited in (2.1), we may take $\alpha_\delta(x) = (\delta z, \delta^2 \omega)$ if $x = (z, \omega)$, and

$$|x| = (|z|^4 + \omega^2)^{n/2}.$$

Armed with the above notions, we come now to some of the results that can be proved. First, there is an elegant analogue of the Hardy-Littlewood maximal theorem. Let K be any bounded subset with non-empty interior on which the dilations α_δ are contractive in the sense that $\alpha_\delta(K) \subset K$, if $\delta \leq 1$; e. g. $K = \{ x, |x| \leq 1 \}$. Write $K_\delta = \alpha_\delta(K)$, and let

$$(3.5) \qquad (Mf)(x) = \sup_{\delta > 0} \frac{1}{m(K_\delta)} \int_{K_\delta} |f(xy)|\, dy$$

where $dy = dm$ denotes Haar measure. Then M satisfies all the usual properties of the maximal function. As a consequence whenever f is integrable

$$(3.6) \qquad \lim_{\delta \to 0} \frac{1}{m(K_\delta)} \int_{K_\delta} |f(xy) - f(x)|\, dy = 0,$$

for a. e. $x \in X$. We shall come to the applications of the maximal function and (3.6) momentarily.

We discuss next a basic class of singular integrals, written in the form

$$(3.7) \qquad \int_X f(x \cdot y) \frac{\Omega(y)}{|y|^{1-s}} dy$$

where the function Ω is homogeneous under α_δ of degree 0, that is $\Omega(\alpha_\delta(x)) = \Omega(x)$, and Ω is suitably smooth away from the group identity. While the integrals have an interest for all complex values of s, and can indeed be studied as meromorphic functions of s, the range when Re $(s) = 0$ is the most critical, and we shall thus impose that restriction for the rest of this section.

Assuming then that Re $(s) = 0$, and f is bounded and sufficiently smooth, then the integral (3.7) can be defined in several ways. First if the mean-value of Ω vanishes, i. e.

$$\int_{a_1 \leq |x| \leq a_2} \Omega(x) dx = 0,$$

then as a principal-value integral

$$(3.7') \qquad \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} f(x \cdot y) \frac{\Omega(y)}{|y|^{1-s}} dy;$$

or more generally, if the mean-value of Ω vanishes or if $s \neq 1$, then the integral exists as

$$(3.7'') \qquad \lim_{\substack{s' \to s \\ \mathrm{Re}(s') > 0}} \int f(x \cdot y) \frac{\Omega(y)}{|y|^{1-s'}} dy \ ({}^{10}).$$

(10) If $s = 1$ and the mean-value of Ω is nonzero, then the integral cannot be defined without a non-trivial normalizing factor; such a factor has the effect of making it a constant multiple of $f(x)$.

The above limits exist for every x and also in the $L^2(X)$ norm. If we denote the limiting operator by $f \to T(f)$, then the first result is its extensibility to a bounded operator on $L^2(X)$,

$$\| T(f) \|_2 \leq A \| f \|_2 . \tag{3.8}$$

Unfortunately this fundamental result cannot be proved by following the standard arguments of the Euclidean case of $\mathbb{R}^n$, because what would amount to a calculation in terms of the Fourier transform (in the sense of the unitary representations of the group X) seems to lead to unmanageable computations. The one attack which has succeeded in proving (3.8) was suggested by a method originally applicable only in $\mathbb{R}^n$. It turns out that even in the general case T can be written, in effect, as an infinite sum of uniformly bounded operators

$$T = \sum_{j=-\infty}^{\infty} T_j , \qquad \| T_j \| \leq A, \tag{3.9}$$

where the T_j are almost orthogonal in the sense

$$\| T_j^* T_k \| \leq a(|j-k|), \qquad \| T_j T_k^* \| \leq a(|j-k|) \tag{3.10}$$

with a sequence $\{ a(k) \}$ which decreases sufficiently rapidly. The two conditions (3.9) and (3.10) suffice to prove the boundedness of T ([11]).

Once the L^2 result (3.8) has been obtained then by using the facts about the maximal function (3.5), and following the broad lines laid down by Calderón and Zygmund for the case of $\mathbb{R}^n$, one can also obtain the L^p theory, and the L^1 results, namely that the operators in question are of weak type (1,1) ([12]).

Some rather immediate generalizations of the above are possible. First, the specific form of the kernel $\dfrac{\Omega(x)}{|x|^{1-s}}$ allows a variety of modifications in form. Secondly, and more interesting, is the fact that the same theory can be carried out in a setting which replaces the existence of dilations by appropriate substitute conditions on the open sets $K_\delta = \{ x : |x| < \delta \}$. This generalization is used if one wants to find the analogues of the above maximal function and singular integrals on the compact version of X, which is of course related to X via a Cayley transform.

4. Some applications.

We shall now discuss several applications of the theory sketched above.

1. One can construct the *intertwining operators* for the principal series of representations by means of the operator (3.7). Let $G = KAN$ as before, then the representations induced by irreducible representations of the subgroup MAN are the principal series. Thus there is natural action of G on the boundary X (where X is isomorphic with N), which action generalizes the usual action of $SL(2, \mathbb{R})$ in $\mathbb{R}^1$ given

([11]) See Knapp and Stein [16]. Earlier ideas of this kind are due to M. Cotlar.

([12]) This is due to Rivière [23], Koranyi and Vagi [18], Coifman and de Guzman [3].

by fractional linear transformations ([13]), and in terms of which the principal series can be defined. Now the action of the elements of M on X are particularly simple, and these transformations have Jacobian determinant equal to one. This allows us to define the Jacobian determinant corresponding to each element of the Weyl group of G. The square roots of the reciprocals of these Jacobian determinants each provide us with an example of a norm function. It is to be emphasized that each satisfies the properties (3.1), (3.2) and (3.3) for appropriate " dilations " coming from the subgroup A, but in general not the crucial compactness property (3.4). However, in the case of *rank one* (when $\dim A = 1$), the non-trivial element of the Weyl group gives us a norm function (satisfying also (3.4)), and the dilations are provided by the conjugations of X given by A. All the intertwining operators are then of the form (3.7), after suitable normalization. This construction provides the basic information as to irreducibility and analytic continuation (that is existence and unitarity of the complementary series). The general case, when G has higher rank, can also be treated to some extent, since the intertwining operators can then be written as products of rank-one intertwining operators ([14]).

2. A special case of the intertwining operators, which arise for a particular representation of the group $SU(n, 1)$ (discussed with its boundary in (2.1)) is the *Cauchy integral* for the complex ball. In the unbounded realization of the ball, if one takes the Cauchy-Szego kernel which represents H^2, then as boundary integrals one is lead the singular integrals (3.7) with $\dfrac{\Omega(x)}{|x|} = \text{constant} \times (|z|^2 + i\omega)^{-n}$, and

$$|x| = (|z|^4 + \omega^2)^{n/2},$$

where $(z, \omega) \in \mathbb{C}^{n-1} \times \mathbb{R}^1$, and $\alpha_\delta(z, \omega) = (\delta z, \delta^2 \omega)$ ([15]).

3. In this application the space $X = \mathbb{R}^n$, but the dilations are not the usual ones. These are now given by $\alpha_\delta(x) = (\delta^{a_1} x_1, \delta^{a_2} x_2, \ldots, \delta^{a_n} x_n)$, with $x = (x_1, \ldots, x_n)$, where $a_i > 0$. We can put

$$|x| = \inf \{ \lambda > 0, \sum_{i=1}^{n} x_i^2/\lambda^{2a_i} \leq 1 \}^{\Sigma a_i}.$$

Then the theory described above reduces essentially to the Euclidean theory of singular integrals with separate homogeneity due to Jones, Fabes and Rivière, Lizorkin and Kree ([16]). Notice that this has many points in common with example 2 just cited, in that the degree of singularity of the kernels depends on the different directions of approach to the group identity. The present application differs from the preceding, however, in that the convolution is commutative.

([13]) This comes about by identifying (modulo sets of measure zero) G/MAN with θN, where θ is the Cartan involution, and then identifying X with θN.

([14]) For details concerning the above application to intertwining operators, see KNAPP and STEIN [16]. Some earlier works in this subject may be found in KUNZE and STEIN [20], and SCHIFFMAN [25]. See also the recent paper of HELGASON in *Advances in Mathematics*, vol. 5, 1970, 1-154.

([15]) See GINDIKIN [9] and KORANYI and VAGI [18].

([16]) See e. g., KREE [19].

Examples 2 and 3 suggest the following problem which, as should be understood, we state only rather vaguely.

PROBLEM 2. — *Construct appropriate algebras of singular integrals (or more generally pseudo-differential operators), together with their symbolic calculus, which algebras are to incorporate such examples as* 2 *and* 3 *as their building blocks.*

It is strongly indicated that such algebras should have applications to various non-elliptic problems, in particular in complex analysis, such as behavior near a pseudo-convex boundary and properties of solutions of $\bar{\partial}$ problems.

4. As a final application, in this case of the maximal function (3.5) we mention some results dealing with harmonic functions on the symmetric space G/K and centering about *Fatou's theorem* and *Poisson integrals.* In the case of bounded functions, the generalization of the boundary behavior guaranteed by the classical Fatou theorem turns out to be a direct consequence of two facts: *a)* Furstenberg's representation of bounded harmonic functions as Poisson integrals, and *b)* the maximal function, and in particular (3.6) ([17]).

However, in the case of Poisson integrals in general (e. g. of L^p functions), much remains to be done. The problems involving Poisson integrals will be discussed more fully when we treat the higher rank case below.

PART II. — ANALYSIS ON THE BOUNDARY; HIGHER RANK CASE

We shall discuss now the situation when the assumption (3.4) concerning the norm function is not satisfied, that is when the sets $\{ x : |x| \leq c \}$ are no longer bounded. Very often in this case the group of automorphisms of X which preserve the measure $\frac{dx}{|x|}$ is larger than a one-parameter group, and so in considering the appropriate dilations it is not entirely natural to limit oneself to a fixed one-parameter group of dilations as we did in Part I. It is for this reason that we refer to the situation when (3.4) is not satisfied as the *higher rank case.*

The rank-one case treated above provides us—at least on the formal level—with an idea of the kind of problems that may be of interest in the general case. However, those results have only a limited applicability in the present context; one instance of this is the decomposition of intertwining operators for the principal series as products of rank-one intertwining operators, already mentioned. In general, however, new and different methods surely need to be developed here.

We shall organize our presentation by discussing several different but related problems which reflect the fragmentary state of our knowledge at this stage.

([17]) HELGASON and KORANYI [12]. This has been superseded by a later results of KORANYI and KNAPP and WILLIAMSON. See [17].

5. **The Siegel upper half-space.**

We are dealing with the example cited in (2.3). X is the space of $n \times n$ real symmetric matrices under addition, which is the Bergman-Shilov boundary of the Siegel domain $= \{ x + iy;\ x, y$ real $n \times n$ symmetric, y pos. def. $\}$. The action of $Sp(n, \mathbb{R})$ imposes the following structure on X. The dilations are provided by the mappings: $x \to ax^t a$, where $a \in GL(n, \mathbb{R})$, and for norm function we take $|x| = |\det(x)|^{\frac{n+1}{2}}$.

Let us first look at the analogues of the integrals (3.7) with the kernels $\dfrac{\Omega(x)}{|x|^{1-s}}$, where Ω is homogeneous in the sense that $\Omega(ax^t a) = \Omega(x)$, $a \in GL(n, \mathbb{R})$. These integrals have a long history, going back to Siegel, Bochner, and others. We indicate an interesting example arising from the Cauchy kernel. Consider the H^2 space of holomorphic functions $f(x + iy)$ on the Siegel upper half-space, those which satisfy

$$\sup_{y>0} \int_X |f(x + iy)|^2 dx < \infty.$$

Such functions have boundary values, namely $\lim_{y \to 0} f(x + iy) = f(x)$ exists in the $L^2(X)$ norm. Their integral representation in terms of their boundary values is then [18]

$$f(x + iy) = c \int_X (\det(t + iy))^{\frac{-n-1}{2}} f(x - t)dt \tag{5.1}$$

where c is an appropriate constant.

The boundary value functions form a closed subspace of $L^2(X)$, and the orthogonal projection on this subspace is formally given by an operator of the form (3.7), where now $\dfrac{\Omega(x)}{|x|} = c\,(\det(x))^{\frac{-n-1}{2}}$. Rigorously the operator is given as the limit as $y \to 0$ in (5.1), and more particularly as

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} c \int_X (\det(t + i\varepsilon I))^{\frac{-n-1}{2}} f(x - t)dt. \tag{5.2}$$

This operator then is clearly a natural generalization of the Hilbert transform to the present context. A host of questions arise for it, but only a few have an answer at present. We indicate one such unsolved problem:

PROBLEM 3. — *The operator* (5.2) *is a projection on* $L^2(X)$. *Is it bounded on any other* $L^p(X)$ *space*?

The close relation of this problem with problem 1 (in section 1) can be aeen as follows. The operator (5.2) is a multiplier operator corresponding to the characteristic function of the cone of positive definite real $n \times n$ matrices. When $n = 2$ this cone is equivalent with a circular cone in $\mathbb{R}^3$, and the intersection of that cone with an appropriate plane is a disc in $\mathbb{R}^2$. Thus by a theorem of de Leeuw, a positive resolution of problem 3 for any p, when $n = 2$, implies the same for problem 1 when $n = 2$.

(18) See BOCHNER [1].

Part of the difficulty in dealing with integrals such as (5.2) lies in the fact that the singularities of the kernel, that is where $|x| = 0$, are a whole variety and not merely a point. However, one is not always stimied by this obstacle. An example of this is the Poisson integral, closely related to (5.2); it is given by

$$\int_X P_y(t) f(x-t)dt \tag{5.3}$$

where

$$P_y(x) = \frac{c\,(\det(2y))^{\frac{n+1}{2}}}{|\det(x+iy)|^{n+1}},$$

and $f \in L^p(X)$.

It can be shown that as $y \to 0$ " regularly ", then the integral (5.3) converges to f almost everywhere, even for $f \in L^1(X)$ ([19]). This result is rather delicate because as $y \to 0$, the singularities of the kernel $P_y(x)$ again appear on the variety $|x| = 0$. It shows us that the hope of carrying out a theory for integrals of the type (5.2) may not be entirely forlorn.

Our discussion for the Siegel upper half-space may be generalized as follows. We consider any bounded symmetric domain of Cartan and realize it as a tube domain when this is possible, or in general as a Siegel domain of type II ([20]). The Cauchy kernel has also been determined ([21]), and we can of course pose the analogue of problem 3 (For the complex ball the answer is in the affirmative for $1 < p < \infty$, by the discussion of section 4). Finally there is an analogue of the Poisson kernel, and the result sketched above is known to hold in that generality ([19]).

6. Poisson integrals.

We have already alluded to Poisson integrals at several occasions, and we shall now discuss them in their generality. Briefly the setting is as follows. For any symmetric space G/K, the class of harmonic functions are those annihilated by all G-invariant differential operators which annihilate constants. Equivalently, these functions can be characterized by the mean-value property. Now every harmonic function which is appropriately bounded at ∞ can be represented as a Poisson integral, which is in effect a convolution on the group X isomorphic to N. By the mean-value property the Poisson kernel P can be described as follows. We have already pointed out the existence of a natural correspondence between X and the compact homogeneous space K/M, if one leaves out an appropriate set of measure zero ([22]). If we transplant Haar measure of K/M to X we get a measure of the form $P(x)dx$, where dx is Haar measure on X.

Now the subgroup A acts on X by automorphisms $x \to axa^{-1}$, $a \in A$. Let α_δ be a one-parameter subgroup of these automorphisms which are dilations in sense defined

([19]) STEIN and N. J. WEISS [33].
([20]) See footnote ([7]).
([21]) See GINDIKIN [9].
([22]) See footnote ([13]).

in section 3. It is then easy to see that for any $f \in L^p(X)$, $1 \leq p < \infty$, the Poisson integral

$$\int_X P(y) f(x \cdot \alpha_\delta(y)) dy \tag{6.1}$$

converges to f in the $L^p(X)$ norm, as $\delta \to 0$.

The main real-variable problem can then be stated as follows.

PROBLEM 4. — *Does the integral* (6.1) *converge almost everywhere, as* $\delta \to 0$, *for any* $f \in L^p(X)$, $1 \leq p$?

One gets an idea of the resistive nature of the problem by observing the increase in difficulty met in passing from the classical case of the upper half-plane, to the case of the product of half-planes contained in the theorem of Marcinkiewicz and Zygmund ([23]).

The farthest advance of the problem at present is the solution of a closely related variant for the symmetric spaces which are bounded domains, already alluded to in section 5. That variant differs from the present one in that it refers to a different boundary of the symmetric space in question, one that can be viewed as a quotient space of the maximal distinguished boundary occurring in problem 4 ([24]).

There is a reason why problem 4 in its general setting seems more complicated than the analogue already obtained for the case of bounded domains. To over-simplify matters a little, it is as follows: the locus of singularities in the latter problem (e. g. $\{ \det(x) = 0 \}$) is generated by straight lines issuing from the origin. Along these lines the theory for $\mathbb{R}^1$ is applicable and then the result follows by a rather delicate calculation which is akin to " integrating " over appropriate lines. In the general case, however, straight lines would have to be replaced by other curves; these curves are the orbits of points under one-parameter groups of dilations. The above raises a simply-stated (and possibly fundamental) problem which we shall discuss only in the context of $\mathbb{R}^n$. Let $\gamma(t)$ be the curve $\gamma(t) = \operatorname{sign}(t)(A_1 |t|^{a_1}, A_2 |t|^{a_2}, \ldots, A_n |t|^{a_n})$ where $A_1, \ldots, A_n$ are real, and $a_i > 0$. Consider the analogue of the Hilbert transform

$$(Tf)(x) = \int_{-\infty}^{\infty} f(x + \gamma(t)) \frac{dt}{t} \tag{6.2}$$

(Notice that if $a_1 = a_2 \ldots = a_n$, then this reduces essentially to the classical Hilbert transform along the direction defined by $(A_1, A_2, \ldots, A_n)$). Consider also the associated maximal operator

$$(Mf)(x) = \sup_{h>0} \frac{1}{h} \int_0^h | f(x + \gamma(t)) | \, dt \tag{6.3}$$

PROBLEM 5. — *Is there an* $L^p(\mathbb{R}^n)$ *theory for* T *and* M?

An analogous result for nilpotent groups (in particular for M) could be applied to the solution of problem 4.

([23]) See ZYGMUND [36], Chapter 17.

([24]) This incidentally raises the question of giving an intrinsic characterization of the functions which arise as Poisson integrals for the other boundaries.

There is one hopeful indication that may be mentioned concerning problem 5. A calculation carried out by Wainger and myself (see [31]) shows that the operators (6.2) when suitably defined is bounded on $L^2(\mathbb{R}^n)$, (and the bound does not depend on $A_1, A_2, \ldots, A_n$).

7. The matrix space $M_n(\mathbb{R})$.

We shall now consider the example (2.4), with $X = M_n(\mathbb{R})$ the $n \times n$ real matrices, and $G = SL(2n, \mathbb{R})$. Here we take as dilations the mappings $x \to axb^{-1}$, with $a, b \in GL(n, \mathbb{R})$, and as norm function $|\det(x)|^n$.

This example has obviously some resemblance to that of the Siegel upper half-space in section 5, but it differs from it in that the space $M_n(\mathbb{R})$ has not only the obvious additive structure (its group structure), but upon removal of a set of measure zero what remains also has a multiplicative structure ($GL(n, \mathbb{R})$). The situation has an analogy with that of a field (e. g. $\mathbb{R}^1$) where one of the concerns is with the interplay of an additive and a multiplicative harmonic analysis. The additive harmonic analysis here is that given by

$$\mathcal{F}(f) = \int_{M_n(\mathbb{R})} e^{2\pi i tr(x^t y)} f(y) dy, \tag{7.1}$$

while the multiplicative analysis (the analogue of the Mellin transform) is given by the unitary (infinite-dimensional) representations of $GL(n, \mathbb{R})$. This interplay is at the bottom of the results detailed below (See also section 8).

The most direct analogue of the integral (3.7) arises if $\Omega \equiv 1$. We consider therefore

$$I_s(f) = \int_{M_n(\mathbb{R})} f(x - y) \frac{dy}{|y|^{1-s}} \tag{7.2}$$

The L^2 theory of this integral is contained in the following statement ([25]). Suppose f is C^∞ and has bounded support. (7.2) initially defined as an absolutely convergent integral when $\operatorname{Re}(s) > 1 - \frac{1}{n}$ has a meromorphic continuation into the whole complex plane, and when $\operatorname{Re}(s) = 0$ the operator $f \to I_s(f)$ is unitary modulo a multiplicative constant. More precisely, with

$$\gamma_*(s) = \prod_{j=1}^{n} \alpha(ns - j + 1), \qquad \alpha(s) = \pi^{1/2-s}\Gamma\left(\frac{s}{2}\right)/\Gamma\left(\frac{1-s}{2}\right),$$

we have that when $\operatorname{Re}(s) = 0$, I_s is a multiplier operator with multiplier $\gamma_*(s)|x|^{-s}$.

The above also has the following consequences:

(*a*) The facts just stated can be reinterpreted by saying that the Fourier transform of the distribution $|x|^{-1+s}$ is $\gamma_*(s)|x|^{-s}$, where both distributions are defined by analytic continuation. This functional equation is closely related to the functional equations of generalizations of the zeta function, and is therefore of interest in several number-theoretic questions (see also the generalizations in (8.3) below).

([25]) See the references cited in footnote ([29]).

(*b*) The operators (7.2) also serve as intertwining operators, but not for the principal series. They arise typically in the " degenerate series ", in this case for the group $SL(2n, \mathbb{R})$.

(*c*) If we write $A(s) = \gamma_*^{-1}(s)I_s$, and $B(s)$ as the multiplication operator by $|x|^{-s}$, then as we have seen $A(s)B(s)$ is unitary when Re $(s) = 0$. In addition $A(s)B(s)$ has an analytic continuation as bounded operators (on $L^2(M_n(\mathbb{R}))$, in the strip $0 \leq \text{Re}\,(s) < 1/2n$. This fact is important in constructing certain uniformly bounded and unitary (complementary series) representations of the group $SL(2n, \mathbb{R})$.

There are many variants and generalizations of the above that can be suggested; we shall discuss briefly one typical of those we have in mind. The underlying space X will be $\mathbb{R}^n$ and we will pick a fixed non-degenerate quadratic form Q on it, which for simplicity we normalize as $Q(x) = x_1^2 + x_2^2 \ldots + x_k^2 - x_{k+1}^2 \ldots - x_n^2$. We introduce the norm function $|x| = |Q(x)|^{n/2}$. The analogue of the integral (7.2) is the integral

$$I_s(f) = \int_{\mathbb{R}^n} f(x-y)\,|Q(y)|^{-\frac{n}{2}(1-s)}dy \tag{7.3}$$

It has well-known analytic continuations, going back to M. Riesz and Gelfand and Graev ([26]). We let $B(s)$ denote the operator of multiplication by $|x|^{-s} = |Q(x)|^{-ns/2}$.

PROBLEM 6. — *Are the $I_sB(s)$ bounded operators on $L^2(X)$ in some strip of the form* $0 < \text{Re}\,(s) < c$? ([27]).

An interesting approach to this problem might be to study the decomposition of the action of $0(n, Q)$ on $L^2(\mathbb{R}^n)$, since after all, the operators $I_sB(s)$ commute with this action ([28]).

PART III. — ANALYSIS ON THE GROUP

8. Euclidean Fourier transform.

The interplay of the additive and multiplicative harmonic analysis on $M_n(\mathbb{R})$, mentioned in the previous section, will now be outlined. We take the additive Fourier transform given by (7.1). A simple change of variables leads to a slight modification of itself, which we shall call $\mathscr{F}^*$ where now

$$\mathscr{F}^*(f) = e * f, \tag{8.1}$$

with the convolution taken on the group $GL(n, \mathbb{R})$, and

$$e(x) = e^{2\pi i tr(x-1)}\,|x|^{-n/2}.$$

([26]) See GELFAND *et al.* [8].

([27]) When Q is definite, the answer is yes, with $c = 1/2$. The cases $n = 1$ and 2 are in KUNZE and STEIN [20]; their method essentially applies to all n, but in the definite case only. When $n = 4$, $k = 2$, we are back to $M_2(\mathbb{R})$, so $c = 1/4$.

([28]) Part of the decomposition of the action of $0(n, Q)$ on $L^2(\mathbb{R}^n)$ is in the book of VILENKIN [34].

The properties of $\mathcal{F}^*$ are then twofold: $\mathcal{F}^*$ is unitary on $L^2(GL(n, \mathbb{R}))$, and secondly $\mathcal{F}^*$ commutes with both left and right group multiplication, i. e. with the action $f(x) \to f(a^{-1}x)$, $f(x) \to f(xb)$, $a, b \in GL(n, \mathbb{R})$. (The original $\mathcal{F}$ had this commutation property only when both a and b were orthogonal). $\mathcal{F}^*$ is therefore a *central* operator on $L^2(GL(n, \mathbb{R}))$. From this it follows by a general form of Schur's lemma that whenever $x \to \rho(x)$ is an irreducible unitary representation of $GL(n, \mathbb{R})$ we may expect that

$$\rho(\mathcal{F}^*(f)) = \gamma(\rho)\rho(f) \tag{8.2}$$

whenever f and $\mathcal{F}^*(f)$ are in $L^1(GL(n, \mathbb{R})) \cap L^2(GL(n, \mathbb{R}))$. Here $\gamma(\rho)$ is a constant factor which depends only on the representation ρ.

This identity is formally equivalent with the statement

$$\mathcal{F}\left(\frac{\rho(x)}{|x|^{1-s}}\right) = \gamma_s(\rho)\rho({}^t x^{-1})\,|x|^{-s} \tag{8.3}$$

where the factor $\gamma_s(\rho)$ can be immediately read off from the factor $\gamma(\rho)$.

When ρ is the trivial representation, then $\gamma_s(\rho)$ reduces to the factor $\gamma_*(s)$ of the previous section. The other cases where the factor $\gamma_s(\rho)$, (and thus $\gamma(\rho)$) has been computed explicitly are those for the representations ρ which arise in the decomposition of $L^2(GL(n, \mathbb{R}))$, (i. e. those which occur in the " Plancherel formula " for the group). In this case, because of the unitary character of $\mathcal{F}^*$, all the factors $\gamma(\rho)$ have absolute value one.

It is particularly simple to describe these factors in the analogous case corresponding to $M_n(\mathbb{C})$. In that case if the representation is induced from the character of the triangular subgroup which has value

$$|\delta_1|^{it_1}\left(\frac{\delta_1}{|\delta_1|}\right)^{m_1} \cdots |\delta_n|^{it_n}\left(\frac{\delta_n}{|\delta_n|}\right)^{m_n}$$

for a triangular matrix with eigenvalues $(\delta_1, \ldots, \delta_n)$, then

$$\gamma(\rho) = \prod_{j=1}^{n} \left\{ i^{|m_j|}\pi^{it_j}\Gamma\left(\frac{|m_j| + 1 + it_j}{2}\right) \Big/ \Gamma\left(\frac{|m_j| + 1 - it_j}{2}\right) \right\}$$

The formulae in the case $M_n(\mathbb{R})$ have a similar appearance but are more complicated because there are now $[n/2] + 1$ different series of representations which occur in the L^2 reduction of $GL(n, \mathbb{R})$ ([29]).

The mapping $f \to \rho(f)$ may be viewed as the natural generalization of the Mellin transform (to which it reduces when $n = 1$). The explicit determination of the factors $\gamma(\rho)$ which occur in (8.2) gives the desired multiplicative analysis of the additive Fourier transform in $M_n(\mathbb{R})$. This " Mellin transform " analysis of $\mathcal{F}$ is the main tool in the proof of several results of the previous section, in particular those stated in paragraph (*c*).

([29]) The results sketched above, and those in section 7, were first obtained in the complex case (corresponding to $M_n(\mathbb{C})$); see Stein [26]. In the real case they were obtained by Gelbart [6], but in the meanwhile several of these problems had been dealt with from a different point of view by Godement (unpublished), and Jacquet and Langlands [14]. These authors have also obtained extensions to the p-adic analogue, when $n = 2$.

A related question arises by analogy with the ordinary Fourier transform on $\mathbb{R}^n$. The fact that the Fourier transform commutes with rotations leads to a well-known decomposition of $L^2(\mathbb{R}^n)$, compatible with the Fourier transform. The various invariant subspaces are defined in terms of spherical harmonics, and the restriction of the Fourier transform to each can be described in terms of appropriate Bessel functions ([30]). The theory of higher Bessel functions, in the setting of matrix spaces, has been started by Bochner ([31]), but much still remains to be done. This discussion is the background for the following problem.

PROBLEM 7. — *Describe the action of the Fourier transform $\mathscr{F}$ on $L^2(M_n(\mathbb{R}))$ when restricted to the subspaces invariant under the action $f(x) \to f(a^{-1}xb)$, $a, b \in O(n)$, in terms of appropriate generalizations of spherical harmonics and Bessel functions.*

9. Other problems on the group manifold.

The last general question we shall deal with is the following. Is it possible to develop a systematic generalization of some of the objects dealt with in Parts I and II, such as Hilbert transforms, boundedness of various convolution operators, multipliers, etc. but on the semi-simple group itself, and not on one of its boundaries.

For compact groups, the answer is surely yes ([32]). However, for non-compact groups, the situation seems to be far from clear. Part of the difficulty of the problem there, and also its interest I believe, is that unlike the classical case the group Fourier transform of an L^p function, $1 \leq p < 2$, is actually analytic in some of its parameters. It is thus more like the classical Laplace transform than the classical Fourier transform. The analyticity of the Fourier transform is intimately connected with the possibility of analytic continuation of the representations of the non-compact semi-simple groups, but even this subject is far from understood ([33]).

To get a better inkling of the nature of these questions, we pose the simplest convolution problems. Suppose we know the L^p classes of two functions f and g, what is the class of $f * g$? There is a very general answer, valid for any locally compact unimodular group, and it is given by Young's inequality and its variants. Young's inequality is

$$\| f * g \|_r \leq \| f \|_p \| g \|_q, \qquad \text{where} \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

The variants of Young's inequality (which include the theorem of fractional integration for $\mathbb{R}^n$ of Hardy, Littlewood and Sobolev) arise when we replace these norms by " weak-type " norms. For $\mathbb{R}^n$ these inequalities are in the nature of best possible; for semi-simple groups this is far from the case. In fact the evidence already at hand, and described below, suggests the following L^2 convolution problem for semi-simple groups.

([30]) See e. g. STEIN and WEISS [32], Chapter IV.

([31]) See BOCHNER [2] and HERZ [13].

([32]) See STEIN [28], where part of this has been carried out; see also COIFMAN and DE GUZMAN [3] and N. J. WEISS [35].

([33]) See KUNZE and STEIN [20], and the survey article, STEIN [30].

PROBLEM 8. — *Suppose G is semi-simple and has finite center. Prove that*

$$\| f * g \|_2 \leq A_p \| f \|_p \| g \|_2, \quad \text{if} \quad 1 \leq p < 2.$$

This problem involves only the relative sizes of $|f|$ and $|g|$, and thus, one would think, should be resolvable without any detailed study of the group Fourier transform of f or of analytic continuation of representations. Paradoxically however, that approach is the only one that has had any substantial success so far. The answer to problem 7 is known to be affirmative in the following cases ([33]).

(i) $G = SL(2, \mathbb{R})$
(ii) G is any complex classical group, i. e. $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, or $Sp(n, \mathbb{C})$
(iii) G is any semi-simple group, but the function f is assumed to be bi-invariant, i. e. $f(k_1 x k_2) = f(x)$, when $k_1, k_2 \in K$, and K is a maximal compact subgroup of G.

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