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HIGHER-ORDER LOGIC

INTRODUCTION

What is nowadays the central part of any introduction to logic, and indeed to some the logical theory par excellence, used to be a modest fragment of the more ambitious language employed in the logicist program of Frege and Russell. 'Elementary' or 'first-order', or 'predicate logic' only became a recognized stable base for logical theory by 1930, when its interesting and fruitful meta-properties had become clear, such as completeness, compactness and Löwenheim-Skolem. Richer higher-order and type theories receded into the background, to such an extent that the (re-) discovery of useful and interesting extensions and variations upon first-order logic came as a surprise to many logicians in the sixties.

In this chapter, we shall first take a general look at first-order logic, its properties, limitations, and possible extensions, in the perspective of so-called 'abstract model theory'. Some characterizations of this basic system are found in the process, due to Lindström, Keisler-Shelah and Fraïssé. Then, we go on to consider the original mother theory, of which first-order logic was the elementary part, starting from second-order logic and arriving at Russell's theory of finite types. As will be observed repeatedly, a border has been crossed here with the domain of set theory; and we proceed, as Quine has warned us again and again, at our own peril. Nevertheless, first-order logic has a vengeance. In the end, it turns out that higher-order logic can be viewed from an elementary perspective again, and we shall derive various insights from the resulting semantics.

Before pushing off, however, we have a final remark about possible pretensions of what is to follow. Unlike first-order logic and some of its less baroque extensions, second and higher-order logic have no coherent well-established theory; the existent material consisting merely of scattered remarks quite diverse with respect to character and origin. As the time available for the present enterprise was rather limited (to say the least) the authors do not therefore make any claims as to complete coverage of the relevant literature.

1 FIRST-ORDER LOGIC AND ITS EXTENSIONS

The starting point of the present story lies somewhere within Hodges's chapter (I.1). We will review some of the peculiarities of first-order logic, in order to set the stage for higher-order logics.

1.1 Limits of Expressive Power

In addition to its primitives *all* and *some*, a first-order predicate language with identity can also express such quantifiers as *precisely one, all but two, at most three*, etcetera, referring to specific finite quantities. What is lacking, however, is the general mathematical concept of *finiteness*.

EXAMPLE. The notion 'finiteness of the domain' is not definable by means of any first-order sentence, or set of such sentences.

It will be recalled that the relevant refutation turned on the *compactness theorem* for first-order logic, which implies that sentences with arbitrarily large finite models will also have infinite ones.

Another striking omission, this time from the perspective of natural language, is that of common quantifiers, such as *most, least*, not to speak of *many* or *few*.

EXAMPLE. The notion 'most A are B' is not definable in a first-order logic with identity having, at least, unary predicate constants A, B. This time, a refutation involves both compactness and the (downward) *Löwenheim-Skolem theorem*: Consider any proposed definition $\mu(A, B)$ together with the infinite set of assertions 'at least n A are B', 'at least n A are not B' $(n = 1, 2, 3, \ldots)$. Any finite subset of this collection is satisfiable in some finite domain with $A - B$ large enough and $A \cap B$ a little larger. By compactness then, the whole collection has a model with infinite $A \cap B$, $A - B$. But now, the Löwenheim-Skolem theorem gives a countably infinite such model, which makes the latter two sets equinumerous — and 'most' A are no longer B: in spite of $\mu(A, B)$.

One peculiarity of this argument is its lifting the meaning of colloquial 'most' to the infinite case. The use of infinite models is indeed vital in the coming sections. Only in Section 1.4.3 shall we consider the purely *finite* case: little regarded in mathematically-oriented model theory, but rather interesting for the semantics of natural language.

In a sense, these expressive limits of first-order logic show up more dramatically in a slightly different perspective. A given *theory* in a first-order language may possess various 'non-standard models', not originally intended. For instance, by compactness, Peano Arithmetic has non-Archimedean models featuring infinite natural numbers. And by Löwenheim-Skolem, Zermelo-Fraenkel set theory has countable models (if consistent), a phenomenon known as 'Skolem's Paradox'. Conversely, a given model may not be defined categorically by its complete first-order theory, as is in fact known for al (infinite) mathematical standard structures such as integers, rationals or reals. (These two observations are sides of the same coin, of course.) Weakness or strength carry no moral connotations in logic, however, as one may turn into the other. Non-standard models for analysis have turned out quite useful for their own sake, and countable models of set theory are at the base of the independence proofs: first-order logic's loss thus can often be the mathematician's or philosopher's gain.

1.2 Extensions

When some reasonable notion falls outside the scope of first-order logic, one rather natural strategy is to add it to the latter base and consider the resulting stronger logic instead. Thus, for instance, the above two examples inspire what is called 'weak second-order logic', adding the quantifier 'there exist finitely many', as well as first-order logic with the added 'generalized quantifier' *most*. But, there is a price to be paid here. Inevitably, these logics lose some of the meta-properties of first-order logic employed in the earlier refutations of definability. Here is a telling little table:

	Compactness	Löwenheim-Sk.
First-order logic	yes	yes
Plus 'there exists finitely many'	no	yes
Plus 'there exist uncountably many'	yes	no
Plus 'most'	no	no

For the second and third rows, cf. [Monk, 1976, Chapter 30]. For the fourth row, here is an argument.

EXAMPLE. Let the most-sentence $\varphi(R)$ express that R is a discrete linear order with end points, possessing a greatest point with more successors than non-successors (i.e. most points in the order are its successors). Such orders can only be finite, though of arbitrarily large size: which contradicts compactness. Next, consider the statement that R is a dense linear order without end points, possessing a point with more successors than predecessors. There are uncountable models of this kind, but no countable ones: and hence Löwenheim-Skolem fails.

As it happens, no proposed proper extension of first-order logic ever managed to retain both the compactness and Löwenheim-Skolem properties. And indeed, in 1969 Lindström proved his famous theorem [Lindström, 1969] that, given some suitable explication of a 'logic', first-order logic is indeed characterizable as the strongest logic to possess these two meta-properties.

1.3 Abstract Model Theory

Over the past two decades, many types of extension of first-order logic have been considered. Again, the earlier two examples illustrate general patterns. First, there are so-called *finitary* extensions, retaining the (effective) finite syntax of first-order logic. The *most* example inspires two general directions of this kind.

First, one may add *generalized quantifiers* Q, allowing patterns

$$Qx \cdot \varphi(x) \text{ or } Qxy \cdot \varphi(x), \psi(y).$$

E.g. 'the φ's fill the universe' (*all*), 'the φ's form the majority in the universe' (*most*), 'the φ's form the majority of the ψ's' (most ψ are φ). But also, one may stick with the old types of quantifier, while employing them with new ranges. For instance, 'most A are B' may be read as an ordinary quantification over functions: 'there exists a 1–1 correspondence between A–B and some subset of $A \cap B$, but not vice versa'. Thus, one enters the domain of *higher-order logic*, to be discussed in later sections.

The earlier example of 'finiteness' may lead to finitary extensions of the above two kinds, but also to an *infinitary* one, where the syntax now allows infinite conjunctions and disjunctions, or even quantifications. For instance, finiteness may be expressed as 'either one, or two, or three, or ... ' in $L_{\omega_1\omega}$: a first-order logic allowing countable conjunctions and disjunctions of formulas (provided that they have only finitely many free variables together) and finite quantifier sequences. Alternatively, it may be expressed as 'there are no $x_1, x_2, \ldots$: all distinct', which would belong to $L_{\omega_1\omega_1}$, having a countably infinite quantifier string. In general, logicians have studied a whole family of languages $L_{\alpha\beta}$; but $L_{\omega_1\omega}$ remains the favourite (cf. [Keisler, 1971]).

Following Lindström's result, a research area of 'abstract model theory' has arisen where these various logics are developed and compared. Here is one example of a basic theme. Every logic L 'casts its net' over the sea of all structures, so to speak, identifying models verifying the same L-sentences (L-equivalence). On the other hand, there is the finest sieve of *isomorphism* between models. One of Lindström's basic requirements on a logic was that the latter imply the former. One measure of strength of the logic is now to which extent the converse obtains. For instance, when L is first-order logic, we know that elementary equivalence implies isomorphism for *finite* models, but not for countable ones. (Cf. the earlier phenomenon of non-categorical definability of the integers.) A famous result concerning $L_{\omega_1\omega}$ is Scott's theorem to the effect that, for *countable* models, $L_{\omega_1\omega}$-equivalence and isomorphism coincide. (Cf. [Keisler, 1971, Chapter 2] or [Barwise, 1975, Chapter VII.6].) That such matches cannot last in the long run follows from a simple set-theoretic consideration, however, first made by Hanf. As long as the L-sentences form a set, they can distinguish at best $2^{\|L\|}$ models, up to L-equivalence — whereas the number of models, even up to isomorphism, is unbounded.

A more abstract line of research is concerned with the earlier meta-properties. In addition to compactness and Löwenheim-Skolem, one also considers such properties as recursive axiomatizability of universally valid sentences ('*completeness*') or *interpolation* (cf. [Hodges, 1983, I.1, Section 14]). Such notions may lead to new characterization results. For instance, Lindström himself proved that elementary logic is also the strongest logic with an effective finitary syntax to possess the Löwenheim-Skolem property and be complete. (The infinitary language $L_{\omega_1\omega}$ has both, without collapsing into elementary logic, however; its countable *admissible fragments* even possess compactness in the sense of [Barwise, 1975].) Similar characterizations for stronger logics have proven rather elusive up till now.

But then, there are many further possible themes in this area which are of a general interest. For instance, instead of haphazardly selecting some particular feature of first-order, or any other suggestive logic, one might proceed to a systematic description of meta-properties.

EXAMPLE. A folklore prejudice has it that interpolation was the 'final elementary property of first-order logic to be discovered'. Recall the statement of this meta-property: if one formula implies another, then (modulo some trivial cases) there exists an *interpolant* in their common vocabulary, implied by the first, itself implying the second. Now, this assertion may be viewed as a (first-order) fact about the two-sorted 'meta-structure' consisting of all first-order formulas, their vocabulary types (i.e. all finite sets of non-logical constants), the relations of implication and type-inclusion, as well as the type-assigning relation. Now, the complete first-order theories of the separate components are easily determined. The pre-order $\langle$formulas, implication$\rangle$ carries a definable Boolean structure, as one may define the connectives ($\wedge$ as greatest lower bound, $\neg$ as some suitable complement). Moreover, this Boolean algebra is countable, and atomless (the latter by the assumption of an infinite vocabulary). Thus, the given principles are complete, thanks to the well-known categoricity and, hence, completeness of the latter theory. The complete logic of the partial order $\langle$finite types, inclusion$\rangle$ may be determined in a slightly more complex way. The vindication of the above conviction concerning the above meta-structure would then consist in showing that interpolation provides the essential link between these two separate theories, in order to obtain a complete axiomatization for the whole.

But as it happens, [Mason, 1985] (in response to the original version of this chapter) has shown that the complete first-order theory of this meta-model is effectively equivalent to True Arithmetic, and hence non-axiomatizable.

Even more revolutionary about abstract model theory is the gradual reversal in methodological perspective. Instead of starting from a given logic and proving some meta-properties, one also considers these properties as such, establishes connections between them, and asks for (the ranges of) logics exemplifying certain desirable combinations of features.

Finally, a warning. The above study by no means exhausts the range of logical questions that can be asked about extensions of first-order logic. Indeed, the perspective of meta-properties is very global and abstract. One more concrete new development is the interest in, e.g. generalized quantifiers from the perspective of linguistic semantics (cf. [Barwise and Cooper, 1981; van Benthem, 1984]), which leads to proposals for reasonable constraints on new quantifiers, and indeed to a semantically-motivated classification of all reasonable additions to elementary logic.

1.4 Characterization Results

A good understanding of first-order logic is essential to any study of its extensions. To this end, various characterizations of first-order definability will be reviewed here in a little more detail than in chapter I.1.

1.4.1 Lindström's Theorem. Lindström's result itself gives a definition of first-order logic, in terms of its global properties. Nevertheless, in practice, it is of little help in establishing or refuting first-order definability. To see if some property Φ of models is elementary, one would have to consider the first-order language with Φ added (say, as a propositional constant), close under the operations that Lindström requires of a 'logic' (notably, the Boolean operations and relativization to unary predicates), and then find out if the resulting logic possesses the compactness and Löwenheim-Skolem properties. Moreover, the predicate logic is to have an *infinite* vocabulary (cf. the proof to be sketched below): otherwise, we are in for surprises.

EXAMPLE. Lindström's theorem fails for the pure identity language. First, it is a routine observation that sentences in this language can only express (negations of) disjunctions 'there are precisely n_1 or . . . or precisely n_k objects in the universe'. Now, add a propositional constant C expressing *countable infinity* of the universe.

This logic retains compactness. For, consider any finitely satisfiable set Σ of its sentences. It is not difficult to see that either $\Sigma \cup \{C\}$ or $\Sigma \cup \{\neg C\}$ must also be finitely satisfiable. In the first case, replace occurrences of C in Σ by some tautology: a set of first-order sentences remains, each of whose finite subsets has a (countably) infinite model. Therefore, it has an infinite model itself and, hence, a countably infinite one (satisfying C) — by ordinary compactness and Löwenheim-Skolem. This model satisfies the original Σ as well. In the second case, replace C in Σ by some contradiction. The resulting set either has a finite model, or an infinite one, and hence an uncountably infinite one: either way, $\neg C$ is satisfied — and again, the original Σ is too.

The logic also retains Löwenheim-Skolem. Suppose that φ has no countably infinite models. Then $\varphi \wedge \neg C$ has a model, if φ has one. Again, replace occurrences of C inside φ by some contradiction: a pure identity sentence remains. But such sentences can always be verified on some finite universe (witness the above description) where $\neg C$ is satisfied too.

1.4.2 Keisler's Theorem. A more local description of first-order definability was given by Keisler, in terms of preservation under certain basic operations on models.

THEOREM. *A property Φ of models is definable by means of some first-order sentence iff both Φ and its complement are closed under the formation of isomorphs and ultraproducts.*

The second operation has not been introduced yet. As it will occur at several other places in this Handbook, a short introduction is given at this point. For convenience, henceforth, our standard example will be that of binary relational models $F = \langle A, R \rangle$ (or $F_i = \langle A_i, R_i \rangle$).

A *logical fable*. A family of models $\{F_i \mid i \in I\}$ once got together and decided to join into a common state. As everyone wanted to be fully represented, it was decided to create new composite individuals as functions f with domain I, picking at each $i \in I$ some individual $f(i) \in A_i$. But now, how were relations to be established between these new individuals? Many models were in favour of consensus democracy:

$$Rfg \text{ iff } R_i f(i) g(i) \textit{ for all } i \in I.$$

But, this lead to indeterminacies as soon as models started voting about whether or *not* Rfg. More often than not, no decision was reached. Therefore, it was decided to ask the gods for an 'election manual' U, saying which sets of votes were to be 'decisive' for a given atomic statement. Thus, votes now were to go as follows:

$$Rfg \text{ iff } \{i \in I \mid R_i f(i) g(i)\} \in U. \tag{$*$}$$

Moreover, although one should not presume in these matters, the gods were asked to incorporate certain requirements of consistency

$$\text{if } X \in U, \text{ then } I - X \notin U$$

as well as democracy

$$\text{if } X \in U \text{ and } Y \supseteq X, \text{ then } Y \in U.$$

Finally, there was also the matter of expediency: the voting procedure for atomic statements should extend to complex decisions:

$$\varphi(f_1, \ldots, f_n) \text{ iff } \{i \in I \mid F_i \vDash \varphi[f_1(i) \ \ldots, f_n(i)]\} \in U$$

for all predicate-logical issues φ.

After having pondered these wishes, the gods sent them an *ultrafilter* U over I, proclaiming the Łoś Equivalence:

THEOREM. *For any ultrafilter* U *over* I*, the stipulation* $(*)$ *creates a structure* $F = \langle \Pi_{i \in I} A_i, R \rangle$ *such that*

$$F \vDash \varphi[f_1, \ldots, f_n] \text{ iff } \{i \in I \mid F_i \vDash \varphi[f_1(i), \ldots, f_n(i)]\} \in U.$$

Proof. The basic case is just $(*)$. The negation and conjunction cases correspond to precisely the defining conditions on ultrafilters, viz. (i) $X \notin U$ iff $I - X \in U$; (ii) $X, Y \in U$ iff $X \cap Y \in U$ (or, alternatively, besides consistency and democracy above: if $X, Y \in U$ then also $X \cap Y \in U$; and: if $I - X \notin U$ then $X \in U$). And finally, the gods gave them the existential quantifier step for free:

- if $\exists x \varphi(x, f_1, \ldots, f_n)$ holds then so does $\varphi(f, f_1, \ldots, f_n)$ for some function f. Hence, by the inductive hypothesis for φ, we have that $\{i \in I \mid F_i \vDash \varphi[f(i), f_1(i), \ldots, f_n(i)]\} \in U$, which set is contained in $\{i \in I \mid F_i \vDash \exists x \varphi[f_1(i), \ldots, f_n(i)]\} \in U$.
- if $\{i \in I \mid F_i \vDash \exists x \varphi[f_1(i), \ldots, f_n(i)]\} \in U$, then choose $f(i) \in A_i$ verifying φ for each of these i (and arbitrary elsewhere): this f verifies $\varphi(x, f_1, \ldots, f_n)$ in the whole product, whence $\exists x \varphi(f_1, \ldots, f_n)$ holds.

■

After a while, an unexpected difficulty occurred. Two functions f, g who did not agree among themselves asked for a public vote, and the outcome was ...

$$\{i \in I \mid f(i) = g(i)\} \in U.$$

Thus it came to light how the gift of the gods had introduced an invisible equality $\sim$. By its definition and the Łoś Equivalence, it even turned out to partition the individuals into equivalence classes, whose members were indistinguishable as to R behaviour:

$$Rfg, f \sim f', g \sim g' \text{ imply } Rf'g'.$$

But then, such classes themselves could be regarded as the building bricks of society, and in the end there were:

DEFINITION. For any family of models $\{F_i \mid i \in I\}$ with an ultrafilter U on I, the *ultraproduct* $\Pi_U F_i$ is the model $\langle A, R \rangle$ with

1. A is the set of classes $f_\sim$ for all functions $f \in \Pi_{i \in I} A_i$, where $f_\sim$ is the equivalence class of f in the above relation,
2. R is the set of couples $\langle f_\sim, g_\sim \rangle$ for which $\{i \in I \mid R_i f(i) g(i)\} \in U$.

By the above observations, the latter clause is well-defined — and indeed the whole Łoś Equivalence remained valid.

Whatever their merits as regards democracy, ultraproducts play an important role in the following fundamental question of model theory:

What structural behaviour makes a class of models *elementary*, i.e. definable by means of some first-order sentence?

First, the Łoś Equivalence implies that first-order sentences φ are preserved under ultraproducts in the following sense:

$$\text{if } F_i \vDash \varphi \text{ (all } i \in I\text{), then } \Pi_U F_i \vDash \varphi.$$

(The reason is that I itself must belong to U.) But conversely, Keisler's theorem told us that this is also enough. *End of fable*.

The proof of Keisler's theorem (subsequently improved by Shelah) is rather formidable: cf. [Chang and Keisler, 1973, Chapter 6]. A more accessible variant will be proved below, however. First, one relaxes the notion of isomorphism to the following partial variant.

DEFINITION. A partial isomorphism between $\langle A, R\rangle$ and $\langle B, S\rangle$ is a set I of coupled finite sequences (s, t) from A resp. B, of equal length, satisfying

$$\begin{array}{lcl} (s)_i = (s)_j & \text{iff} & (t)_i = (t)_j \\ (s)_i R (s)_j & \text{iff} & (t)_i S (t)_j \end{array}$$

which possesses the *back-and-forth property*, i.e. for every $(s, t) \in I$ and every $a \in A$ there exists some $b \in B$ with $(s^\frown a, t^\frown b) \in I$; and vice versa.

Cantor's zig-zag argument shows that partial isomorphism coincides with total isomorphism on the countable models. Higher up, matters change; e.g. $\langle \mathbb{Q}, <\rangle$ and $\langle \mathbb{R}, <\rangle$ are partially isomorphic by the obvious I without being isomorphic.

First-order formulas φ are preserved under partial isomorphism in the following sense:

$$\text{if } (s, t) \in I, \text{ then } \langle A, R\rangle \vDash \varphi[s] \text{ iff } \langle B, S\rangle \vDash \varphi[t].$$

Indeed, this equivalence extends to formulas from arbitrary infinitary languages $L_{\alpha\omega}$: cf. [Barwise, 1977, Chapter A.2.9] for further explanation.

THEOREM. *A property Φ of models is first-order definable iff both Φ and its complement are closed under the formation of partial isomorphs and countable ultraproducts.*

1.4.3 Fraïssé's Theorem. Even the Keisler characterization may be difficult to apply in practice, as ultraproducts are such abstract entities. In many cases, a more combinatorial method may be preferable; in some, it's even necessary.

EXAMPLE. As was remarked earlier, colloquial 'most' only seems to have natural meaning on the *finite* models. But, as to first-order definability on this restricted class, both previous methods fail us completely, all relevant notions being tied up with infinite models. Nevertheless, *most* A *are* B is not definable on the finite models in the first-order language with A, B and identity. But this time, we need a closer combinatorial look at definability.

First, a natural measure of the 'pattern complexity' of a first-order formula φ is its *quantifier depth* $d(\varphi)$, which is the maximum length of quantifier nestings inside φ. (Inductively, $d(\varphi = 0$ for atomic φ, $d(\neg\varphi) = d(\varphi)$, $d(\varphi \wedge \psi) =$ maximum$(d(\varphi), d(\psi))$, etcetera, $d(\exists x \varphi) = d(\forall x \varphi) = d(\varphi + 1.)$ Intuitively, structural complexity beyond this level will escape φ's notice. We make this precise.

Call two sets X, Y n*-equivalent* if either $|X| = |Y| < n$ or $|X|, |Y| \geq n$. By extension, call two models $\langle D, A, B\rangle, \langle D', A', B'\rangle$ n-equivalent if all four 'state descriptions' $A \cap B$, $A \backslash B$, $B \backslash A$, $D \backslash (A \cup B)$ are n-equivalent to their primed counterparts.

LEMMA. *If* $\langle D, A, B\rangle, \langle D', A', B'\rangle$ *are* n*-equivalent then all sequences* d, d' *with corresponding points in corresponding states verify the same first-order formulas with quantifier depth not exceeding* n.

COROLLARY. *'Most* A *are* B*' is not first-order definable on the finite models.*

Proof. For no finite number n, 'most A are B' exhibits the required n-insensitivity. ■

This idea of insensitivity to structural complexity beyond a certain level forms the core of our third and final characterization, due to Fraïssé. Again, only the case of a binary relation R will be considered, for ease of demonstration.

First, on the linguistic side, two models are n*-elementarily equivalent* if they verify the same first-order sentences of quantifier depth not exceeding n. Next, on the structural side, a matching notion of n*-partial isomorphism* may be defined, by postulating the existence of a chain $I_n \supseteq \cdots \supseteq I_0$ of sets of matching couples (s, t), as in the earlier definition of partial isomorphism. This time, the back-and-forth condition is index-relative, however:

> if $(s, t) \in I_{i+1}$ and $a \in A$, then for some $b \in B$, $(s^\frown a, t^\frown b) \in I_i$, and vice versa.

PROPOSITION. *Two models are* n*-elementarily equivalent iff they are* n*-partially isomorphic.*

The straightforward proof uses the following auxiliary result, for first-order languages with a finite non-logical vocabulary of relations and individual constants.

LEMMA. *For each depth n and for each fixed number of free variables $x_1, \ldots, x_m$, there exist only finitely many formulas $\varphi(x_1, \ldots, x_m)$, up to logical equivalence.*

This lemma allows us to describe all possible n-patterns in a single first-order formula, a purpose for which one sometimes uses explicit 'Hintikka normal forms'.

THEOREM. *A property Φ of models is first-order definable iff it is preserved under n-partial isomorphism for some natural number n.*

Proof. The invariance condition is obvious for first-order definable properties. Conversely, for n-invariant properties, the disjunction of all complete n-structure descriptions for models satisfying Φ defines the latter property. ∎

Applications. Now, from the Fraïssé theorem, both the weak Keisler and the Lindström characterization may be derived in a perspicuous way. Here is an indication of the proofs.

EXAMPLE. (*Weak Keisler from Fraïssé*) First-order definable properties are obviously preserved under partial isomorphism and (countable) ultraproducts. As for the converse, suppose that Φ is not thus definable. By Fraïssé, this implies the existence of a sequence of n-partially isomorphic model pairs $\mathfrak{A}_n, \mathfrak{B}_n$ of which only the first verify Φ.

The key observation is now simply this. Any *free* ultrafilter U on $\mathbb{N}$ (containing all tails of the form $[n, \infty)$) will make the countable ultraproducts $\Pi_U \mathfrak{A}_n$, $\Pi_U \mathfrak{B}_n$ partially isomorphic. The trick here is to find a suitable set I of partial isomorphisms, and this is accomplished by setting, for sequences of functions s, t of length m

$$((s)_U, (t)_U) \in I \text{ iff } \{n \geq m \mid (s(n), t(n)) \in I^n_{n-m}\} \in U$$

where '$I^n_n, \ldots, I^n_0$' is the sequence establishing the n-partial isomorphism of $\mathfrak{A}_n$, $\mathfrak{B}_n$.

So, by the assumed preservation properties, Φ would hold for $\Pi_U \mathfrak{A}_n$ and hence for $\Pi_U \mathfrak{B}_n$. But, so would not-Φ: a contradiction.

EXAMPLE. (*Lindström from Fraïssé*) Let L be a logic whose non-logical vocabulary consists of infinitely many predicate constants of all arities. L is completely specified by its *sentences* S, each provided with a finite 'type' (i.e. set of predicate constants), its *models* M (this time: ordinary first-order models) and its *truth relation* T between sentences and models. We assume four basic conditions on L: the truth relation is invariant for *isomorphs*, the sentence set S is closed under *negations* and *conjunctions* (in the obvious semantic sense), and each sentence φ can be relativized by arbitrary unary predicates A, such that a model verifies φ^A iff its

A-submodel verifies φ. Finally, we say that L 'contains elementary logic' if each first-order sentence is represented by some sentence in S having the same models. 'Compactness and 'Löwenheim-Skolem' are already definable in this austere framework. (By the latter we'll merely mean: 'sentences with any model at all have countable models'.)

THEOREM. *Any logic containing elementary logic has compactness and Löwenheim-Skolem iff it coincides with elementary logic.*

The non-evident half of this assertion again starts from Fraïssé's result. Suppose that $\Phi \in S$ is not first-order. Again, there is a sequence $\mathfrak{A}_n$, $\mathfrak{B}_n$ as above. For a natural number n, consider the complex model (an expanded "model pair") $\mathfrak{M}_n = (\mathfrak{A}_n, \mathfrak{B}_n, R_0, \ldots, R_n)$, where the $2i$-ary relations $R_i \subseteq A_n^i \times B_n^i$ $(i = 0, \ldots, n)$ are defined by

$$R_i(a_1, \ldots, a_i, b_1, \ldots, b_i) :\equiv (\mathfrak{A}_n, a_1, \ldots, a_i) \equiv^{n-i} (\mathfrak{B}_n, b_1, \ldots, b_i)$$

($\equiv^{n-i}$ denoting $(n-i)$-equivalence here). The model $\mathfrak{M}_n$ satisfies sentences expressing that

1. Φ is true in its first component $\mathfrak{A}_n$ but false in its second one $\mathfrak{B}_n$,
2. if $i \leq n$ and $R_i(a_1, \ldots, a_i, b_1, \ldots, b_i)$ holds, then the relation $\{(a_1, b_1), \ldots, (a_i, b_i)\}$ between the component-models $\mathfrak{A}_n$ and $\mathfrak{B}_n$ has the properties of a partial isomorphism (preservation of equality and relations) introduced earlier,
3. (a) R_0 (which has 0 arguments) is true (of the empty sequence),
   (b) if $i < n$ and $R_i(a_1, \ldots, a_i, b_1, \ldots, b_i)$ holds, then for all $a \in A_n$ there exists $b \in B_n$ such that $R_{i+1}(a_1, \ldots, a_i, a, b_1, \ldots, b_i, b)$, and vice versa.

By the Downward Löwenheim-Skolem and Compactness property, there is a *countable* complex $(\mathfrak{A}, \mathfrak{B}, R_0, R_1, R_2, \ldots)$ with an *infinite* sequence $R_0, R_1, R_2, \ldots$ that satisfies these requirements for *every* i. By requirements 2 and 3 and Cantor's zig-zag argument it follows that $\mathfrak{A} \cong \mathfrak{B}$. However, this contradicts requirement 1. ■

2 SECOND-ORDER LOGIC

2.1 Language

Quantification over properties and predicates, rather than just objects, has a philosophical pedigree. For instance, Leibniz's celebrated principle of Identity of Indiscernibles has the natural form

$$\forall xy(\forall X(X(x) \leftrightarrow X(y)) \rightarrow x = y).$$

There also seems to be good evidence for this phenomenon from natural language, witness Russell's example 'Napoleon had all the properties of a great general'

$$\forall X(\forall y(GG(y) \rightarrow X(y)) \rightarrow X(n)).$$

Moreover, of course, mathematics abounds with this type of discourse, with its explicit quantification over relations and functions. And indeed, logic itself seems to call for this move. For, there is a curious asymmetry in ordinary predicate logic between individuals: occurring both in constant and variable contexts, and predicates: where we are denied the power of quantification. This distinction seems arbitrary: significantly, Frege's *Begriffsschrift* still lacks it. We now pass on to an account of second-order logic, with its virtues and vices.

The language of second-order logic distinguishes itself from that of first-order logic by the addition of variables for subsets, relations and functions of the universe and the possibility of quantification over these. The result is extremely strong in expressive power; we list a couple of examples in Section 2.2. As a consequence, important theorems valid for first-order languages fail here; we mention the compactness theorem, the Löwenheim-Skolem theorems (Section 2.2) and the completeness theorem (Section 2.3). With second-order logic, one really enters the realm of set theory. This state of affairs will be illustrated in Section 2.4 with a few examples. What little viable logic can be snatched in the teeth of these limitations usually concerns special fragments of the language, of which some are considered in Section 2.5.

2.2 Expressive Power

2.2.1. An obvious example of a second-order statement is Peano's induction axiom according to which every set of natural numbers containing 0 and closed under immediate successors contains all natural numbers. Using S for successor, this might be written down as

$$\forall Y[Y(0) \wedge \forall x(Y(x) \rightarrow Y(S(x))) \rightarrow \forall x Y(x)] \qquad (1)$$

(The intention here is that x stands for numbers, Y for sets of numbers, and $Y(x)$ says, as usual, that x is an element of Y.)

Dedekind already observed that the axiom system consisting of the induction axiom and the two first-order sentences

$$\forall x \forall y(S(x) = S(y) \rightarrow x = y) \qquad (2)$$

and

$$\forall x(S(x) \neq 0) \tag{3}$$

is categorical. Indeed suppose that $\langle A, f, a\rangle$ models (1)–(3). Let $A' = \{a, f(a), f(f(a)), \ldots\}$. Axioms (2) and (3) alone imply that the submodel $\langle A', f \restriction A', a\rangle$ is isomorphic with $\langle \mathbb{N}, S, 0\rangle$ (the isomorphism is clear). But, (1) implies that $A' = A$ (just let X be A').

This result should be contrasted with the first-order case. *No* set of first-order sentences true of $\langle \mathbb{N}, S, 0\rangle$ is categorical. This can be proved using either the upward Löwenheim-Skolem theorem or the compactness theorem. As a result, neither of these two extend to second-order logic. The nearest one can come to (1) in first-order terms is the 'schema'

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x) \tag{4}$$

where φ is any first-order formula in the vocabulary under consideration. It follows that in models $\langle A, f, a\rangle$ of (4) the set A' above cannot be defined in first-order terms: otherwise one could apply (4) showing $A' = A$ just as we applied (1) to show this before. (This weakness of first-order logic becomes its strength in so-called 'overspill arguments', also mentioned in [Hodges, 1983, Chapter I.1].) We will use the categoricity of (1)–(3) again in Section 2.3 to show non-axiomatizability of second-order logic.

2.2.2. The next prominent example of a second-order statement is the one expressing 'Dedekind completeness' of the order of the reals: every set of reals with an upper bound has a least upper bound. Formally

$$\begin{aligned}&\forall X[\exists x \forall y(X(y) \rightarrow y \leq x) \rightarrow \\ &\quad \rightarrow \exists x(\forall y(X(y) \rightarrow y \leq x) \wedge \forall x'[\forall y(X(y) \rightarrow y \leq x') \rightarrow x \leq x'])]\end{aligned} \tag{5}$$

It is an old theorem of Cantor's that (5) together with the first-order statements expressing that $\leq$ is a dense linear order without endpoints plus the statement 'there is a countable dense subset', is categorical. The latter statement of so-called 'separability' is also second-order definable: cf. Section 2.2.5. Without it a system is obtained whose models all *embed* $\langle \mathbb{R}, \leq\rangle$. (For, these models must embed $\langle \mathbb{Q}, \leq\rangle$ for first-order reasons; and such an embedding induces one for $\langle \mathbb{R}, \leq\rangle$ by (5).) Thus, the downward Löwenheim-Skolem theorem fails for second-order logic.

2.2.3. A relation $R \subseteq A^2$ is *well-founded* if every non-empty subset of A has an R-minimal element. In second-order terms w.r.t. models $\langle A, R, \ldots\rangle$

$$\forall X[\exists x X(x) \rightarrow \exists x(X(x) \wedge \forall y[X(y) \rightarrow \neg R(y, x)])] \tag{6}$$

This cannot be expressed in first-order terms. For instance, every first-order theory about R which admits models with R-chains of arbitrary large but finite length must, by compactness, admit models with infinite R-chains which decrease, and such a chain has no minimal element.

2.2.4. Every first-order theory admitting arbitrarily large, finite models has infinite models as well: this is one of the standard applications of compactness. On the other hand, higher-order terms enable one to define finiteness of the universe. Probably the most natural way to do this uses *third-order* means: a set is finite iff it is in every *collection of* sets containing the empty set and closed under the addition of one element. Nevertheless, we can define finiteness in second-order terms as well: A is finite iff every relation $R \subseteq A^2$ is well-founded; hence, a second-order definition results from (6) by putting a universal quantifier over R in front. Yet another second-order definition of finiteness uses Dedekind's criterion: every injective function on A is surjective. Evidently, such a quantification over functions on A may be simulated using suitable predicates. By the way, to see that these second-order sentences do indeed define finiteness one needs the axiom of choice.

2.2.5 Generalized Quantifiers. Using Section 2.2.4, it is easy to define the quantifier $\exists_{<\aleph_0}$ (where $\exists_{<\aleph_0} x\varphi(x)$ means: there are only finitely many x s.t. $\varphi(x)$) in second-order terms; $\exists_{\geq\aleph_0}$ simply is its negation. (In earlier terminology, weak second-order logic is part of second-order logic.) What about higher cardinalities? Well, e.g. $|X| \geq \aleph_1$ iff X has an infinite subset Y which cannot be mapped one–one onto X. This can obviously be expressed using function quantifiers. And then of course one can go on to $\aleph_2, \aleph_3, \ldots$

Other generalized quantifiers are definable by second-order means as well. For instance, the standard example of Section 1 has the following form. *Most* A *are* B becomes 'there is no injective function from $A \cap B$ into $A - B$'.

A highly successful generalised quantifier occurs in *stationary logic*, cf. [Barwise *et al.*, 1978]. Its language is second-order in that it contains monadic second-order variables; but the only quantification over these allowed is by means of the *almost all* quantifier aa. A sentence $\mathit{aa}X\varphi(X)$ is read as: there is a collection C of countable sets X for which $\varphi(X)$, which is closed under the formation of countable unions and has the property that every countable subset of the universe is subset of a member of C. (We'll not take the trouble explaining what 'stationary' means here.) The obvious definition of aa in higher-order terms employs third-order means. Stationary logic can define the quantifier $\exists_{\geq\aleph_1}$. It has a complete axiomatization and, as a consequence, obeys compactness and downward Löwenheim-Skolem (in the form: if a sentence has an uncountable model, it has one of power $\aleph_1$).

Other compact logics defining $\exists_{\geq\aleph_1}$ have been studied by Magidor and Malitz [1977].

2.2.6. The immense strength of second-order logic shows quite clearly when set theory itself is considered.

Zermelo's *separation axiom* says that the elements of a given set sharing a given property form again a set. Knowing of problematic properties occurring in the paradoxes, he required 'definiteness' of properties to be used. In later times, Skolem replaced this by 'first-order definability', and the axiom became a first-order schema. Nevertheless, the intended axiom quite simply is the second-order statement

$$\forall X \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge X(z)) \tag{8}$$

Later on, Fraenkel and Skolem completed Zermelo's set theory with the substitution axiom: the complete image of a set under an operation is again a set, resulting from the first by 'substituting' for its elements the corresponding images. Again, this became a first-order schema, but the original intention was the second-order principle

$$\forall F \forall a \exists b \forall y (y \in b \leftrightarrow \exists x [x \in a \wedge y = F(x)]) \tag{9}$$

Here F is used as a variable for arbitrary operations from the universe to itself; $F(x)$ denotes application. The resources of set theory allow an equivalent formulation of (9) with a set (i.e. class) variable, of course. Together with the usual axioms, (9) implies (8).

It must be considered quite a remarkable fact that the first-order versions of (8) and (9) have turned out to be sufficient for every mathematical purpose. (By the way, in ordinary mathematical practice, (9) is seldom used; the proof that Borel-games are determined is a notable exception. Cf. also Section 2.4.)

The Zermelo-Fraenkel axioms intend to describe the cumulative hierarchy with its membership structure $\langle V, \in \rangle$, where $V = \cup_\alpha V_\alpha$ (α ranging over all ordinals) and $V_\alpha = \cup_{\beta < \alpha} \mathcal{P} V_\beta$. For the reasons mentioned in Section 1, the first-order version ZF^1 of these axioms does not come close to this goal, as it has many non-standard models as well. The second-order counterpart ZF^2 using (9) has a much better score in this respect:

THEOREM. *$\langle A, E \rangle$ satisfies ZF^2 iff for some strongly inaccessible cardinal κ : $\langle A, E \rangle \cong \langle V_\kappa, \in \rangle$.*

It is generally agreed that the models $\langle V_\kappa, \in \rangle$ are 'standard' to a high degree.

If we add an axiom to ZF^2 saying there are no inaccessibles, the system even becomes categorical, defining $\langle V_\kappa, \in \rangle$ for the first inaccessible κ.

2.3 Non-axiomatizability

First-order logic has an effective notion of proof which is complete w.r.t. the intended interpretation. This is the content of Gödel's completeness theorem. As a result, the set of (Gödel numbers of) universally valid first-order formulas is recursively enumerable. Using Example 2.2.1, it is not hard to show that the set of second-order validities is not arithmetically definable, let alone recursively enumerable, and hence that an effective and complete axiomatization of second-order validity is impossible.

Let P^2 be Peano arithmetic in its second-order form, i.e. the theory in the language of $\mathfrak{N} = \langle \mathbb{N}, S, 0, +, \times \rangle$ consisting of (1)–(3) above plus the (first-order) recursion equations for $+$ and $\times$. P^2 is a categorical description of $\mathfrak{N}$, just as (1)–(3) categorically describe $\langle \mathbb{N}, S, 0 \rangle$. Now, let φ be any first-order sentence in the language of $\mathfrak{N}$. Then clearly

$$\mathfrak{N} \vDash \varphi \text{ iff } P^2 \rightarrow \varphi \text{ is valid.}$$

(Notice that P^2 may be regarded as a single second-order sentence.)

Now the left-hand side of this equivalence expresses a condition on (the Gödel number of) φ which is not arithmetically definable by Tarski's theorem on non-definability of truth (cf. Section 3.2 or, for a slightly different setting, see [Hodges, 1983, Section 20]). Thus, second-order validity cannot be arithmetical either. ■

Actually, this is still a very weak result. We may take φ second-order and show that second-order truth doesn't fit in the analytic hierarchy (again, see Section 3.4). Finally, using Section 2.2.6, we can replace in the above argument $\mathfrak{N}$ by $\langle V_\kappa, \in \rangle$, where κ is the smallest inaccessible, and P^2 by ZF$^2+$ 'there are no inaccessibles', and find that second-order truth cannot be (first-order) defined in $\langle V_\kappa, \in \rangle$, etc. This clearly shows how frightfully complex this notion is.

Not to end on too pessimistic a note, let it be remarked that the logic may improve considerably for certain fragments of the second-order language, possibly with restricted classes of models. An early example is the decidability of second-order monadic predicate logic (cf. [Ackermann, 1968]). A more recent example is Rabin's theorem (cf. [Rabin, 1969]) stating that the monadic second-order theory (employing only second-order quantification over subsets) of the structure $\langle 2^{\overset{\omega}{\smile}}, P_0, P_1 \rangle$ is still decidable. Here, $2^{\overset{\omega}{\smile}}$ is the set of all finite sequences of zeros and ones, and P_i is the unary operation 'post-fix i' ($i = 0, 1$).

Many decidability results for monadic second-order theories have been derived from this one by showing their models to be definable parts of the Rabin structure. For instance, the monadic second-order theory of the natural numbers $\langle \mathbb{N}, < \rangle$ is decidable by this method.

The limits of Rabin's theorem show up again as follows. The *dyadic* second-order theory of $\langle \mathbb{N}, < \rangle$ is already non- arithmetical, by the previous type of consideration. (Briefly, $\mathfrak{N} \vDash \varphi$ iff $\langle \mathbb{N}, < \rangle$ verifies $P \rightarrow \varphi$ for all those choices of $0, S, +, \times$ whose defined relation 'smaller than' coincides with the actual $<$. Here, P is first-order Peano Arithmetic minus induction. In this formulation, ternary predicates are employed (for $+, \times$), but this can be coded down to the binary case.)

2.4 Set-Theoretic Aspects

Even the simplest questions about the model theory of second-order logic turn out to raise problems of set theory, rather than logic. Our first example of this phenomenon was a basic theme in Section 1.3.

If two models are first-order (elementarily) equivalent and one of them is finite, they must be isomorphic. What, if we use second-order equivalence and relax finiteness to, say, countability? Ajtai [1979] contains a proof that this question is undecidable in ZF (of course, the *first-order* system is intended here). One of his simplest examples shows it is consistent for there to be two countable well-orderings, second-order (or indeed higher-order) equivalent but not isomorphic.

The germ of the proof is in the following observation. If the Continuum Hypothesis holds, there must be second-order (or indeed higher-order) equivalent well-orderings of power $\aleph_1$: for, up to isomorphism, there are $\aleph_2$ such well orderings (by the standard representation in terms of ordinals), whereas there are only $2^{\aleph_0} = \aleph_1$ second-order theories. The consistency-proof itself turns on a refined form of this cardinality-argument, using 'cardinal collapsing'. On the other hand, Ajtai mentions the 'folklore' fact that countable second-order equivalent models *are* isomorphic when the axiom of constructibility holds. In fact, this may be derived from the existence of a second-order definable well-ordering of the reals (which follows from this axiom).

Another example belongs to the field of second-order cardinal characterization (cf. [Garland, 1974]). Whether a sentence without non-logical symbols holds in a model or not depends only on the cardinality of the model. If a sentence has models of one cardinality only, it is said to *characterize* that cardinal. As we have seen in Section 1, first-order sentences can only characterize single finite cardinals. In the meantime, we have seen how to characterize, e.g. $\aleph_0$ in a second-order way: let φ be the conjunct of (1)–(3) of Section 2.2.1 and consider $\exists S \exists 0 \varphi$ — where S and 0 are now being considered as variables. Now, various questions about the simplest second-order definition of a given cardinal, apparently admitting of 'absolute' answers, turn out to be undecidable set theoretic problems; cf. [Kunen, 1971].

As a third example, we finally mention the question of cardinals characterizing, conversely, a logic L. The oldest one is the notion of *Hanf number* of a logic, alluded to in Section 1.3. This is the least cardinal γ such that, if an L-sentence has a model of power $\geq \gamma$, it has models of arbitrarily large powers. The *Löwenheim number* λ of a language L compares to the *downward* Löwenheim-Skolem property just as the Hanf number does to the *upward* notion: it is the least cardinal with the property that every satisfiable L-sentence has a model of power $\leq \lambda$. It exists by a reasoning similar to Hanf's: for satisfiable φ, let $|\varphi|$ be the least cardinal which is the power of some model of φ. Then λ clearly is the sup of these cardinals. (By the way, existence proofs such as these may rely heavily on ZF's substitution-axiom. Cf. [Barwise, 1972].)

How large are these numbers pertaining to second-order logic? From Section 2.2.6 it follows, that the first inaccessible (if it exists) can be second-order characterized; thus the Löwenheim and Hanf numbers are at least bigger still. By similar reasoning, they are not smaller than the second, third, $\ldots$ inaccessible. And we can go on to larger cardinals; for instance, they must be larger than the first *measurable*. The reason is mainly that, like inaccessibility, defining measurability of κ only needs reference to sets of rank not much higher than κ. (In fact, inaccessibility of κ is a first-order property of $\langle V_{\kappa+1}, \in\rangle$; measurability one of $\langle V_{\kappa+2}, \in\rangle$.) Only when large cardinal properties refer in an essential way to the *whole* set theoretic universe (the simplest example being that of *strong compactness*) can matters possibly change. Thus, [Magidor, 1971] proves that the Löwenheim number of universal second-order sentences (and hence, by 4.3, of higher-order logic in general) is less than the first *supercompact* cardinal.

As these matters do bring us a little far afield (after all, this is a handbook of *philosophical* logic) we stop here.

In this light, the recommendation in the last problem of the famous list 'Open problems in classical model theory' in Chang and Keisler [1973] remains as problematic as ever: 'Develop the model theory of second and higher-order logic'.

Additional evidence for the view that second-order logic (and, a fortiori, higher-order logic in general) is not so much logic as set theory, is provided by looking directly at existing set-theoretic problems in second-order terms.

Let κ be the first inaccessible cardinal. In Section 2.2.6 we have seen that every ZF^2 model contains (embeds) $\langle V_\kappa, \in\rangle$. As this portion is certainly (first-order) definable in all ZF^2 models in a uniform way, ZF^2 decides every set theoretic problem that mentions sets in V_κ only. This observation has led Kreisel to recommend this theory to our lively attention, so let us continue.

Indeed, already far below κ, interesting questions live. Foremost is the *continuum problem*, which asks whether there are sets of reals in cardinality strictly between $\mathbb{N}$ and $\mathbb{R}$. (Cantor's famous *continuum hypothesis* (CH) says there are not.) Thus, ZF^2 decides CH: either it or its negation follows logically from ZF^2. Since ZF^2 is correct, in the former case CH is true, while it is false in the latter. But of course, this reduction of the continuum problem to second-order truth really begs the question and is of no help whatsoever.

It does refute an analogy, however, which is often drawn between the continuum hypothesis and the Euclidean postulate of parallels in geometry. For, the latter axiom is not decided by second-order geometry. Its independence is of a different nature; there are different 'correct' geometries, but only one correct set theory (modulo the addition of large cardinal axioms): ZF^2. (In view of Section 2.2.6, a better formal analogy would be that between the parallel postulate and the existence of inaccessibles — though it has shortcomings as well.)

Another example of a set-theoretic question deep down in the universe is whether there are non-constructible reals. This question occurs at a level so low that, using a certain amount of coding, it can be formulated already in the language of P^2.

ZF^2 knows the answers — unfortunately, we're not able to figure out exactly what it knows.

So, what is the practical use of second-order set theory? To be true, there are *some* things we do know ZF^2 proves while ZF^1 does not; for instance, the fact that ZF^1 is consistent. Such metamathematical gains are hardly encouraging, however, and indeed we can reasonably argue that there is no way of knowing something to follow from ZF^2 unless it is provable in the two-sorted set/class theory of Morse-Mostowski, a theory that doesn't have many advantages over its subtheory ZF $=$ ZF^1. (In terms of Section 4.2 below, Morse-Mostowski can be described as ZF^2 under the general-models interpretation with full comprehension-axioms added.)

We finally mention that sometimes, higher-order notions find application in the theory of sets. In Myhill and Scott [1971] it is shown that the class of hereditarily ordinal-definable sets can be obtained by iterating second-order (or general higher-order) definability through the ordinals. (The constructible sets are obtained by iterating first-order definability; they satisfy the ZF-axioms only by virtue of their first-order character.) Also, interesting classes of large cardinals can be obtained by their reflecting higher-order properties; cf. for instance [Drake, 1974, Chapter 9].

2.5 Special Classes: Σ_1^1 *and* Π_1^1

In the light of the above considerations, the scarcity of results forming a subject of 'second-order logic' becomes understandable. (A little) more can be said, however, for certain fragments of the second-order language. Thus, in Section 2.3, the monadic quantificational part was considered, to which belong, e.g. second-order Peano arithmetic P^2 and Zermelo-Fraenkel set theory ZF^2. The more fruitful restriction for general model-theoretic purposes employs quantificational pattern complexity, however. We will consider the two simplest cases here, viz. prenex forms with only existential second-order quantifiers (Σ_1^1 formulas) or only universal quantifiers (Π_1^1 formulas). For the full prenex hierarchy, cf. Section 3.2; note however that we restrict the discussion here to formulas all of whose free variables are first-order. One useful shift in perspective, made possible by the present restricted language, is the following.

If $\exists X_1, \ldots, \exists X_k \varphi$ is a Σ_1^1 formula in a vocabulary L, we sometimes consider φ as a first-order formula in the vocabulary $L \cup \{X_1, \ldots, X_k\}$ — now suddenly looking upon the $X_1, \ldots, X_k$ not as second-order *variables* but as non-logical *constants* of the extended language. Conversely, if φ is a first-order L formula containing a relational symbol R, we may consider $\exists R\varphi$ as a Σ_1^1 formula of $L - \{R\}$ — viewing R now as a second-order *variable*. As a matter of fact, this way of putting things has been used already (in Section 2.4).

2.5.1 Showing Things to be Σ^1_1 *or* Π^1_1. Most examples of second-order formulas given in Section 2.2 were either Σ^1_1 or Π^1_1; in most cases, it was not too hard to translate the given notion into second-order terms.

A simple result is given in Section 3.2 which may be used in showing things to be Σ^1_1 or Π^1_1-expressible: any formula obtained from a Σ^1_1 (Π^1_1) formula by prefixing a series of first-order quantifications still has a Σ^1_1 (Π^1_1) equivalent.

For more intricate results, we refer to Kleene [1952] and Barwise [1975]. The first shows that if ϕ is a recursive set of first-order formulas, the infinitary conjunct $\bigwedge \phi$ has a Σ^1_1 equivalent (on infinite models). Thus, $\exists X_1, \ldots, \exists X_k \bigwedge \phi$ is also Σ^1_1. This fact has some relevance to resplendency, cf. Section 2.5.4 below. Kleene's method of proof uses absoluteness of definitions of recursive sets, coding of satisfaction and the integer structure on arbitrary infinite models. (It is implicit in much of Barwise [1975, Chapter IV 2/3], which shows that we are allowed to refer to integers in certain ways when defining Σ^1_1 and Π^1_1 notions.)

We now consider these concepts one by one.

2.5.2 Σ^1_1*-sentences.* The key quantifier combination in Frege's predicate logic expresses dependencies beyond the resources of traditional logic: $\forall\exists$. This dependency may be made explicit using a Σ^1_1 formula:

$$\forall x \exists y \varphi(x, y) \leftrightarrow \exists f \forall x \varphi(x, f(x)).$$

This introduction of so-called *Skolem functions* is one prime source of Σ^1_1 statements. The quantification over functions here may be reduced to our predicate format as follows:

$$\exists X(\forall xyz(X(x, y) \wedge X(x, z) \rightarrow y = z) \wedge \forall x \exists y(X(x, y) \wedge \varphi(x, y))).$$

Even at this innocent level, the connection with set theory shows up (Section 2.4): the above equivalence itself amounts to the assumption of the Axiom of Choice (Bernays).

Through the above equivalence, all first-order sentences may be brought into 'Skolem normal form'. E.g., $\forall x \exists y \forall z \exists u A(x, y, z, u)$ goes to $\exists f \forall x \forall z \exists u A(x, f(x), z, u)$, and thence to $\exists f \exists g \forall x \forall z A(x, f(x), z, g(x, z))$. For another type of Skolem normal form (using relations instead), cf. [Barwise, 1975, Chapter V 8.6].

Conversely, Σ^1_1 sentences allow for many other patterns of dependency. For instance, the variant $\exists f \exists g \forall x \forall z A(x, f(x), z, g(z))$, with g only dependent on z, is not equivalent to any first-order formula, but rather to a so-called 'branching' pattern (first studied in [Henkin, 1961])

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists u \end{pmatrix} A(x, y, z, u).$$

For a discussion of the linguistic significance of these ‘branching quantifiers’, cf. [Barwise, 1979]. One sentence which has been claimed to exhibit the above pattern is ‘some relative of each villager and some relative of each townsman hate each other’ (Hintikka). The most convincing examples of first-order branching to date, however, rather concern quantifiers such as (precisely) *one* or *no*. Thus, ‘one relative of each villager and one relative of each townsman hate each other’ seems to lack any linear reading. (The reason is that any linear sequence of *precisely one’s* creates undesired dependencies. In this connection, recall that ‘one sailor has discovered one sea’ is not equivalent to ‘one sea has been discovered by one sailor’.) An even simpler example might be ‘no one loves no one’, which has a linear reading $\neg\exists x \neg\exists y L(x, y)$ (i.e. everyone loves someone), but also a branching reading amounting to $\neg\exists x \exists y L(x, y)$. (Curiously, it seems to lack the inverse scope reading $\neg\exists y \neg\exists x L(x, y)$ predicted by Montague Grammar.) Actually, this last example also shows that the phenomenon of branching does not lead inevitably to second-order readings.

The preceding digression has illustrated the delicacy of the issue whether second-order quantification actually occurs in natural language. In any case, if branching quantifiers occur, then the logic of natural language would be extremely complex, because of the following two facts. As Enderton [1970] observes, universal validity of Σ^1_1 statements may be effectively reduced to that of branching statements. Thus, the complexity of the latter notion is at least that of the former. And, by inspection of the argument in Section 2.3 above, we see that

THEOREM. *Universal validity of* Σ^1_1*-sentences is non-arithmetical, etc.*

Proof. The reduction formula was of the form $P^2 \to \varphi$, where P^2 is Π^1_1 and φ is first-order. By the usual prenex operation, the universal second-order quantifier in the antecedent becomes an existential one in front. ■

Indeed, as will be shown in Section 4.3, the complexity of Σ^1_1-universal validity is essentially that of universal validity for the whole second-order (or higher-order) language. Nevertheless, one observation is in order here.

These results require the availability of non-logical constants and, e.g. universal validity of $\exists X \varphi(X, R)$ really amounts to universal validity of the Π^1_2-statement $\forall Y \exists X \varphi(X, Y)$. When attention is restricted to ‘pure’ cases, it may be shown that universal validity of Σ^1_1 statements is much less complex, amounting to truth in all finite models (cf. [van Benthem, 1977]). Thus, in the arithmetical hierarchy (cf. Section 3.2.) its complexity is only Π^0_1.

When is a Σ^1_1 sentence, say of the form $\exists X_1, \ldots, \exists X_k \varphi(X_1, \ldots, X_k, R)$, equivalent to a first-order statement about its parameter R? An answer follows from Keisler’s theorem (Section 1.4.2), by the following observation.

THEOREM. *Truth of* Σ^1_1 *sentences is preserved under the formation of ultra-products.*

(This is a trivial corollary of the preservation of first-order sentences, cf. Section 1.4.2.)

COROLLARY. *A Σ_1^1 sentence is first-order definable iff its negation is preserved under ultraproducts.*

(That Σ_1^1 sentences, and indeed all higher-order sentences are preserved under isomorphism should be clear.)

Moreover, there is a consequence analogous to Post's theorem in recursion theory:

COROLLARY. *Properties of models which are both Σ_1^1 and Π_1^1 are already elementary.*

(Of course, this is also immediate from the interpolation theorem which, in this terminology, says that disjoint Σ_1^1 classes an be separated by an elementary class.)

Next, we consider a finer subdivision of Σ_1^1 sentences, according to their first-order matrix. The simplest forms are the following (φ quantifier-free):

1. ($\exists\exists$) $\exists X_1 \dots X_k \exists y_1 \dots y_m \varphi(X_1, \dots, X_k, y_1, \dots, y_m, R)$
2. ($\exists\forall$) $\exists X_1 \dots X_k \forall y_1 \dots y_m \varphi(X_1, \dots, X_k, y_1, \dots, y_m, R)$
3. ($\exists\forall\exists$) $\exists X_1 \dots X_k \forall y_1 \dots y_m \exists z_1 \dots z_n \varphi(X_1, \dots y_1, \dots z_1, \dots R)$.

We quote a few observations from [van Benthem, 1983]:

- all forms (1) have a first-order equivalent,
- all forms (2) are preserved under elementary (first-order) equivalence, and hence are equivalent to some (infinite) disjunction of (infinite) conjunctions of first-order sentences,
- the forms (3) harbour the full complexity of Σ_1^1.

The first assertion follows from its counterpart for Π_1^1 sentences, to be stated below. A proof sketch of the second assertion is as follows. If (2) holds in a model $\mathfrak{A}$, then so does its first-order matrix $(2)^*$ in some expansion $\mathfrak{A}^+$ of $\mathfrak{A}$. Now suppose that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$. By a standard compactness argument, $(2)^*$ is satisfiable together with the elementary diagram of $\mathfrak{B}$, i.e. in some elementary extension of $\mathfrak{B}$. But, restricting $X_1, \dots, X_k$ to B, a substructure arises giving the same truth values to formulas of the specific form $(2)^*$; and hence we have an expansion of $\mathfrak{B}$ to a model for $(2)^*$ — i.e. $\mathfrak{B}$ satisfies (2).

Finally, the third assertion follows from the earlier Skolem reduction: with proper care, the Skolem normal form of the first-order matrix will add some predicates to $X_1, \dots, X_k$, while leaving a first-order prefix of the form $\forall\exists$. ■

Lastly, we mention the Svenonius characterization of Σ^1_1-sentences in terms of quantifiers of infinite length. In chapter I.1 an interpretation is mentioned of finite formulas in terms of games. This is a particularly good way of explaining infinite sequences of quantifiers like

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \forall x_3 \exists y_3 \ldots \varphi(x_1, y_1, x_2, y_2, \ldots). \qquad (1)$$

Imagine players $\forall$ and $\exists$ alternatively picking $x_1, x_2, \ldots$ resp. $y_1, y_2, \ldots$: $\exists$ *wins* iff $\varphi(x_1, y_1, x_2, \ldots)$. (1) is counted as *true* iff $\exists$ has a *winning strategy*, i.e. a function telling him how to play, given $\forall$'s previous moves, in order to win. Of course, a winning strategy is nothing more than a bunch of Skolem functions.

Now, Svenonius' theorem says that, on countable models, every Σ^1_1 sentence is equivalent to one of the form (1) where φ is the conjunction of an (infinite) recursive set of first-order formulas. The theorem is in Svenonius [Svenonius, 1965]; for a more accessible exposition, cf. [Barwise, 1975, Chapter VI.6].

2.5.3 Π^1_1*-sentences.* Most examples of second-order sentences in Section 2.2 were Π^1_1: full induction, Dedekind completeness, full substitution. Also, our recurrent example *most* belonged to this category — and so do, e.g. the modal formulas of intensional logic (compare van Benthem [II.4]).

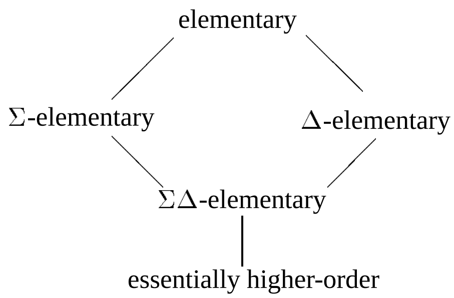
Results about Π^1_1 sentences closely parallel those for Σ^1_1. (One notable exception is universal validity, however: that notion is recursively axiomatizable here, for the simple reason that $\vDash_2 \forall X \varphi(X, Y)$ iff $\vDash_1 \varphi(X, Y)$.) For instance, we have

THEOREM. *A* Π^1_1*-sentence has a first-order equivalent iff it is preserved under ultraproducts.*

This time, we shall be little more explicit about various possibilities here. The above theorem refers to *elementary* definitions of Π^1_1-sentences, i.e. in terms of *single* first-order sentences. The next two more liberal possibilities are Δ*-elementary* definitions (allowing an infinite conjunction of first-order sentences) and Σ*-elementary* ones (allowing an infinite disjunction). As was noted in Section 1.2, the non-first-order Π^1_1 notion of finiteness is also Σ-elementary: 'precisely one or precisely two or $\ldots$'. The other possibility does not occur, however: all Δ-elementary Π^1_1 sentences are already elementary. (If the conjunction $\bigwedge S$ defines $\forall X_1, \ldots, X_n \varphi$, then the following first-order implication holds: $S \vDash \varphi(X_1, \ldots, X_n)$. Hence, by compactness $S_0 \vDash \varphi$ for some finite $S_0 \subseteq S$ — and $\bigwedge S_0$ defines $\forall X_1, \ldots, X_n \varphi$ as well.) The next levels in this more liberal hierarchy of first-order definability are $\Sigma\Delta$ and $\Delta\Sigma$. (Unions of intersections and intersections of unions, respectively.) These two, and in fact all putative 'higher' ones collapse, by the following observation from Bell and Slomson [1969].

PROPOSITION. *A property of models is preserved under elementary equivalence iff it is* $\Sigma\Delta$*-elementary.*

Thus, essentially, there remains a hierarchy of the following form:



Now, by a reasoning similar to the above, we see that $\Sigma\Delta$-elementary Π^1_1-sentences are Σ-elementary already. Thus, in the Π^1_1-case, the hierarchy collapses to 'elementary, Σ-elementary, essentially second-order'. This observation may be connected up with the earlier syntactic classification of Σ^1_1. Using negations, we get for Π^1_1-sentences the types $\forall\forall(1)$, $\forall\exists(2)$ and $\forall\exists\forall(3)$. And these provide precisely instances for each of the above remaining three stages. For instance, that all types (1) are elementary follows from the above characterization theorem, in combination with the observation that type (1) Π^1_1-sentences are preserved under *ultraproducts* (cf. [van Benthem, 1983]).

We may derive another interesting example of failure of first-order model theory here. One of the classical mother results is the Łoś-Tarski theorem: preservation under submodels is syntactically characterized by definability in universal prenex form. But now, consider well-foundedness (Section 2.2.3). This property of models is preserved in passing to submodels, but it cannot even be defined in the universal form (1), lacking first-order definability.

Our last result shows that even this modest, and basic topic of connections between Π^1_1 sentences and first-order ones is already fraught with complexity.

THEOREM. *The notion of first-order definability for Π^1_1-sentences is not arithmetical.*

Proof. Suppose, for the sake of reduction, that it were. We will then derive the arithmetical definability of arithmetical truth — again contradicting Tarski's theorem. Actually, it is slightly more informative to consider a set-theoretic reformulation (involving only one, binary relation constant): truth in $\langle V_\omega, \in\rangle$ cannot be arithmetical for first-order sentences ψ.

Now, consider any categorical Π^1_1-definition Φ for $\langle V_\omega, \in\rangle$. Truth in $\langle V_\omega, \in\rangle$ of ψ then amounts to the implication $\Phi \vDash_2 \psi$. It now suffices to show that this statement is effectively equivalent to the following one: '$\Phi \vee \psi$ is first-order definable'.

Here, the Π^1_1 statement $\Phi \vee \psi$ is obtained by pulling ψ into the first-order matrix of Φ.

'$\Longrightarrow$': If $\Phi \vDash_2 \psi$, then $\Phi \vee \psi$ is defined by ψ.

'$\Longleftarrow$': Assume that some first-order sentence α defines $\Phi \vee \psi$. Consider $\langle V_\omega, \in\rangle$: Φ holds here, and hence so does α. Now let $\mathfrak{A}$ be any proper elementary extension of $\langle V_\omega, \in\rangle$: Φ fails there, while α still holds. Hence ($\Phi \vee \psi$ and so) ψ holds in $\mathfrak{A}$. But then, ψ holds in the elementary submodel $\langle V_\omega, \in\rangle$, i.e. $\Phi \vDash_2 \psi$. ■

2.5.4 Resplendent Models. One tiny corner of 'higher-order model theory' deserves some special attention. Models on which Σ^1_1 formulas are equivalent with their set of first-order consequences recently acquired special interest in model theory. Formally, $\mathfrak{A}$ is called *resplendent* if for every first-order formula $\varphi = \varphi(x_1, \ldots, x_n)$ in the language of $\mathfrak{A}$ supplemented with some relation symbol R:

$$\mathfrak{A} \vDash_2 \forall x_1 \ldots x_n (\bigwedge \psi \rightarrow \exists R\varphi),$$

where ψ is the set of all first-order $\psi = \psi(x_1, \ldots, x_n)$ in the language of $\mathfrak{A}$ logically implied by φ. Thus, $\mathfrak{A}$ can be expanded to a model of φ as soon as it satisfies all first-order consequences of φ in its own language.

Resplendency was introduced, in the setting of infinitary admissible languages by Ressayre [1977] under the name *relation-universality*. A discussion of its importance for the first-order case can be found in Barwise and Schlipf [1976]. The notion is closely related to *recursive saturation* (i.e. saturation w.r.t. recursive types of formulas): every resplendent model is recursively saturated, and, for countable models, the converse obtains as well. In fact, Ressayre was led to (the infinitary version of) this type of saturation by looking at what it takes to prove resplendency.

The importance of resplendent models is derived from the fact that they exist in abundance in all cardinals and can be used to trivialize results in first-order model theory formerly proved by means of saturated and special models of awkward cardinalities. Besides, Ressayre took the applicability of the infinitary notions to great depth, deriving results in descriptive set theory as well.

We only mention two easy examples.

Proof. (*Craig interpolation theorem*) Suppose that $\vDash \varphi(R) \rightarrow \psi(S)$; let Φ be the set of R-less consequences of φ. When $\Phi \vDash \psi$, we are finished by one application of compactness. Thus, let $(\mathfrak{A}, S)$ be a *resplendent* model of Φ. By definition, we can expand $(\mathfrak{A}, S)$ to a model $(\mathfrak{A}, R, S)$ of φ. Hence, $(\mathfrak{A}, S) \vDash \psi$. But then, $\Phi \vDash \psi$, as *every* model has a resplendent equivalent. ■

As is the case of saturated and special models, many global definability theorems have local companions for resplendent ones. We illustrate this fact again with the interpolation theorem, which in its local version takes the following form: if the resplendent model $\mathfrak{A}$ satisfies $\forall x(\exists R\varphi(x) \rightarrow \forall S\psi(x))$, then there exists a first-order formula $\eta(x)$ in the $\mathfrak{A}$-language such that $\mathfrak{A}$ satisfies both $\forall x(\exists R\varphi \rightarrow \eta)$ and $\forall x(\eta \rightarrow \forall S\psi)$. To make the proof slightly more perspicuous, we make the statement more symmetrical. Let $\varphi' = \neg\psi$. The first sentence then is equivalent with $\forall x(\neg\exists R\varphi \vee \neg\exists S\varphi')$ (*), while the last amounts then to $\forall x(\exists S\varphi' \rightarrow \neg\eta)$. Hence, interpolation takes the (local) 'Robinson-consistency' form: disjoint Σ^1_1-definable sets on $\mathfrak{A}$ can be separated by a first-order definable one.

Now for the proof, which is a nice co-operation of both resplendency and recursive saturation. Suppose that our resplendent model $\mathfrak{A} = \langle A, \ldots \rangle$ satisfies (*). By resplendency, the set of logical consequences of either φ or φ' in the language of $\mathfrak{A}$ is not satisfiable in $\mathfrak{A}$.

Applying recursive saturation (the set concerned is only recursively enumerable according to first-order completeness — but we can use Craig's 'pleonasm' trick to get a recursive equivalent), some finite subset $\Phi \cup \Phi'$ is non-satisfiable already, where we've put the φ consequences in Φ and the φ' consequences in Φ'. We now have $\models \varphi \rightarrow \bigwedge \Phi$, $\models \varphi' \rightarrow \bigwedge \Phi'$, and, by choice of $\Phi \cup \Phi'$, $\mathfrak{A} \models \forall x \neg \bigwedge(\Phi \cup \Phi')$, which amounts to $\mathfrak{A} \models \forall x(\neg \bigwedge \Phi \vee \neg \bigwedge \Phi')$; hence we may take either $\bigwedge \Phi$ or $\bigwedge \Phi'$ as the 'separating' formula. The local Beth theorem is an immediate consequence: if the disjoint Σ^1_1-definable sets are each other's complement, they obviously coincide with the first-order definable separating set and its complement, respectively. In other words, sets which are both Σ^1_1 and Π^1_1-definable on $\mathfrak{A}$ are in fact first-order definable. ■

This situation sharply contrasts with the case for (say) $\mathfrak{N}$ discussed in Section 3.2, where we mention that arithmetic truth is both Σ^1_1 and Π^1_1-definable, but not arithmetical.

3 HIGHER-ORDER LOGIC

Once upon the road of second-order quantification, higher predicates come into view. In mathematics, one wants to quantify over functions, but also over functions defined on functions, etcetera. Accordingly, the type theories of the logicist foundational program allowed quantification over objects of all finite orders, as in *Principia Mathematica*. Bt also natural language offers such an ascending hierarchy, at least in the types of its lexical items. For instance, nouns (such as 'woman') denote properties, but then already adjectives become higher-order phrases ('blond woman'), taking such properties to other properties. In fact, the latter type of motivation has given type theories a new linguistic lease of life, in so-called 'Montague Grammar', at a time when their mathematical functions had largely been taken over by ordinary set theory (cf. [Montague, 1974]).

In this section, we will consider a stream-lined relational version of higher-order logic, which leads to the basic logical results with the least amount of effort. Unfortunately for the contemporary semanticist, it does not look very much like the functional Montagovian type theory. In fact, we will not even encounter such modern highlights as lambda-abstraction, because our language can do all this by purely traditional means. Moreover, in Section 4, we shall be able to derive partial completeness results from the standard first-order ones for many-sorted logic in an extremely simple fashion. (In particular, the complicated machinery of [Henkin, 1950] seems unnecessary.) It's all very elegant, simple, and exasperating. A comparison with the more semantic, categorial grammar-oriented type theories will be given at the end.

3.1 Syntax and Semantics

As with first-order languages, higher-order formulas are generated from a given set L of non-logical constants, among which we can distinguish individual constants, function symbols and relation signs. (Often, we will just think of the latter.) Formulas will be interpreted in the same type of models $\mathfrak{A} = \langle A, * \rangle$ as used in the first-order case, i.e. $A \neq \emptyset$, and $*$ assigns something appropriate to every L symbol: ('distinguished') elements of A to individual constants, functions over A to function symbols (with the proper number of arguments) and relations over A to relation signs (again, of the proper arity).

Thus, fix any such set L. Patterns of complexity are now recorded in *types*, defined inductively by

1. 0 is a type
2. a finite sequence of types is again a type.

Here, 0 will be the type of *individuals*, $(\tau_1, \ldots, \tau_n)$ that of *relations* between objects of types $\tau_1, \ldots, \tau_n$. Notice that, if we read clause (2) as also producing the *empty* sequence, we obtain a type of relation without arguments; i.e. of propositional constants, or *truth values*. Higher up then, we will have propositional functions, etcetera. This possibility will not be employed in the future, as our metatheory would lose some of its elegance. (But see Section 3.3 for a reasonable substitute.)

The *order* of a type is a natural number defined as follows: the order of 0 is 1 (individuals are 'first-order' objects), while the order or $(\tau_1, \ldots, \tau_n)$ equals $1 + \max \text{order}(\tau_i)$ $(1 \leq i \leq n)$. Thus, the terminology of 'first-order', 'second-order', etcetera, now becomes perfectly general.

For each type τ, the language has a countably infinite set of variables. The order of a variable is the order of its type. Thus, there is only one kind of first-order variable, because the only order 1-type is 0. The second-order variables all have types $(0,\ldots,0)$. Next, the *terms* of type 0 are generated from the type 0 variables and the individual constants by applying function symbols in the proper fashion. A term of type $\neq 0$ is just a variable of that type. Thus, for convenience, non-logical constants of higher-orders have been omitted: we are really thinking of our former first-order language provided with a quantificational higher-order apparatus. Finally, one might naturally consider a relation symbol with n places as a term of type $(0,\ldots,0)$ (n times); but the resulting language has no additional expressive power, while it becomes a little more complicated. Hence, we refrain from utilising this possibility.

Atomic formulas arise as follows:

1. $R(t_1,\ldots,t_n)$ where R is an n-place relation symbol and $t_1,\ldots,t_n$ terms of type 0,

2. $X(t_1,\ldots,t_n)$, where X is a variable of type $(\tau_1,\ldots,\tau_n)$ and t_i a type τ_i-term $(1 \leq i \leq n)$.

We could have added identities $X = Y$ here for all higher types; but these may be thought of as defined through the scheme $\forall X_1 \ldots \forall X_n (X(X_1,\ldots,X_n) \leftrightarrow Y(X_1,\ldots,X_n))$, with appropriate types.

Formulas are defined inductively from the atomic ones using propositional connectives and quantification with respect to variables of all types. The resulting set, based on the vocabulary L is called L_ω. L_n is the set of formulas all of whose variables have order $\leq n$ ($n = 1,2,\ldots$). Thus, we can identify L_1 with the first-order formulas over L, and L_2 with the second-order ones. (A more sophisticated classification of orders is developed in Section 3.2 below, however.) The reader is requested to formulate the examples of Section 2.2 in this language; especially the L_3-definition of finiteness.

Again, let us notice that we have opted for a rather austere medium: no higher-order constants or identities, no conveniences such as *function quantifiers*, etcetera. One final omission is that of relational *abstracts* taking formulas $\varphi(X_1,\ldots,X_n)$ to terms $\lambda X_1 \ldots X_n \cdot \varphi(X_1,\ldots,X_n)$ denoting the corresponding relation. In practice, these commodities do make life a lot easier; but they are usually dispensable in theory. For instance, the statement $\varphi(\lambda X \cdot \psi(X))$ is equally well expressed by means of $\exists Y(Y = \lambda X \cdot \psi(X) \wedge \varphi(Y))$, and this again by $\exists Y(\forall Z(Y(Z) \leftrightarrow \psi(Z)) \wedge \varphi(Y))$.

From the syntax of our higher-order language, we now pass on to its semantics. Let $\mathfrak{A}$ be an ordinary L-model $\langle A, * \rangle$ as described above. To interpret the L_ω-formulas in $\mathfrak{A}$ we need the *universes of type τ over A* for all types τ:

1. $D_0(A) = A$

2. $D_{(\tau_1,\ldots,\tau_n)}(A) = \mathcal{P}(D_{\tau_1}(A) \times \cdots \times D_{\tau_n}(A))$.

An A-*assignment* is a function α defined on all variables such that, if X has type $\tau, \alpha(X) \in D_\tau(A)$.

We now lift the ordinary satisfaction relation to L_ω-formulas φ in the obvious way. For instance, for an L-model $\mathfrak{A}$ and an A-assignment α,

$$\mathfrak{A} \vDash_\omega X(t_1, \dots, t_n)[\alpha] \text{ iff } \alpha(X)(t_1^{\mathfrak{A}}[\alpha], \dots, t_n^{\mathfrak{A}}[\alpha]);$$

where $t^{\mathfrak{A}}[\alpha]$ is the *value* of the term t under α in $\mathfrak{A}$ defined as usual. Also, e.g. $\mathfrak{A} \vDash_\omega \forall X \varphi[\alpha]$ iff for all assignments α' differing from α at most in the value given to $X : \mathfrak{A} \vDash_\omega \varphi[\alpha']$.

The other semantical notions are derived from satisfaction in the usual fashion.

3.2 The Prenex Hierarchy of Complexity

The logic and model theory of L_ω exhibit the same phenomena as those of L_2: a fluid border line with set theory, and a few systematic results. Indeed, in a sense, higher-order *logic* does not offer anything new. It will be shown in Section 4.2 that there exists an effective reduction from universal validity for L_ω formulas to that for second-order ones, indeed to monadic Σ_1^1 formulas [Hintikka, 1955].

As for the connections between L_ω and set theory, notice that the present logic is essentially that of arbitrary models A provided with a natural set-theoretic super-structure $\bigcup_n V^n(A)$; where $V^0(A) = A$, and $V^{n+1}(A) = V^n(A) \cup \mathcal{P}(V^n(A))$. As a 'working logic', this is a sufficient setting for many mathematical purposes. (But cf. [Barwise, 1975] for a smaller, *constructible* hierarchy over models, with a far more elegant metatheory.)

We will not go into the exact relations between the logic of L_ω and ordinary set theory, but for the following remark.

Given a structure $\mathfrak{A} = \langle A, * \rangle$, the structure $\mathfrak{A}^+ = \langle \bigcup_n V^n(A), \in, * \rangle$ is a model for a set theory with atoms. There is an obvious translation from the L_ω-theory of $\mathfrak{A}$ into a fragment of the ordinary first-order theory of $\mathfrak{A}^+$. The reader may care to speculate about a converse (cf. [Kemeny, 1950]).

What will be considered instead in this section, is one new topic which is typical for a hierarchical language such as the present one. We develop a prenex classification of formulas, according to their patterns of complexity; first in general, then on a specific model, viz. the natural numbers. This is one of the few areas where a coherent body of higher-order theory has so far been developed.

There exists a standard classification of first-order formulas in prenex form. $\Sigma_0 = \Pi_0$ is the class of quantifier-free formulas; Σ_{m+1} is the class of formulas $\exists x_1 \dots \exists x_k \varphi$ where $\varphi \in \Pi_m$; and dually, Π_{m+1} is the class of formulas $\forall x_1 \dots \forall x_k \varphi$ with $\varphi \in \Sigma_m$. The well-known Prenex Normal Form Theorem now says that every first-order formula is logically equivalent to one in $\bigcup_m(\Sigma_m \cup \Pi_m)$; i.e. to one in prenex form.

The above may be generalized to arbitrary higher-order formulas as follows. We classify quantificational complexity with respect to the $n+1$st order. $\Sigma_0^n = \Pi_0^n$ is the class of L_ω-formulas all of whose quantified variables have order $\leq n$. Thus, Σ_0^0 is the class of quantifier-free L_ω-formulas. (Notice that the above Σ_0 is a proper subclass of Σ_0^0, as we allow free variables of higher type in Σ_0^0 formulas. Also, it is not true that $\Sigma_0^1 \subseteq L_1$, or even $\Sigma_0^1 \subseteq L_n$ for some $n > 1$.)

Next, Σ_{m+1}^n is the class of formulas $\exists X_1 \ldots \exists X_k \varphi$, where $\varphi \in \Pi_m^n$ and $X_1, \ldots, X_k$ have order $n+1$; and dually, Π_{m+1}^n consists of the formulas $\forall X_1 \ldots \forall X_k \varphi$ with $\varphi \in \Sigma_m^n$ and $X_1, \ldots, X_k$ $(n+1)$st order. (Notice the peculiar, but well-established use of the upper index n: a Σ_2^1 formula thus has quantified *second*-order variables.)

The reader may wonder why we did not just take Σ_0^n to be L_n. The reason is that we do not consider the mere occurrence of, say, second-order variables in a formula a reason to call it (at least) second-order. (Likewise, we do not call first-order formulas 'second-order' ones, because of the occurrence of second-order relational constants.) It is *quantification* that counts: we take a formula to be of order n when its interpretation in a model $\langle A, \ldots \rangle$ presupposes complete knowledge about some nth order universe $D_\tau(A)$ over A. And it is the quantifier over some order n variable which presupposes such knowledge, not the mere presence of free variables of that order. (After all, we want to call, e.g. a property of type $((0))$ 'first-order' definable, even if its first-order definition contains a second-order free variable — and it must.) There is an interesting historical analogy here. One way to think of the prenex hierarchy is as one of *definitional complexity*, superimposed upon one of *argument type complexity* (given by the free variable pattern of a formula). This move is reminiscent of Russell's passage from *ordinary* to *ramified* type theory.

THEOREM. *Every L_{n+1}-formula has an equivalent in $\bigcup_m (\Sigma_m^n \cup \Pi_m^n)$.*

Proof. Let $\varphi \in L_{n+1}$ be given. First, manipulate it into prenex form, where the order of the quantifiers is immaterial — just as in the first-order case. Now, if we can manage to get quantifiers over $n+$ 1st order variables to the front, we are done. But, this follows by repeated use of the valid equivalence below and its dual.

Let x have type τ_0 and order less than $n+1$: the order of the type $(\tau_1, \ldots, \tau_k)$ of the variable X. Let Y be some type $(\tau_0, \ldots, \tau_k)$ variable; its order is then $n+1$ too, and we have the equivalence

$$\forall x \exists X \psi \leftrightarrow \exists Y \forall x \psi'.$$

Here ψ' is obtained from ψ by replacing subformulas $X(t_1, \ldots, t_k)$ by $Y(x, t_1, \ldots, t_k)$; where Y does not occur in ψ. Thanks to the restriction to L_{n+1}, the *only* atomic subformulas of ψ containing X are of the above form and, hence, ψ' does not contain X any longer. (If X could occur in argument positions, it would have to be defined away using suitable Y *abstracts*. But, this addition to the language would bring about a revised account of complexity in any case.)

To show intuitively that the above equivalence is valid, assume that $\forall x \exists X\, \psi(x, X)$. For every x, choose X_x such that $\psi(x, X_x)$. Define Y by setting $Y(x, y_1, \ldots, y_n) := X_x(y_1, \ldots, y_n)$. Then clearly $\forall x\, \psi(x, \{\langle y_1, \ldots, y_n\rangle \mid Y(x, y_1, \ldots, y_n)\}$ and, hence, $\exists Y \forall x\, \psi'$. The converse is immediate. ■

We will now pass on to more concrete hierarchies of higher-order definable relations on specific models.

Let $\mathfrak{A} = \langle A, \ldots\rangle$ be some model, $R \in D_\tau(A)$, $\tau = (\tau_1, \ldots, \tau_n)$, and let $\varphi \in L_\omega$ have free variables $X_1, \ldots, X_n$ of types (respectively) $\tau_1, \ldots, \tau_n$. φ is said to *define* R on $\mathfrak{A}$ if, whenever $S_1 \in D_{\tau_1}(A), \ldots, S_n \in D_{\tau_n}(A)$,

$$R(S_1, \ldots, S_n) \text{ iff } \mathfrak{A} \vDash_\omega \varphi[S_1, \ldots, S_n].$$

R is called Σ^n_m (Π^n_m) on $\mathfrak{A}$ if it has a defining formula of this kind. It is Δ^n_m if it is both Σ^n_m and Π^n_m. We denote these classes of definable relations on $\mathfrak{A}$ by $\Sigma^n_m(\mathfrak{A})$, etcetera.

Now, let us restrict attention to $\mathfrak{A}$ = the natural numbers $\mathfrak{N} : \langle \mathbb{N}, +, \times, 0\rangle$. (In this particular case it is customary to let $\Sigma^0_0(\mathfrak{N}) = \Pi^0_0(\mathfrak{N})$ be the wider class of relations definable using formulas in which *restricted* quantification over first-class variables is allowed.) For any type τ, $\Delta^0_1(\mathfrak{N}) \cap D_\tau(\mathbb{N})$ is the class of *recursive* relations of type τ; the ones in $\Sigma^0_1(\mathfrak{N}) \cap D_\tau(\mathbb{N})$ are called *recursively enumerable*. These are the simplest cases of the *arithmetical hierarchy*, consisting of all Σ^0_n and Π^0_n-definable relations on $\mathfrak{N}$. Evidently, these are precisely the first-order-definable ones, in any type τ.

At the next level, the *analytic hierarchy* consists of the Σ^1_n and Π^1_n-definable relations on $\mathfrak{N}$. Those in $\Delta^1_1(\mathfrak{N})$ are called *hyperarithmerical*, and have a (transfinite) hierarchy of their own. One reason for the special interest in this class is the fact that *arithmetic truth* for first-order sentences is hyper-arithmerical (though not arithmetical, by Tarski's Theorem).

These hierarchies developed after the notion of recursiveness had been identified by Gödel, Turing and Church, and were studied in the fifties by Kleene, Mostowski and others.

Just to give an impression of the more concrete type of investigation in this area, we mention a few results. Methods of proof are rather uniform: positive results (e.g. '$\varphi \in \Sigma^1_n$') by actual inspection of possible definitions, negative results ('$\varphi \notin \Sigma^1_n$') by diagonal arguments reminiscent of the mother example in Russell's Paradox.

1. The satisfaction predicate 'the sequence (coded by) s satisfies the first-order formula (coded by) φ in $\mathfrak{N}$' is in $\Delta^1_1(\mathfrak{N}) \cap D_{(0,0)}(\mathbb{N})$.

2. This predicate is not in $\Sigma^1_0(\mathfrak{N})$.

3. The *Analytic Hierarchy Theorem* for $D_{(0)}(\mathbb{N})$ relations. All inclusions in the following scheme are proper (for all m):

$$\begin{array}{ccccc} & & \Sigma^1_m(\mathfrak{R}) & & \\ & \subseteq & & \subseteq & \\ \Delta^1_m(\mathfrak{R}) & & & & \Delta^1_{m+1}(\mathfrak{R}) \\ & \subseteq & & \subseteq & \\ & & \Pi^1_m(\mathfrak{R}) & & \end{array}$$

   These results may be generalized to higher orders.

4. Satisfaction for Σ^n_0-formulas (with first-order free variables only) on $\mathfrak{N}$ is in $\Delta^n_1(\mathfrak{N}) - \Sigma^n_0(\mathfrak{N})$.

5. The Hierarchy Theorem holds in fact for any upper index ≥ 1.

By allowing second-order parameters in the defining formulas, the analytic hierarchy is transformed into the classical hierarchy of *projective* relations. Stifled in set-theoretic difficulties around the twenties, interest in this theory was revived by the set-theoretic revolution of the sixties. The reader is referred to the modern exposition [Moschovakis, 1980].

3.3 Two Faces of Type Theory

As was observed earlier, the above language L_ω is one elegant medium of description for one natural type superstructure on models with relations. Nevertheless, there is another perspective, leading to a more function-oriented type theory closer to the categorial system of natural language. In a sense, the two are equivalent through codings of functions as special relations, or of relations through characteristic functions. It is this kind of *sous entendu* which would allow an ordinary logic text book to suppress all reference to functional type theories in the spirit of [Church, 1940; Henkin, 1950] or [Montague, 1974]. (It is this juggling with codings and equivalences also, which makes advanced logic texts so impenetrable to the outsider lacking that frame of mind.)

For this reason, we give the outline of a functional type theory, comparing it with the above. As was observed earlier on, in a first approximation, the existential part of natural language can be described on the model of a *categorial grammar*, with basic *entity expressions* (e.g. proper names; type e) and *truth value expressions* (sentences; type t), allowing arbitrary binary couplings (a, b): the type of functional expressions taking an a-type expression to a b-type one. Thus, for instance, the intransitive verb 'walk' has type (e, t), the transitive verb 'buy' type $(e, (e, t))$, the sentence negation 'not' has (t, t) while sentence conjunction has $(t, (t, t))$. More complicated examples are quantifier phrases, such as 'no man', with type $((e, t), t)$, or determiners, such as ‘no’, with type $((e,t),((e,t),t))$. Again, to a first approximation, there arises the picture of natural language as a huge jigsaw puzzle, in which the interpretable sentences are those for which the types of their component words can be fitted together, step by step, in such a way that the end result for the whole is type t.

Now, the natural matching type theory has the above types, with a generous supply of variables and constants for each of these. Its basic operations will be, at least, *identity* (between expressions of the same type), yielding truth value expressions, and *functional application* combining B with type (a,b) and A with type a to form the expression $B(A)$ of type b. What about the logical constants? In the present light, these are merely constants of specific categories. Thus, binary connectives (‘and’, ‘or’) are in $(t,(t,t))$, quantifiers (‘all’, ‘some’) in the above determiner type $((e,t),((e,t),t))$. (Actually, this makes them into binary relations between properties: a point of view often urged in the logical folklore.) Nevertheless, one can single them out for special treatment, as was Montague’s own strategy. On the other hand, a truly natural feature of natural language seems to be the phenomenon of *abstraction*: from any expression of type b, we can make a functional one of type (a,b) by varying some occurrence(s) of component a expressions. Formally then, our type theory will have so-called ‘lambda abstraction’: if B is an expression of type b, and x a variable of type a, then $\lambda x \cdot B$ is an expression of type (a,b).

Semantic structures for this language form a function hierarchy as follows:

1. D_e is some set (of ‘entitities’ or ‘individuals’),
2. D_t is the set of truth values $\{0,1\}$ (or some generalization thereof),
3. $D_{(a,b)} = D_b^{D_a}$

Given a suitable interpretation for constants and assignments for variables, values may be computed for terms of type a in the proper domain D_a through the usual compositional procedure. Thus, in particular, suppressing indices,

$$\text{val}(B(A)) = \text{val}(B)(\text{val}(A))$$
$$\text{val}(\lambda x \cdot B) = \lambda a \in D_a \cdot \text{val}(B)_{x \to a}.$$

(Just this once, we have refrained from the usual pedantic formulation.)

In Montague’s so-called ‘intensional type theory’, this picture is considerably complicated by the addition of a realm of possible world-times, accompanied by an auxiliary type s with restricted occurrences. This is a classical example of an unfortunate formalization. Actually, the above set-up remains exactly the same with one additional basic type s (or two, or ten) with corresponding semantic domains D_s (all world-times, in Montague’s case). In the terms of [Gallin, 1975]: once we move up from Ty to $Ty2$, simplicity is restored.

We return to the simplest case, as all relevant points can be made here. What is the connection with the earlier logic L_ω? Here is the obvious translation, simple in content, a little arduous in combinatorial detail.

First, let us embed the Montague hierarchy of domains D_a over a given universe A into our previous hierarchy $D_\tau(A)$. In fact, we shall *identify* the D_a with certain subsets of the $D_\tau(A)$. There seems to be one major problem here, viz. what is to correspond to $D_t = \{0, 1\}$. (Recall that we opted for an L_ω-hierarchy without truth-value types.) We choose to define $D_t \subseteq D_{(0)}(A) : 0$ becoming $\emptyset$, and 1 becoming the whole A. Next, of course $D_e = D_0(A)$. The rule $D_{(a,b)} = D_b^{D_a}$ then generates the other domains. Thus, every Montague universe D_a has been identified with a subset of a certain $D_{\underline{a}}(A)$; where $\underline{a}$ is obviously determined by the rules $\underline{e} := 0$, $\underline{t} := (0)$ and $\underline{(a,b)} := (\underline{a}, \underline{b})$. (Thus, functions have become identified with their graphs; which are binary relations in this case.)

Next, for each Montague type a, one can write down an L_ω-formula $T_a(x)$ (with x of type $\underline{a}$) which *defines* D_a in $D_{\underline{a}}(A)$, i.e. for $b \in D_{\underline{a}}(A)$, $\mathfrak{A} \vDash_\omega T_a[b]$ iff $b \in D_a$.

When $E = E(x_1, \ldots, x_n)$ is any type a_0 expression in the Montague system, with the free variables $x_1, \ldots, x_n$ (with types $a_1, \ldots, a_n$, respectively) and $b_i \in D_{a_i}$ $(1 \leq i \leq n)$, an object $E^{\mathfrak{A}}[b_1, \ldots, b_n] \in D_{a_0}$ has been defined which is the *value* of E under $b_1, \ldots, b_n$ in $\mathfrak{A}$. We shall indicate now how to write down an L_ω-formula $V(x_0, E)$ with free variables $x_0, \ldots, x_n$ (where *now* x_i has type $\underline{a}_i$ $(1 \leq i \leq n)$), which says that x_0 is the value of E under $x_1, \ldots, x_n$. To be completely precise, we will have

$$\mathfrak{A} \vDash_\omega V(x_0, E)[b_0, \ldots, b_n] \text{ iff } b_0 = E^{\mathfrak{A}}[b_1, \ldots, b_n]$$

for objects $b_0, \ldots, b_n$ of the appropriate types.

As a consequence of this, we obtain

$$\mathfrak{A} \vDash_\omega \exists x(V(x, E_1) \wedge V(x, E_2)) \text{ iff } E_1^{\mathfrak{A}} = E_2^{\mathfrak{A}}$$

for closed expressions E_1, E_2. Thus, the characteristic assertions of Montagovian type theory have been translated into our higher-order logic.

It remains to be indicated how to construct the desired V. For perspicuity, three shorthands will be used in L_ω. First, $x(y)$ stands for the unique z such that $x(y, z)$, if it exists. (Elimination is always possible in the standard fashion.) Furthermore, we will always have $\forall x_1 \ldots x_n \exists! x_0 V(x_0, E)$ valid when relativized to the proper types. Therefore, instead of $V(x_0, E)$, one may write $x_0 = V(E)$. Third, quantifier relativization to T_a will be expressed by $\forall x \in T_a$ $(\exists x \in T_a)$ (where x has type $\underline{a}$). Finally, in agreement with the above definition of the truth values, we abbreviate $\forall y \in T_e x(y)$ and $\forall y \in T_e \neg x(y)$ by $x = \top$, $x = \bot$, respectively (where x has type (0)).

Here are the essential cases:

1. E is a two-place relation symbol of the base vocabulary L.

$$V(x,E) := x \in T_{(e,(e,t))} \wedge \forall yz \in T_e((x(y))(z) = \top \leftrightarrow E(y,z)).$$

2. $E = E_1(E_2)$.

$$V(x,E) := x = V(E_1)(V(E_2)).$$

3. $E = \lambda y \cdot F$ (y of type a, F of type b).

$$V(x,E) := x \in T_{(a,b)} \wedge \forall y \in T_a(x(y) = V(F)).$$

4. $E = (E_1 = E_2)$.

$$V(x,E) := x \in T_t \wedge (x = \top \leftrightarrow V(E_1) = V(E_2)).$$

That these clauses do their job has to be demonstated by induction, of course; but this is really obvious.

It should be noted that the procedure as it stands does not handle higher-order constants: but, a generalization is straightforward.

For further details, cf. [Gallin, 1975, Chapter 13]. Gallin also has a converse translation from L_ω into functional type theory, not considered here.

The reduction to L_ω makes some prominent features of functional type theory disappear. Notably, lambda abstraction is simulated by means of ordinary quantification. It should be mentioned, however, that this also deprives us of some natural and important questions of functional type theory, such as the search for unique *normal forms*. This topic will be reviewed briefly at the end of the following Section.

4 REDUCTION TO FIRST-ORDER LOGIC

One weak spot in popular justifications for employing higher-order logic lies precisely in the phrase 'all predicates'. When we say that Napoleon has all properties of the great generals, we surely mean to refer to some sort of relevant human properties, probably even definable ones. In other words, the lexical item 'property' refers to some sort of 'things', just like other common nouns. Another, more philosophical illustration of this point is Leibniz' Principle, quoted earlier, of the identity of indiscernibles. Of course, when x, y share *all* properties, they will share that of being identical to x and, hence, they coincide. But this triviality is not what the great German had in mind — witness the charming anecdote about the ladies at court, whom Leibniz made to search for autumn leaves, promising them noticeable differences in colour or shape for any two merely distinct ones.

Thus, there arises the logical idea of re-interpreting second-order, or even higher-order logic as some kind of *many-sorted* first-order logic, with various distinct kinds of objects: a useful, though inessential variation upon first-order logic itself. To be true, properties and predicates are rather abstract kinds of 'things'; but then, so are many other kinds of 'individual' that no one would object to. The semantic net effect of this change in perspective is to allow a greater variety of models for L_ω, with essentially smaller ranges of predicates than the original 'full ones'. Thus, more potential counter-examples become available to universal truths, and the earlier set of L_ω-validities decreases; so much so, that we end up with a recursively axiomatizable set. This is the basic content of the celebrated introduction of 'general models' in [Henkin, 1950]: the remainder is frills and laces.

4.1 General Models

The type structure $\langle D_\tau(A) \mid \tau \in \mathcal{T}\rangle$ ($\mathcal{T}$ the set of types) over a given non-empty set A as defined in Section 3.1 is called the principal or *full* type structure over A; the interpretation of L_ω by means of $\vDash_\omega$ given there the *standard* interpretation. We can generalize these definitions as follows.

$E = \langle E_\tau \mid \tau \in \mathcal{T}\rangle$ is called a *type structure* over A when

1. $E_0 = A$ (as before)
2. $E_{(\tau_1,\ldots,\tau_n)} \subseteq \mathcal{P}(E_{\tau_1} \times \cdots \times E_{\tau_n})$.

Thus, not every relation on $E_{\tau_1} \times \cdots \times E_{\tau_n}$ need be in $E_{(\tau_1,\ldots,\tau_n)}$ any more. Restricting assignments to take values in such more general type structures, satisfaction can be defined as before, leading to a notion of truth with respect to arbitrary type structures. This so-called *general models interpretation* of L_ω admits of a complete axiomatisation, as we shall see in due course.

First, we need a certain transformation of higher-order logic into first- order terms. Let L be a given vocabulary. L^+ is the *first-order* language based on the vocabulary

$$L \cup \{\varepsilon_\tau \mid 0 \neq \tau \in \mathcal{T}\} \cup \{T_\tau \mid \tau \in \mathcal{T}\};$$

where ε_τ is an $n+1$ary relation symbol when $\tau = (\tau_1,\ldots,\tau_n)$, and the T_τ are unary relation symbols. Now, define the translation $^+ : L_\omega \to L^+$ as follows. Let $\varphi \in L_\omega$. First, replace every atom $X(t_1,\ldots,t_n)$ in it by $\varepsilon_\tau(X, t_1,\ldots,t_n)$ when X has type τ. Second, relativize quantification with respect to type τ variables to T_τ. Third, consider all variables to be (type 0) variables of L^+. This defines φ^+. (For those familiar with many-sorted thinking (cf. [Hodges, 1983, Chapter I.1]), the unary predicates T_τ may even be omitted, and φ^+ just becomes φ, in a many-sorted reading.)

On the model-theoretic level, suppose that $(\mathfrak{A}, E)$ is a general model for L; i.e. $\mathfrak{A}$ is an L-model with universe A and E is a type structure over A. We indicate how $(\mathfrak{A}, E)$ can be transformed into an ordinary (first-order) model $(\mathfrak{A}, E)^+$ for L^+:

1. the universe of $(\mathfrak{A}, E)^+$ is $\bigcup_{\tau \in \mathcal{T}} E_\tau$
2. the interpretation of L-symbols is the same as in $\mathfrak{A}$
3. ε_τ is interpreted by ($\tau = (\tau_1, \ldots, \tau_n)$): $\varepsilon^*_\tau(R, S_1, \ldots, S_n)$ iff $R \in E_\tau$, $S_i \in E_{\tau_i}$ ($1 \leq i \leq n$) and $R(S_1, \ldots, S_n)$
4. T_τ is interpreted by E_τ.

There is a slight problem here. When L contains function symbols, the corresponding functions in $\mathfrak{A}$ should be extended on $\bigcup_{\tau \in \mathcal{T}} E_\tau$. It is irrelevant how this is done, as arguments outside of E_0 will not be used.

The connection between these transformations is the

LEMMA. *Let α be an E assignment, and let $\varphi \in L_\omega$. Then $(\mathfrak{A}, E) \vDash_\omega \varphi[\alpha]$ iff $(\mathfrak{A}, E)^+ \vDash \varphi^+[\alpha]$.*

The proof is a straightforward induction on φ.

There is semantic drama behind the simple change in clause (2) for $E_{(\tau_1, \ldots, \tau_n)}$ from identity to inclusion. Full type structures are immense; witness their cardinality, which increases exponentially at each level. In stark contrast, a general model may well have an empty type structure, not ascending beyond the original universe. Evidently, the interesting general models lie somewhere in-between these two extremes.

At least two points of view suggest themselves for picking out special candidates, starting from either boundary.

'From above to below', the idea is to preserve as much as possible of the global type structure; i.e. to impose various principles valid in the full model, such as Comprehension or Choice (cf. the end of Section 4.2). In the limit, one might consider general models which are L_ω-elementarily equivalent to the full type model. Notice that, by general logic, only Π^1_1 truths are automatically preserved in passing from the full model to its general submodels. Such preservation phenomena were already noticed in [Orey, 1959], which contains the conjecture that a higher-order sentence is first-order definable if and only if it has the above persistence property, as well as its converse. (A proof of this assertion is in van Benthem [1977].)

Persistence is of some interest for the semantics of natural language, in that some of its 'extensional' fragments translate into persistent fragments of higher-order logic (cf. [Gallin, 1975, Chapter 1.4]). Although the main observation (due to Kamp and Montague) is a little beyond the resources of our austere L_ω, it may be stated quite simply. Existential statements $\exists X A(X)$ may be lost in passing from full standard models to their general variants (cf. the example given below). But, *restricted* existential statements $\exists X(P(X,Y) \wedge A(X))$ with all their parameters (i.e. $P(!), Y$) in the relevant general model, are thus preserved — and the above-mentioned extensional fragments of natural language translate into these restricted forms, which are insensitive, in a sense, to the difference between a general model and its full parent. Therefore, the completeness of L_ω with respect to the general models interpretation (Section 4.2) extends to these fragments of natural language, despite their prima facie higher-order nature.

Conversely, one may also look 'from below to above', considering reasonable constructions for filling the type universes without the above explosive features. For instance, already in the particular case of L_2, a natural idea is to consider predicate ranges consisting of all predicates *first-order definable* in the base vocabulary (possibly with individual parameters). Notice that this choice is stable, in the sense that iteration of the construction (plugging in newly defined predicates into first-order definitions) does not yield anything new. (By the way, the simplest proof that, e.g. von Neumann-Bernays-Gödel set theory is conservative over ZF uses exactly this construction.)

EXAMPLE. The first-order definable sets on the base model $\langle \mathbb{N}, < \rangle$ are precisely all finite and co-finite ones; and a similar characterization may be given for arbitrary predicates. This general model for L_2 is not elementarily equivalent to the standard model, however, as it fails to validate

$$\exists X \forall y((\exists z(X(z) \wedge y < z) \wedge \exists z(\neg X(z) \wedge y < z)).$$

Second-order general models obtained in this way only satisfy the so-called 'predicative' comprehension axioms. (Referring to the end of Section 4.2, these are the sentences (1) where φ does not *quantify* over second-order variables, but may contain them freely.) We can, however, obtain general models of full ('impredicative') comprehension if we iterate the procedure as follows. For any second-order general model $\langle \mathfrak{A}, E \rangle$, let E^+ consist of all relations on $\mathfrak{A}$ parametrically second-order definable in $\langle \mathfrak{A}, E \rangle$. Thus, the above 'predicative' extension is just $\langle \mathfrak{A}, \emptyset^+ \rangle$. This time, define E_α for ordinals α by $E_\alpha = \bigcup_{\beta < \alpha} E_\beta^+$. By cardinality considerations, the hierarchy must stop at some γ (by first-order Löwenheim-Skolem, it can in fact be proved that γ has the same cardinal as $\mathfrak{A}$), which obviously means that $\langle \mathfrak{A}, E_\gamma \rangle$ satisfies full comprehension.

For $\mathfrak{A} = \mathfrak{N}$, the above transfinite hierarchy is called *ramified analysis*, γ is Church-Kleene ω_1 and there is an extensive literature on the subject. Barwise [1975] studies related things in a more set-theory oriented setting for arbitrary models.

4.2 General Completeness

As a necessary preliminary to a completeness theorem for L_ω with its new semantics, we may ask which L^+-sentences hold in every model of the form $(\mathfrak{A}, E)^+$, where $\mathfrak{A}$ is an L-model and E a type structure over its universe A. As it happens, these are of six kinds.

1. $\exists x T_0 x$. This is because T_0 is interpreted by $E_0 = A$, which is not empty. The other type levels of E might indeed be empty, if E is not full.

2. The next sentences express the fact that the L-symbols stand for distinguished elements, functions and relations over the set denoted by T_0:

   (a) $T_0(c)$, for each individual constant of L.

   (b) $\forall x_1 \ldots \forall x_n (T_0(x_1) \wedge \cdots \wedge T_0(x_n) \rightarrow T_0(F(x_1, \ldots, x_n)))$, for all n-place function symbols F of L.

   (c) $\forall x_1 \ldots \forall x_n (R(x_1, \ldots, x_n) \rightarrow T_0(x_1) \wedge \cdots \wedge T_0(x_n))$, for all n-place relation symbols R of L.

Finally, there are sentences about the type levels.

3. $\forall x (T_\tau(x) \rightarrow \neg T_{\tau'}(x))$, whenever $\tau \neq \tau'$.

   As a matter of fact, there is a small problem here. If A has elements which are sets, then we might have simultaneously $a \in A$ and $a \subseteq A$. It could happen then that $a \in E_{(0)}$ also, and hence $E_0 \cap E_{(0)} \neq \emptyset$. To avoid inessential sophistries, we shall resolutely ignore these eventualities.

4. $\forall x \bigvee_{\tau \in \mathcal{T}} T_\tau(x)$.

   The content of this statement is clear; but unfortunately, it is not a first- order sentence of L^+, having an infinite disjunction. We shall circumvent this problem eventually.

5. $\forall x \forall y_1 \ldots \forall y_n (\varepsilon_\tau(x, y_1, \ldots, y_n) \rightarrow T_\tau(x) \wedge T_{\tau_1}(y_1) \wedge \cdots \wedge T_{\tau_n}(y_n))$, whenever $\tau = (\tau_1, \ldots, \tau_n)$. (Compare the earlier definition of $(\mathfrak{A}, E)^+$: especially the role of ε^*_τ.)

The sentences (1)–(5) are all rather trivial constraints on the type framework. The following *extensionality axioms* may be more interesting:

6. $\forall x \forall y (T_\tau(x) \wedge T_\tau(y) \wedge \forall z_1 \ldots \forall z_n (T_{\tau_1}(z_1) \wedge \cdots \wedge T_{\tau_n}(z_n) \rightarrow$ $\rightarrow (\varepsilon_\tau(x, z_1, \ldots, z_n) \leftrightarrow \varepsilon_\tau(y, z_1, \ldots, z_n))) \rightarrow x = y)$; whenever $\tau = (\tau_1, \ldots, \tau_n)$.

   That this holds in $(\mathfrak{A}, E)^+$ when E is full, is due to the extensionality axiom of set theory. But it is also easily checked for general type structures.

This exhausts the obvious validities. Now, we can ask whether, conversely, every L^+ model of (1)–(6) is of the form $(\mathfrak{A}, E)^+$, at least, up to isomorphism. (Otherwise, trivial counter-examples could be given.) The answer is positive, by an elementary argument. For any L^+-model $\mathfrak{B}$ of our six principles, we may construct a general model $(\mathfrak{A}, E)$ and an isomorphism $h : \mathfrak{B} \to (\mathfrak{A}, E)^+$ as follows.

Writing $h_\tau := h \restriction T_\tau^{\mathfrak{B}}$, we shall construct h_τ and E_τ simultaneously by induction on the order of τ, relying heavily on (6). (This construction is really a particular case of the Mostowski collapsing lemma in set theory.)

First, let $A = E_0 := T_0^{\mathfrak{B}}$, while h_0 is the identity of $T_0^{\mathfrak{B}}$. (1) says that $A \neq \emptyset$, and (2) adds that we can define $\mathfrak{A}$ by taking over the interpretations that $\mathfrak{B}$ gave to the L-symbols. Trivially then, h_0 preserves L-structure. Next, suppose $\tau = (\tau_1, \ldots, \tau_n)$, where $E_{\tau_i}, h_{\tau_i} (1 \leq i \leq n)$ have been constructed already. Define h_τ on $T_\tau^{\mathfrak{B}}$ by setting

$$h_\tau(b) := \{(h_{\tau_1}(a_1), \ldots, h_{\tau_n}(a_n)) \mid \varepsilon_\tau^{\mathfrak{B}}(b, a_1, \ldots, a_n)\}$$

(by (5), this stipulation makes sense); putting $E_\tau := h_\tau[T_\tau^{\mathfrak{B}}]$. Clearly, $E_\tau \subseteq \mathcal{P}(E_{\tau_1} \times \cdots \times E_{\tau_n})$. We are finished if it can be shown that h_τ is one-one, while $\varepsilon_\tau^{\mathfrak{B}}(b, a_1, \ldots, a_n)$ iff $h_\tau(b)(h_{\tau_1}(a_1), \ldots, h_{\tau_n}(a_n))$. But, the first assertion is immediate from (6), and it implies the second. Finally, put $h := \bigcup_{\tau \in \mathcal{T}} h_\tau$. (3) is our licence to do this. That h is defined on all of $\mathfrak{B}$ is implied by (4).

The previous observations yield a conclusion:

LEMMA. *An L_ω-sentence φ is true in all general models if its translation φ^+ logically follows from (1)–(6) above.*

Proof. The direction from right to left is immediate from the definition of the translation $^+$, and its semantic behaviour. From left to right, we use the above representation. ■

The value of the Lemma is diminished by the fact that (4) has an infinite disjunction, outside of L^+. But we can do better.

THEOREM. *$\varphi \in L_\omega$ is true in all general models iff φ^+ follows from (1), (2), (3), (5) and (6).*

Proof. The first half is as before. Next, assume that φ is true in all general models, and consider any L^+-model $\mathfrak{B}$ satisfying the above five principles. Now, its submodel $\mathfrak{B}^*$ with universe $\bigcup_{\tau \in \mathcal{T}} T_\tau^{\mathfrak{B}}$ satisfies these principles as well, but in addition, it also verifies (4). Thus, as before, $\mathfrak{B}^* \vDash \varphi^+$. But then, as all quantifiers in φ^+ occur restricted to the levels T_τ, $\mathfrak{B} \vDash \varphi^+$, and we are done after all. ■

This theorem effectively reduces L_ω-truth under the general model interpretation to first-order consequence from a recursive set of axioms: which shows it to be recursively enumerable and, hence, recursively axiomatisable (by Craig's Theorem). This strongly contrasts with the negative result in Section 2.3. We conclude with a few comments on the situation.

Henkin's original general models (defined, by the way, with respect to a richer language) form a proper subclass of ours. This is because one may strengthen the theorem a little (or much — depending on one's philosophy) by adding to (1)–(6) translations of L_ω-sentences obviously true in the *standard model* interpretation, thereby narrowing the class of admissible general models. Of course, Section 2.3 prevents an effective narrowing down to *exactly* the standard models!

Here are two examples of such additional axioms, bringing the general models interpretation closer to the standard one.

1. Comprehension Axioms for type $\tau = (\tau_1, \ldots, \tau_n)$:

   $$\forall X_1 \ldots \forall X_m \exists Y \forall Z_1 \ldots \forall Z_n (Y(Z_1, \ldots, Z_n) \leftrightarrow \varphi),$$

   where Y has type τ, Z_i type $\tau_i (1 \leq i \leq n)$ and the free variables of φ are among $X_1, \ldots, X_m, Z_1, \ldots, Z_n$. Thus, all definable predicates are to be actually present in the model.

2. Axioms of Choice for type $\tau = (\tau_1, \ldots, \tau_n, \tau_{n+1})$:

   $$\forall Z_1 \exists Z_2 \forall X_1 \ldots \forall X_n (\exists Y Z_1(X_1, \ldots, X_n, Y) \rightarrow \\ \rightarrow \exists! Y Z_2(X_1, \ldots, X_n, Y));$$

   where Z_1, Z_2 have type τ, X_i has τ_i ($1 \leq i \leq n$) and Y has type τ_{n+1}. Thus, every relation contains a function: cf. Bernay's Axiom of Choice mentioned in Section 2.5.1.

There is also a more 'deductive' motivation for these axioms. When one ponders which principles of deduction should enter into any reasonable higher-order logic, one immediate candidate is the ordinary complete first-order axiom set, with quantifiers now also of higher orders (cf. [Enderton, 1972] last chapter, for this line). All usual principles are valid in general models without further ado, except for Universal Instantiation, or equivalently, Existential Generalization:

$$\forall X \varphi(X) \rightarrow \varphi(T) \text{ or } \varphi(T) \rightarrow \exists X \varphi(X).$$

These two axioms are valid in all general models when T is any variable or constant of the type of X. But, in actual practice, one wants to substitute further instances in higher-order reasoning. For example, from $\forall X \varphi(X, R)$, with X of type (0), one wants to conclude $\varphi(\psi)$ for any *first-order* definable property ψ in R, = (cf. van

Benthem, Chapter II.4). In terms of Comprehension, this amounts to closure of predicate ranges under first-order definability, mentioned in Section 4.1. A further possibility is to allow *predicative* substitutions, where ψ may be higher-order, but with its quantifiers all ranging over orders lower than that of X. Finally, no holds barred, there is the use of *arbitrary substitutions*, whether predicative or not; as in the above Comprehension Schema.

One consequence of Comprehension is the following Axiom of Descriptiveness:

$$\forall x \exists ! y \varphi(x, y) \rightarrow \exists f \forall x \varphi(x, f(x)).$$

If we want to strengthen this to the useful existence of Skolem functions (cf. Section 2.5.2), we have to postulate

$$\forall x \exists y \varphi(x, y) \rightarrow \exists f \forall x \varphi(x, f(x));$$

and this motivates the above Axioms of Choice.

No further obvious logical desiderata seem to have been discovered in the literature.

By the way, our above formulation of the Axiom of Choice cannot be strengthened when all types are present, assuming the comprehension axioms. If this is not the case, it can be. For instance, in the second-order language, the strongest possible formulation is just the implication $\forall x \exists X \psi \rightarrow \exists Y \forall x \psi'$ (where ψ' is obtained from ψ by substituting $Y(x, t_1, \ldots, t_n)$ for $X(t_1, \ldots, t_n)$) used to prove the prenex theorem in Section 3.2.

In a sense, this form gives more than just choice; conceived of set-theoretically, it has the flavour of a 'collection' principle. It plays a crucial role in proving reflectivity of second-order theories containing it, similar to the role the substitution (or collection) axiom has in proving reflection principles in set theory.

The general picture emerging here is that of an ascending range of recursively axiomatized higher-order logics, formalizing most useful fragments of L_ω-validity that one encounters in practice.

4.3 Second-Order Reduction

The general completeness theorem, or rather, the family of theorems in Section 4.2, by no means exhausts the uses of the general model idea of Section 4.1. For instance, once upon this track, we may develop a 'general model theory' which is much closer to the first-order subject of that description. A case in point are the 'general ultraproducts' of van Benthem [1983], which allow for an extension of the fundamental characterization theorems of Section 1.4 to higher-order logic. This area remains largely unexplored.

Here we present a rather more unexpected application, announced in Section 3.2: L_ω-standard validity is effectively reducible to standard validity in monadic L_2, in fact in the monadic Σ^1_1-fragment.

Consider the *first-order* language L^+ (relative to a given base language L) introduced in Section 4.1. Extend it to a *second-order* language L^+_2 by adding second-order variables of all types $(0, \ldots, 0)$, with which we can form atoms $X(t_1, \ldots, t_n)$. Consider the following L^+_2-principles ($\tau = (\tau_1, \ldots, \tau_n)$):

Plenitude(τ)

$$\forall X \exists x \forall y_1 \ldots \forall y_n (T_\tau(x) \wedge (T_{\tau_1}(y_1) \wedge \cdots \wedge T_{\tau_n}(y_n) \rightarrow \\ \rightarrow (\varepsilon_\tau(x, y_1, \ldots, y_n) \leftrightarrow X(y_1, \ldots, y_n)))).$$

Evidently, Plenitude holds in all $^+$-transforms of all standard models of L_ω. Conversely, if the L^+-model $\mathfrak{B}$ satisfies Plenitude(τ) for all types τ, then its submodel $\mathfrak{B}^*$ (cf. the proof of the main theorem in Section 4.2) is isomorphic to a model of the form $(\mathfrak{A}, E)^+$ with a full type structure E.

THEOREM. *$\varphi \in L_\omega$ is true in all standard models iff φ^+ follows from (1), (2), (3), (5), (6), and the Plenitude axioms.*

As φ can only mention a finite number of types and non-logical constants, the relevant axioms of the above-mentioned kinds can be reduced to a finite number and hence to a single sentence ψ.

THEOREM. *With every $\varphi \in L_\omega$, a Π^1_1-sentence ψ of L^+_2 can be associated effectively, and uniformly, such that*

$$\vDash_\omega \psi \textit{ iff } \vDash_2 \psi \rightarrow \varphi^+.$$

As $\psi \in \Pi^1_1$ and φ^+ is first-order, this implication is equivalent to a Σ^1_1-sentence; and the promised reduction is there.

But Plenitude has been formulated using second-order variables of an arbitrary type. We finally indicate how this may be improved to the case of only monadic ones. Consider the variant

Plenitude$^*(\tau)$

$$\forall X \exists x \forall y_1 \ldots \forall y_n (T_\tau(x) \wedge (T_{\tau_1}(y_1) \wedge \cdots \wedge T_{\tau_n}(y_n) \rightarrow \\ \rightarrow (\varepsilon_\tau(x, y_1, \ldots, y_n) \leftrightarrow \exists y (T_\tau(y) \wedge X(y) \wedge \varepsilon_\tau(y, y_1, \ldots, y_n)))).$$

When E is full, this will obviously hold in $(\mathfrak{A}, E)^+$. To make this monadic variant do its job, it has to be helped by the following first-order principle stating the existence of singleton sets of ordered sequences:

Singletons(τ)

$$\forall z_1 \ldots \forall z_n \exists x \forall y_1 \ldots \forall y_n (T_{\tau_1}(z_1) \wedge \cdots \wedge T_{\tau_n}(z_n) \rightarrow (T_\tau(x) \wedge$$

$$\wedge(T_{\tau_1}(y_1) \wedge \cdots \wedge T_{\tau_n}(y_n) \rightarrow (\varepsilon_\tau(x, y_1, \ldots, y_n) \leftrightarrow \\ \leftrightarrow y_1 = z_1 \wedge \cdots \wedge y_n = z_n)))).$$

Suppose now that $\mathfrak{B}$ satisfies all these axioms and $(\mathfrak{A}, E)^+ \cong \mathfrak{B}^*$. Let $S \subseteq E_{\tau_1} \times \cdots \times E_{\tau_n}$ be arbitrary: we must show that $S \in E_\tau$. Notice that Singletons(τ) implies that, if $s \in E_{\tau_1} \times \cdots \times E_{\tau_n}$ (in particular, if $s \in S$), then $\{s\} \in E_\tau$. Now let $S' := \{\{s\} \mid s \in S\}$. Clearly, $S = \bigcup S'$ and $S' \subseteq E_\tau$. That $S \in E_\tau$ follows from one application of Plenitude$^*(\tau)$, taking S' as value for X. ■

4.4 Type Theory and Lambda Calculus

Readers of Section 4.2 may have been a little disappointed at finding no preferred *explicit* axiomatized 'first-order' version of L_ω-logic. And indeed, an extreme latitude of choices was of the essence of the situation. Indeed, there exist various additional points of view leading to, at least, interesting logics. One of these is provided by the earlier functional type theory of Section 3.3. We will chart the natural road from the perspective of its basic primitives.

Identity and *application* inspire the usual identity axioms, including replacement of identicals. *Lambda abstraction* really contributes only one further principle, viz. the famous 'lambda conversion'

$$\lambda x \cdot B(A) = [A/x]B;$$

for x, B, A of suitable types, and modulo obvious conditions of freedom and bondage. Thus, there arises a simple kind of *lambda calculus*. (Actually, a rule of 'alphabetic bound variants' will have to be added in any case, for domestic purposes.)

Lambda conversion is really a kind of simplification rule, often encountered in the semantics of natural, or programming languages. One immediate question then is if this process of simplification ever stops.

THEOREM. *Every lambda reduction sequence stops in a finite number of steps.*

Proof. Introduce a suitable measure of type complexity on terms, so that each reduction lowers complexity ■

This theorem does not hold for the more general *type free* lambda calculi of [Barendregt, 1980]; where, e.g. $\lambda x \cdot x(x)(\lambda x \cdot x(x))$ runs into an infinite regress.

Another immediate follow-up question concerns the *unicity* (in addition to the above *existence*) of such irreducible 'normal forms'. This follows in fact from the 'diamond property':

THEOREM. (Church-Rosser) *Every two lambda reduction sequences starting from the same terms can be continued to meet in a common term (up to alphabetic variance).*

Stronger lambda calculi arise upon the addition of further principles, such as *extensionality*:

$$\lambda x \cdot A(x) = \lambda x \cdot B(x) \text{ implies } A = B \ \text{ (for } x \text{ not free in } A, B).$$

This is the lambda analogon of the earlier principle (6) in Section 4.2.

Still further additions might be made reflecting the constancy of the truth value domain D_t. Up till now, all principles considered would also be valid for arbitrary truth value structures. (In some cases, this will be a virtue, of course.)

Let us now turn to traditional logic. Henkin has observed how all familiar logical constants may be *defined* (under the standard interpretation) in terms of the previous notions. Here is the relevant list [Henkin, 1963]:

$\top$ (a tautology) $:= \lambda x \cdot x = \lambda x \cdot x$

$\bot$ (a contradiction) $:= \lambda x_t \cdot x_t = \lambda x_t \cdot \top$

$\neg$ (negation) $:= \lambda x_t \cdot x_t = \bot$

The most tricky case is that of conjunction:

$$\wedge := \lambda x_t \cdot \lambda y_t(\lambda f_{(t,t)} \cdot (f_{(t,t)}(x_t) = y_t) = \lambda f_{(t,t)} \cdot f_{(t,t)}\top))$$

One may then define $\vee, \rightarrow$ in various ways. Finally, as for the quantifiers,

$$\forall x A := \lambda x \cdot A = \lambda x \cdot \top.$$

The induced logic has not been determined yet, as far as we know.

With the addition of the axiom of *bivalence*, we are on the road to classical logic:

$$\forall x_t \cdot f_{(t,t)} \cdot x_t = f_{(t,t)}\top \wedge f_{(t,t)}\bot.$$

For a fuller account, cf. [Gallin, 1975, Chapter 1.2].

One may prove a general completeness theorem for the above identity, application, abstraction theory in a not inelegant direct manner, along the lines of Henkin's original completeness proof. (Notably, the familiar 'witnesses' would now be needed in order to provide instances $f(c) \neq g(c)$ when $f \neq g$.) But, the additional technicalities, especially in setting up the correct account of general models for functional-type theory, have motivated exclusion here.

Even so, the differences between the more 'logical' climate of functional- type theory and the more 'set-theoretic' atmosphere of the higher-order L_ω will have become clear.

5 REFLECTIONS

Why should a Handbook of (after all) Philosophical Logic contain a chapter on extensions of first-order logic; in particular, on higher-order logic? There are some very general, but also some more specific answers to this (by now) rather rhetorical question.

One general reason is that the advent of competitors for first-order logic may relativize the intense preoccupation with the latter theory in philosophical circles. No specific theory is sacrosanct in contemporary logic. It is rather a certain logical perspective in setting up theories, weaker or stronger as the needs of some specific application require, that should be cultivated. Of course, this point is equally valid for *alternatives* to, rather than *extensions* of classical first-order logic (such as intuitionistic logic).

More specifically, two themes in Section 1 seem of a wider philosophical interest: the role of limitative results such as the Löwenheim-Skolem, or the Compactness theorem for scientific theory construction; but also the new systematic perspective upon the nature of logical constants (witness the remarks made about generalized quantifiers). Some authors have even claimed that proper applications of logic, e.g. in the philosophy of science or of language, can only get off the ground now that we have this amazing diversity of logics, allowing for conceptual 'fine tuning' in our formal analyses.

As for the specific case study of higher-order logic, there was at least a convincing prima facie case for this theory, both from the (logicist) foundations of mathematics and the formal semantics of natural language. Especially in the latter area, there have been recurrent disputes about clues from natural language urging higher-order descriptions. (The discussion of branching quantifiers in Section 2.5.1 has been an example; but many others could be cited.) This subject is rather delicate, however, having to do with philosophy as much as with linguistics. (Cf. [van Benthem, 1984] for a discussion of some issues.) For instance, the choice between a standard model or a general model approach to higher-order quantification is semantically highly significant and will hopefully undercut at present rather dogmatic discussions of the issue. For instance, even on a Montagovian type theoretic semantics, we are not committed to a non-axiomatizable logic, or models of wild cardinalities: contrary to what is usually claimed. (General models on a countable universe may well remain countable throughout, no matter how far the full type structure explodes.)

One might even hazard the conjecture that natural language is partial to restricted predicate ranges which are *constructive* in some sense. For instance, [Hintikka, 1973] contains the suggestion to read branching quantifier statements on countable domains in terms of the existence of Skolem functions which are recursive in the base predicates. If so, our story might end quite differently: for, the higher-order logic of constructive general models might well lapse into non-axiomatizability again. Thus, our chapter is an open-ended one, as far as the philosophy and semantics of language are concerned. It suggests possibilities for semantic description; but on the other hand, this new area of application may well inspire new directions in logical research.

ADDENDA

This chapter was written in the summer of 1982, in response to a last-minute request of the editors, to fill a gap in the existing literature. No standard text on higher-order logic existed then, and no such text has emerged in the meantime, as far as our information goes. We have decided to keep the text of this chapter unchanged, as its topics still seem to the point. Nevertheless, there have been quite a few developments concerning different aspects of our exposition. We provide a very brief indication — without any attempt at broad coverage.[1]

Ehrenfeucht-Fraïssé Games

Game methods have become a common tool in logic for replacing compactness arguments to extend standard meta-properties beyond first-order model theory. Cf. [Hodges, 1993], [Doets, 1996]. They extend to many variations and extensions of first-order logic (cf. [Barwise and van Benthem, 1996]).

Finite Model Theory

Model theory over finite models has become a topic in its own right. Cf. [Ebbinghaus and Flum, 1995]. For connections with data base theory, cf. [Kanellakis, 1990]. In particular, over finite models, logical definability links up with computational complexity: cf. [Immerman, 1996].

General Models

[Henkin, 1996] is an exposition by the author of the original discovery. [Manzano, 1996] develops a broad spectrum of applied higher-order logics over general models with partial truth values. [van Benthem, 1996] gives a principled defense of general models in logical semantics, as a 'geometric' strategy of replacing predicates by objects.

Order-Independent Properties of Logics

The distinction 'first-order'/'higher-order' is sometimes irrelevant. Many logical properties hold independently of the division into logical 'orders'. Examples are

[1] The following people were helpful in providing references: Henk Barendregt, Philip Kremer, Godehard Link, Maria Manzano, Marcin Mostowski, Reinhard Muskens, Mikhail Zakhariashev.

monotonicity (upward preservation of positive statements) or relativization (quantifier restriction to definable subdomains), whose model-theoretic statements have nothing to do with orders. There is an emerging linguistic interest in such 'transcendental' logical properties: cf. [van Benthem, 1986b], [Sanchez Valencia, 1991].

Generalized Quantifier Theory

The theory of generalized quantifiers has had a stormy development in the 80s and 90s, both on the linguistic and the mathematical side. Cf. [van Benthem, 1986a], [Westerståhl, 1989]. In particular, the latter has systematic game-based (un-)definability results for hierarchies of generalized quantifiers. [van Benthem and Westerståhl, 1995] is a survey of the current state of the field, [Keenan and Westerståhl, 1996] survey the latest linguistic applications, many of which involve the polyadic quantifiers first introduced by [Lindström, 1966].

Higher-Order Logic in Computer Science

Higher-order logics have been proposed for various applications in computer science. Cf. [Leivant, 1994].

Higher-Order Logic in Natural Language

Much discussion has centered around the article [Boolos, 1984], claiming that plurals in natural language form a plausible second-order logic. Strong relational higher-order logics have been proposed by [Muskens, 1995]. The actual extent of higher-order phenomena is a matter of debate: cf. [Lönning, 1996], [Link, 1997, Chapter 14]. In particular, there is a continuing interest in better-behaved 'bounded fragments' that arise in natural language semantics.

Higher-Order Logic in the Philosophy of Science

Higher-order logic has been used essentially in the philosophy of time (cf. various temporal postulates and open questions in [van Benthem, 1992]), the foundations of physics and measurement (cf. the higher-order physical theories of [Field, 1980]) and mathematics (cf. [Shapiro, 1991]).

Infinitary Logic

Infinitary logics have become common in computer science: cf. [Harel, 1984], [Goldblatt, 1982]. In particular, fixed-point logics are now a standard tool in the theory of data bases and query languages: cf. [Kanellakis, 1990]. Recently, [Barwise and van Benthem, 1996] have raised the issue just what are the correct formulations of the first-order meta-properties that should hold here. (For instance, the standard interpolation theorem fails for $L_{\infty\omega}$, but more sophisticated variants go through.) Similar reformulation strategies might lead to interesting new meta-properties for second-order logic.

Lambda Calculus and Type Theories

There is an exploding literature on (typed) lambda calculus and type theories, mostly in computer science. Cf. [Hindley and Seldin, 1986], [Barendregt, 1980; Barendregt, 1992], [Mitchell, 1996; Gunter and Mitchell, 1994]. In natural language, higher-order logics and type theories have continued their influence. Cf. [Muskens, 1995] for a novel use of relational type theories, and [Lapierre, 1992; Lepage, 1992] for an alternative in partial functional ones. [van Benthem, 1991] develops the mathematical theory of 'categorial grammars', involving linear fragments of a typed lambda calculus with added Booleans.

Modal Definability Theory

First-order reductions of modal axioms viewed as Π^1_1-sentences have been considerably extended in [Venema, 1991], [de Rijke, 1993]. In the literature on theorem proving, these translations have been extended to second-order logic itself: cf. [Ohlbach, 1991], [Doherty *et al.*, 1994]. [Zakhariashev, 1992; Zakhariashev, 1996] provides a three-step classification of all second-order forms occurring in modal logic.

Propositional Quantification in Intensional Logic

Modal Logic. [Kremer, 1996] considers the obvious interpretation of propositional quantification in the topological semantics for S4, and defines a system S4πt, related to the system S4π^+ of [Fine, 1970]. He shows that second-order arithmetic can be recursively embedded in S4πt, and asks whether second order logic can.

[Fine, 1970] is the most comprehensive early piece on the topic of propositional quantifiers in modal logic. (Contrary to what is stated therein, decidability of S4.3π^+ is open.)

Intuitionistic Logic. References here are [Löb, 1976], [Gabbay, 1981], [Kreisel, 1981] and [Pitts, 1992].

Relevance Logic. Cf. [Kremer, 1994].

Higher-Order Proof Theory

Cf. [Troelstra and Schwichtenberg, 1996, Chapter 11], for a modern exposition of relevant results.

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