

The use of homotopy methods for solving nonlinear foam drainage equation

Sh. Sadigh Behzadi*

Department of Mathematics, Islamic Azad University, Qazvin Branch, Qazvin, Iran.

Copyright 2014 © Sh. Sadigh Behzadi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Foaming occurs in many distillation and absorption processes. The drainage of liquid foams involves the interplay of gravity, surface tension, and viscous forces. In this paper, the nonlinear foam drainage equation is solved by using the Adomian's decomposition method, modified Adomian's decomposition method, variational iteration method, modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method. The existence and uniqueness of the solution and convergence of the proposed methods are proved in details. Finally an example shows the accuracy of these methods.

Keywords: Nonlinear foam drainage equation, Adomian decomposition method (ADM), Modified Adomian decomposition method (MADM), Variational iteration method (VIM), Modified variational iteration method (MVIM), Homotopy perturbation method (HPM), Modified homotopy perturbation method (MHPM), Homotopy analysis method (HAM).

1 Introduction

Foams are of great importance in many technological processes and applications, and their properties are subject of intensive studies from both practical and scientific points of view [1]. Liquid foam is an example of soft matter (or complex fluid) with a very well defined structure that first clearly described by Joseph plateau in the 19th century. Weaire et al. [2] showed in their work simple answers to many such questions exist, but no going experiments continue to challenge our understanding. Foams and emulsions are well known to scientists and the general public alike because of their everyday occurrence [3, 4]. Foams are common in foods and personal care products such as creams and lotions, and foams often occur, even when not desired, during cleaning (clothes, dishes, scrubbing) and dispensing processes [5]. They have important applications in the food and chemical industries, firefighting, mineral processing, and structural material science [6]. Less obviously, they appear in acoustic cladding, lightweight mechanical components, and impact absorbing parts on cars, heat exchangers, and textured wallpapers (incorporated as foaming inks) and even have an analogy in cosmology. The packing of bubbles or cells can form both random and symmetrical arrays, such as sea foam and bees honeycomb. History connects foams with a number of eminent scientists, and foams continue to excite imaginations [7]. There are now many applications of polymeric foams [8] and more recently metallic foams, which are foams made of metals such as aluminum [9]. Some commonly mentioned applications include the use of foams for reducing the impact of explosions and for cleaning up oil spills. In addition, industrial applications of polymeric foams and porous metals include their use for structural purposes and as heat

*Corresponding author. Email address: shadan.bhezadi@yahoo.com.

exchange media analogous to common finned structures [10]. Polymeric foams are used in cushions and packing and structural materials [11]. Glass, ceramic, and metal foams [12] can also be made and find an increasing number of new applications. In addition, mineral processing utilizes foam to separate valuable products by flotation. Finally, foams enter geophysical studies of the mechanics of volcanic eruptions [5]. Recent research in foams and emulsions has centered on three topics which are often treated separately but are, in fact, interdependent: drainage, coarsening, and rheology, see Figure 1. We focus here on a quantitative description of the coupling of drainage and coarsening. Foam drainage is the flow of liquid through channels (plateau borders) and nodes (intersections of four channels) between the bubbles, driven by gravity and capillarity [13, 14, 15]. During foam production, the material is in the liquid state, and fluid can rearrange while the bubble structure stays relatively unchanged. The flow of liquid relative to the bubbles is called drainage. Generally, drainage is driven by gravity and/or capillary (surface tension) forces and is resisted by viscous forces [5]. Because of their limited time stability and despite the numerous studies reported in the literature, many of their properties are still not well understood, in particular the drainage of the liquid in between the bubbles under the influence of gravity [16, 17]. Drainage plays an important role in foam stability. Indeed, when foam dries, its structure becomes more fragile; the liquid films between adjacent bubbles being thinner, then can break, leading to foam collapse. In the case of aqueous foams, surfactant is added into water, and it adsorbs at the surface of the films, protecting them against rupture [18]. Most of the basic rules that explain the stability of liquid gas foams were introduced over 100 years ago by the Belgian Joseph Plateau who was blind before he completed his important book on the subject. This modern-day book by Weaire and Hutzler provides valuable summaries of Plateau's work on the laws of equilibrium of soap films, and it is especially useful since the original 1873 French text does not appear to be in a fully translated English version. Weaire and Hutzler note that Sir W. Thompson (Lord Kelvin) was stimulated by Plateau's book to examine the optimum packing of free space by regular geometrical cells. His solution to the problem remained the best until quite recently. Why does this area of theoretical research, still active today, have connections with the apparently frivolous theme of bubbles? It is because the packing of free space involves the minimization of the surface energy of the structure (i.e., least amount of boundary material). Thus, one might ask why such an often-observed medium as a foam has not provided the optimum solution to this problem much earlier, perhaps, this shows that observation is often biased towards what one expects to see, rather than to the unexpected. Also, in nature, there are packing problems, such as the bees' honeycomb. Its shaped ends provide a nice example of Plateau's rules in a natural environment [7]. Recent theoretical studies by Verbist and Weaire describe the main features of both free drainage [19, 20], where liquid drains out of a foam due to gravity, and forced drainage [21], where liquid is introduced to the top of a column of foam. In the latter case, a solitary wave of constant velocity is generated when liquid is added at a constant rate [22]. Forced foam drainage may well be the best prototype for certain general phenomena described by nonlinear differential equations, particularly the type of solitary wave which is most familiar in tidal bores.

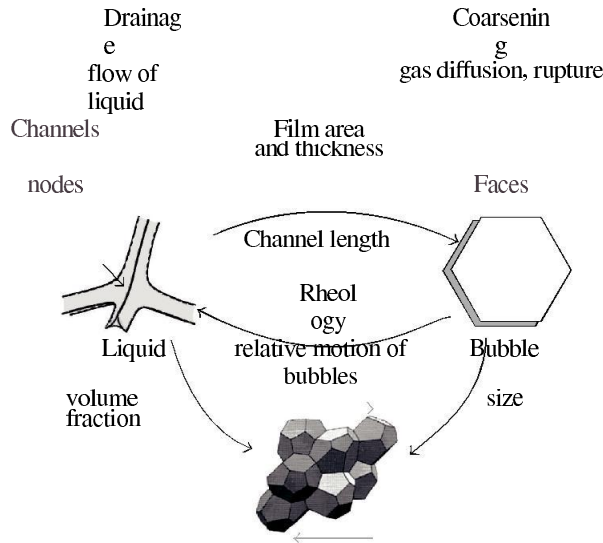


Figure 1: Schematic illustration of the interdependence of drainage, coarsening, and rheology of foams

In this work, we develop the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM to solve this equation as follows [21]:

$$u_t(x,t) + 2u^2(x,t)u_x(x,t) - u_x^2(x,t) - \frac{1}{2}u_{xx}(x,t)u(x,t) = 0. \tag{1.1}$$

With the initial condition:

$$u(x,0) = g(x) = -\sqrt{c} \tanh(\sqrt{c}x), \tag{1.2}$$

where c is the velocity of the wave front.

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1.1). In section 3 we prove the existence , uniqueness of the solution and convergence of the proposed methods. Finally, the numerical example is shown in section 4.

In order to obtain an approximate solution of Eq.(1.1), let us integrate one time Eq.(1.1) with respect to t using the initial condition we obtain,

$$u(x,t) = g(x) - 2 \int_0^t F_1(u(x,\tau)) d\tau + \int_0^t F_2(u(x,\tau)) d\tau + \frac{1}{2} \int_0^t F_3(u(x,\tau)) d\tau, \tag{1.3}$$

where,

$$\begin{aligned} F_1(u(x,t)) &= u^2(x,t)u_x(x,t), \\ F_2(u(x,t)) &= u_x^2(x,t), \\ F_3(u(x,t)) &= u_{xx}(x,t)u(x,t). \end{aligned}$$

In Eq.(1.3), we assume $g(x)$ is bounded for all x in $J = [0, T](T \in \mathbb{R})$.

The terms $F_1(u(x,t))$, $F_2(u(x,t))$ and $F_3(u(x,t))$ are Lipschitz continuous with $| F_1(u) - F_1(u^*) | \leq L_1 | u - u^* |$, $| F_2(u) - F_2(u^*) | \leq L_2 | u - u^* |$ and $| F_3(u) - F_3(u^*) | \leq L_3 | u - u^* |$.

2 The iterative methods

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu + Ru + Nu = f, \tag{2.4}$$

where $u(x, t)$ is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L , Nu represents the nonlinear terms, and f is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(2.4), and using the given conditions we obtain

$$u(x, t) = z(x) - L^{-1}(Ru) - L^{-1}(Nu), \quad (2.5)$$

where the function $z(x)$ represents the terms arising from integrating the source term f . The nonlinear operator $Nu = G_1(u)$ is decomposed as

$$G_1(u) = \sum_{n=0}^{\infty} A_n, \quad (2.6)$$

where A_n , $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \right]_{\lambda=0}. \quad (2.7)$$

The first Adomian polynomials (introduced in [23, 24, 25]) are:

$$\begin{aligned} A_0 &= G_1(u_0), \\ A_1 &= u_1 G_1'(u_0), \\ A_2 &= u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0), \\ A_3 &= u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), \dots \end{aligned} \quad (2.8)$$

2.1.1 Adomian decomposition method

The standard decomposition technique represents the solution of $u(x, t)$ in Eq.(2.4) as the following series,

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \quad (2.9)$$

where, the components u_0, u_1, \dots which can be determined recursively

$$\begin{aligned} u_0(x, t) &= g(x), \\ u_1(x, t) &= -2 \int_0^t A_0(x, t) dt + \int_0^t B_0(x, t) dt + \frac{1}{2} \int_0^t Z_0(x, t) dt, \\ &\vdots \\ u_{n+1}(x, t) &= -2 \int_0^t A_n(x, t) dt + \int_0^t B_n(x, t) dt + \frac{1}{2} \int_0^t Z_n(x, t) dt, \quad n \geq 0. \end{aligned} \quad (2.10)$$

Substituting Eq.(2.8) into Eq.(2.10) leads to the determination of the components of u .

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [26]. The modified forms was established on the assumption that the function $g(x)$ can be divided into two parts, namely $g_1(x)$ and $g_2(x)$. Under this assumption we set

$$g(x) = g_1(x) + g_2(x). \quad (2.11)$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1 . The suggestion was that only the part g_1 be assigned to the zeroth component u_0 , whereas the remaining part g_2 be combined with the other terms given in Eq.(2.11) to define u_1 . Consequently, the modified recursive relation

$$\begin{aligned} u_0 &= g_1(x), \\ u_1 &= g_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0), \\ &\vdots \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1, \end{aligned} \quad (2.12)$$

was developed.

To obtain the approximation solution of Eq.(1.1), according to the MADM, we can write the iterative formula Eq.(2.12) as follows:

$$\begin{aligned} u_0 &= g_1(x), \\ u_1 &= g_2(x) - 2 \int_0^t A_0(x,t) dt + \int_0^t B_0(x,t) dt + \frac{1}{2} \int_0^t Z_0(x,t) dt, \\ &\vdots \\ u_{n+1} &= -2 \int_0^t A_n(x,t) dt + \int_0^t B_n(x,t) dt + \frac{1}{2} \int_0^t Z_n(x,t) dt, \quad n \geq 1. \end{aligned} \quad (2.13)$$

The operators $F_i(u(x,t))$ ($i = 1, 2, 3$) are usually represented by the infinite series of the Adomian polynomials as follows:

$$\begin{aligned} F_1(u) &= \sum_{i=0}^{\infty} A_i, \\ F_2(u) &= \sum_{i=0}^{\infty} B_i, \\ F_3(u) &= \sum_{i=0}^{\infty} Z_i. \end{aligned}$$

where A_i , B_i and Z_i are the Adomian polynomials.

Also, we can use the following formula for the Adomian polynomials [27]:

$$\begin{aligned} A_n &= F_1(s_n) - \sum_{i=0}^{n-1} A_i, \\ B_n &= F_2(s_n) - \sum_{i=0}^{n-1} B_i, \\ Z_n &= F_3(s_n) - \sum_{i=0}^{n-1} Z_i. \end{aligned} \quad (2.14)$$

Where $s_n = \sum_{i=0}^n u_i(x,t)$ is the partial sum.

2.2 Description of the VIM and MVIM

In the VIM [28, 29, 30, 31, 32, 33, 34, 35], it has been considered the following nonlinear differential equation:

$$Lu + Nu = g, \quad (2.15)$$

where L is a linear operator, N is a nonlinear operator and g is a known analytical function. In this case, the functions u_n may be determined recursively by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau) \{L(u_n(x,\tau)) + N(u_n(x,\tau)) - g(x,\tau)\} d\tau, \quad n \geq 0, \quad (2.16)$$

where λ is a general Lagrange multiplier which can be computed using the variational theory. Here the function $u_n(x,\tau)$ is a restricted variations which means $\delta u_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions.

With λ determined, then several approximation $u_n(x, t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \quad (2.17)$$

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converge rapidly to accurate solutions.

To obtain the approximation solution of Eq.(1.1), according to the VIM, we can write Eq.(2.16) as follows:

$$u_{n+1}(x, t) = u_n(x, t) + L_t^{-1}(\lambda[u_n(x, t) - g(x) + 2 \int_0^t F_1(u_n(x, t)) dt - \int_0^t F_2(u_n(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_n(x, t)) dt]), \quad n \geq 0. \quad (2.18)$$

Where,

$$L_t^{-1}(\cdot) = \int_0^t (\cdot) d\tau.$$

To find the optimal λ , we proceed as

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta L_t^{-1}(\lambda[u_n(x, t) - g(x) + 2 \int_0^t F_1(u_n(x, t)) dt - \int_0^t F_2(u_n(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_n(x, t)) dt]). \quad (2.19)$$

From Eq.(2.19), the stationary conditions can be obtained as follows: $\lambda' = 0$ and $1 + \lambda = 0$. Therefore, the Lagrange multipliers can be identified as $\lambda = -1$ and by substituting in Eq.(2.18), the following iteration formula is obtained.

$$\begin{aligned} u_0(x, t) &= g(x), \\ u_{n+1}(x, t) &= u_n(x, t) - L_t^{-1}(u_n(x, t) - g(x) + 2 \int_0^t F_1(u_n(x, t)) dt - \int_0^t F_2(u_n(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_n(x, t)) dt), \quad n \geq 0. \end{aligned} \quad (2.20)$$

To obtain the approximation solution of Eq.(1.1), based on the MVIM [36, 37, 38], we can write the following iteration formula:

$$\begin{aligned} u_0(x, t) &= g(x), \\ u_{n+1}(x, t) &= u_n(x, t) - L_t^{-1}(2 \int_0^t F_1(u_n(x, t) - u(x, t)) dt - \int_0^t F_2(u_n(x, t) - u(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_n(x, t) - u(x, t)) dt), \quad n \geq 0. \end{aligned} \quad (2.21)$$

Eq.(2.20) and Eq.(2.21) will enable us to determine the components $u_n(x, t)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x, t)$ is an unknown function and x is an independent variable. let $u_0(x, t)$ denote an initial guess of the exact solution $u(x, t)$, $h \neq 0$ an auxiliary parameter, $H_1(x, t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x, t)] = 0$ when $s(x, t) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1 - q)L[\phi(x, t; q) - u_0(x, t)] - qhH_1(x, t)N[\phi(x, t; q)] = \hat{H}[\phi(x, t; q); u_0(x, t), H_1(x, t), h, q]. \quad (2.22)$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x, t)$, the auxiliary linear operator L , the non-zero auxiliary parameter h , and the auxiliary function $H_1(x, t)$.

Enforcing the homotopy Eq.(2.22) to be zero, i.e.,

$$\hat{H}[\phi(x, t; q); u_0(x, t), H_1(x, t), h, q] = 0, \quad (2.23)$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH_1(x,t)N[\phi(x,t;q)]. \quad (2.24)$$

When $q = 0$, the zero-order deformation Eq.(2.24) becomes

$$\phi(x;0) = u_0(x,t), \quad (2.25)$$

and when $q = 1$, since $h \neq 0$ and $H_1(x,t) \neq 0$, the zero-order deformation Eq.(2.24) is equivalent to

$$\phi(x,t;1) = u(x,t). \quad (2.26)$$

Thus, according to Eq.(2.25) and Eq.(2.26), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy [39, 40, 41, 42, 43].

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \quad (2.27)$$

where,

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \Big|_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary linear parameter L , the nonzero auxiliary parameter h and the auxiliary function $H_1(x,t)$ be properly chosen so that the power series Eq.(2.27) of $\phi(x,t;q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \quad (2.28)$$

From Eq.(2.27), we can write Eq.(2.24) as follows

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = (1-q)L[\sum_{m=1}^{\infty} u_m(x,t)q^m] = qhH_1(x,t)N[\phi(x,t;q)] \Rightarrow \\ L[\sum_{m=1}^{\infty} u_m(x,t)q^m] - qL[\sum_{m=1}^{\infty} u_m(x,t)q^m] = qhH_1(x,t)N[\phi(x,t;q)] \quad (2.29)$$

By differentiating Eq.(2.29) m times with respect to q , we obtain

$$\{L[\sum_{m=1}^{\infty} u_m(x,t)q^m] - qL[\sum_{m=1}^{\infty} u_m(x,t)q^m]\}^{(m)} = \{qhH_1(x,t)N[\phi(x,t;q)]\}^{(m)} = \\ m! L[u_m(x,t) - u_{m-1}(x,t)] = hH_1(x,t) m \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \Big|_{q=0}.$$

Therefore,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH_1(x,t)\mathfrak{R}_m(u_{m-1}(x,t)), \quad (2.30)$$

where,

$$\mathfrak{R}_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \Big|_{q=0}, \quad (2.31)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Note that the high-order deformation Eq.(3.7) is governing the linear operator L , and the term $\mathfrak{R}_m(u_{m-1}(x,t))$ can be expressed simply by Eq.(2.31) for any nonlinear operator N .

To obtain the approximation solution of Eq.(1.1), according to HAM, let

$$N[u(x,t)] = u(x,t) - g(x) + 2 \int_0^t F_1(u(x,t)) dt - \int_0^t F_2(u(x,t)) dt - \frac{1}{2} \int_0^t u(x,t) dt,$$

so,

$$\Re_m(u_{m-1}(x,t)) = u_{m-1}(x,t) - g(x) + 2 \int_0^t F_1(u_{m-1}(x,t)) dt - \int_0^t F_2(u_{m-1}(x,t)) dt - \frac{1}{2} \int_0^t F_3(u_{m-1}(x,t)) dt. \quad (2.32)$$

Substituting Eq.(2.32) into Eq.(3.7)

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h H_1(x,t) [u_{m-1}(x,t) - g(x) + 2 \int_0^t F_1(u_{m-1}(x,t)) dt - \int_0^t F_2(u_{m-1}(x,t)) dt - \frac{1}{2} \int_0^t F_3(u_{m-1}(x,t)) dt + (1 - \chi_m)g(x)]. \quad (2.33)$$

We take an initial guess $u_0(x,t) = g(x)$, an auxiliary linear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H_1(x,t) = 1$. This is substituted into Eq.(2.33) to give the recurrence relation

$$u_0(x,t) = g(x), \quad (2.34)$$

$$u_{n+1}(x,t) = -2 \int_0^t F_1(u_n(x,t)) dt + \int_0^t F_2(u_n(x,t)) dt + \frac{1}{2} \int_0^t F_3(u_n(x,t)) dt, \quad n \geq 0.$$

Therefore, the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = g(x) + \sum_{n=1}^{\infty} \left(-2 \int_0^t F_1(u_n(x,t)) dt + \int_0^t F_2(u_n(x,t)) dt + \frac{1}{2} \int_0^t F_3(u_n(x,t)) dt \right). \quad (2.35)$$

Which is the method of successive approximations. If

$$|u_n(x,t)| < 1,$$

then the series solution Eq.(2.35) convergence uniformly.

2.4 Description of the HPM and MHPM

To explain HPM [44, 45, 46, 47, 48], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u), \quad (2.36)$$

with initial conditions

$$u(x,0) = f(x).$$

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p Lv_0 + p [Nu - f(u)] = 0, \quad (2.37)$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(2.37) is expressed as

$$u(x,t) = u_0(x,t) + p u_1(x,t) + p^2 u_2(x,t) + \dots \quad (2.38)$$

Hence the approximate solution of Eq.(2.36) can be expressed as a series of the power of p , i.e.

$$u = \lim_{p \rightarrow 1} u = u_0 + u_1 + u_2 + \dots$$

where,

$$\begin{aligned} u_0(x,t) &= g(x), \\ &\vdots \\ u_m(x,t) &= \sum_{k=0}^{m-1} -2 \int_0^t F_1(u_{m-k-1}(x,t)) dt + \int_0^t F_2(u_{m-k-1}(x,t)) dt + \frac{1}{2} \int_0^t F_3(u_{m-k-1}(x,t)) dt, \quad m \geq 1. \end{aligned} \quad (2.39)$$

To explain MHPM [49, 50, 51], we consider Eq.(1.1) as

$$L(u) = u(x, t) - g(x) + 2 \int_0^t F_1(u_{m-k-1}(x, t)) dt - \int_0^t F_2(u_{m-k-1}(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_{m-k-1}(x, t)) dt.$$

Where $F_1(u(x, t)) = g_1(x)h_1(t)$, $F_2(u(x, t)) = g_2(x)h_2(t)$ and $F_3(u(x, t)) = g_3(x)h_3(t)$. We can define homotopy $H(u, p, m)$ by

$$H(u, 0, m) = f(u), \quad H(u, 1, m) = L(u),$$

where, m is an unknown real number and

$$f(u(x, t)) = u(x, t) - z(x, t).$$

Typically we may choose a convex homotopy by

$$H(u, p, m) = (1 - p)f(u) + p L(u) + p (1 - p)[m(g_1(x) + g_2(x) + g_3(x))] = 0, \quad 0 \leq p \leq 1. \quad (2.40)$$

Where m is called the accelerating parameters, and for $m = 0$ we define $H(u, p, 0) = H(u, p)$, which is the standard HPM.

The convex homotopy Eq.(2.40) continuously trace an implicity defined curve from a starting point $H(u(x, t) - f(u), 0, m)$ to a solution function $H(u(x, t), 1, m)$. The embedding parameter p monotonically increase from 0 to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$.

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \quad (2.41)$$

when $p \rightarrow 1$, Eq.(2.37) corresponds to the original one and Eq.(2.41) becomes the approximate solution of Eq.(1.1), i.e.,

$$u = \lim_{p \rightarrow 1} v = \sum_{m=0}^{\infty} u_m.$$

Where,

$$\begin{aligned} u_0(x, t) &= g(x), \\ u_1(x, t) &= -2 \int_0^t F_1(u_0(x, t)) dt + \int_0^t F_2(u_0(x, t)) dt + \frac{1}{2} \int_0^t F_3(u_0(x, t)) dt - m(g_1(x) + g_2(x) + g_3(x)), \\ u_2(x, t) &= -2 \int_0^t F_1(u_1(x, t)) dt + \int_0^t F_2(u_1(x, t)) dt + \frac{1}{2} \int_0^t F_3(u_1(x, t)) dt + \\ &\quad m(g_1(x) + g_2(x) + g_3(x)), \\ &\vdots \\ u_m(x, t) &= \sum_{k=0}^{m-1} -2 \int_0^t F_1(u_{m-k-1}(x, t)) dt + \int_0^t F_2(u_{m-k-1}(x, t)) dt + \frac{1}{2} \int_0^t F_3(u_{m-k-1}(x, t)) dt, \quad m \geq 3. \end{aligned} \quad (2.42)$$

3 Existence and convergency of iterative methods

We set,

$$\alpha_1 := T(2L_1 + L_2 + \frac{1}{2}L_3),$$

$$\beta_1 := 1 - T(1 - \alpha_1), \quad \gamma_1 := 1 - T\alpha_1.$$

Theorem 3.1. Let $0 < \alpha_1 < 1$, then Eq.(1.1), has a unique solution.

Proof. Let u and u^* be two different solutions of Eq.(1.3) then

$$\begin{aligned} |u - u^*| &= |-2 \int_0^t [F_1(u(x, t)) - F_1(u^*(x, t))] dt + \int_0^t [F_2(u(x, t)) - F_2(u^*(x, t))] dt \\ &\quad + \frac{1}{2} \int_0^t [F_3(u(x, t)) - F_3(u^*(x, t))] dt| \\ &\leq 2 \int_0^t |F_1(u(x, t)) - F_1(u^*(x, t))| dt + \int_0^t |F_2(u(x, t)) - F_2(u^*(x, t))| dt + \\ &\quad \frac{1}{2} \int_0^t |F_3(u(x, t)) - F_3(u^*(x, t))| dt \\ &\leq T(2L_1 + L_2 + \frac{1}{2}L_3) |u - u^*| = \alpha_1 |u - u^*|. \end{aligned}$$

From which we get $(1 - \alpha_1) |u - u^*| \leq 0$. Since $0 < \alpha_1 < 1$, then $|u - u^*| = 0$. Implies $u = u^*$ and completes the proof. \square

Theorem 3.2. The series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ of Eq.(1.1) using MADM convergence when

$$0 < \alpha_1 < 1, |u_1(x, t)| < \infty.$$

Proof. Denote as $(C[J], \|\cdot\|)$ the Banach space of all continuous functions on J with the norm $\|g(t)\| = \max |g(t)|$, for all t in J . Define the sequence of partial sums s_n , let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\begin{aligned} \|s_n - s_m\| &= \max_{t \in J} |s_n - s_m| = \max_{t \in J} |\sum_{i=m+1}^n u_i(x, t)| = \\ &= \max_{t \in J} |-2 \int_0^t (\sum_{i=m}^{n-1} A_i) dt + \int_0^t (\sum_{i=m}^{n-1} B_i) dt + \\ &\quad \frac{1}{2} \int_0^t (\sum_{i=m}^{n-1} Z_i) dt|. \end{aligned}$$

From [27], we have

$$\begin{aligned} \sum_{i=m}^{n-1} A_i &= F_1(s_{n-1}) - F_1(s_{m-1}), \\ \sum_{i=m}^{n-1} B_i &= F_2(s_{n-1}) - F_2(s_{m-1}), \\ \sum_{i=m}^{n-1} Z_i &= F_3(s_{n-1}) - F_3(s_{m-1}). \end{aligned}$$

So,

$$\begin{aligned} \|s_n - s_m\| &= \max_{t \in J} |-2 \int_0^t [F_1(s_{n-1}) - F_1(s_{m-1})] dt + \int_0^t [F_2(s_{n-1}) - F_2(s_{m-1})] dt + \\ &\quad \frac{1}{2} \int_0^t [F_3(s_{n-1}) - F_3(s_{m-1})] dt| \leq \\ &\leq 2 \int_0^t |F_1(s_{n-1}) - F_1(s_{m-1})| dt + \int_0^t |F_2(s_{n-1}) - F_2(s_{m-1})| dt \\ &\quad + \frac{1}{2} \int_0^t |F_3(s_{n-1}) - F_3(s_{m-1})| dt \leq \alpha_1 \|s_n - s_m\|. \end{aligned}$$

Let $n = m + 1$, then

$$\|s_n - s_m\| \leq \alpha_1 \|s_m - s_{m-1}\| \leq \alpha_1^2 \|s_{m-1} - s_{m-2}\| \leq \dots \leq \alpha_1^m \|s_1 - s_0\|.$$

From the triangle inequality we have

$$\begin{aligned} \|s_n - s_m\| &\leq \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \dots + \|s_n - s_{n-1}\| \\ &\leq [\alpha_1^m + \alpha_1^{m+1} + \dots + \alpha_1^{n-m-1}] \|s_1 - s_0\| \\ &\leq \alpha_1^m [1 + \alpha_1 + \alpha_1^2 + \dots + \alpha_1^{n-m-1}] \|s_1 - s_0\| \leq \alpha_1^m \left[\frac{1 - \alpha_1^{n-m}}{1 - \alpha_1} \right] \|u_1(x, t)\|. \end{aligned}$$

Since $0 < \alpha_1 < 1$, we have $(1 - \alpha_1^{n-m}) < 1$, then

$$\|s_n - s_m\| \leq \frac{\alpha_1^m}{1 - \alpha_1} \max_{t \in J} |u_1(x, t)|. \quad (3.43)$$

But $|u_1(x, t)| < \infty$, so, as $m \rightarrow \infty$, then $\|s_n - s_m\| \rightarrow 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete. \square

Theorem 3.3. The maximum absolute truncation error of the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ to Eq.(1.1) by using MADM is estimated to be

$$\max |u(x, t) - \sum_{i=0}^m u_i(x, t)| \leq \frac{k\alpha_1^m}{1 - \alpha_1}. \quad (3.44)$$

Proof. From inequality Eq.(3.43), when $n \rightarrow \infty$, then $s_n \rightarrow u$ and

$$\max |u_1(x, t)| \leq T(\max_{\forall t \in J} |2F_1(u_0(x, t))| + \max_{\forall t \in J} |F_2(u_0(x, t))| + \frac{1}{2} \max_{\forall t \in J} |F_3(u_0(x, t))|).$$

Therefore,

$$\|u(x, t) - s_m\| \leq \frac{\alpha_1^m}{1 - \alpha_1} T(\max_{\forall t \in J} |2F_1(u_0(x, t))| + \max_{\forall t \in J} |F_2(u_0(x, t))| + \frac{1}{2} \max_{\forall t \in J} |F_3(u_0(x, t))|).$$

Finally the maximum absolute truncation error in the interval J is obtained by Eq.(3.44). \square

Theorem 3.4. The solution $u_n(x, t)$ obtained from the relation Eq.(2.20) using VIM converges to the exact solution of the Eq.(1.1) when $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$.

Proof.

$$u_{n+1}(x, t) = u_n(x, t) - L_t^{-1}([u_n(x, t) - g(x) + 2 \int_0^t F_1(u_n(x, t)) dt - \int_0^t F_2(u_n(x, t)) dt - \frac{1}{2} \int_0^t F_3(u_n(x, t)) dt]) \quad (3.45)$$

$$u(x, t) = u(x, t) - L_t^{-1}([u(x, t) - g(x) + 2 \int_0^t F_1(u(x, t)) dt - \int_0^t F_2(u(x, t)) dt - \frac{1}{2} \int_0^t F_3(u(x, t)) dt]) \quad (3.46)$$

By subtracting relation Eq.(3.45) from Eq.(3.46),

$$\begin{aligned} u_{n+1}(x, t) - u(x, t) &= u_n(x, t) - u(x, t) - L_t^{-1}(u_n(x, t) - u(x, t) \\ &+ 2 \int_0^t [F_1(u_n(x, t)) - F_1(u(x, t))] dt - \int_0^t [F_2(u_n(x, t)) - F_2(u(x, t))] dt - \\ &\frac{1}{2} \int_0^t [F_3(u_n(x, t)) - F_3(u(x, t))] dt), \end{aligned}$$

if we set, $e_{n+1}(x, t) = u_{n+1}(x, t) - u_n(x, t)$, $e_n(x, t) = u_n(x, t) - u(x, t)$, $|e_n(x, t^*)| = \max_t |e_n(x, t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write,

$$\begin{aligned} e_{n+1}(x, t) &= e_n(x, t) + L_t^{-1}(-e_n(x, t) + 2 \int_0^t [F_1(u_n(x, t)) - F_1(u(x, t))] dt \\ &- \int_0^t [F_2(u_n(x, t)) - F_2(u(x, t))] dt - \frac{1}{2} \int_0^t [F_3(u_n(x, t)) - F_3(u(x, t))] dt) \\ &\leq e_n(x, t) + L_t^{-1}[-e_n(x, t) + L_t^{-1} |e_n(x, t)| (T(2L_1 + L_2 + \frac{1}{2}L_3))] \\ &\leq e_n(x, t) - Te_n(x, \eta) + T(2L_1 + L_2 + \frac{1}{2}L_3)L_t^{-1}L_t^{-1} |e_n(x, t)| \\ &\leq (1 - T(1 - \alpha_1)) |e_n(x, t^*)|, \end{aligned}$$

where $0 \leq \eta \leq t$. Hence, $e_{n+1}(x, t) \leq \beta_1 |e_n(x, t^*)|$.

Therefore,

$$\|e_{n+1}\| = \max_{\forall t \in J} |e_{n+1}| \leq \beta_1 \max_{\forall t \in J} |e_n| \leq \beta_1 \|e_n\|.$$

Since $0 < \beta_1 < 1$, then $\|e_n\| \rightarrow 0$. So, the series converges and the proof is complete. \square

Theorem 3.5. The solution $u_n(x, t)$ obtained from the Eq.(2.22) using MVIM for the Eq.(1.1) converges when $0 < \alpha_1 < 1$, $0 < \gamma_1 < 1$.

Proof. The Proof is similar to the previous theorem. \square

Theorem 3.6. *The maximum absolute truncation error of the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ to Eq.(1.1) by using VIM is estimated to be*

$$\|e_n\| \leq \frac{\beta_1^n k'}{1 - \beta_1}, \quad k' = \max |u_1(x, t)|.$$

Proof.

$$\begin{aligned} u_{n+1} - u_n &= (u_{n+1} - u) + (u - u_n) = e_n - e_{n+1} \\ \rightarrow e_n &= e_{n+1} - (u_{n+1} - u_n) \\ \|e_n\| &= \|e_{n+1} - (u_{n+1} - u_n)\| \leq \|e_{n+1}\| + \|u_{n+1} - u_n\| \leq \beta_1 \|e_n\| + \|u_{n+1} - u_n\| \\ \rightarrow \|e_n\| &\leq \frac{\|u_{n+1} - u_n\|}{1 - \beta_1} \leq \frac{\beta_1^n k'}{1 - \beta_1}. \end{aligned}$$

□

Theorem 3.7. *If the series solution Eq.(2.34) of Eq.(1.1) using HAM convergent then it converges to the exact solution of the Eq.(1.1).*

Proof. We assume:

$$\begin{aligned} u(x, t) &= \sum_{m=0}^{\infty} u_m(x, t), \\ \hat{F}_1(u(x, t)) &= \sum_{m=0}^{\infty} F_1(u_m(x, t)), \\ \hat{F}_2(u(x, t)) &= \sum_{m=0}^{\infty} F_2(u_m(x, t)), \\ \hat{F}_3(u(x, t)) &= \sum_{m=0}^{\infty} F_3(u_m(x, t)). \end{aligned}$$

Where,

$$\lim_{m \rightarrow \infty} u_m(x, t) = 0.$$

We can write,

$$\sum_{m=1}^n [u_m(x, t) - \chi_m u_{m-1}(x, t)] = u_1 + (u_2 - u_1) + \dots + (u_n - u_{n-1}) = u_n(x, t). \quad (3.47)$$

Hence, from Eq.(3.47),

$$\lim_{n \rightarrow \infty} u_n(x, t) = 0. \quad (3.48)$$

So, using and the definition of the linear operator L , we have

$$\sum_{m=1}^{\infty} L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = L\left[\sum_{m=1}^{\infty} [u_m(x, t) - \chi_m u_{m-1}(x, t)]\right] = 0.$$

therefore from , we can obtain that,

$$\sum_{m=1}^{\infty} L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h H_1(x, t) \sum_{m=1}^{\infty} \mathfrak{R}_{m-1}(u_{m-1}(x, t)) = 0.$$

Since $h \neq 0$ and $H_1(x, t) \neq 0$, we have

$$\sum_{m=1}^{\infty} \mathfrak{R}_{m-1}(u_{m-1}(x, t)) = 0. \quad (3.49)$$

By substituting $\mathfrak{R}_{m-1}(u_{m-1}(x, t))$ into the relation Eq.(3.49) and simplifying it , we have

$$\begin{aligned} \sum_{m=1}^{\infty} \mathfrak{R}_{m-1}(u_{m-1}(x, t)) &= \sum_{m=1}^{\infty} \left[-2 \int_0^t F_1(u_{m-1}(x, t)) dt \right. \\ &\quad \left. + \int_0^t F_2(u_{m-1}(x, t)) dt + \frac{1}{2} \int_0^t F_3(u_{m-1}(x, t)) dt + (1 - \chi_m) g(x) \right] \\ &= u(x, t) - g(x) + 2 \int_0^t \hat{F}_1(u(x, t)) dt - \int_0^t \hat{F}_2(u(x, t)) dt - \frac{1}{2} \int_0^t \hat{F}_3(u(x, t)) dt. \end{aligned} \quad (3.50)$$

From Eq.(3.49) and Eq.(3.50), we have

$$u(x, t) = g(x) - 2 \int_0^t \widehat{F}_1(u(x, t)) dt + \int_0^t \widehat{F}_2(u(x, t)) dt + \frac{1}{2} \int_0^t \widehat{F}_3(u(x, t)) dt.$$

Therefore, $u(x, t)$ must be the exact solution. \square

Theorem 3.8. The maximum absolute truncation error of the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ to Eq.(1.1) by using HAM is estimated to be

$$\|e_n\| \leq \frac{\alpha_1^n k'}{1 - \alpha_1}, \quad k' = \max |u_1(x, t)|.$$

Proof. The Proof is similar to the 3.6 theorem \square

Theorem 3.9. If $|u_m(x, t)| \leq 1$, then the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ of Eq.(1.1) converges to the exact solution by using HPM.

Proof. We set,

$$\phi_n(x, t) = \sum_{i=1}^n u_i(x, t),$$

$$\phi_{n+1}(x, t) = \sum_{i=1}^{n+1} u_i(x, t).$$

$$\begin{aligned} |\phi_{n+1}(x, t) - \phi_n(x, t)| &= D(\phi_{n+1}(x, t), \phi_n(x, t)) = D(\phi_n + u_n, \phi_n) \\ &= D(u_n, 0) \leq \sum_{k=0}^{m-1} -2 \int_0^t |F_1(u_{m-k-1}(x, t))| dt + \int_0^t |F_2(u_{m-k-1}(x, t))| dt \\ &\quad + \frac{1}{2} \int_0^t \int_0^t |F_3(u_{m-k-1}(x, t))| dt. \end{aligned}$$

$$\rightarrow \sum_{n=0}^{\infty} \|\phi_{n+1}(x, t) - \phi_n(x, t)\| \leq m\alpha_1 |g(x)| \sum_{n=0}^{\infty} (m\alpha_1)^n.$$

Therefore,

$$\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t).$$

\square

Theorem 3.10. If $|u_m(x, t)| \leq 1$, then the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ of Eq.(1.1) converges to the exact solution by using MHPM.

Proof. The Proof is similar to the previous theorem. \square

Theorem 3.11. The maximum absolute truncation error of the series solution $u(x, t) = \sum_{i=0}^{\infty} u_i(x, t)$ to Eq.(1.1) by using HPM is estimated to be

$$\|e_n\| \leq \frac{(n\alpha_1)^n nk'}{1 - \alpha_1}, \quad k' = \max |u_1(x, t)|.$$

Proof. The Proof is similar to the 3.6 theorem \square

4 Numerical example

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, HPM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm 1:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations Eq.(2.10) for ADM , Eq.(2.13) for MADM, Eq.(2.34) for HAM, Eq.(2.39) for HPM and Eq.(2.42) for MHPM.
Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 4,
else $n \leftarrow n + 1$ and go to step 2.
Step 4. Print $u(x,t) = \sum_{i=0}^n u_i(x,t)$ as the approximate of the exact solution.

Algorithm 2:
Step 1. Set $n \leftarrow 0$.
Step 2. Calculate the recursive relations Eq.(2.20) for VIM and Eq.(2.21) for MVIM.
Step 3. If $|u_{n+1} - u_n| < \varepsilon$ then go to step 4,
else $n \leftarrow n + 1$ and go to step 2.
Step 4. Print $u_n(x,t)$ as the approximate of the exact solution.

Example 4.1. Consider the nonlinear foam drainage equation as follows:

$$u_t(x,t) + 2u^2(x,t)u_x(x,t) - u_x^2(x,t) - \frac{1}{2}u_{xx}(x,t)u(x,t) = 0..$$

With initial condition:

$$g(x) = -\sqrt{2}\tanh(\sqrt{2}x).$$

$\varepsilon = 10^{-3}$.

(x,t)	Errors			
	ADM(n=24)	MADM(n=22)	VIM(n=17)	MVIM(n=14)
(0.13,0.14)	0.0073348	0.0062219	0.0044609	0.0031315
(0.23,0.17)	0.0074426	0.0063547	0.0045113	0.0032231
(0.31,0.24)	0.0075605	0.0064606	0.0046718	0.0033322
(0.44,0.37)	0.0076421	0.0065736	0.0047839	0.0034212
(0.52,0.45)	0.0077515	0.0066877	0.0048618	0.0035368
(0.7,0.47)	0.0078105	0.0067136	0.0049857	0.0036493

Table 1: Numerical results for Example 4.1

(x,t)	Errors		
	HPM(n=15)	MHPM(n=13)	HAM(n=9)
(0.13,0.14)	0.0054265	0.0033171	0.0025829
(0.23,0.17)	0.00552491	0.0034596	0.0026534
(0.31,0.24)	0.0056742	0.0035277	0.0027509
(0.44,0.37)	0.0057822	0.0036288	0.0027593
(0.52,0.45)	0.0058157	0.0037344	0.0028902
(0.7,0.47)	0.0059739	0.0038482	0.0029085

Table 1, shows that, approximate solution of the nonlinear foam drainage equation is convergence with 9 iterations by using the HAM . By comparing the results of Table 1 , we can observe that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

5 Conclusion

The homotopy analysis method has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the nonlinear foam drainage equation. For this purpose, we showed that the HAM is more rapid convergence than the ADM, MADM, VIM, MVIM, HPM and MHPM.

References

- [1] S. D. Stoyanov, V. N. Paunov, E. S. Basheva, I. B. Ivanov, A. Mehreteab, G. Broze, Motion of the front between thick and thin film: hydrodynamic theory and experiment with vertical foam films, *Langmuir*, 13 (1997) 1400-1407.
<http://dx.doi.org/10.1021/la9608019>
- [2] D. Weaire, S. Hutzler, S. Cox, N. Kern, M. D. Alonso, D. D. Drenckhan, The fluid dynamics of foams, *Journal of Physics Condensed Matter*, 15 (2003) 65-73.
<http://dx.doi.org/10.1088/0953-8984/15/1/307>
- [3] R. K. Prud'homme, S. A. Khan, *Foams: Theory, Measurements and Applications*, Marcel Dekker, New York, NY, USA, (1996).
- [4] D. L. Weaire, S. Hutzler, *The Physics of Foams*, Oxford University Press, Oxford, UK, (2000).
- [5] H. A. Stone, S. A. Koehler, S. Hilgenfeldt, M. Durand, Perspectives on foam drainage and the influence of interfacial rheology, *Journal of Physics Condensed Matter*, 15 (2003) 283-290.
<http://dx.doi.org/10.1088/0953-8984/15/1/338>
- [6] S. Hilgenfeldt, S. A. Koehler, H. A. Stone, Dynamics of coarsening foams: accelerated and selflimiting drainage, *Physical Review Letters*, 86 (2001) 4704-4707.
<http://dx.doi.org/10.1103/PhysRevLett.86.4704>
- [7] J. I. B. Wilson, Essay review, scholarly froth and engineering skeletons, *Contemporary Physics*, 44 (2003) 153-155.
<http://dx.doi.org/10.1080/0010751031000095107>
- [8] L. J. Gibson, M. F. Ashby, *Cellular Solids: Structure and Properties*, Cambridge University Press, Cambridge, UK, (1997).
- [9] J. Banhart, *Metallschaume*, MIT, Bermen, Germany, (1997).
- [10] S. A. Koehler, H. A. Stone, M. P. Brenner, J. Eggers, Dynamics of foam drainage, *Physical Review E*, 58 (1998) 2097-2106.
<http://dx.doi.org/10.1103/PhysRevE.58.2097>
- [11] L. J. Gibson, M. F. Ashby, *Cellular Solids: Structure and Properties*, Cambridge University Press, Cambridge, UK, (1999).
- [12] M. F. Ashby, A. G. Evans, N. A. Fleck, L. J. Gibson, J. w. Hutchinson, H. N. G. Wasley, *Metal Foams: A Design Guide*, Society of Automotive Engineers, Boston, Mass, USA, (2000).

- [13] R. A. Leonard, R. Lemlich, A study of interstitial liquid flow in foam. Part I. Theoretical model and application to foam fractionation, *AIChE Journal*, 11 (1965) 18-29.
<http://dx.doi.org/10.1002/aic.690110108>
- [14] I. I. Gol'dfarb, K. B. Kann, I. R. Shreiber, Liquid flow in foams, *Fluid Dynamics*, 23 (1988) 244-249.
<http://dx.doi.org/10.1007/BF01051894>
- [15] J. J. Bikerman, *Foams; theory and industrial applications*, Reinhold, (1953).
- [16] D. Weaire, S. Hutzler, G. Verbist, E. A. J. Peters, A review of foam drainage, *Advances in Chemical Physics*, 102 (1997) 315-374.
<http://dx.doi.org/10.1002/9780470141618.ch5>
- [17] A. Bhakta, E. Ruckenstein, Decay of standing foams: drainage, coalescence and collapse, *Advances in Colloid and Interface Science*, 70 (1997) 1-123.
[http://dx.doi.org/10.1016/S0001-8686\(97\)00031-6](http://dx.doi.org/10.1016/S0001-8686(97)00031-6)
- [18] M. Durand, D. Langevin, Physicochemical approach to the theory of foam drainage, *European Physical Journal E*, 7 (2002) 35-44.
<http://dx.doi.org/10.1007/s10189-002-8215-0>
- [19] G. Verbist, D. Weaire, Soluble model for foam drainage, *Europhysics Letters*, 26 (1994) 631-634.
<http://dx.doi.org/10.1209/0295-5075/26/8/013>
- [20] G. Verbist, D. Weaire, A. M. Kraynik, The foam drainage equation, *Journal of Physics Condensed Matter*, 8 (1996) 3715-3731.
<http://dx.doi.org/10.1088/0953-8984/8/21/002>
- [21] G. Verbist, D. Weaire, Soluble model for foam drainage, *Europhysics Letters*, 26 (1994) 631-641.
<http://dx.doi.org/10.1209/0295-5075/26/8/013>
- [22] D. Weaire, S. Hutzler, N. Pittet, D. Pardal, Steady-state drainage of an aqueous foam, *Physical Review Letters*, 71 (1993) 2670-2673.
<http://dx.doi.org/10.1103/PhysRevLett.71.2670>
- [23] S. H. Behriy, H. Hashish, I. L. E-Kalla, A. Elsaid, A new algorithm for the decomposition solution of nonlinear differential equations, *App.Math.Comput*, 54 (2007) 459-466.
<http://dx.doi.org/10.1016/j.camwa.2006.12.027>
- [24] M. A. Fariborzi Araghi, Sh. S. Behzadi, Solving nonlinear Volterra-Fredholm integral differential equations using the modified Adomian decomposition method, *Comput. Methods in Appl. Math*, 9 (2009) 1-11.
- [25] A. M. Wazwaz, Construction of solitary wave solution and rational solutions for the KdV equation by ADM, *Chaos, Solution and fractals*, 12 (2001) 2283-2293.
- [26] A. M. Wazwaz, *A first course in integral equations*, WSPC, New Jersey; (1997).
<http://dx.doi.org/10.1142/3444>
- [27] I. L. El-Kalla, Convergence of the Adomian method applied to a class of nonlinear integral equations, *Appl.Math.Comput*, 21 (2008) 372-376.
<http://dx.doi.org/10.1016/j.aml.2007.05.008>
- [28] S. Abbasbandy, Numerical method for non-linear wave and diffusion equations by the variational iteration method, *Q1 Int. J. Numer. Methods Eng*, 73 (2008) 1836-1843.
<http://dx.doi.org/10.1002/nme.2150>

- [29] S. Abbasbandy, A. Shirzadi, The variational iteration method for a class of eight-order boundary value differential equations, *Z. Naturforsch*, 63 (2008) 745-751.
- [30] M. A. Fariborzi Araghi, Sh. S. Behzadi, Solving nonlinear Volterra-Fredholm integro-differential equations using He's variational iteration method, *International Journal of Computer Mathematics*, 88 (2011) 829-838.
<http://dx.doi.org/10.1080/00207161003770394>
- [31] M. Ghasemi, M. Tavasoli, E. Babolian, Application of He's homotopy perturbation method of nonlinear integro-differential equation, *Appl. Math. Comput*, 188 (2007) 538-548.
<http://dx.doi.org/10.1016/j.amc.2006.10.016>
- [32] J. H. He, X. H. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons and Fractals*, 30 (2006) 700-708.
<http://dx.doi.org/10.1016/j.chaos.2006.03.020>
- [33] J. H. He, Variational principle for some nonlinear partial differential equations with variable coefficients, *Chaos, Solitons and Fractals*, 19 (2004) 847-851.
[http://dx.doi.org/10.1016/S0960-0779\(03\)00265-0](http://dx.doi.org/10.1016/S0960-0779(03)00265-0)
- [34] J. H. He, Wang. Shu-Qiang, Variational iteration method for solving integro-differential equations, *Physics Letters A*, 367 (2007) 188-191.
<http://dx.doi.org/10.1016/j.physleta.2007.02.049>
- [35] J. H. He, Variational iteration method some recent results and new interpretations, *J. Comp. and Appl. Math*, 207 (2007) 3-17.
<http://dx.doi.org/10.1016/j.cam.2006.07.009>
- [36] A. Yildirim, S. T. Mohyud-Din, D. H. Zhang, Analytical solutions to the pulsed Klein-Gordon equation using Modified Variational Iteration Method (MVIM) and Boubaker Polynomials Expansion Scheme (BPES), *Computers and Mathematics with Applications*, 59 (2010) 2473-2477.
<http://dx.doi.org/10.1016/j.camwa.2009.12.026>
- [37] T. A. Abassy, El-Tawil, H. El. Zoheiry, Toward a modified variational iteration method (MVIM), *J. Comput. Appl. Math*, 207 (2007) 137-147.
<http://dx.doi.org/10.1016/j.cam.2006.07.019>
- [38] T. A. Abassy, El-Tawil, H. El. Zoheiry, Modified variational iteration method for Boussinesq equation, *Comput. Math. Appl*, 54 (2007) 955-965.
<http://dx.doi.org/10.1016/j.camwa.2006.12.040>
- [39] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall/CRC Press, Boca Raton, (2003).
<http://dx.doi.org/10.1201/9780203491164>
- [40] S. J. Liao, Notes on the homotopy analysis method: some definitions and theorems, *Communication in Nonlinear Science and Numerical Simulation*, 14 (2009) 983-997.
<http://dx.doi.org/10.1016/j.cnsns.2008.04.013>
- [41] Sh. S. Behzadi, The convergence of homotopy methods for nonlinear Klein-Gordon equation, *J. Appl. Math. Informatics*, 28 (2010) 1227-1237.
- [42] Sh. S. behzadi, M. A. Fariborzi Araghi, The use of iterative methods for solving Naveir-Stokes equation, *J. Appl. Math. Informatics*, 29 (2011) 1-15.

- [43] M. A. Fariborzi Araghi, Sh. S. Behzadi, Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Homotopy analysis method, *Journal of Applied Mathematics and Computing*, 37 (2011) 1-12.
<http://dx.doi.org/10.1007/s12190-010-0417-4>
- [44] J. Biazar, H. Ghazvini, Convergence of the homotopy perturbation method for partial differential equations, *Nonlinear Analysis: Real World Application*, 10 (2009) 2633-2640.
<http://dx.doi.org/10.1016/j.nonrwa.2008.07.002>
- [45] M. A. Fariborzi Araghi, S. S. Behzadi, Numerical solution for solving Burger's-Fisher equation by using Iterative Methods, *Mathematical and Computational Applications*, 16 (2011) 443-455.
- [46] S. A. Sezer, A. Yildirim, S. T. Mohyud-Din, Hes homotopy perturbation method for solving the fractional KdV-Burgers-Kuramoto equation, *International Journal of Numerical Methods for Heat and Fluid Flow*, 21 (2011) 448-458.
<http://dx.doi.org/10.1108/09615531111123119>
- [47] A. A. Yildirim, Solution of BVPs for Fourth-Order Integro-Differential Equations by using Homotopy Perturbation Method, *Computers and Mathematics with Applications*, 56 (2008) 3175-3180.
<http://dx.doi.org/10.1016/j.camwa.2008.07.020>
- [48] A. A. Yildirim, The Homotopy Perturbation Method for Approximate Solution of the Modified KdV Equation, *Zeitschrift fr Naturforschung A, A Journal of Physical Sciences*, 63 (2008) 621-626.
- [49] S. Abbasbandy, Modified homotopy perturbation method for nonlinear equations and comparison with Adomian decomposition method, *Appl. Math. Comput.*, 172 (2006) 431-438.
<http://dx.doi.org/10.1016/j.amc.2005.02.015>
- [50] A. Golbabai, B. Keramati, Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method, *Chaos Solitons and Fractals*, 5 (2009) 2316-2321.
<http://dx.doi.org/10.1016/j.chaos.2007.06.120>
- [51] M. Javidi, Modified homotopy perturbation method for solving system of linear Fredholm integral equations, *Mathematical and Computer Modelling*, 50 (2009) 159-165.
<http://dx.doi.org/10.1016/j.mcm.2009.02.003>