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Analytical solutions for a vertically loaded pile in multilayered soil

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Explicit elastic solutions for a vertically loaded single pile embedded in multilayered soil are presented. Solutions are also provided for the case in which the pile base rests on a rigid material. Energy principles are used in the derivation of the governing differential equations. The solutions, which satisfy compatibility of displacement in the vertical and radial directions, were obtained by determining the unknown integration constants using the boundary conditions, Cramer's rule, and a recurrence formula. The solutions provide the pile vertical displacement as a function of depth, the load-transfer curves, and the vertical soil displacement as a function of the radial distance from the pile axis at any depth. The use of the analysis is illustrated for a pile that was load-tested under well-documented conditions by obtaining load–transfer and load–settlement curves that are then compared with those obtained from the load test.

Keywords: Analytical solutions; Elasticity; Piles; Multilayered soil; Vertical load; Settlement

1. Introduction

Analytical solutions for a vertically loaded pile in a homogeneous single soil layer have been obtained by a number of authors. Poulos and Davis (1968) and Butterfield and Banerjee (1971) studied the response of axially loaded piles by integrating Mindlin's point load solution (Mindlin 1936). Motta (1994) and Kodikara and Johnston (1994) obtained closed-form solutions for homogeneous soil or rock using the load–transfer (*t*–*z*) method of analysis to incorporate idealized elastic-plastic behaviour.

In reality, piles are rarely installed in an ideal homogeneous single-soil layer. For this reason, analytical solutions for axially loaded piles embedded in a nonhomogeneous soil deposit have been sought. Randolph and Wroth (1978) proposed an approximate closed-form solution for the settlement of a pile in a Gibson soil (Gibson 1967). Poulos (1979) also presented a series of solutions for the settlement of a pile in a Gibson soil and applied the analysis to a pile in a three-layered soil. Rajapakse (1990) performed a parametric study for the problem of an axially loaded single pile in a Gibson soil based on a variational formulation coupled with a boundary integral representation. Lee (1991) and Lee and Small (1991) proposed solutions for axially loaded piles in finite-layered soil using a discrete layer analysis. Chin and Poulos (1991) presented solutions for an axially loaded vertical pile embedded in a Gibson soil and a two-layered soil using the *t*-*z* method. Guo and Randolph (1997) and Guo (2000) obtained elastic–plastic solutions for the axial response of piles in a Gibson soil. Most of the analytical studies have been performed for a Gibson soil rather than for a multilayered soil because the mathematical treatment is easier.

Vallabhan and Mustafa (1996) proposed a simple closed-form solution for a drilled pier embedded in a two-layer elastic soil based on energy principles. Lee and Xiao (1999) expanded the solution of Vallabhan and Mustafa (1996) to multilayered soil and compared their solution with the results obtained by Poulos (1979) for a three-layered soil. Although Lee and Xiao (1999) suggested an analytical method for a vertically loaded pile in a multilayered soil, they did not obtain explicit analytical solutions.

In this paper we present explicit analytical solutions for a vertically loaded pile in a multilayered soil. The soil is assumed to behave as a linear elastic material. The governing differential equations are derived based on energy principles and calculus of variations. The integration constants are determined using Cramer's rule and a recurrence formula. In addition, solutions for a pile embedded in a multilayered soil with the base resting on a rigid material are obtained by changing the boundary conditions of the problem. We first review the mathematical formulation and the derivation of the equations using energy principles. We then compare our solutions with others from the literature. Finally, we use the results of a pile load test from the literature to verify the results obtained using the solutions proposed in this paper.

2. Mathematical formulation

2.1 *Definition of the problem and basic assumptions*

We consider a cylindrical pile of length L_p and circular cross section of radius r_p. The pile, which is under an axial

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load P, is embedded in a total of N horizontal soil layers. The pile itself crosses m layers, and there are $N - m$ layers below its base. All soil layers extend to infinity in the radial direction, and the bottom (N^{th}) layer extends to infinity downwards in the vertical direction. As shown in Figure 1, H_i denotes the vertical depth from the ground surface to the bottom of any layer i, which implies that the thickness of layer i is $H_i - H_{i-1}$ with $H_0 = 0$.

We refer to the pile cross section at the top of the pile as the pile head and to the pile cross section at the base of the pile as the pile base. Since the problem is axisymmetric, we choose a system of cylindrical coordinates with the origin coinciding with the centre of the pile cross section at the pile head and the z axis coinciding with the pile axis (z is positive in the downward direction). One of the assumptions we have made is that the pile and the surrounding soil have perfect compatibility of displacement at the pile–soil interface and at the boundaries between soil layers. In other words, it is assumed that there is no slippage or separation between the pile and the surrounding soil and between soil layers. Furthermore, the soil medium within each layer is assumed to be isotropic, homogeneous, and linear elastic. Since radial and tangential strains are very small compared with the vertical strains, they can be neglected. The vertical displacement $u_z(r, z)$ at any point in the soil is represented as follows:

$$u_z(r,z) = w(z) \cdot \phi(r) \tag{1}$$

where $w(z)$ is the vertical displacement of the pile at a depth equal to z, and $\phi(r)$ is the soil displacement dissipation function in the radial direction. The function $\phi(r)$ is a shape function that determines the rate at which the vertical soil displacement decreases in the radial direction with increasing distance from the pile. Since the vertical displacements within any given cross section of the pile are the same, we assume that $\phi(r) = 1$ from $r = 0$ to $r = r_p$. As the vertical soil displacement is zero as r approaches infinity, we assume that $\phi(r) = 0$ as $r \to \infty$.

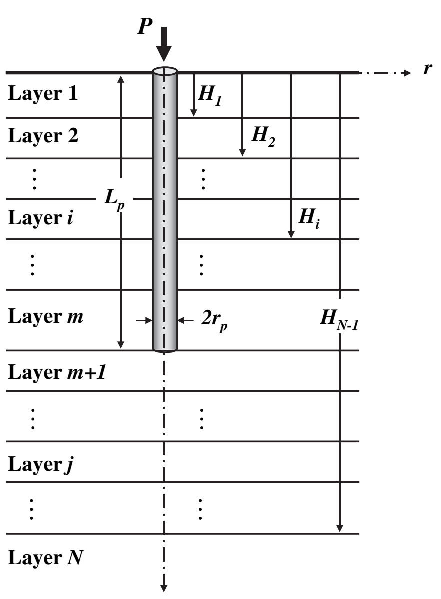

Figure 1. Geometry of the pile–soil system.

2.2 *Stress–strain–displacement relationships*

The stress–strain relationship in an isotropic elastic soil medium can be expressed as:

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{rz} \\ \tau_{\theta z} \end{bmatrix} = \begin{bmatrix} \lambda_s + 2G_s & \lambda_s & \lambda_s & 0 & 0 & 0 \\ \lambda_s & \lambda_s + 2G_s & \lambda_s & 0 & 0 & 0 \\ \lambda_s & \lambda_s & \lambda_s + 2G_s & 0 & 0 & 0 \\ 0 & 0 & 0 & G_s & 0 & 0 \\ 0 & 0 & 0 & 0 & G_s & 0 \\ 0 & 0 & 0 & 0 & 0 & G_s \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{bmatrix} \tag{2}$$

where G_s and λ_s are the elastic constants of the soil. The strain–displacement relationship is given by:

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{bmatrix} = \begin{bmatrix} -\frac{\partial u_r}{\partial r} \\ -\frac{u_r}{r} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\ -\frac{\partial u_z}{\partial z} \\ -\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \\ -\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z} \\ -\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\phi(r)\frac{dw(z)}{dz} \\ 0 \\ -w(z)\frac{d\phi(r)}{dr} \\ 0 \end{bmatrix}. \tag{3}$$

By substituting (3) into (2), we obtain the strain energy density function $W = \sigma_{ij}\varepsilon_{ij}/2$, with summation implied by the repetition of the indices i and j as required in indicial notation:

$$\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left[(\lambda_s + 2G_s)\left(\phi\frac{dw}{dz}\right)^2 + G_s\left(w\frac{d\phi}{dr}\right)^2\right] \tag{4}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors.

2.3 *Governing differential equation for the pile and soil beneath the pile*

The total potential energy Π of an elastic body is defined as the sum of the internal potential energy (the sum of the strain energy U of the pile and soil) and the external potential energy (equal to minus the work done by the external forces applied to the pile in taking it from the at-rest condition to its configuration under load). The total potential energy of the soil–pile system subjected to an axial force P is given by:

$$\begin{aligned} \Pi &= U_{pile} + U_{soil} - Pw(0) \\ &= \frac{1}{2}\int_0^{L_p} E_p A_p \left(\phi\frac{dw}{dz}\right)^2 dz \\ &+ \frac{1}{2}\int_0^{L_p}\int_0^{2\pi}\int_{r_p}^{\infty} \sigma_{ij}\varepsilon_{ij} r dr d\theta dz \\ &+ \frac{1}{2}\int_{L_p}^{\infty}\int_0^{2\pi}\int_0^{\infty} \sigma_{ij}\varepsilon_{ij} r dr d\theta dz - Pw(0) \end{aligned} \tag{5}$$

Substituting (4) into (5) and integrating with respect to θ, we obtain:

$$\Pi = \frac{1}{2}\int_0^{L_p} E_pA_p\left(\phi\frac{dw}{dz}\right)^2 dz + \pi\int_0^{L_p}\int_{r_p}^{\infty}\left[(\lambda_s+2G_s)\left(\phi\frac{dw}{dz}\right)^2 + G_s\left(w\frac{d\phi}{dr}\right)^2\right] rdrdz + \pi\int_{L_p}^{\infty}\int_0^{\infty}\left[(\lambda_s+2G_s)\left(\phi\frac{dw}{dz}\right)^2 + G_s\left(w\frac{d\phi}{dr}\right)^2\right] rdrdz - Pw(0) \quad (6)$$

We can now use calculus of variations to obtain the equilibrium equations. In order for a stable equilibrium state to be reached, the first variation of the potential energy must be zero ($\delta\Pi = 0$). We then take variations on w and ϕ. We obtain the governing differential equations for each domain by equating these variations to zero.

The following differential equation for the pile displacement in any layer i is obtained for $0 \le z \le L_p$:

$$-(E_pA_p + 2t_i)\frac{d^2w_i}{dz^2} + k_iw_i = 0 \text{ for } 0 \le z \le L_p \quad (7)$$

where

$$k_i = 2\pi G_{si}\int_{r_p}^{\infty} r\left(\frac{d\phi}{dr}\right)^2 dr \quad (8)$$

$$t_i = \pi(\lambda_{si} + 2G_{si})\int_{r_p}^{\infty} r\phi^2 dr. \quad (9)$$

Since there are m layers in this interval, equation (7) is valid for $i = 1, \ldots, m$. The parameter k_i has units of FL^{-2} (F and L denote force and length, respectively) and represents the shearing resistance of the soil in the vertical direction and hence the change in shear stress along the radial direction. The parameter t_i has units of force and accounts for the soil resistance to vertical compression.

Similarly, we obtain the following differential equation for the soil displacement in any layer j beneath the pile:

$$-\left[\pi r_p^2(\lambda_{sj} + 2G_{sj}) + 2t_j\right]\frac{d^2w_j}{dz^2} + k_jw_j = 0 \quad \text{for } L_p \le z \le \infty \quad (10)$$

where k_j and t_j are also defined by equations (8) and (9) with j replacing i. Equation (10) is valid for $j = m+1, \ldots, N$.

Equations (7) and (10), which were obtained for different domains, can be consolidated into a single governing differential equation. This can be done by noting that $\lambda_{si} + 2G_{si}$ is a function of the Poisson's ratio ν_{si} and the Young's modulus E_{si} of the soil. This leads to:

$$\lambda_{si} + 2G_{si} = \frac{E_{si}(1-\nu_{si})}{(1+\nu_{si})(1-2\nu_{si})} = \overline{E}_{si} \quad (11)$$

where $\overline{E}_{si}$ is the constrained modulus of the soil for a given layer i. Using this notation, we obtain the governing differential equation for the pile and soil below it:

$$-(E_iA_i + 2t_i)\frac{d^2w_i}{dz^2} + k_iw_i = 0 \quad (12)$$

where $E_i = E_p$ and $A_i = A_p$ when $1 \le i \le m$, $E_i = \overline{E}_{si}$ and $A_i = \pi r_p^2$ when $(m+1) \le i \le N$. This notation for E_i and A_i will be used hereafter unless otherwise stated. Note that both k_i and t_i are a function of ϕ and the shear modulus of the soil.

2.4 *Governing differential equation for the soil surrounding the pile*

As done earlier, we obtain the governing differential equation for the soil surrounding the pile by taking the variation of ϕ and then equating it to zero:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \beta^2\phi = 0 \quad (13)$$

where

$$\beta = \sqrt{\frac{n_s}{m_s}} \quad (14)$$

and m_s and n_s are given by

$$m_s = \sum_{i=1}^{N} G_{si}\int_{H_{i-1}}^{H_i} w_i^2 dz \quad (15)$$

$$n_s = \sum_{i=1}^{N}(\lambda_{si} + 2G_{si})\int_{H_{i-1}}^{H_i}\left(\frac{dw_i}{dz}\right)^2 dz. \quad (16)$$

The parameter m_s has units of FL, and n_s has units of FL^{-1}. Therefore β has units of L^{-1}, and it determines the rate at which the vertical soil displacement diminishes in the radial direction.

3. Solutions for the governing differential equations

3.1 *Solution for the displacement dissipation function* ϕ

Equation (13) is a form of the modified Bessel differential equation, and its general solution is given by:

$$\phi(r) = c_1I_0(\beta r) + c_2K_0(\beta r) \quad (17)$$

where $I_0(\cdot)$ is a modified Bessel function of the first kind of zero order, and $K_0(\cdot)$ is a modified Bessel function of the second kind of zero order.

As discussed earlier, $\phi(r) = 1$ at $r = r_p$, and $\phi = 0$ at $r \to \infty$. These boundary conditions lead to:

$$\phi(r) = \frac{K_0(\beta r)}{K_0(\beta r_p)}. \quad (18)$$

3.2 *Solution for the pile displacement function* w

The general solution of equation (12), which is a second-order linear differential equation, is given by:

$$w_i(z) = B_i e^{\lambda_i z} + C_i e^{-\lambda_i z} \tag{19}$$

where

$$\lambda_i = \sqrt{\frac{k_i}{E_i A_i + 2t_i}} \tag{20}$$

and B_i and C_i are integration constants. We obtain the pile axial strain by differentiating (19) with respect to z. Based on the relationship between axial strain and axial force, we obtain:

$$Q_i(z) = -(E_i A_i + 2t_i)\frac{dw_i}{dz} \tag{21}$$

where $Q_i(z)$ is the axial load acting on the pile at a depth z in the i^{th} layer. Then we obtain the following equation for the axial load in the pile at a given cross section:

$$Q_i(z) = -a_i B_i e^{\lambda_i z} + a_i C_i e^{-\lambda_i z} \tag{22}$$

where

$$a_i = \lambda_i (E_i A_i + 2t_i) = \sqrt{k_i(E_i A_i + 2t_i)}. \tag{23}$$

As we have $2N$ unknown integration constants (B_1, C_1, B_2, $C_2, \ldots, B_N$, C_N), we need to identify $2N$ boundary conditions in order to determine their values. First, the vertical soil displacement at an infinite depth below the pile base must be zero. Secondly, the magnitude of the load at the pile head should be equal to the applied external load. Finally, displacement and force should be the same at the interface between any two layers when calculated with the properties of either layer. These give us the $2N$ boundary conditions, and we can determine all the integration constants. These boundary conditions can be expressed as follows:

$$w_N(z)|_{z\to\infty} = 0 \tag{24}$$

$$Q_1(z)|_{z=0} = P \tag{25}$$

$$e^{\lambda_i H_i} B_i + e^{-\lambda_i H_i} C_i - e^{\lambda_{i+1} H_i} B_{i+1} - e^{-\lambda_{i+1} H_i} C_{i+1} = 0 \quad \text{for } 1 \le i \le N-1 \tag{26}$$

$$-a_i e^{\lambda_i H_i} B_i + a_i e^{-\lambda_i H_i} C_i + a_{i+1} e^{\lambda_{i+1} H_i} B_{i+1} - a_{i+1} e^{-\lambda_{i+1} H_i} C_{i+1} = 0 \quad \text{for } 1 \le i \le N-1. \tag{27}$$

From equations (19) and (24) and equations (22) and (25), we obtain:

$$B_N = 0 \tag{28}$$

$$-a_1 B_1 + a_1 C_1 = P. \tag{29}$$

No matter how many layers we have, equations (28) and (29) always apply and remain unchanged. Equations (26) to (29) can be expressed in matrix form as follows:

$$[M][X] = [V] \tag{30}$$

where

$$[M] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 \\ -a_1 & a_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{\lambda_1 H_1} & e^{-\lambda_1 H_1} & -e^{\lambda_2 H_1} & -e^{-\lambda_2 H_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 e^{\lambda_1 H_1} & a_1 e^{-\lambda_1 H_1} & a_2 e^{\lambda_2 H_1} & -a_2 e^{-\lambda_2 H_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_2 H_2} & e^{-\lambda_2 H_2} & -e^{\lambda_3 H_2} & -e^{-\lambda_3 H_2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 e^{\lambda_2 H_2} & a_2 e^{-\lambda_2 H_2} & a_3 e^{\lambda_3 H_2} & -a_3 e^{-\lambda_3 H_2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & e^{\lambda_{N-1} H_{N-1}} & e^{-\lambda_{N-1} H_{N-1}} & -e^{\lambda_N H_{N-1}} & -e^{-\lambda_N H_{N-1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{N-1} e^{\lambda_{N-1} H_{N-1}} & a_{N-1} e^{-\lambda_{N-1} H_{N-1}} & a_N e^{\lambda_N H_{N-1}} & -a_N e^{-\lambda_N H_{N-1}} \end{bmatrix} \tag{31}$$

$$[X] = \begin{bmatrix} B_1 \\ C_1 \\ B_2 \\ C_2 \\ \vdots \\ B_{N-1} \\ C_{N-1} \\ B_N \\ C_N \end{bmatrix} \tag{32}$$

$$[V] = \begin{bmatrix} 0 \\ P \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{33}$$

The dimensions of $[M]$, $[X]$, and $[V]$ are $[2N \times 2N]$, $[2N \times 1]$, and $[2N \times 1]$, respectively. If we solve equation (30), which can be done either analytically or numerically, we can determine the integration constants. However, a more efficient way of determining all the integration constants is to find a recurrence relation based on the boundary conditions. For this purpose, we rewrite equations (19) and (22) in matrix form:

$$\begin{bmatrix} w_i(z) \\ Q_i(z) \end{bmatrix} = \begin{bmatrix} e^{\lambda_i z} & e^{-\lambda_i z} \\ -a_i e^{\lambda_i z} & a_i e^{-\lambda_i z} \end{bmatrix} \begin{bmatrix} B_i \\ C_i \end{bmatrix} \tag{34}$$

From the continuity condition of displacement and force at the interface between layers, we obtain:

$$\begin{bmatrix} w_i(H_i) \\ Q_i(H_i) \end{bmatrix} = \begin{bmatrix} w_{i+1}(H_i) \\ Q_{i+1}(H_i) \end{bmatrix} \tag{35}$$

Equations (34) and (35) give us the following recurrence formula for the integration constants:

$$\begin{bmatrix} B_i \\ C_i \end{bmatrix} = \frac{1}{2a_i} \begin{bmatrix} (a_i + a_{i+1}) e^{-(\lambda_i H_i - \lambda_{i+1} H_i)} & (a_i - a_{i+1}) e^{-(\lambda_i H_i + \lambda_{i+1} H_i)} \\ (a_i - a_{i+1}) e^{(\lambda_i H_i + \lambda_{i+1} H_i)} & (a_i + a_{i+1}) e^{(\lambda_i H_i - \lambda_{i+1} H_i)} \end{bmatrix} \begin{bmatrix} B_{i+1} \\ C_{i+1} \end{bmatrix}$$

for $1 \le i \le N-1$. (36)

Therefore, if we determine B_N and C_N, we can determine all the B_i and C_i, in sequence.

Using Cramer's rule, the constants B_i and C_i are obtained from

$$B_i = \frac{|M_{2i-1}|}{|M|} \tag{37}$$

$$C_i = \frac{|M_{2i}|}{|M|} \tag{38}$$

where $|M|$ is the determinant of $[M]$, and $|M_k|$ is the determinant of $[M]$ with the k^{th} column replaced by the vector $[V]$. In order for a given problem to have physical meaning, $|M|$ must not be zero. Therefore, from (28) and (37), we obtain:

$$|M_{2N-1}| = 0. \tag{39}$$

Similarly, C_N is given by

$$C_N = \frac{|M_{2N}|}{|M|} \tag{40}$$

where

$$|M_{2N}| = \begin{vmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ -a_1 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & P \\ e^{\lambda_1 H_1} & e^{-\lambda_1 H_1} & -e^{\lambda_2 H_1} & -e^{-\lambda_2 H_1} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{N-1} e^{\lambda_{N-1} H_{N-1}} & a_{N-1} e^{-\lambda_{N-1} H_{N-1}} & a_N e^{\lambda_N H_{N-1}} & 0 \end{vmatrix} \tag{41}$$

The determinant of $[M_{2N}]$ is:

$$|M_{2N}| = 2^{N-1} P \prod_{i=1}^{N-1} a_i \tag{42}$$

where the symbol Π indicates a product:

$$\prod_{i=1}^{k} x_i = x_1 x_2 x_3 \cdots x_k.$$

If we substitute equations (37) and (38) into equation (36), we obtain:

$$\begin{bmatrix} |M_{2i-1}| \\ |M_{2i}| \end{bmatrix} = \frac{1}{2a_i} \begin{bmatrix} (a_i + a_{i+1}) e^{-(\lambda_i H_i - \lambda_{i+1} H_i)} & (a_i - a_{i+1}) e^{-(\lambda_i H_i + \lambda_{i+1} H_i)} \\ (a_i - a_{i+1}) e^{(\lambda_i H_i + \lambda_{i+1} H_i)} & (a_i + a_{i+1}) e^{(\lambda_i H_i - \lambda_{i+1} H_i)} \end{bmatrix} \begin{bmatrix} |M_{2i+1}| \\ |M_{2i+2}| \end{bmatrix}$$

for $1 \le i \le N-1$. (43)

In order to obtain $|M|$, we use the boundary condition at the pile head. By substituting $B_1 = |M_1|/|M|$ and $C_1 = |M_2|/|M|$ into equation (29), we obtain the following relationship:

$$|M| = \frac{a_1}{P}(|M_2| - |M_1|). \tag{44}$$

The numerators in equations (37) and (38) can be determined recurrently from equation (43) by using equations (39) and (42) as its ignition terms. The denominators in equations (37) and (38) are obtained from equation (44). Finally, we determine all the integration constants using equations (37) and (38). The displacement and force at each layer follow from equations (19) and (22), respectively.

Using this procedure, we can obtain explicit analytical solutions for a vertically loaded pile installed in a soil with N layers. Table 1 shows the analytical solutions for the cases of two, three, and four layers. An example of the three-layer case is given in the Appendix.

In design, we are interested in estimating the settlement at the pile head when the pile is subjected to the design load. This can be obtained from the solution for the displacement within the first layer:

$$w_t = w_1(0) = B_1 + C_1 = \frac{|M_1|}{|M|} + \frac{|M_2|}{|M|} \tag{45}$$

The explicit analytical solutions presented in this paper have the advantage that they can easily be computer coded.

3.3 *Solution for a pile embedded in a layered soil resting on a rigid base*

Piles are often socketed in a competent layer or rock to obtain a large base capacity. If we know the elastic properties of such a layer, we can use the solution presented in the previous section. We can also obtain analytical solutions for a vertically loaded pile with the base resting on a rigid material which can be used when the elastic properties of the bearing layer are unknown but it is known that it is very stiff. We can do this by restricting the vertical displacement at the base of the pile to zero. The pile–soil system considered here is shown in Figure 2.

In this case, we have zero displacement at the base of the pile instead of at infinity. All other boundary conditions remain the same. Therefore only equation (24) changes:

Table 1. Analytical solutions for a vertically loaded pile installed in profiles consisting of two, three, and four layers

	Number of layers		
$\lvert M_i \rvert$	Two layers	Three layers	Four layers
$\lvert M_1 \rvert$	$P(a_1 - a_2)e^{-(\lambda_1 H_1 + \lambda_2 H_1)}$	$P[(a_1 + a_2)(a_2 - a_3)e^{-(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 + \lambda_3 H_2)}$ $+ (a_1 - a_2)(a_2 + a_3)e^{-(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 + \lambda_3 H_2)}]$	$P[(a_1 + a_2)(a_2 + a_3)(a_3 - a_4)e^{-(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 + \lambda_4 H_3)}$ $+ (a_1 + a_2)(a_2 - a_3)(a_3 + a_4)e^{-(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 + \lambda_4 H_3)}$ $+ (a_1 - a_2)(a_2 - a_3)(a_3 - a_4)e^{-(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 + \lambda_4 H_3)}$ $+ (a_1 - a_2)(a_2 + a_3)(a_3 + a_4)e^{-(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 + \lambda_4 H_3)}]$
$\lvert M_2 \rvert$	$P(a_1 + a_2)e^{(\lambda_1 H_1 - \lambda_2 H_1)}$	$P[(a_1 - a_2)(a_2 - a_3)e^{(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 - \lambda_3 H_2)}$ $+ (a_1 + a_2)(a_2 + a_3)e^{(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 - \lambda_3 H_2)}]$	$P[(a_1 - a_2)(a_2 + a_3)(a_3 - a_4)e^{(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 - \lambda_4 H_3)}$ $+ (a_1 - a_2)(a_2 - a_3)(a_3 + a_4)e^{(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 - \lambda_4 H_3)}$ $+ (a_1 + a_2)(a_2 - a_3)(a_3 - a_4)e^{(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 - \lambda_4 H_3)}$ $+ (a_1 + a_2)(a_2 + a_3)(a_3 + a_4)e^{(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 - \lambda_4 H_3)}]$
$\lvert M_3 \rvert$	0	$2Pa_1(a_2 - a_3)e^{-(\lambda_2 H_2 + \lambda_3 H_2)}$	$2a_1 P[(a_2 + a_3)(a_3 - a_4)e^{-(\lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 + \lambda_4 H_3)}$ $+ (a_2 - a_3)(a_3 + a_4)e^{-(\lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 + \lambda_4 H_3)}]$
$\lvert M_4 \rvert$	$2Pa_1$	$2Pa_1(a_2 + a_3)e^{(\lambda_2 H_2 - \lambda_3 H_2)}$	$2a_1 P[(a_2 - a_3)(a_3 - a_4)e^{(\lambda_2 H_2 + \lambda_3 H_2 - \lambda_3 H_3 - \lambda_4 H_3)}$ $+ (a_2 + a_3)(a_3 + a_4)e^{(\lambda_2 H_2 - \lambda_3 H_2 + \lambda_3 H_3 - \lambda_4 H_3)}]$
$\lvert M_5 \rvert$	–	0	$4a_1 a_2 P(a_3 - a_4)e^{-(\lambda_3 H_3 + \lambda_4 H_3)}$
$\lvert M_6 \rvert$	–	$4Pa_1 a_2$	$4a_1 a_2 P(a_3 + a_4)e^{(\lambda_3 H_3 - \lambda_4 H_3)}$
$\lvert M_7 \rvert$	–	–	0
$\lvert M_8 \rvert$	–	–	$8Pa_1 a_2 a_3$
$\lvert M \rvert$	$\lvert M \rvert = \frac{a_1}{P}(\lvert M_2 \rvert - \lvert M_1 \rvert)$	$\lvert M \rvert = \frac{a_1}{P}(\lvert M_2 \rvert - \lvert M_1 \rvert)$	$\lvert M \rvert = \frac{a_1}{P}(\lvert M_2 \rvert - \lvert M_1 \rvert)$

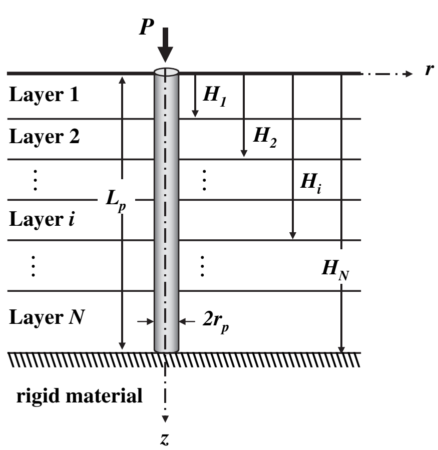

Figure 2. Pile embedded in a multilayered soil with the base resting on a rigid material.

$$w_N(z)|_{z \to L_p} = 0. \quad (46)$$

This gives us:

$$e^{\lambda_N L_p} B_N + e^{-\lambda_N L_p} C_N = 0. \quad (47)$$

We now have a new matrix $[M]$ for the case of a vertically loaded pile with the base over a rigid material. As before, we can calculate $|M_{2N-1}|$ and $|M_{2N}|$:

$$|M_{2N-1}| = -2^{N-1} e^{-\lambda_N L_p} P \prod_{i=1}^{N-1} a_i \quad (48)$$

$$|M_{2N}| = 2^{N-1} e^{\lambda_N L_p} P \prod_{i=1}^{N-1} a_i. \quad (49)$$

Using these two ignition terms and (43), we can obtain explicit analytical solutions for this case as well.

3.4 *Solution scheme*

Since the parameter β appearing in equation (18) is not known a priori, an iterative procedure is required to obtain the exact solutions. First an initial value is assumed for β, and the k_i and t_i values of equations (8) and (9) are calculated. The pile displacements in each layer are calculated from these values using the analytical solutions given in Table 1. The calculated pile displacements in each layer are then used to obtain β using equation (14), and the resulting value is compared with the assumed initial value. Since β has units of L^{-1}, iterations are repeated until the difference between the values of the dimensionless parameter βr_p obtained from two consecutive iterations falls below the prescribed convergence tolerance. We found that an exact solution is generally obtained in less than 10 iterations, regardless of the initial value of β assumed. The details of the solution steps are given in the form of a flowchart in Figure 3.

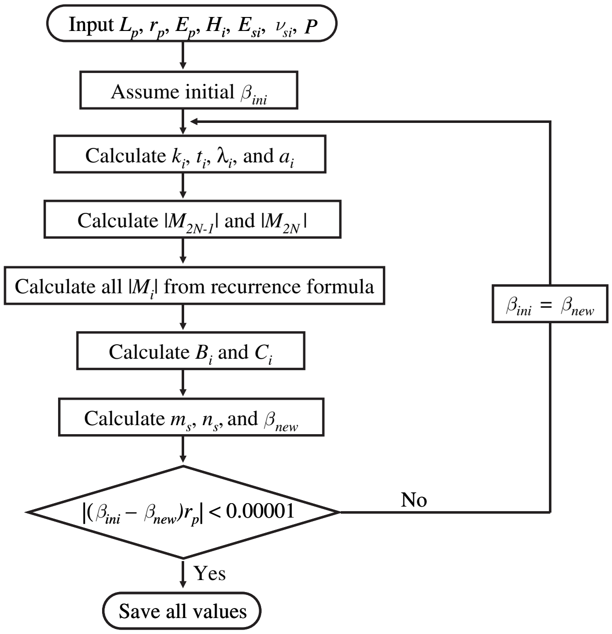

Figure 3. Flowchart for the iterative procedure used to determine the parameter β.

4. Comparison with previous analyses

It is useful to compare our analysis and its results with previous analyses of the same problem. We consider the analyses of Poulos (1979) and Lee (1991).

Poulos (1979) analysed the settlement of a single pile in nonhomogeneous soil using the method proposed by Mattes and Poulos (1969). In this analysis, the pile is divided into a number of equal cylindrical elements, with any element j being acted upon by a shear stress τ_j. The expressions for the pile displacements are obtained from the vertical equilibrium of a small cylindrical element of the pile, assuming that the pile deforms in simple axial compression. The vertical displacements of the soil due to the shear stress along the pile shaft are obtained by double integration of the Mindlin equation for vertical displacement. To calculate the displacement of the soil at any element i due to the shear stress τ_j on element j, the average of the Young's moduli of soil elements i and j was used for analysis of nonhomogeneous soils. By imposing a no-slippage condition at the pile–soil interface, the shear stresses and the displacements along the pile can be calculated. The solutions obtained were compared with those from finite element analysis for the three idealized cases shown in Figure 4. The solutions are given in terms of a settlement influence factor I_w defined by:

$$I_w = \frac{E_{s,ref} B w_t}{P} \tag{50}$$

where $E_{s,ref}$ is the reference Young's modulus of soil, B ($=2r_p$) is the pile diameter, w_t is the settlement at the pile head, and P is the applied load.

Lee (1991) expanded the approach of Randolph and Wroth (1978) to layered soil. Lee's analysis accounts for the effect of the change in shear stress in the radial direction. Like the analysis of Randolph and Wroth (1978), it relies on the concept of the magical radius r_m at which the displacement becomes negligible. Both Randolph and Wroth (1978) and Lee (1991) did not include the vertical compressive stress in the equilibrium equation.

To consider the differences between the three analyses, we performed calculations for the same cases proposed by Poulos (1979) which were also used by Lee (1991) for validation of his analysis. To use our analysis for these cases, we divided the soil profile into five layers, with the bottom of the third layer flush with the base of the pile. The fourth layer extends from a depth of L to $2L$, and the fifth layer extends from $2L$ to infinity. The same value for the Young's modulus of the soil was used for the third and fourth layers. For the rigid base (fifth layer), $E_{s5} = 10^{10}E_p$ was used. Our results are given together with those of Poulos (1979) and Lee (1991) in Table 2. The Poulos's analysis produces the largest values of I_w, while the results from the solutions proposed in this paper yield the smallest values. Our solution is slightly stiffer than that of Poulos (1979) who neglected the boundary condition in the radial direction (zero soil displacement at $r = \infty$) and considered only the shear stress at the pile–soil interface (the resistance of the soil to compression was ignored). Lee (1991) also neglected the resistance of the soil to compression.

The solutions proposed in this paper consider both the resistance offered by the soil to shearing along the radial direction and the resistance of the soil to vertical compression. These shear and compression soil resistances are reflected in the parameters k_i and t_i, respectively, which appear in equations (8) and (9). The smallest values of I_w are obtained with the solutions presented in this paper because they account for both these soil resistances.

5. Case study

Russo (2004) presented a case history of micropiles used for underpinning a historical building in Naples, Italy. The micropiles were installed in a complex soil profile (there are thick layers of man-made materials accumulated over millennia at the site). According to Russo (2004), the installation steps were as follows: (1) drilling a 200-mm-diameter hole using a continuous flight auger, (2) inserting a steel pipe equipped with injection valves, (3) filling the annular space between the pipe and the soil with grout, (4) grouting the pile shaft through each valve using a double packer, and (5) filling the steel pipe with grout. A micropile (0.2 m in diameter and 19 m long) was load tested. Two anchor piles were used to provide reaction to the loading frame, and the compressive load was applied to the test pile with a hydraulic jack. The vertical displacement of the pile head was measured using linear variable differential transformers (LVDTs), and the axial strain along the shaft was measured with vibrating-wire strain gages.

Russo (2004) compared the pile load test results with those obtained from finite element analysis. The soil profile and the

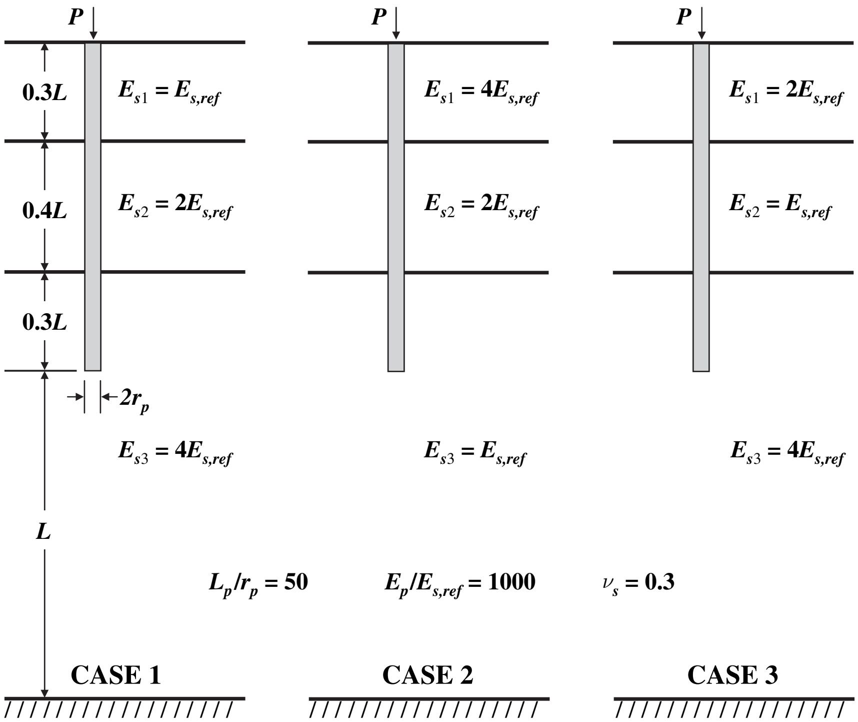

Figure 4. Layered soil profiles for analyses (modified from Poulos 1979).

Table 2. Comparison of solutions in layered soil

	Settlement influence factor $I_w = E_{s,ref}Bw_t/P$			
Case	Poulos 1979	Poulos 1979 (FEM)	Lee 1991	Present solution
1	0.0386	0.0377	0.0361	0.0336
2	0.0330	0.0430	0.0372	0.0309
3	0.0366	0.0382	0.0358	0.0323

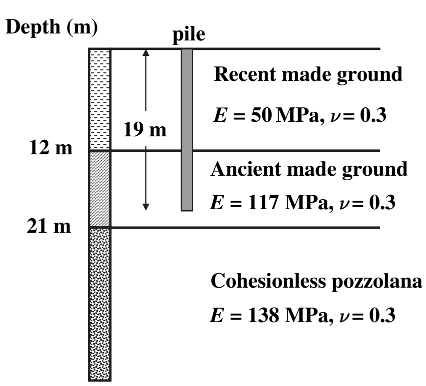

Figure 5. Soil profile and elastic properties of each layer (after Russo 2004).

elastic properties of each soil layer are shown in Figure 5. Young's moduli of each soil layer were back-calculated from the finite element analysis. The Poisson's ratio was assumed to be 0.3 for all the soil layers. Although Russo (2004) did not provide information on the geometry and properties of the steel pipe left inside the micropile, its outer diameter and inner diameter were assumed to be 33.4 mm and 25.4 mm, respectively. Accordingly, assuming that The Young's moduli of the steel and grout are 200 GPa and 25 GPa, respectively, the equivalent Young's modulus of the composite steel–grout cross section is calculated to be approximately 27 GPa.

Table 3 shows the input values used in the analysis. As the soil profile consists of four layers, we used the analytical solution given in Table 1. By following the iterative procedure outlined in Figure 3, we obtained the results shown in Table 4. All these calculations can be performed easily with mathematical software such as MathCAD or MATLAB.

Table 3. Input values for analysis ($r_p = 0.1$ m, $E_p = 27$ GPa*)

Layer (i)	H_i (m)	E_{si} (MPa)	ν_{si}*
1	12	50	0.3
2	19	117	0.3
3	21	117	0.3
4	∞	138	0.3

*Assumed values.

Table 4. Results of analysis ($P = 542$ kN)

Layer	$\phi(r)$	$w_i(z)$	$Q_i(z)$
1		$-1.239 \times 10^{-5}e^{0.1719z} + 3.146 \times 10^{-3}e^{-0.1719z}$	$-2.126 \times 10^{3}e^{0.1719z} + 5.399 \times 10^{5}e^{-0.1719z}$
2	$\frac{K_0(0.3344r)}{K_0(0.03344)}$	$1.650 \times 10^{-7}e^{0.2399z} + 5.327 \times 10^{-3}e^{-0.2399z}$	$4.748 \times 10^{1}e^{0.2399z} + 1.533 \times 10^{6}e^{-0.2399z}$
3		$-2.408 \times 10^{-10}e^{0.4400z} + 3.105 \times 10^{-1}e^{-0.4400z}$	$-3.777 \times 10^{-2}e^{0.4400z} + 4.871 \times 10^{7}e^{-0.4400z}$
4		$2.849 \times 10^{-1}e^{-0.4400z}$	$5.272 \times 10^{7}e^{-0.4400z}$

Figure 6 shows measured and calculated load-transfer curves for applied loads of 51, 253, and 542 kN. Figure 7 shows both the measured load–settlement curve and that calculated using the solution presented in this paper. These figures show that there is very good agreement between the calculated and measured values, although the calculated pile head settlement is smaller than the measured value for loads greater than about 400kN.

Figure 8 shows the calculated vertical pile displacement versus depth. The vertical pile displacement decreases exponentially with increasing depth. Figure 9 shows the vertical soil displacement as a function of the radial distance from the pile.

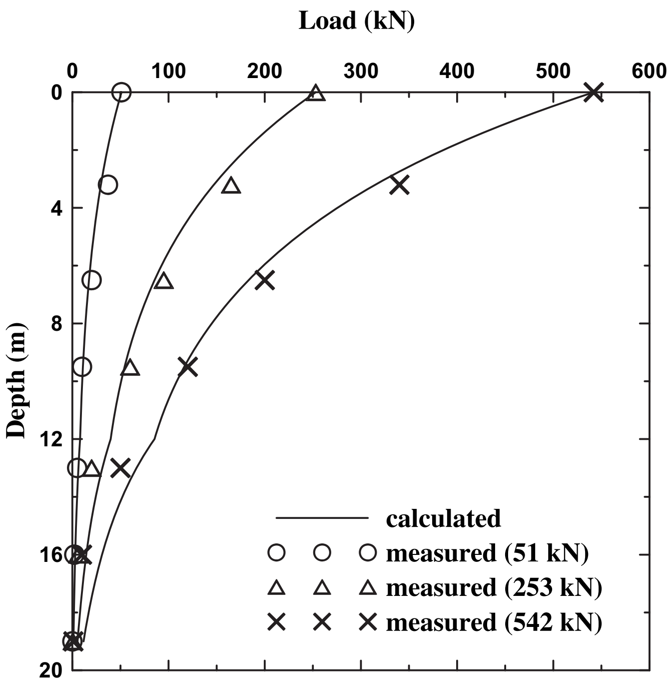

Figure 6. Measured and calculated load–transfer curves for applied loads of 51, 253, and 542 kN.

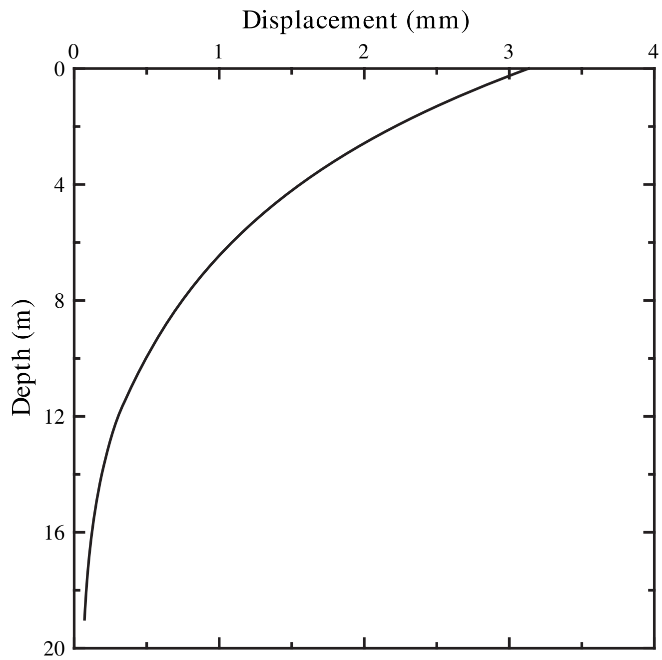

Figure 8. Calculated vertical pile displacement versus depth.

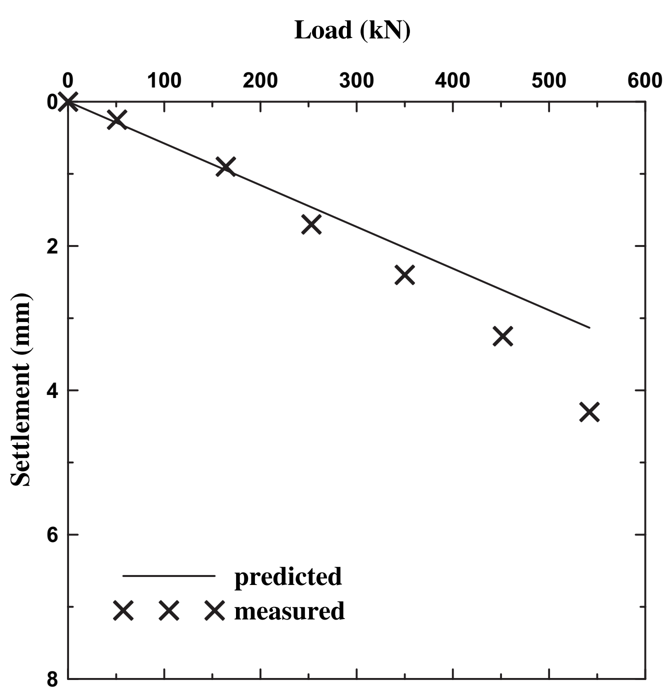

Figure 7. Comparison of the calculated and measured pile head settlement values as a function of the load applied at the pile head.

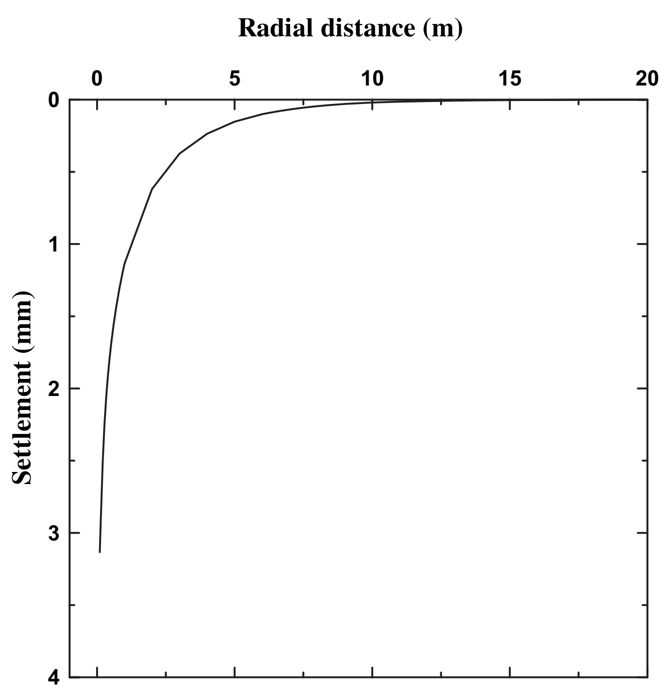

Figure 9. Calculated vertical soil displacement at the level of the pile head as a function of the radial distance from the pile axis.

Since the present solution satisfies radial boundary conditions, the vertical soil displacement at the level of the pile head becomes negligible at a distance of about 10 m from the pile axis, which corresponds to 50 times the pile diameter.

6. Conclusions

Explicit analytical solutions for a vertically loaded pile embedded in a multilayered soil have been presented in this paper. The solutions satisfy the boundary conditions of the problem. The soil is assumed to behave as a linear elastic material. The governing differential equations are derived based on the principle of minimum potential energy and calculus of variations. The integration constants are determined using Cramer's rule and a recurrence formula. In addition, solutions for a pile embedded in a multilayered soil with the base resting on a rigid material are obtained by changing the boundary conditions of the problem. The solutions provide the vertical pile displacement as a function of depth, the load-transfer curves, and the vertical soil displacement as a function of the radial direction at any depth if the following are known: radius, length, and Young's modulus of the pile, Poisson's ratio and Young's modulus of the soil in each layer, thickness of each soil layer, number of soil layers, and applied load. The use of the analysis was illustrated by obtaining load–transfer and load–settlement curves for a case available in the literature for which pile load test results are available.

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Appendix. Analytical solution for a profile consisting of three layers

We will illustrate how we can obtain analytical solutions using the recurrence procedures described previously. Since we have three layers, $N = 3$. From (39) and (42), we obtain:

$$|M_5| = 0 \tag{A1}$$

$$|M_6| = 2^2 P \prod_{i=1}^{2} a_i = 4Pa_1a_2 \tag{A2}$$

If we substitute equations (A1) and (A2) into equation (43), we obtain:

$$\begin{aligned} \begin{bmatrix} |M_3| \\ |M_4| \end{bmatrix} &= \frac{1}{2a_2} \begin{bmatrix} (a_2+a_3)e^{-(\lambda_2 H_2 - \lambda_3 H_2)} & (a_2-a_3)e^{-(\lambda_2 H_2 + \lambda_3 H_2)} \\ (a_2-a_3)e^{(\lambda_2 H_2 + \lambda_3 H_2)} & (a_2+a_3)e^{(\lambda_2 H_2 - \lambda_3 H_2)} \end{bmatrix} \begin{bmatrix} 0 \\ 4Pa_1a_2 \end{bmatrix} \\ &= 2a_1 P \begin{bmatrix} (a_2-a_3)e^{-(\lambda_2 H_2 + \lambda_3 H_2)} \\ (a_2+a_3)e^{(\lambda_2 H_2 - \lambda_3 H_2)} \end{bmatrix} \end{aligned} \tag{A3}$$

$$\begin{aligned} \begin{bmatrix} |M_1| \\ |M_2| \end{bmatrix} &= \frac{1}{2a_1} \begin{bmatrix} (a_1+a_2)e^{-(\lambda_1 H_1 - \lambda_2 H_1)} & (a_1-a_2)e^{-(\lambda_1 H_1 + \lambda_2 H_1)} \\ (a_1-a_2)e^{(\lambda_1 H_1 + \lambda_2 H_1)} & (a_1+a_2)e^{(\lambda_1 H_1 - \lambda_2 H_1)} \end{bmatrix} \begin{bmatrix} |M_3| \\ |M_4| \end{bmatrix} \\ &= P \begin{bmatrix} (a_1+a_2)(a_2-a_3)e^{-(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 + \lambda_3 H_2)} \\ \quad +(a_1-a_2)(a_2+a_3)e^{-(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 + \lambda_3 H_2)} \\ (a_1-a_2)(a_2-a_3)e^{(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 - \lambda_3 H_2)} \\ \quad +(a_1+a_2)(a_2+a_3)e^{(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 - \lambda_3 H_2)} \end{bmatrix} \end{aligned} \tag{A4}$$

Finally, $|M|$ is obtained from equation (44):

$$\begin{aligned} |M| &= \frac{a_1}{P}(|M_2| - |M_1|) \\ &= a_1(a_1-a_2)(a_2-a_3)e^{(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 - \lambda_3 H_2)} \\ &\quad + a_1(a_1+a_2)(a_2+a_3)e^{(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 - \lambda_3 H_2)} \\ &\quad - a_1(a_1+a_2)(a_2-a_3)e^{-(\lambda_1 H_1 - \lambda_2 H_1 + \lambda_2 H_2 + \lambda_3 H_2)} \\ &\quad - a_1(a_1-a_2)(a_2+a_3)e^{-(\lambda_1 H_1 + \lambda_2 H_1 - \lambda_2 H_2 + \lambda_3 H_2)} \end{aligned} \tag{A5}$$

Now, by simply calculating $|M_i|/|M|$, we obtain the analytical solutions for the vertical settlement of the pile and the axial load in the pile at a given cross section.