Proof Complexity of Propositional Default Logic

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What is Default Logic?

- a non-monotone logic, introduced 1980 by Reiter
- models common-sense reasoning
- extends classical logic with *default rules*
- we work with propositional logic

Definition (Reiter 80)

A default rule is a triple $\frac{\alpha : \beta}{\gamma}$, where

- α is called the prerequisite,
- β is called the justification, and
- γ is called the consequent,

for α, β, γ propositional formulae.

Informally, we can infer a formula γ from a set of formulae W by a default rule $\frac{\alpha : \beta}{\gamma}$, if $\alpha \in W$ and $\neg\beta \notin W$.

Default Theories

Definition (Reiter 80)

A default theory is a tuple $\langle W, D\rangle$, where W is a set of formulae and D is a set of default rules.

Example: Playing Football with Default Rules

$$W := \{football, rain, cold \land rain \to snow\}$$
$$D := \left\{\frac{football : \neg snow}{takesPlace}\right\}$$

$\neg snow$ is consistent with W. Hence we can infer $takesPlace$.

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snow is consistent with W. Hence we cannot infer *takesPlace*.

Default logics are non-monotone!

Definition (Reiter 80)

For default theory $\langle W, D\rangle$ and set of formulae E, we define $\Gamma(E)$ as the smallest set, s.t.

1. $W \subseteq \Gamma(E)$,
2. $\Gamma(E)$ is deductively closed, and
3. for all defaults $\frac{\alpha\colon \beta}{\gamma}$ with $\alpha \in \Gamma(E)$ and $\neg\beta \notin E$, it holds that $\gamma \in \Gamma(E)$.

A stable extension of $\langle W, D\rangle$ is a set E s.t. $E = \Gamma(E)$.

Stable Extensions

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Motivation

Stable extensions correspond to possible views of an agent on the basis of $\langle W, D \rangle$.

Semantics: A Stage Construction (Reiter 80)

For default theory $\langle W, D \rangle$ and set of formulae E let

- $E_0 := W$ and
- $E_{i+1} := \mathrm{Th}\,(E_i) \cup \{\gamma \mid \frac{\alpha : \beta}{\gamma} \in D, \alpha \in E_i$ and $\neg\beta \notin E\}$.

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Two Important Problems

Credulous Reasoning Problem

Instance: a formula φ and a default theory $\langle W, D\rangle$

Question: Is there a stable extension of $\langle W, D\rangle$ that includes φ?

Skeptical Reasoning Problem

Instance: a formula φ and a default theory $\langle W, D\rangle$

Question: Does every stable extension of $\langle W, D\rangle$ include φ?

- Semantics and complexity of default logic have been intensively studied.
- Credulous Reasoning is Σ_2^p-complete. [Gottlob 92]
- Skeptical Reasoning is Π_2^p-complete. [Gottlob 92]
- There are many proof-theoretic methods for default logic. [Gabbay 85, Makinson 89, Kraus et al. 90, Risch & Schwind 94, Amati et al. 96]
- Bonatti and Olivetti (ACM ToCL'02) introduced the first purely axiomatic formalism using sequent calculi.
- Generalized to first-order default logic. [Egly & Tompits 01]

Our Results

- We give the first proof-theoretic analysis of the sequent calculi of Bonatti and Olivetti.
- The calculus for credulous default reasoning obeys almost the same bounds on the proof size as Gentzen's system LK, i. e., proof lengths are polynomially related.
- For the calculus for skeptical default reasoning we show an exponential lower bound to the proof size (even to the number of steps).

The Proof Systems

Bonatti and Olivetti's sequent calculi for default logic consist of four main ingredients:

- classical sequents and rules from LK,
- antisequents to refute non-tautologies,
- a residual calculus for simple, justification-free default rules, and
- sequents and rules with proper defaults.

Axioms: $\Gamma \nvdash \Delta$ where Γ and Δ are disjoint sets of variables.

$$\frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma, \neg\alpha \nvdash \Sigma}\,(\neg \nvdash) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma \nvdash \Sigma, \neg\alpha}\,(\nvdash \neg)$$

$$\frac{\Gamma, \alpha, \beta \nvdash \Sigma}{\Gamma, \alpha \wedge \beta \nvdash \Sigma}\,(\wedge \nvdash) \qquad \frac{\Gamma \nvdash \Sigma, \alpha}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\,(\nvdash \bullet\wedge) \qquad \frac{\Gamma \nvdash \Sigma, \beta}{\Gamma \nvdash \Sigma, \alpha \wedge \beta}\,(\nvdash \wedge\bullet)$$

$$\frac{\Gamma \nvdash \Sigma, \alpha, \beta}{\Gamma \nvdash \Sigma, \alpha \vee \beta}\,(\nvdash \vee) \qquad \frac{\Gamma, \alpha \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\,(\bullet\vee \nvdash) \qquad \frac{\Gamma, \beta \nvdash \Sigma}{\Gamma, \alpha \vee \beta \nvdash \Sigma}\,(\vee\bullet \nvdash)$$

Theorem (Bonatti 93)

The antisequent calculus is sound and complete, i. e., $\Gamma \nvdash \Sigma$ is derivable iff there is an assignment satisfying Γ, but falsifying Σ.

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Observation

The antisequent calculus is polynomially bounded.

Definition

A residual rule is a default rule $\frac{\alpha}{\gamma}$ without justification.

Rules

$$(\mathbf{Re1})\ \frac{\Gamma \vdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \vdash \Delta} \qquad (\mathbf{Re2})\ \frac{\Gamma \vdash \alpha \qquad \Gamma, \gamma \vdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \vdash \Delta}$$

$$(\mathbf{Re3})\ \frac{\Gamma \nvdash \Delta \qquad \Gamma \nvdash \alpha}{\Gamma, \frac{\alpha}{\gamma} \nvdash \Delta} \qquad (\mathbf{Re4})\ \frac{\Gamma, \gamma \nvdash \Delta}{\Gamma, \frac{\alpha}{\gamma} \nvdash \Delta}$$

The Residual Calculus

Theorem (Bonatti, Olivetti 02)

The residual calculus is sound and complete, i. e., for all default theories $\langle W, R\rangle$ with only residual rules

1. $\langle W, R\rangle \vdash \Delta$ *is derivable iff* $\bigvee \Delta$ *is in some stable extension of* $\langle W, R\rangle$*;*
2. $\langle W, R\rangle \nvdash \Delta$ *is derivable iff no stable extension of* $\langle W, R\rangle$ *contains* $\bigvee \Delta$*.*

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Lemma

1. *The minimal proof lengths in the residual calculus and in LK are polynomially related.*
2. *Antisequents* $\langle W, R\rangle \nvdash \Delta$ *even have polynomial-size proofs.*

The Credulous Default Calculus

Definition

- A provability constraint is of the form $\mathbf{L}\alpha$ or $\neg\mathbf{L}\alpha$ with a formula α.
- A set E of formulas satisfies a constraint $\mathbf{L}\alpha$ if $\alpha \in Th(E)$.
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Definition

- A credulous default sequent $\Sigma; \Gamma \mid\!\sim \Delta$ consists of a set Σ of provability constraints, a default theory Γ, and a set Δ of propositional sentences.
- Semantically, $\Sigma; \Gamma \mid\!\sim \Delta$ is true, if there exists a stable extension of Γ which satisfies all constraints in Σ and contains $\bigvee \Delta$.

The credulous default calculus uses rules from LK, the anti-sequent calculus, the residual calculus and

Rules for residual theories

$$\text{(cD1)}\ \frac{\Gamma \vdash \Delta}{;\ \Gamma \mid\!\sim \Delta}$$

$$\text{(cD2)}\ \frac{\Gamma \vdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{\mathbf{L}\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta} \qquad \text{(cD3)}\ \frac{\Gamma \nvdash \alpha \qquad \Sigma;\ \Gamma \mid\!\sim \Delta}{\neg\mathbf{L}\alpha,\ \Sigma;\ \Gamma \mid\!\sim \Delta}$$

Rules for default theories with justifications

$$\text{(cD4)}\ \frac{\mathbf{L}\neg\beta_i,\ \Sigma;\ \Gamma \mid\!\sim \Delta}{\Sigma;\ \Gamma, \frac{\alpha:\ \beta_1 \dots \beta_n}{\gamma} \mid\!\sim \Delta} \qquad \text{(cD5)}\ \frac{\neg\mathbf{L}\neg\beta_1 \dots \neg\mathbf{L}\neg\beta_n,\ \Sigma;\ \Gamma, \frac{\alpha}{\gamma} \mid\!\sim \Delta}{\Sigma;\ \Gamma, \frac{\alpha:\ \beta_1 \dots \beta_n}{\gamma} \mid\!\sim \Delta}$$

Theorem (Bonatti, Olivetti 02)

The calculus is sound and complete, i.e., a credulous default sequent is true iff it is derivable.

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Theorem

The length of proofs in the credulous default calculus and in LK are polynomially related. The same holds for the number of steps.

A Typical Derivation

$$
\begin{array}{c}
\dfrac{RC \qquad \begin{array}{c} \dfrac{\dfrac{RC \qquad \dfrac{RC}{\Gamma' \mathrel{|}\!\sim \Delta}\,(\mathbf{cD1})}{\sigma; \Gamma' \mathrel{|}\!\sim \Delta}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3})}{\vdots}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3}) \\ \Sigma''; \Gamma' \mathrel{|}\!\sim \Delta \end{array}}{\Sigma'; \Gamma' \mathrel{|}\!\sim \Delta}\,(\mathbf{cD2})\text{ or }(\mathbf{cD3}) \\
\overline{\phantom{\Sigma'; \Gamma' \mathrel{|}\!\sim \Delta}}\,(\mathbf{cD4})\text{ or }(\mathbf{cD5}) \\
\vdots \\
\Sigma; \Gamma \mathrel{|}\!\sim \Delta
\end{array}
$$

Summary

Proof complexity for credulous default reasoning

is tightly connected to length of proofs in classical logic:

- Bonatti and Olivetti's sequent calculus obeys the same bounds as LK.
- This connection also extends to (non-)automatizability.
- Even holds for stronger proof systems:
  For each propositional proof system we construct a proof system of the same strength for credulous reasoning.

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- Even holds for stronger proof systems:
  For each propositional proof system we construct a proof system of the same strength for credulous reasoning.

For skeptical default reasoning

- we obtain an exponential lower bound.
- Are there better proof systems?