

## Hybrid ARQ with Random Transmission Assignments

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**ABSTRACT.** In mobile packet data and voice networks, a special coding scheme, known as the incremental redundancy hybrid ARQ (IR HARQ), achieves higher throughput efficiency than ordinary turbo codes by adapting its error correcting code redundancy to fluctuating channel conditions characteristic for this application. An IR HARQ protocol operates as follows. Initially, the information bits are encoded by a “mother” code and a selected number of parity bits are transmitted. If a retransmission is requested, only additional selected parity bits are transmitted. At the receiving end, the additional parity bits are combined with the previously received bits, allowing for an increase in the error correction capacity. This procedure is repeated after each subsequent retransmission request until the entire codeword of the mother code is transmitted. A number of important issues such as error rate performance after each transmission on time varying channels, and rate and power control are difficult to analyze in a network employing a particular HARQ scheme, *i.e.*, a given mother code and given selection of bits for each transmission. By relaxing only the latter constraint, namely, by allowing random selection of the bits for each transmission, we provide very good estimates of error rates allowing us to address to a certain extent the rate and power control problem.

### 1. Introduction

In conventional *automatic repeat request* (ARQ) schemes, frame errors are examined at the receiving end by an error detecting (usually cyclic redundancy check (CRC)) code. If a frame passes the CRC, the receiving end sends an acknowledgement (ACK) of successful transmission to the receiver. If a frame does not pass the CRC, the receiving end sends a negative acknowledgement (NAK), requesting retransmission. User data and its CRC bits may be additionally protected by an error correcting code which increases the probability of successful transmission. Such ARQ schemes which combine the ARQ principle with error control coding are known as *hybrid ARQ* schemes.

The standard measure of ARQ protocol efficiency is *throughput*, defined as the average number of user data bits accepted at the receiving end in the time required for transmission of a single bit. Therefore the level of redundancy of the error correcting code employed in an HARQ scheme has two opposing effects on the scheme

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efficiency, namely, with increased redundancy the probability of successful transmission increases but the percentage of user data in the frame decreases. Usually, a fixed rate code which is well suited to the channel characteristics and throughput requirements is selected.

In applications with fluctuating channel conditions within a range of signal-to-noise ratios (SNRs), such as mobile and satellite packet data transmission, the so called *incremental redundancy* (IR) HARQ schemes exhibit higher throughput efficiency by adapting their error correcting code redundancy to different channel conditions. An IR HARQ protocol operates as illustrated by the example in Fig. 1. At the transmitter, the information and CRC bits are encoded by a systematic

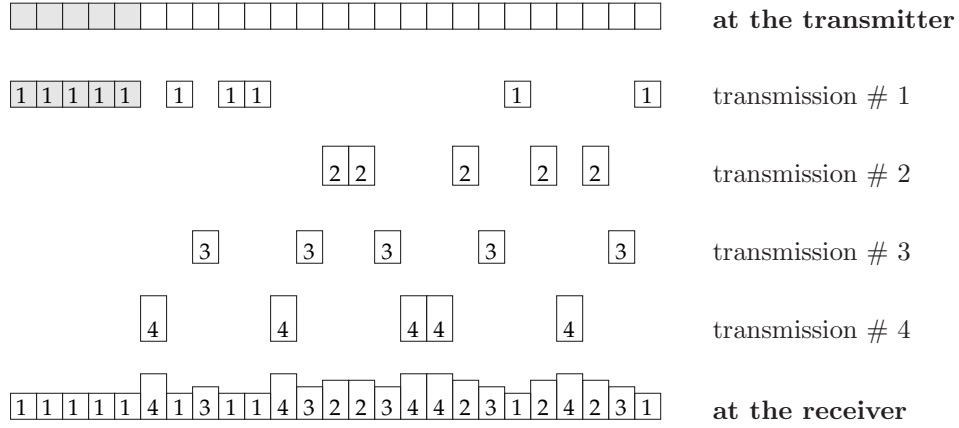


FIGURE 1. Incremental redundancy HARQ protocol.

“mother” code. In Fig. 1, the codeword at the transmitter is represented by a string of 25 boxes. The first 5 (gray in the figure) correspond to the systematic bits. Initially, only the systematic part of the codeword and a selected number of parity bits are transmitted (transmission #1 in the figure). The selected parity bits together with the systematic bits form a codeword of a punctured mother code. Decoding of this code is performed at the receiving end. If a retransmission is requested, the transmitter sends additional parity bits possibly under different channel conditions or at different power (depicted as taller boxes in transmission #2 in the figure). Decoding is again attempted at the receiving end, where the new parity bits are combined with those previously received. The procedure is repeated after each subsequent retransmission request until all the parity bits of the mother code are transmitted.

First ideas for systems combining error correction and ARQ date back to the work of Wozencraft and Horstein in 1960 [1], [2]. A historic overview of further development of such techniques, now known as hybrid ARQ, can be found in [3]. Recent interest in the scheme comes from the quest for reliable and efficient transmission under fluctuating conditions in wireless networks. An information-theoretic analysis of some HARQ protocols, concerning throughput and average delay for block-fading Gaussian collision channels have been reported in [4]–[7]. Another

line of recent work on HARQ is concerned with the mother code and its puncturing. Given the number of parity bits which are at each stage omitted from the mother code (*i.e.*, punctured and not transmitted), their identity is determined by a *puncturing pattern*. The throughput of HARQ schemes is strongly affected by the power of the mother code used in the system and the family of codes obtained by puncturing. Thus recently proposed HARQ schemes use powerful turbo codes, and design of puncturing patterns is an important issue [8]–[16].

We here have two goals motivated directly by questions in practice. The first is to evaluate the error rate performance after each transmission, which is equivalent to evaluating performance of punctured turbo codes on time varying channels. The second is to show how one should go about choosing the signal power and the number of bits for transmission  $j$  after a failed transmission  $j - 1$ . To approach and solve both problems, we introduce the idea of random transmission assignments of the mother code bits, which is related to our previous work on randomly punctured turbo codes [20], [21].

## 2. Channel Model and Performance of the Mother Code

**2.1. Time-Invariant Channel.** We first consider a binary input memoryless channel with output alphabet  $\Omega$  and transition probabilities  $W(y|0)$  and  $W(y|1)$ ,  $y \in \Omega$ . Performance of turbo codes on such channels has been widely studied by various techniques; particularly successful are those based on density evolution [17] and EXIT charts [18]. For our analysis of the mother turbo code and IR HARQ scheme performance on time-varying channels (described in the following section), it is especially convenient to use the coding theorems for turbo code ensembles established recently by Jin and McEliece in [19]. We briefly review their results below.

When a codeword  $\mathbf{x} \in \mathcal{C} \subseteq \{0, 1\}^n$  has been transmitted, the probability that the maximum likelihood (ML) detector finds codeword  $\mathbf{x}'$  at Hamming distance  $h$  from  $\mathbf{x}$  more likely can be bounded as follows:

$$(2.1) \quad P_e(\mathbf{x}, \mathbf{x}') \leq \gamma^h,$$

where  $\gamma$  is the *Bhattacharyya noise parameter* defined as

$$(2.2) \quad \gamma = \sum_{y \in \Omega} \sqrt{W(y|x=0)W(y|x=1)}$$

if  $\Omega$  is discrete and as

$$\gamma = \int_{\Omega} \sqrt{W(y|x=0)W(y|x=1)} dy$$

if  $\Omega$  is a measurable subset of  $\mathcal{R}$ .

For an  $(n, k)$  binary linear code  $\mathcal{C}$  with the weight enumerator  $A_h$  (*i.e.*,  $A_h$  codewords of weight  $h$ ), we have the well known *union-Bhattacharyya bound* on the ML decoder word error probability

$$P_W^{\mathcal{C}} \leq \sum_{h=1}^n A_h \gamma^h.$$

To derive their coding theorems, Jin and McEliece use the union-Bhattacharyya bound on the ML decoder word error probability averaged over the set of all turbo codes with identical component codes but different interleavers.

Suppose that the mother turbo code consists of  $L$  pseudorandom interleavers, and  $L$  recursive convolutional encoders. There are  $k!$  possible choices for each interleaver. Consequently, for a given set of  $L$  recursive convolutional encoders, there are  $(k!)^L$  different  $(n, k)$  turbo codes, corresponding to all different interleavers. We denote this *set* by  $\mathcal{C}^{(n)}$ . By the mother turbo code *ensemble*  $[\mathcal{C}]$ , we will mean a *sequence* of turbo code sets  $\{\mathcal{C}^{(n)}\}$  with a common rate. We are interested in asymptotic performance of  $\mathcal{C}^{(n)}$  when  $n \rightarrow \infty$ .

For a turbo code ensemble  $[\mathcal{C}]$ , the average number of codewords of weight  $h$  in  $\mathcal{C}^{(n)}$  is denoted by  $\bar{A}_h^{[\mathcal{C}](n)}$ . The bound on the ML decoder word error probability averaged over the ensemble is obtained by averaging the (additive) union bound:

$$(2.3) \quad \bar{P}_W^{[\mathcal{C}]} \leq \sum_{h=1}^n \bar{A}_h^{[\mathcal{C}](n)} \gamma^h.$$

To further analyze this expression, we write

$$(2.4) \quad \bar{P}_W^{[\mathcal{C}]} \leq \sum_{h=1}^n \bar{A}_h^{[\mathcal{C}](n)} \gamma^h \leq \sum_{h=1}^{D_n} \bar{A}_h^{[\mathcal{C}](n)} + \sum_{h=D_n+1}^n \bar{A}_h^{[\mathcal{C}](n)} \gamma^h,$$

where  $D_n$  is a sequence of numbers such that

$$(2.5) \quad D_n \rightarrow \infty \text{ and } \frac{D_n}{n^\epsilon} \rightarrow 0 \quad \forall \epsilon > 0.$$

The average number of codewords of weight  $h > D_n$  in the ensemble  $[\mathcal{C}]$  can be bounded in terms of the ensemble *noise threshold* defined as follows [19]:

$$c_0^{[\mathcal{C}]} = \limsup_{n \rightarrow \infty} \max_{D_n < h \leq n} \frac{\log \bar{A}_h^{[\mathcal{C}](n)}}{h}.$$

It was shown in [19] that for a turbo code ensemble  $[\mathcal{C}]$  with the number of component codes  $L \geq 2$ , the ensemble noise threshold  $c_0^{[\mathcal{C}]}$  is a finite positive number. Therefore, we have

$$(2.6) \quad \bar{A}_h^{[\mathcal{C}](n)} \leq_n \exp(h c_0^{[\mathcal{C}]}), \quad D_n < h \leq n,$$

where  $\leq_n$  means that the inequality holds for large enough  $n$ . By using the inequality (2.6), we see that on a binary-input memoryless channel whose Bhattacharyya parameter  $\gamma < \exp(-c_0^{[\mathcal{C}]})$ , the second term of the right-hand-side of the bound (2.4) goes to zero as  $n$  increases. The analysis of the first term of (2.4) is related to the analysis of the minimum distance of turbo codes [22]–[26]. For turbo codes with  $L > 2$  interleaver branches, Kahale and Urbanke proved that the minimum distance of a turbo code with a random interleaver is at least  $\Omega(n^{1-2/L})$  with high probability [22]. For this case, it was shown in [19] that we have the so-called *interleaver gain*:

$$(2.7) \quad \bar{P}_W^{[\mathcal{C}](n)} = O(n^{-L+2+\epsilon}) \quad \text{for any } \epsilon > 0.$$

For turbo codes with  $L = 2$  interleaver branches, Breiling in [23] proved that the minimum distance of turbo codes is upper bounded by  $O(\log n)$ , and three groups of authors in [24, 25, 26] showed that, for some specially constructed interleavers,

the minimum distance of turbo codes grows as  $O(\log n)$ . Thus, for an ensemble of two-branch turbo codes with “good” interleavers, we have

$$\lim_{n \rightarrow \infty} \sum_{h=1}^{D_n} \bar{A}_h^{[C](n)} = 0$$

when  $D_n < n \cdot \text{const} \cdot \log n$ . It follows that for both  $L > 2$  with random interleavers and  $L = 2$  with specially designed interleavers, we have

$$(2.8) \quad \lim_{n \rightarrow \infty} \bar{P}_W^{[C](n)} = 0$$

when  $\gamma < \exp(-c_0^{[C]})$ .

**2.2. Time-Varying Channel.** We again consider a DMC with the binary input alphabet  $\{0, 1\}$  and output alphabet  $\mathcal{Y}$ , but this time we assume that the channel varies during the transmission of a single codeword, namely, channel transition probabilities at time  $i$  are  $W_i(b|0)$  and  $W_i(b|1)$ ,  $b \in \mathcal{Y}$ . When codeword  $\mathbf{x} \in \{0, 1\}^n$  has been transmitted, the probability that the maximum likelihood (ML) detector finds codeword  $\mathbf{x}' \in \{0, 1\}^n$  more likely can be bounded as follows:

$$P_e(\mathbf{x}, \mathbf{x}') \leq \sum_{\mathbf{y} \in \mathcal{Y}^n} \sqrt{W^n(\mathbf{y}|\mathbf{x})W^n(\mathbf{y}|\mathbf{x}')},$$

where we denote

$$W^n(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^n W_i(y_i|x_i).$$

It is easy to see that

$$(2.9) \quad \begin{aligned} P_e(\mathbf{x}, \mathbf{x}') &\leq \sum_{\mathbf{y} \in \mathcal{Y}^n} \sqrt{W^n(\mathbf{y}|\mathbf{x})W^n(\mathbf{y}|\mathbf{x}')} \\ &= \prod_{i=1}^n \left( \sum_{b \in \mathcal{Y}} \sqrt{W_i(b|x_i)W_i(b|x'_i)} \right). \end{aligned}$$

Note that when  $x_i = x'_i$ , the corresponding factor  $\sum_{b \in \mathcal{Y}} \sqrt{W_i(b|x_i)W_i(b|x'_i)}$  in the product (2.9) equals 1 and can be omitted. When  $x_i \neq x'_i$ , the corresponding factor  $\sum_{b \in \mathcal{Y}} \sqrt{W_i(b|x_i)W_i(b|x'_i)}$  equals to the Bhattacharyya noise parameter  $\gamma_i$  of the channel at time  $i$ -th:

$$\gamma_i = \sum_{b \in \mathcal{Y}} \sqrt{W_i(b|0)W_i(b|1)}$$

Therefore, the bound (2.9) can be written as

$$(2.10) \quad P_e(\mathbf{x}, \mathbf{x}') \leq \prod_{i: x_i \neq x'_i} \gamma_i.$$

Note that when all  $\gamma_i$  have the same value  $\gamma$  (time-invariant channel case), the above bound reduces to the well known  $\gamma^h$  bound (2.1), where  $h$  is the Hamming distance between  $\mathbf{x}$  and  $\mathbf{x}'$ .

We now assume that the codewords of the mother code are transmitted in  $m$  transmissions, and the decoding is performed after the last transmission has been received. This will help us to later analyze an IR HARQ protocol with at most  $m$  transmissions. Let  $\mathcal{I} = \{1, \dots, n\}$  denote the set indexing the bit positions in a codeword. For the  $m$  transmissions, set  $\mathcal{I}$  is partitioned in  $m$  subsets  $\mathcal{I}(j)$ ,

for  $1 \leq j \leq m$ . During the  $j$ -th transmission, only bits at positions  $i$  where  $i \in \mathcal{I}(j)$  are transmitted. We assume that the channel is slowly time-varying, namely that  $W_i(y|0)$  and  $W_i(y|1)$  remain constant for all bits at positions  $i$  taking part in the same transmission. Consequently, the Bhattacharyya noise parameter for transmission  $j$  depends only on  $j$ :

$$\gamma_i = \gamma(j) \text{ for all } i \in \mathcal{I}(j).$$

Let  $h_j = d_H(\mathbf{x}, \mathbf{x}', \mathcal{I}(j))$  denote the Hamming distance between sequences  $\mathbf{x}$  and  $\mathbf{x}'$  over the index set  $\mathcal{I}(j)$ . The bound (2.10) can be written as

$$P_e(\mathbf{x}, \mathbf{x}') \leq \prod_{j=1}^m \gamma(j)^{h_j}, \quad h_j = d_H(\mathbf{x}, \mathbf{x}', \mathcal{I}(j)).$$

In the case of only two transmissions, we have

$$P_e(\mathbf{x}, \mathbf{x}') \leq \gamma(1)^{d_H(\mathbf{x}, \mathbf{x}', \mathcal{I}(1))} \cdot \gamma(2)^{d_H(\mathbf{x}, \mathbf{x}', \mathcal{I}(2))} = \gamma(1)^{h_1} \gamma(2)^{h-h_1},$$

where  $h$  is the Hamming distance between  $\mathbf{x}$  and  $\mathbf{x}'$ .

Let  $A_{h_1 \dots h_m}$  denote the number of codewords with weight  $h_j$  over the index set  $\mathcal{I}(j)$ , for  $1 \leq j \leq m$ . The union bound on the ML decoder word error probability is given by

$$(2.11) \quad P_W \leq \sum_{h_1=1}^{|\mathcal{I}(1)|} \cdots \sum_{h_m=1}^{|\mathcal{I}(m)|} A_{h_1 \dots h_m} \prod_{j=1}^m \gamma(j)^{h_j}.$$

Further direct analysis of this expression seems formidable, even in the case of only two transmissions for which we have

$$\begin{aligned} P_W &\leq \sum_{h_1=1}^{|\mathcal{I}(1)|} \sum_{h_2=1}^{|\mathcal{I}(2)|} A_{h_1 h_2} \gamma(1)^{h_1} \gamma(2)^{h_2} \\ &= \sum_{h=1}^n \sum_{h_1=1}^{|\mathcal{I}(1)|} A_{h_1 h-h_1} \gamma(1)^{h_1} \gamma(2)^{h-h_1}. \end{aligned}$$

We thus resort to finding the expected performance over all possible transmission assignments where a bit of a mother code is assigned to transmission  $j$  with probability  $\alpha_j$ ,  $\alpha_j > 0$ ,  $\sum_j \alpha_j = 1$ . The expected (and asymptotic as  $n \rightarrow \infty$ ) number of bits assigned to transmission  $j$  equals to  $\alpha_j n$ . In the place of (2.11), we derive an expression for the expected word error probability to which the results of [19] can be directly applied.

**2.3. Random Transmission Assignment.** We will assume that a bit of a mother code is assigned to transmission  $j$  with probability  $\alpha_j$ ,  $\alpha_j > 0$ ,  $\sum_j \alpha_j = 1$ . Such scheme can actually be implemented as follows:

- (1) For each bit position  $i$ ,  $i = 1, 2, \dots, n$ , generate a number  $\theta_i$  independently and uniformly at random over  $[0, 1)$ .
- (2) Compute  $m$  numbers  $\lambda_j$  as follows:

$$\lambda_j = 1 - \sum_{i=1}^j \alpha_i \text{ for } 1 \leq j \leq m.$$

Note that  $0 = \lambda_m < \lambda_{m-1} < \dots < \lambda_2 < \lambda_1 < 1$ .

- (3) Make the transmission assignment for each bit  $i$ ,  $i = 1, 2, \dots, n$ , as follows:

- (a) if  $\theta_i \geq \lambda_1$ , assign bit  $i$  to transmission 1, otherwise
- (b) if  $\lambda_j \leq \theta_i < \lambda_{j+1}$ , for some  $j$  s.t.  $2 \leq j \leq m$ , assign bit  $i$  to transmission  $j$ .

We are interested in the expected performance of the mother code under the above probabilistic model. If each bit of a codeword with Hamming weight  $h$  is randomly assigned to transmission  $j$  with probability  $\alpha_j$ , then the probability that the sub-word corresponding to the  $j$ -th transmission has weight  $h_j$  for  $1 \leq j \leq m$  is given by

$$(2.12) \quad \binom{h}{h_1} \binom{h-h_1}{h_2} \cdots \binom{h-h_1-\cdots-h_{m-1}}{h_m} \alpha_1^{h_1} \alpha_2^{h_2} \cdots \alpha_m^{h_m}.$$

Therefore, for a given codeword with Hamming weight  $h$ , the expected value of  $A_{h_1, \dots, h_m}$  is given by

$$\bar{A}_{h_1, \dots, h_m} = A_h \binom{h}{h_1} \binom{h-h_1}{h_2} \cdots \binom{h-h_1-\cdots-h_{m-1}}{h_m} \alpha_1^{h_1} \alpha_2^{h_2} \cdots \alpha_m^{h_m},$$

and consequently, the expected value of the union bound (2.11) is

$$\begin{aligned} \bar{P}_W &\leq \sum_{h_i \geq 0; \sum h_i \leq n} \bar{A}_{h_1, \dots, h_m} \gamma(1)^{h_1} \gamma(2)^{h_2} \cdots \gamma(m)^{h_m} \\ &= \sum_h A_h \left\{ \sum_{\sum h_i = h} \binom{h}{h_1} \binom{h-h_1}{h_2} \cdots \binom{h-h_1-\cdots-h_{m-1}}{h_m} \prod_{j=1}^m (\gamma(j) \alpha_j)^{h_j} \right\} \\ &= \sum_h A_h \left( \sum_{j=1}^m \gamma(j) \alpha_j \right)^h. \end{aligned}$$

We define the average Bhattacharyya noise parameter seen by the mother code as

$$(2.13) \quad \bar{\gamma} = \sum_{j=1}^m \gamma(j) \alpha_j.$$

Then, we have

$$(2.14) \quad \bar{P}_W^{[C]} \leq \sum_{h=1}^n \bar{A}_h^{[C](n)} \bar{\gamma}^h.$$

Following the results of [19], if

$$(2.15) \quad \bar{\gamma} < \exp(-c_0^{[C]}),$$

we have  $\lim_{n \rightarrow \infty} \bar{P}_W^{[C](n)} = 0$ .

### 3. A Random Transmission Assignment IR HARQ Scheme

We consider an IR HARQ scheme with at most  $m$  transmissions where a bit is assigned to transmission  $j$  with probability  $\alpha_j$ . Transmission  $j$  takes place if transmission  $j-1$  fails. The rates  $\alpha_j$  may be predetermined (*e.g.*, specified by a standard) or determined based on current network conditions. In both cases, we are interested in evaluating performance after  $j$  transmissions,  $1 \leq j \leq m$ . In the latter case, we are interested in determining  $\alpha_j$  to achieve some required performance.

The random transmission assignment for an incremental redundancy HARQ protocol can be implemented by an "on-the-fly" dynamic version of the algorithm described in the beginning of Sec. 2.3:

**Before the IR HARQ protocol starts:**

- (1) For each bit position  $i$ ,  $i = 1, 2, \dots, n$ , generate a number  $\theta_i$  independently and uniformly at random over  $[0, 1)$ .
- (2) Compute  $\lambda_1$  as  $\lambda_1 = 1 - \alpha_1$ .
- (3) If  $\theta_i \geq \lambda_1$ , assign bit  $i$  to transmission 1.

**If transmission  $j - 1$  fails for  $2 \leq j < m$ :**

- (1) Determine  $\alpha_j$ , if it was not predetermined.
- (2) Compute  $\lambda_j$  as  $\lambda_j = \lambda_{j-1} - \alpha_j$ .
- (3) If  $\lambda_j \leq \theta_i < \lambda_{j-1}$ , assign bit  $i$  to transmission  $j$ .

**If transmission  $m - 1$  fails:**

transmit all remaining bits.

We denote by  $\bar{\gamma}(j)$  the average Bhattacharyya noise parameter seen by the mother code after  $j$ -th channel transmission. For  $j < m$ , not all bits of the mother code are transmitted. A bit is punctured (*i.e.*, not transmitted) with probability  $(1 - \alpha_1 - \dots - \alpha_j)$ . To be able to use the results of Sec. 2.3, we will assume that the punctured bits are transmitted over a really bad channel, *i.e.*, a channel with  $\gamma(j + 1) = 1$ . This assumption allows us to compute  $\bar{\gamma}(j)$  as

$$\bar{\gamma}(j) = \alpha_1 \cdot \gamma(1) + \dots + \alpha_j \cdot \gamma(j) + (1 - \alpha_1 - \dots - \alpha_j) \cdot 1.$$

This equation holds for  $1 \leq j < m$  and for  $j = m$  as well since the last term becomes equal to 0. Note that  $\gamma(m) = \bar{\gamma}$ , where  $\bar{\gamma}$  is defined by (2.13).

In this paper, by evaluating performance after transmission  $j$ , we will mean computing  $\bar{\gamma}(j)$ . Recall that the knowledge of  $\bar{\gamma}(j)$  allows only the computation of upper bounds on the average error probability of the turbo code ensembles. However, it has been conjectured in [19], based on experimental evidence, that those upper bounds are asymptotically close to the best possible, and that probability of error of an ensemble exhibits a threshold behavior, *i.e.*, either approaches zero or goes to 1 as  $n \rightarrow \infty$ . For the same reasons, we will require that  $\bar{\gamma}(j) < \exp(-c_0^{[C]})$ , which in turn implies  $\lim_{n \rightarrow \infty} \bar{P}_W^{[C](n)} = 0$ .

**3.1. First Transmission and Punctured Turbo Code Ensembles.** If a bit is assigned to the first transmission with probability  $\alpha_1$ , the transmitter receives on the average  $\alpha_1 \cdot n$  bits of the mother code over the channel with Bhattacharyya noise parameter  $\gamma(1)$ . Parameter  $\gamma(1)$  is determined by the current channel model and level of noise, which depend on the speed of the mobile and other conditions in a wireless network, and by the signal power which is chosen by the transmitter. The remaining  $(1 - \alpha_1) \cdot n$  bits of the mother code are not transmitted. As mentioned above, we can equivalently assume that they are transmitted over a really bad channel, *i.e.*, a channel with  $\gamma(2) = 1$ . This assumption allows us to compute  $\bar{\gamma}(1)$ , the average Bhattacharyya noise parameter after the first transmission, as

$$\bar{\gamma}(1) = \alpha_1 \cdot \gamma(1) + (1 - \alpha_1) \cdot 1.$$

Our goal is to guarantee  $\lim_{n \rightarrow \infty} \bar{P}_W^{[C](n)} = 0$ , and we have seen that can be done by choosing  $\alpha_1$  or  $\gamma(1)$  or both so that  $\bar{\gamma}(1) < \exp(-c_0^{[C]})$ , *i.e.*,

$$(3.1) \quad \alpha_1 \cdot \gamma(1) + (1 - \alpha_1) \cdot 1 < \exp(-c_0^{[C]}).$$

Condition (3.1) can be written in a form which clearly shows the tradeoff between the rate of the first transmission code and the signal power:

$$\alpha_1(1 - \gamma(1)) > 1 - \exp(-c_0^{[C]})$$

To satisfy this lower bound on the product of  $\alpha_1$  and  $1 - \gamma(1)$ , the transmitter can either increase the code redundancy  $\alpha_1$  or increase the signal power which results in a decrease of  $\gamma(1)$  and increase of  $1 - \gamma(1)$ . An increase in redundancy results in the lower throughput of the user while an increase in the power results in a higher interference level experienced by other users in the network. Since  $\gamma(1)$  is positive, there is a minimal redundancy requirement:

$$(3.2) \quad \alpha_1 > 1 - \exp(-c_0^{[C]}).$$

In the case of predetermined  $\alpha_1$ , the required signal power is specified by

$$(3.3) \quad \gamma(1) < \frac{\exp(-c_0^{[C]}) - (1 - \alpha_1)}{\alpha_1}.$$

It is interesting to compare the above results with those obtained by the authors in [20], where randomly punctured turbo codes were studied. We briefly present the connection. The general structure of a punctured turbo code is shown in Fig. 2. The puncturing device punctures each codeword symbol independently

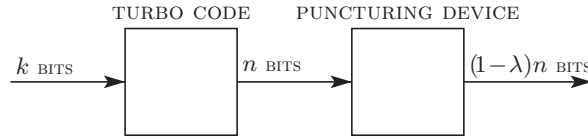


FIGURE 2. Punctured turbo codes. The puncturing device punctures each bit independently with probability  $\lambda$ . The expected number of remaining bits equals  $(1 - \lambda)n$ .

with probability  $\lambda$ . Therefore, if an  $(n, k)$  rate  $R$  code is punctured, the expected number of punctured bits equals  $\lambda n$  and the resulting code expected rate, as well as the asymptotic rate when  $n \rightarrow \infty$ , equals  $R/(1 - \lambda)$ . For each mother code in the set  $\mathcal{C}^{(n)}$ , we will have a set of  $2^n$  punctured codes corresponding to the  $2^n$  possible puncturing patterns. Each code in this set appears with the probability determined by the puncturing pattern. The set of codes obtained by puncturing all codes in the set  $\mathcal{C}^{(n)}$  is denoted  $\mathcal{C}_P^{(n)}$ . By the punctured code turbo code *ensemble*  $[\mathcal{C}_P]$ , we will mean a *sequence* of randomly punctured turbo code sets  $\{\mathcal{C}_P^{(n)}\}$ .

In [20], we analyzed the expected error rate performance of the punctured codes by bounding their expected weight enumerators. If a codeword of weight  $d$  enters a puncturing device, the codeword at the output will have a weight  $h$  with probability  $\binom{d}{h}(1 - \lambda)^h \lambda^{d-h}$ . Since on the average  $\bar{A}_d^{[C](n)}$  codewords of weight  $d$  enter the puncturing device, the expected number of punctured codewords of weight  $h$  is given by

$$(3.4) \quad \bar{A}_h^{[C_P](n)} = \sum_{d \geq h} \bar{A}_d^{[C](n)} \binom{d}{h} \lambda^{d-h} (1 - \lambda)^h.$$

By applying the bound (2.6) on  $\overline{A}_h^{[C_P](n)}$  in the above expression and using the Chernoff bound, we showed in [20] that if the codeword symbols of a turbo code ensemble  $[C]$  with a finite noise threshold  $c_0^{[C]}$  are punctured with probability  $\lambda$  satisfying

$$(3.5) \quad \log \lambda < -c_0^{[C]},$$

then there exists a finite *noise threshold for the punctured code ensemble*  $c_0^{[C_P]}$  such that

$$(3.6) \quad c_0^{[C_P]} \leq \log \left[ \frac{1 - \lambda}{\exp(-c_0^{[C]}) - \lambda} \right]$$

and

$$(3.7) \quad \overline{A}_h^{[C_P](n)} \leq_n \exp(hc_0^{[C_P]}), \quad D_n < h \leq n.$$

Similarly as in [19], by using the inequality (3.7), in the bound (2.3), it can be shown (see [20] for details) that for a turbo code ensemble  $[C]$  with  $L \geq 2$  component encoders, punctured at rate  $\lambda$  and a binary-input memoryless channel whose Bhattacharyya parameter satisfies

$$(3.8) \quad \gamma < \exp(-c_0^{[C_P]}),$$

we have

$$(3.9) \quad \lim_{n \rightarrow \infty} \overline{P}_W^{[C_P](n)} = 0.$$

Here  $\lambda$  satisfies (3.5) and  $c_0^{[C_P]}$  satisfies (3.6). Now, we can compare these results with those obtained by the random transmission assignment approach in the beginning of the section. Note that since  $\lambda = 1 - \alpha_1$ , condition (3.2) is equivalent to (3.5), and condition (3.8) together with (3.6) is equivalent to (3.3).

**3.2. Subsequent Transmissions.** We now assume that the decoding after transmission  $j - 1$  failed. On the average  $n\alpha_j$  bits will participate in the  $j$ -th transmission, and the remaining  $(1 - \alpha_1 - \dots - \alpha_j) \cdot n$  bits of the mother code will not be transmitted. We again assume that they are transmitted over a really bad channel, *i.e.*, a channel with  $\gamma(j + 1) = 1$ , and compute  $\overline{\gamma}(j)$ , the average Bhattacharyya noise parameter after the  $j$ -th transmission, as

$$\overline{\gamma}(j) = \alpha_1 \cdot \gamma(1) + \dots + \alpha_j \cdot \gamma(j) + (1 - \alpha_1 - \dots - \alpha_j) \cdot 1.$$

Again, our goal is to guarantee  $\lim_{n \rightarrow \infty} \overline{P}_W^{[C](n)} = 0$ , and we have seen that can be done by choosing  $\alpha_j$  or  $\gamma(j)$  or both so that

$$(3.10) \quad \overline{\gamma}(j) < \exp(-c_0^{[C]}).$$

Condition (3.10) can be written in a form which clearly show the tradeoff between the rate of the  $j$ -th transmission code and the signal power:

$$\alpha_j(1 - \gamma(j)) > 1 - \exp(-c_0^{[C]}) - \sum_{i=1}^{j-1} \alpha(i)(1 - \gamma(i)).$$

Similar conclusions as those of the analysis for the first transmission hold: to satisfy the above lower bound on the product of  $\alpha_j$  and  $1 - \gamma(j)$ , the transmitter can either increase the code redundancy  $\alpha_1$  or increase the signal power which results in a decrease of  $\gamma(j)$  and increase of  $1 - \gamma(j)$ . An increase in redundancy results in

the lower throughput of the user while an increase in the power results in a higher interference level experienced by other users in the network. Since  $\gamma(j)$  is positive, there is a minimum redundancy requirement:

$$\alpha_j > 1 - \exp(-c_0^{[C]}) - \sum_{i=1}^{j-1} \alpha(i)(1 - \gamma(i)).$$

In the case of predetermined  $\alpha_j$ , the required signal power is specified by

$$\gamma(j) < \frac{\exp(-c_0^{[C]}) - (1 - \alpha_j) - \sum_{i=1}^{j-1} \alpha(i)(1 - \gamma(i))}{\alpha_j}.$$

#### 4. An Example

Here we study an example  $R = 1/3$  turbo code specified by the 1xEV-DV wireless standard as a mother code. The turbo encoder consists of  $L = 2$  recursive convolutional encoders with rates  $R_1 = 1/2$  and  $R_2 = 1$  connected in parallel through an “S-random” interleaver. The component code transfer functions are

$$G_1(D) = \left[ 1, \frac{1+D+D^3}{1+D^2+D^3} \right] \text{ and } G_2(D) = \left[ \frac{1+D+D^3}{1+D^2+D^3} \right].$$

For an IR HARQ scheme, the bits of the mother code are assigned randomly to 15 transmissions so that the rates of the resulting punctured codes are

$$1, 0.95, 0.9, 0.85, \dots, 0.4, 0.35, \frac{1}{3}.$$

Note that the rate of the punctured mother code after transmission  $j$  is given by  $R_p(j) = R / \sum_{i=1}^j \alpha_i$ . Thus we can compute transmission rates  $\alpha_i$  corresponding to the above set of punctured code rates. The assignment of bits to transmissions is implemented as described in Sec. 3.

We determined the throughput and the average FER performance of this IR HARQ scheme by simulation. The simulations were done for the standard interleaver lengths of  $k = 384$  and  $3840$ , and BPSK over an AWGN channel for a range of  $E_s/N_0$ . The average throughput performance for this HARQ scheme is plotted in Fig. 3. We can see that, for a range of puncturing rates, random puncturing delivered codes whose performance ensured that the throughput be in the region between the cutoff rate and the capacity. After a certain rate the performance of the punctured codes leaves this region. The analysis presented in this paper shows how to estimate this point. We next show that for this particular example, the estimate is in a very good agreement with the numerical value observed in the simulation.

We have computed the average weight enumerator  $\overline{A}_d^{[C](n)}$  of the mother codes in the set  $\mathcal{C}^{(n)}$  by applying the technique of [27] for code length  $n = 1152$  and code rate  $R = 1/3$ , and obtained

$$c_0^{[C](n)} = \max_{D_n < d \leq n} \frac{\log \overline{A}_d^{[C](n)}}{d} = 0.5198.$$

Since  $c_0^{[C]} >_n c_0^{[C](n)}$ , the minimal redundancy requirement (3.2) for the first transmission gives

$$(4.1) \quad \alpha_1 > 0.4054 \text{ and thus } R_p = \frac{R}{\alpha_1} < 0.822.$$

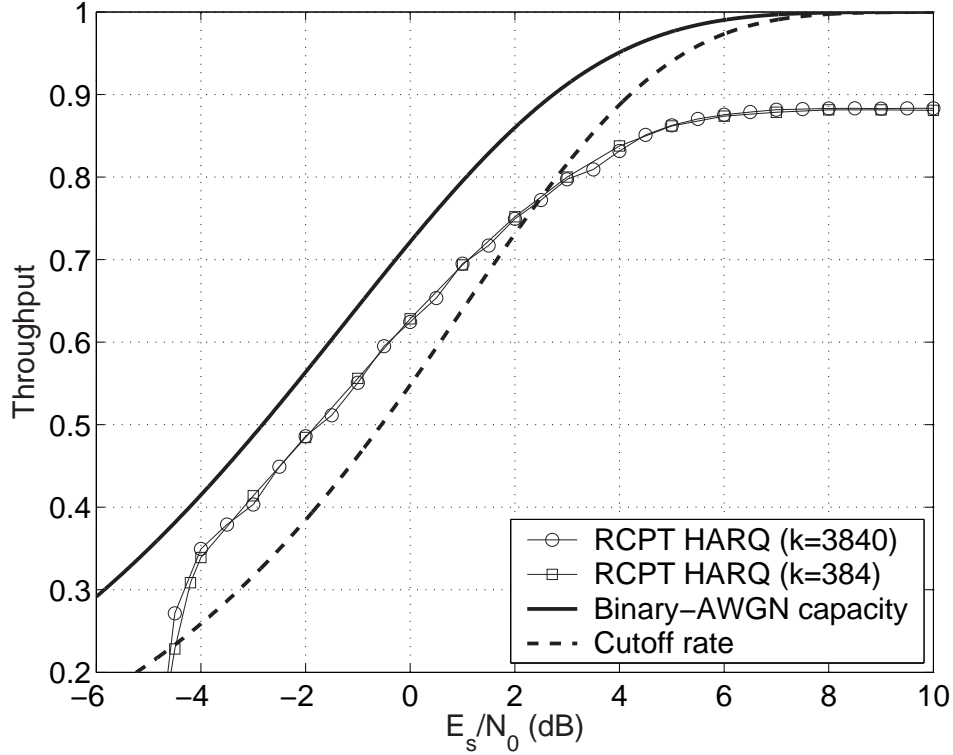


FIGURE 3. Average throughput performance of an BPSK IR HARQ scheme on AWGN channel for two interleaver lengths.

Although our results give only a necessary condition on the minimal  $\alpha_1$ , the simulation results of Fig. 3 show that an abrupt loss in performance occurs roughly for the code rate of 0.82.

Fig. 4 compares the average FER performance of the punctured turbo codes with rates 0.7, 0.8 and 0.9 and interleaver lengths  $k = 384$  and 3840. Condition (3.2) on the minimal punctured code redundancy,  $\alpha_1 > 1 - \exp(-c_0^{[C]})$ , implies that the punctured code rates below  $R/\alpha_1 < 0.82$  are necessary to guarantee  $\bar{\gamma}(1) < \exp(-c_0^{[C]})$ . From Fig. 4, we see that when rates are 0.7 and 0.8, the FER of the  $k = 3840$  code drops noticeably faster to  $10^{-3}$  than that of the  $k = 384$  code. We also see that, when the rate is 0.9, the FERs of both the  $k = 384$  and the  $k = 3840$  codes do not go below 0.1.

## 5. Conclusions

Because of their good performance, HARQ schemes are included in current proposals for wireless data/voice networks. A number of important issues such as error rate performance after each transmission and rate and power control are difficult to analyze in a network employing a particular HARQ scheme, *i.e.*, a given mother code and given selection of bits for each transmission. We here addressed two issues motivated directly by questions in practice. The first was to evaluate the

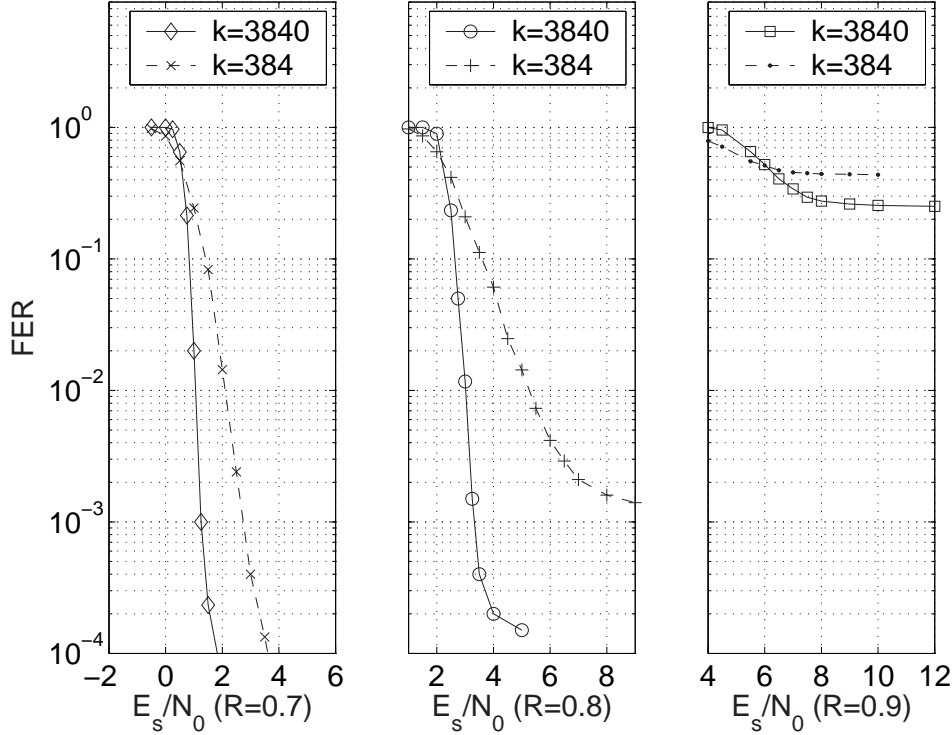


FIGURE 4. Average FER performance of punctured turbo codes on an AWGN channel for two interleaver lengths  $k = 384$  and  $3840$ . The mother turbo code has rate  $1/3$ , and the punctured codes have rates  $0.7$ ,  $0.8$ , and  $0.9$ .

error rate performance after each transmission, which is equivalent to evaluating performance of punctured turbo codes on time varying channels. The second was to show how one should go about choosing the signal power and the number of bits for transmission  $j$  after a failed transmission  $j - 1$ . To approach and solve both problems, we introduced the idea of random transmission assignments of the mother code bits. Although theoretical results we obtained are mainly upper bounds and necessary conditions, simulation results show that they can be very useful tools for predicting and evaluating performance of practical schemes.

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