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On the non-trivial dynamics of complex networks

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Abstract

Some phenomena are characterized by a non-trivial network dynamics exhibiting self-organized criticality or discontinuous transitions, coexistence and hysteresis. After a short review, we show that a similar approach suggests that social communities stabilized by network interactions may become unstable if they grow too large.
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Many real systems cannot be fully understood without accounting for their complex network structure. For example, static properties of networks—such as their scale-free nature [1] or the small world property [2]—are not only ubiquitous, but bear dramatic consequences on simple processes—such as epidemics [3]—taking place on them. Networks are relevant also for their dynamic properties.[1] The complexity of many systems arises precisely from the fact that the network of

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[1]Our focus here is not on dynamical processes which generate particular statistical regularities, nor on the processes which take place on the network, but rather on the evolution of the network itself.

interactions itself co-evolves with the system defined on it in a non-trivial way. For example congestion "storms" in the Internet exhibit intermittent behavior [4] whereas financial crises and recessions are often explained in terms of avalanches (the so-called domino effect) [5]. This suggests that the evolution of the network of interaction in these systems is not smooth, but rather has the "punctuated-equilibrium" feature of a self-organized critical (SOC) process. This conclusion has been recently put on a more firm basis by the introduction of a stylized models which allows for a detailed theoretical [9] approach. The model describes a network whose evolution takes place in avalanches of rewiring events. These shape the network which, in the stationary state, acquires statistical properties which can be related to the microscopic dynamics and which include scale-free as well as single-scale networks. The model also features a non-trivial sharp phase transition to the complete network. We refer the interested reader to Ref. [9].

Here we shall focus on a qualitatively different way in which network dynamics can be non-trivial, which is mostly relevant for socio-economic networks. It has been realized that the network in which socio-economic phenomena are "embedded" plays a key role [6,7]. In addition, many phenomena, such as the emergence of economic districts or the outburst of crime [8], for which networks are important evolve in an abrupt, sometimes explosive, manner. In addition, some realities seem to be characterized by a sparse network and some by a dense one, without an evident reason (see e.g. Ref. [8] on crime). Ref. [10] found that a very simple dynamics of a social network in a changing environment reproduces these features: the underlying network evolves in a discontinuous manner with an abrupt transition from a disconnected population state to a strongly interconnected society. In addition, for a range of parameters, the sparse and the dense network phase coexist, and the system exhibits hysteresis. In what follows, we shall first briefly review the findings of Ref. [10] and show that a similar modelling approach gives interesting insights on the stability of communities of finite size.

1. Searching partners in volatile networks

In a well-networked society, individuals are linked through a dense pattern of interaction which results in both high payoffs and a brisk and broad dissemination of information [6]. Think, to fix ideas, to the job contact network. It has been consistently shown by sociologists and economists alike [7] that personal acquaintances or neighborhood effects play a prominent role in the way in which individuals find new job opportunities. Information flow through the network has important consequences in the long run if the underlying environment is volatile. In this case, former choices tend to become obsolete and individuals must swiftly search for new opportunities to offset such a negative trend. This can be seen as a manifestation of the so-called *Red Queen principle* [11]: "... it takes all the running you can do, to keep in the same place". Ref. [10] presents a simple model that

addresses these issues. The model, which is a continuum stochastic process for the state of the network, captures three main processes:

1. *Long distance search*: At rate η, each node i gets the opportunity to establish a link to a node j randomly selected.
2. *Short distance search*: At rate ξ, each node i picks at random one of its neighbors j and j then randomly selects one of its other neighbors $k \neq i$. The link between i and k is formed (if k is not already a neighbor of i). Nothing happens if i has no neighbor or if j has no other neighbor but i.
3. *Link decay*: At rate λ, each existing link decays and it is randomly deleted.

Let us briefly recall the main results of Ref. [10]. The key quantities are the average number of neighbors c and the clustering coefficient q, i.e. the fraction of pairs of neighbors of a node who are also neighbors among themselves. Their typical behavior is shown in Fig. 1.

For $\xi = 0$, the dynamics is very simple and the stationary network is a random graph with average degree $c = 2\eta/\lambda$. For $\eta \ll \lambda$ the network is composed of many disconnected parts. When the local search rate ξ is turned on, local search opportunities are rapidly saturated. Indeed the clustering coefficient q raises quickly to values close to one. Clusters of more than 2 nodes are rare and when they form local search quickly saturates the possibilities of forming new links. Suddenly, at a critical value ξ_2, a giant component connecting a finite fraction of the nodes emerges. The average degree c indeed jumps abruptly at ξ_2. The network becomes more and more densely connected as ξ increases further. But when ξ decreases, we observe that the giant component remains stable also beyond the transition point ($\xi < \xi_2$). Only at
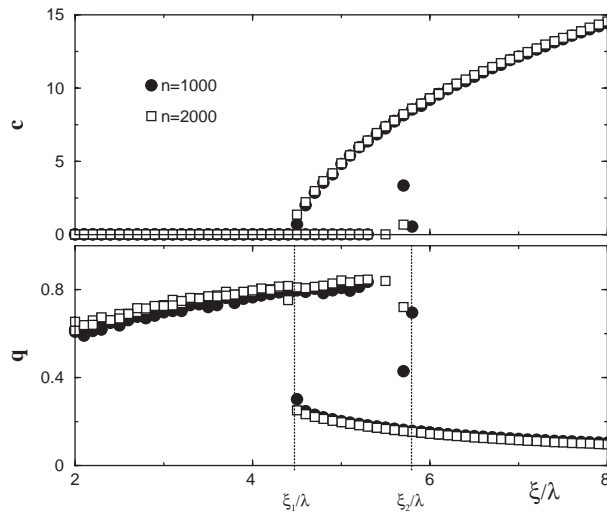

Fig. 1. Average degree c (top) and clustering coefficient q (bottom) from numerical simulations with $\eta/\lambda = 0.01$ for populations of size $n = 1000$ and 2000. Runs were equilibrated for a time $t_{eq} = 3000/\lambda$ before taking averages.

a second point ξ_1 the population reverts to the sparse network phase. There is a whole interval $[\xi_1, \xi_2]$ where both a dense-network phase and one with a nearly empty network coexist. This behavior is typical of first-order phase transitions. Notice that, loosely speaking, ξ is the rate at which "triangles" form and q is the density of "triangles". One would then expect that q increases with ξ, which is indeed what happens in the sparse network phase. Instead, we find that q decreases with ξ in the high-density phase. This, combined to the fact that q is small, is very important because it makes local search very effective in the high-density phase. Furthermore, from a theoretical point of view, Ref. [10] shows that it is possible to reproduce qualitatively the findings of Fig. 1 within a mean field theory only if one accounts for the dynamical behavior of q.

These results suggest that the rise of a vigorous and lively society may be a discontinuous process. Furthermore, it suggests that the appearance of a dense network is related to the emergence of key network features—such as a decreasing clustering with increasing density—which result from its social dynamics. The occurrence of a discontinuous phase transition and hysteresis, suggests that valuable network features (high-density and low clustering) may not necessarily materialize even under favorable conditions while, by contrast, they may display a significant resilience to deteriorating conditions.

2. The fall of large networked communities

Let us now consider a polar situation where agents, rather than using the network to search for new partners, use it to gather information on the reliability of potential new partners. Cooperation between two agents—as modelled by a link of our graph—in many cases requires trust or reliable information on the reputation of the partners. Common neighbors on the graph can provide such information or enforce trust between the parties [12]. We model this type of situation by a networked population of N agents subject to the following two processes:

- Each existing link decays at rate λ.
- Each agent i receives an opportunity to form a link with an agent j, randomly drawn from the population, at a rate $\frac{1}{2}$. The attempt is successful if i and j share a common neighbor. Otherwise it is only successful with probability η.

The model aims at describing informal social network whose cohesion is sustained by a common acquaintance reference system. The key intuition is that, in a large population, the probability that two nodes have a common neighbor is of order $1/N$, if the average degree c is finite.[2] Hence common acquaintance cannot sustain dense networks of many agents. If densely networked communities are possible for some N at all, they are bound to disappear (or fall) as N increases.

We explored the behavior of the model through numerical simulations (see Fig. 2) for $\eta \ll \lambda < 1$. We find a low-density network state with $c \simeq \eta/\lambda$ which coexists, for

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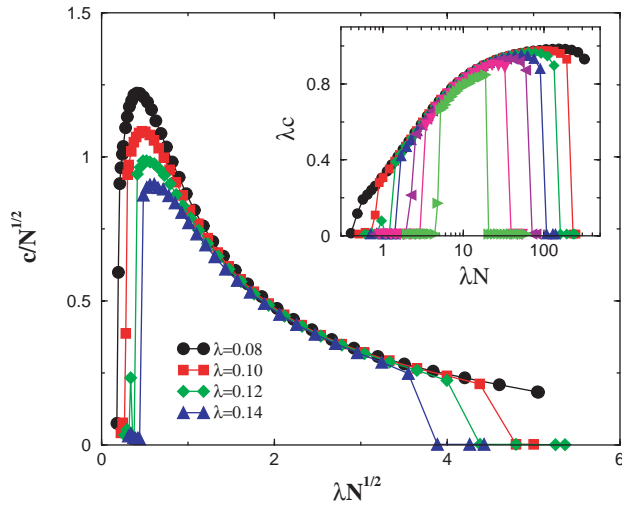[2]This is the case because $c \leqslant 1/\lambda$.

Fig. 2. Average degree c as a function of N for $\lambda = 0.08, 0.1, 0.12, 0.14$ (in the inset also data for $\lambda = 0.16, 0.18, 0.2$ is shown). Simulations were run for a time $t = 10^5$, starting from the complete graph, before taking averages over a similar time interval. The scaling suggested by the random graph theory ($q \approx 0$) and that holding in the high clustering region are shown in the main figure and in the inset, respectively.

intermediate values of N, with a high-density state. Both the "rise" and the "fall" of the high-density network takes place in an abrupt manner.[3]

In order to shed light on this result, let us first approximate the network by a random graph where each link is present with probability c/N, where c is the average degree. Then the transition rates for nodes with k neighbors are

$$w_{k \to k-1} = \lambda k, \qquad w_{k \to k+1} = \eta + (1 - \eta)\left[1 - \left(1 - \frac{c}{N}\right)^k\right], \tag{1}$$

where the second term in $w_{k \to k+1}$ is $1 - \eta$ times the probability that at least one of the neighbors of i has a link to j (which occur with probability c/N). Averaging the link creation and decay rates over the Poisson degree distribution, and then equalizing them we get in the stationary state

$$\lambda c = 1 + (1 - \eta)e^{-c^2/N} . \tag{2}$$

Notice that, setting $x = c/\sqrt{N}$, this becomes $\lambda\sqrt{N}x = 1 - (1 - \eta)e^{-x^2}$. which shows that a dense network dissolves for $N > N_c$ where $N_c \sim \lambda^{-2}$. Indeed a plot of $x = c/\sqrt{N}$ versus $\lambda\sqrt{N}$ shows an approximate collapse of the curves for N large (see Fig. 2).

The agreement of this simple theory with numerical results is only qualitative. In particular it fails when $x = c/\sqrt{N}$ approaches one because then the average number

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[3]The high-density network is only reached if simulations start from a highly connected initial state. This decays to a sparse network when N increases beyond the upper or decreases below the lower critical value.
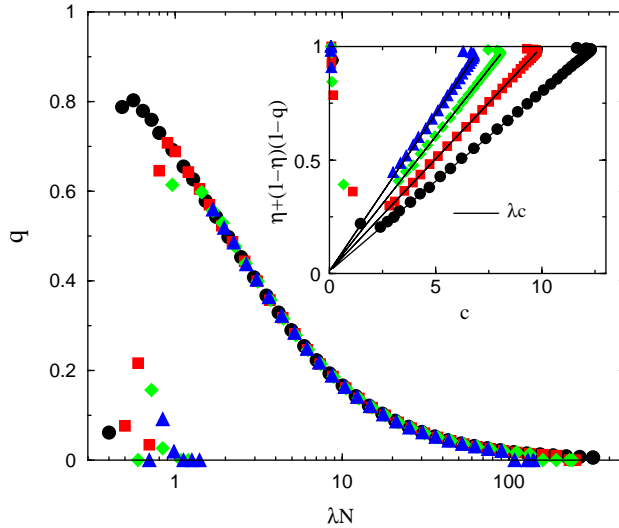
Fig. 3. Clustering coefficient q as a function of λN (same data set of the previous figure). Inset: $\eta + (1 - \eta)(1 - q)$ as a function of c. Lines are the theoretical prediction λc.

of second neighbors of any node—which is c^2 in the present approximation—becomes of order N. This means that any two randomly taken nodes have, with high probability, a common neighbor. Therefore, in this region of N the link decay process is balanced by a creation process which is limited mostly by clustering, i.e. by the fact that new links are not formed simply because they are already in place. This implies

$$\lambda c \cong \eta + (1 - \eta)(1 - q)$$

which indeed holds to a remarkable degree of accuracy (see inset of Fig. 3). Fig. 3 shows that for intermediate values of N the clustering coefficient q is not small. Indeed the random graph theory above is qualitatively correct in the region where q is small (N large), but it fails for smaller N. Furthermore, numerical simulations suggests that q requires a different scaling with λ, i.e. curves collapse (for both q and λc) is obtained against the variable λN.

Summarizing, we have shown that in a simple model of network dynamics, social cohesion can be maintained only in finite size groups. While the behavior for relatively large networks is well described by a random graph theory ($q \approx 0$) the intermediate range requires a strongly correlated network theory. This, in particular, seems to be characterized by a different scaling limit.

Acknowledgements

References

[1] R. Albert, A.-L. Barabási, Rev. Mod. Phys. 74 (2002) 47;
    S.N. Dorogovtsev, J.F.F. Mendes, Adv. Phys. 51 (2002) 1079.
[2] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440–442.
[3] R. Pastor-Satorras, A. Vespignani, Phys. Rev. Lett. 86 (2001) 3200.
[4] M. Magnasco, preprint nlin.AO/0010051, 2000.
[5] A. Aleksejuk, J.A. Holyst, G. Lossinets, Int. J. Mod. Phys. C 13 (2002) 333.
[6] R. Putnam, Making Democracy Work, Princeton University Press, Princeton, NJ, 1993.
[7] M. Granovetter, Getting a Job: A Study of Contacts and Careers, Chicago University Press, Chicago,
    1995;
    G. Topa, Rev. Econ. Stud. 68 (2001) 261.
[8] J. Crane, Am. J. Soc. 96 (1991) 1226.
[9] G. Bianconi, M. Marsili, Phys. Rev. E 70 (2004) 035105.
[10] M. Marsili, F. Slanina, F. Vega-Redondo, Proc. Nat. Acad. Sci. 101 (2004) 1439.
[11] L. Van Valen, Evol. Theory 1 (1973) 1.
[12] F. Vega-Redondo, Building up social capital in a changing world, WP-AD 2002–26, Instituto
    Valenciano de Investigaciones Económicas, 2002.