Pricing Growth-Rate Risk

Lars Peter Hansen
University of Chicago

José A. Scheinkman
Princeton University

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Abstract

We characterize the compensation demanded by investors in equilibrium for incremental exposure to growth-rate risk. Given an underlying Markov diffusion that governs the state variables in the economy, the economic model implies a stochastic discount factor process S and a reference stochastic growth process G for the macroeconomy. Both are modeled conveniently as multiplicative functionals of a multidimensional Brownian motion. To study pricing we consider the pricing implications of parameterized family of growth processes G^{ϵ}, with $G^0 = G$, as ϵ is made small. This parameterization defines a direction of growth-rate risk exposure that is priced using the stochastic discount factor S. By changing the investment horizon we trace a *term structure* of risk prices that shows how the valuation of risky cash flows depends on the investment horizon. Using methods of Hansen and Scheinkman (2009), we characterize the limiting behavior of the risk prices as the investment horizon is made arbitrarily long.

1 Introduction

A standard result from asset pricing theories is the characterization of the local risk-return tradeoff. This tradeoff is particularly simple in the case of Brownian information structures. In mathematical finance the risk prices are embedded in the transformation to a risk-neutral measure. Applying Girsanov's Theorem, this change of measure adds a drift vector to the multivariate standard Brownian motion. The vector of local risk prices is the negative of the drift vector used in constructing the risk neutral transformation. This price vector reflects the local compensation in terms of the drift for exposure to alternative components of the Brownian motion. With these local prices, we price exposure to linear combinations of the Brownian risks by forming the corresponding linear combination of prices.

While derivative claims are often priced using the risk neutral measure, structural models of asset prices interpret these prices in terms of the fundamentals of the underlying economy. In this paper, as in Hansen and Scheinkman (2009) and Hansen (2008), we characterize the compensation demanded by investors for a added risk at different time horizons, that is a *term-structure* of risk prices. This compensation will typically depend on how investors discount risky payoffs and the risk they already face. Our approach is as follows. There is an underlying Markov diffusion X that governs the state variables in the economy. The economic model implies a stochastic discount factor process S and a reference stochastic growth process G for the macroeconomy. Both are modeled conveniently as multiplicative functionals of a multi-dimensional Brownian motion. To feature the role of price dynamics, we normalize the reference growth functional to a be a martingale. More generaly this martingale can be the martingale component in a factorization of the growth functional (as in Hansen and Scheinkman (2009)). To study pricing we consider a parameterized family of growth processes G^ϵ, with $G^0 = G$ and study its pricing implications for payoffs at different horizons. We define the price of growth-rate risk as:

$$\rho_t = -\frac{d}{d\epsilon}\frac{1}{t}\log E\left[G_t^\epsilon S_t | X_0 = x\right]|_{\epsilon=0}.$$

It is the elasticity of the expected rate of return (per unit of time) with respect to the exposure to growth-rate risk. The expected return implicit in this calculation is the reciprocal of the price $E\left[G_t^\epsilon S_t | X_0 = x\right]$ since G_t^ϵ has expectation one by construction.

The resulting prices of growth-rate risk extend the local prices to arbitrary investment horizons. While we focus on scalar parameterizations, we can interpret our calculations as producing prices for an arbitrary linear combination of exposure to the Brownian motion risks. By changing the exposure weights, we feature alternative components of the Brownian increments and thus construct the counterpart to the local risk-price vector.

For a given investment horizon, we characterize our risk prices by applying tools that are used to compute sensitivities of option prices (the "Greeks"). The prices we compute reveal the local risk prices as the horizon t shrinks to zero:

$$\lim_{t\downarrow 0}\rho_t = \rho_0$$

We add to this a characterization of the limit prices as the investment horizon tends to ∞:

$$\lim_{t\uparrow\infty}\rho_t = \rho_\infty,$$

along with formulas for the intermediate investment horizons.

2 Mathematical setup

The underlying state vector process X is n-dimensional and satisfies,

$$dX_t = \beta(X_t)dt + \alpha(X_t)dW_t.$$

We use multiplicative functionals M of the form

$$M_t = \exp\left[\int_0^t \delta(X_u)du + \int_0^t \gamma(X_u)dW_u\right]$$

where

$$\int_0^t |\delta(X_u)|du < \infty$$
$$\int_0^t |\gamma(X_u)|^2 du < \infty$$

for all t with probability one.[1] Let G and S be two such multiplicative functionals. The process G captures stochastic growth and the process S stochastic discounting. Asset valuation over a horizon t is represented as:

$$E\left[S_t G_t | X_0 = x\right]$$

where G_t is the asset payoff that is priced. Thus there are two channels by which the term structure of risk premia the associated prices are altered over alternative investment horizons: a stochastic discount factor channel and a stochastic growth channel. Our aim is to focus on the latter channel.

Hansen and Scheinkman (2009) (Corollary 6.1) establishes a multiplicative factorization of G:

$$G_t = \exp(\eta t) G_t^* \left[\frac{f(X_0)}{f(X_t)}\right]$$

where G^* is a multiplicative martingale.[2] The exponential growth term $\exp(\eta t)$ is of no consequence for risk prices and can be omitted. Since predictability in S and G alter the term structure of risk premia, one possibility is to feature the role of pricing dynamics by abstracting from the the "transient term" to stochastic growth: $\left[\frac{f(X_0)}{f(X_t)}\right]$. Alternatively, we may absorb this term into the stochastic discount factor S (replace S_t by $S_t \left[\frac{f(X_0)}{f(X_t)}\right]$). In what

[1] Contrary to the usual treatment, we allow for multiplicative functionals that do not have bounded variation.

[2] Strictly speaking, this corollary produces a local martingale rather than a martingale.

follows we will assume that G is a martingale and use it to change probability measures.

To construct risk prices for any given payoff horizon, we parameterize a family of growth functionals as G^ϵ with $G = G^0$ where G^ϵ is a martingale for each ϵ. The parameterized martingale is constructed to feature exposures to specific combination of shocks. By altering the parameterization, we can feature sensitivity to alternative shocks thereby constructing counterparts to local risk prices.

Recall that the stochastic exponential of a semi-martingale N is a semi-martingale $\mathcal{E}(N)$ that solves $\mathcal{E}(N)_t = 1 + \int_0^t \mathcal{E}(N)_{s_-} dNs$. Since sample paths are continuous,

$$\mathcal{E}(N) = \exp\left(N - \frac{1}{2}[N, N]\right). \tag{1}$$

We assume that the positive martingale G is the stochastic exponential $\mathcal{E}(Z^o)$ of a martingale $Z_t^o = \int_0^t \gamma_g(X_t) dW_t$. Consider a family of perturbabtions G^ϵ of the form:

$$G^\epsilon = \mathcal{E}(Z^o + \epsilon Z), \tag{2}$$

$\epsilon \in (-1, 1)$ where $Z_t = \int_0^t \gamma_d(X_t) dW_t$. For the stochastic integrals to be well behaved, $\int_0^t |\gamma_g(X_u)|^2 < \infty$ and $\int_0^t |\gamma_d(X_u)|^2 du < \infty$ with probability one.

The process Z used to construct the perturbation can feature any of the individual components of the underlying Brownian motion. The resulting parameterized family expressed in logarithms is:

$$\log G_t^\epsilon = \int_0^t \gamma_g(X_u) dW_u + \epsilon \int_0^t \gamma_d(X_u) dW_u - \frac{1}{2} \int_0^t |\gamma_g(X_u) + \epsilon \gamma_d(X_u)|^2 du$$

In this specification $\epsilon \int_0^t \gamma_d(X_u) dW_u$ parameterizes the (growth rate) risk exposure. By changing γ_d we alter which Brownian increments are featured in the pricing.

3 Finite-Horizon Prices

In this section we apply an approach developed by Fournia et al. (1999, 2001) to show that

$$\rho_t(x) = -\frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) dW_u - \int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du\right) | X_0 = x\right]}{t E\left(S_t G_t | X_0 = x\right)}. \tag{3}$$

We start by using the multiplicative martingale G to change measure. Then Girsanov's Theorem guarantees that $\frac{G^\epsilon}{G} = \mathcal{E}[\epsilon \tilde{Z}]$, and $\tilde{Z}_t = \int_0^t \gamma_d(X_u) d\tilde{W}_u$, with $\tilde{W}$ a Brownian motion under the changed measure $\tilde{\mathsf{Pr}}$. Let $\tilde{E}$ denote the associated expectations operator. Hence,

$$\begin{aligned}\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} &= \int_0^t \frac{G_u^\epsilon}{G_u} d\tilde{Z}_u, \text{ or} \\ &= \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u \end{aligned} \tag{4}$$

If the right-hand side has a well-defined limit, then necessarily this limit is:

$$\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} \to \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u$$

We may write the price of an asset as a function of the perturbation on the growth factor as:

$$\begin{aligned} U(\epsilon) &= E\left[G_t^\epsilon S_t | X_0 = x\right] \\ &= \tilde{E}\left[\frac{G_t^\epsilon}{G_t} S_t | X_0 = x\right]. \end{aligned}$$

Hence,

$$U'(0) = \lim_{\epsilon \to 0} \frac{\tilde{E}\left[(\frac{G_t^\epsilon}{G_t} - 1) S_t | X_0 = x\right]}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \tilde{E}\left[S_t \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u | X_0 = x\right]$$

Next we impose two assumptions that are sufficient for the main result in this section. After establishing this result, we provide sufficient conditions for the second of these assumptions.

Assumption 3.1. $E\left[(S_t)^2 G_t | X_0 = x\right] < \infty$.

Imposing this restriction is equivalent to assuming that S_t has a finite conditional second moment (in the $\tilde{\mathsf{Pr}}$ measure) for each x.

Assumption 3.2.

$$\lim_{\epsilon \to 0} \frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} \to \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u.$$

in mean-square.

Proposition 3.3. *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then*

$$\begin{aligned} U'(0) &= \tilde{E}\left[S_t \int_0^t \gamma_d(X_u) d\tilde{W}_u | X_0 = x\right] \\ &= E\left(S_t G_t \left[\int_0^t \gamma_d(X_u) dW_u - \int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du\right] | X_0 = x\right) \end{aligned}$$

Proof. This follows directly from Holder's Inequality. □

The elasticity of interest is the ratio of $U'(0)/U(0)$ and is given by (3).

We now provide sufficient conditions for Assumption 3.2. To insure that $\frac{G^\epsilon}{G}$ is a martingale we assume Novikov's condition:

Assumption 3.4.

$$\tilde{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma_d(X_u)|^2 du\right) | X_0 = x\right] < \infty.$$

Equation (1) implies that,

$$\frac{G^\epsilon}{G} = \mathcal{E}(\epsilon \tilde{Z}) = [\mathcal{E}(\tilde{Z})]^\epsilon \exp\left(\frac{1}{2}(\epsilon - \epsilon^2)[\tilde{Z}, \tilde{Z}]\right). \tag{5}$$

Notice that for a fixed t, $\mathcal{E}(\epsilon\tilde{Z})$ converges to 1 for each ω as $\epsilon \to 0$. For fixed integer $1 \le m < \infty$ and $t > 0$, consider the space L^m of adapted stochastic processes $f(u, \omega)$, $0 \le u \le t$ with norm $||f|| = \left(\tilde{E}\int_0^t |f(u,\omega)|^m du\right)^{1/m}$ A stronger form of convergence is established in the following Lemma.

Lemma 3.5. *Suppose Assumption 3.4 is satisfied. Then* $\lim_{\epsilon\to 0} \frac{G_t^\epsilon}{G_t} = 1$ *in* L^m *for any* $m \ge 1$ *and any* $t > 0$.

Proof. : We first consider the limit for $\epsilon > 0$. We will show that we can choose $\bar{\epsilon}$ small such that for $\epsilon \le \bar{\epsilon}$ there is a bound to the $2m$ moment of each of the terms in the RHS of (5) that holds for all $t < T$. From Jensen's Inequality, provided $\epsilon \le \frac{1}{2m}$, for all $t \le T$,

$$E[[\mathcal{E}(\tilde{Z})_t]^{4m\epsilon}] \le 1$$

Also for ϵ small ,

$$\exp[\frac{1}{2}(\epsilon - \epsilon^2)[\tilde{Z}, \tilde{Z}]_t] \le \exp\left(\frac{1}{4m}[\tilde{Z}, \tilde{Z}]_t\right).$$

Hence the bound on the $2m$-th moment follows from Assumption 3.4.

To deal with the limit from the left, we note that from our assumptions there exists an $c > 0$ such that $\mathcal{E}(-c\tilde{Z})$ is a martingale in the measure formed using G, for some $c > 0$.

Thus for $\epsilon < c$,

$$\mathcal{E}(-\epsilon\tilde{Z}) = [\mathcal{E}(-c\tilde{Z})]^{\frac{\epsilon}{c}} \exp\left(\frac{1}{2}(\epsilon c - \epsilon^2)[\tilde{Z}, \tilde{Z}]\right) \tag{6}$$

The proof now proceeds as before. □

To control the term in $\gamma_d(X_t)$ we need to assume

Assumption 3.6. *There exists a constant* Γ *such that*

$$\tilde{E}|\gamma_d(X_u)|^4 \leq \Gamma$$

for $u \leq t$.

Lemma 3.7. *Suppose Assumptions 3.4 and 3.6 are satisfied. Then Assumption 3.2 holds.*

Proof. Use (4) to represent

$$\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} = \int_0^t \left(\frac{G_u^\epsilon}{G_u}\right) \gamma_d(X_u) d\tilde{W}_u.$$

Thus we must show that $\int_0^t \left(\frac{G_u^\epsilon}{G_u} - 1\right) \gamma_d(X_u) d\tilde{W}_u$ converges in mean-square to zero. Notice that the stochastic integral $\int_0^t \left(\frac{G_u^\epsilon}{G_u} - 1\right) \gamma_d(X_u) d\tilde{W}_u$ has second moment

$$\tilde{E} \int_0^t \left(\frac{G_u^\epsilon}{G_u} - 1\right)^2 |\gamma_d(X_u)|^2 dt. \tag{7}$$

As $\epsilon \to 0$, expression (7) converges to zero from the assumptions, Lemma 3.7 for $m = 4$ and Holder's inequality. □

Finally, we consider a sufficient condition for Assumption 3.6. When the functions $|\gamma_d(x)|$ are bounded by a polynomial in $|x|$ Assumption 3.6 follows from the following assumption on the coefficients of the diffusion X

Assumption 3.8.

$$dX_t = \beta(X_t)dt + \alpha(X_t)dW_t \tag{8}$$

with the coefficients β *and* γ *satisfying sublinear growth*

$$|\beta(x)|^2 + \|\alpha(x)\|^2 \leq K(1 + |x|^2),$$

for some constant K.

When Assumption 3.8 holds, for each for each $m \geq 1$ there exists a $C = C(d, K, T, m)$ such that $E[\max_{s \leq t} ||X_t||^{2m}] \leq C(1 + E||X_0||^{2m})e^{Ct}$, if $t \leq T$. (A more general result than this is problem 3.15 in Karatzas and Shreve (1991) page 306). Hence Assumption 3.6 obtains from the polynomial bound in $|\gamma^i(X_t)|$.

4 Short-term limits

We now use the formula:

$$\rho_t(x) = \frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right) |X_0 = x\right]}{tE\left(S_t G_t | X_0 = x\right)}$$

to study valuation over short time intervals. Formally we calculate short-horizon limits by computing the drift of an Ito process .

Recall that the SG has continuous sample paths that converge to one as t declines to zero. We add to this the assumption

Assumption 4.1. $\lim_{t \downarrow 0} E\left(S_t G_t | X_0 = x\right) = 1.$

This assumption follows from the Dominated Convergence Theorem provided that we can dominate SG uniformly for small t.

With this restriction, we are lead to compute

$$\rho(x) = \lim_{t \downarrow 0} \frac{1}{t} E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right) |X_0 = x\right].$$

We calculate this limit as the drift of the Ito process

$$S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right)$$

at $t = 0$. Since

$$\rho(0) = \gamma_d(x) \cdot \gamma_g(x) - \gamma_d(x) \cdot [\gamma_g(x) + \gamma_s(x)],$$

the following proposition holds.

Proposition 4.2. *Suppose Assumption 4.1 is satisfied. Then*

$$\rho(0) = -\gamma_s(x) \cdot \gamma_d(x). \tag{9}$$

As we vary the risk exposure vector γ_d, we trace out the local risk prices. This results in the interpretation of $-\gamma_s$ a vector of local risk prices.[3] As is well known, the local risk price vector is the risk exposure of the stochastic discount factor S. The risk exposure of the stochastic growth process plays no role in this calculation.

5 Long-term limits

Following Hansen and Scheinkman (2009), use the factorization:

$$S_t G_t = \exp(\delta t) \hat{M}_t \frac{e(X_0)}{e(X_t)}$$

where $\hat{M}$ is a multiplicative martingale. Change measure using this martingale and express the finite t derivative of interest as:

$$\begin{aligned} \rho_t(x) &= \frac{E\left[S_t G_t \left(\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u) du - \int_0^t \gamma_d(X_u) dW_u\right) | X_0 = x\right]}{tE\left(S_t G_t | X_0 = x\right)} \\ &= -\frac{\hat{E}\left(\frac{1}{e(X_t)}\left[\int_0^t \gamma_d(X_u) \cdot \kappa(X_u) du + \int_0^t \gamma_d(X_u) d\hat{W}_u\right] | X_0 = x\right)}{t\hat{E}\left(\frac{1}{e(X_t)} | X_0 = x\right)}. \end{aligned}$$

Under the $\hat{\cdot}$ change of measure,

$$dW_u = [\kappa(X_u) + \gamma_g(X_u)] du + d\hat{W}_u$$

where $\hat{W}$ is a multivariate standard Brownian motion.

As a precursor to studying the large horizon behavior of ρ_t, consider the class functions $\hat{f}$ that are in the space

$$\hat{\mathsf{L}}^2 \doteq \left\{\hat{f} : \int \hat{f}^2 d\hat{Q} < \infty\right\}$$

where $\hat{Q}$ is a stationary distribution for X under this change of measure.

Assumption 5.1. *The process X has a stationary distribution under the $\hat{\mathsf{Pr}}$ probability measure.*

There are many well known results for the existence of stationary distributions. See for example Meyn and Tweedie (1993).

[3] In general this limit is computed as in Ito's Lemma by using stopping times. When the associated local martingale is in fact a square integrable martingale, stopping times can be dispensed with.

Assumption 5.2. *The only eigenfunctions associated with the generator $\hat{\mathcal{A}}$ defined on a dense subset of $\hat{\mathsf{L}}^2$ are the constant functions.*[4]

As discussed in Hansen and Scheinkman (1995), under Assumption 5.2, the Markov process X is ergodic.

To study limiting behavior we impose an even stronger restriction. Later we will comment on its relaxation. Let

$$\hat{\mathsf{Z}} \doteq \left\{ \hat{f} \in \hat{\mathsf{L}}^2 : \int \hat{f} d\hat{Q} = 0 \right\}$$

Assumption 5.3. *The semigroup of conditional expectation operators associated with X under the change of measure implied by $\hat{M}$ is a strong contraction semigroup.*

As discussed by Rosenblatt (1971) and Hansen and Scheinkman (1995), Assumption 5.3 is ρ-mixing with mixing coefficients that necessarily decay exponentially to zero.

Proposition 5.4. *Suppose that $\gamma_d \cdot \kappa$, γ_d and $\frac{1}{e}$ are in $\hat{\mathsf{L}}^2$. Then*

$$\lim_{t \to \infty} \rho_t(x) \to - \int \gamma_d \cdot \kappa d\hat{Q}.$$

Thus long-term risk prices are obtained by changing the state-dependent risk exposure γ_d in the representation given by (5.4). As in local counterpart given in Proposition 4.2, we think of γ_d as parameterizing the exposure to (growth-rate) risk, which we allow to be state dependent. The vector $(\kappa + \gamma_g)$ is the risk exposure of the martingale component of SG and γ_g is the risk exposure of the multiplicative martingale growth functional. In effect, the state dependent vector κ in conjunction with the probability distribution $\hat{Q}$ determine the long-term counterpart to the local risk price vector $-\gamma_s$ given in Proposition 4.2.

Proof. Recall that if $\hat{e} = \frac{1}{e}$ then,

$$\rho_t(x) = \frac{\frac{1}{t}\hat{E}\left(\hat{e}(X_t)\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right)}{\hat{E}\left(\hat{e}(X_t)|X_0 = x\right)}.$$

First notice that

$$\begin{aligned} \frac{1}{t}\hat{E}\left(\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du + \int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right) &= \frac{1}{t}\hat{E}\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du|X_0 = x\right) \\ &\to \int \gamma_d \cdot \kappa d\hat{Q} \end{aligned} \tag{10}$$

[4]See Hansen and Scheinkman (1995) for a discussion of the $\hat{\mathsf{L}}^2$ construction of the semigroup of conditional expectation operators associated with the Markov process and the construction of its associated generator.

given the ergodicity of X under the $\hat{\cdot}$ probability measure.

It remains to show that

$$\frac{\frac{1}{t}\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)|X_0=x\right)\right]\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du+\int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0=x\right)}{\hat{E}\left(\frac{1}{e(X_t)}|X_0=x\right)}\to 0.$$

We consider this in three parts.

i)

$$\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)|X_0=x\right)\right]\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)du\right]|X_0=x\right)$$
$$=\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)|X_0=x\right)\right]\int_0^t\left[\gamma_d(X_u)\cdot\kappa(X_u)-\hat{E}\left(\gamma_d(X_u)\cdot\kappa(X_u)\right)\right]du|X_0=x\right)$$

Notice that

$$\begin{aligned}\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)|X_0=x\right)\right]^2|X_0=x\right)&\leq\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)\right)\right]^2|X_0=x\right)\\&\leq\mathsf{c}_1(x)\\&<\infty\end{aligned}$$

when $\hat{e}(X_t)$ has finite second moment under the $\hat{\cdot}$ stationary distribution. The bound $\mathsf{c}_1(x)$ can be chosen to be independent of t. Moreover,

$$\frac{1}{t}\hat{E}\left[\left(\int_0^t\left[\gamma_d(X_u)\cdot\kappa(X_u)-\hat{E}\gamma_d(X_u)\cdot\kappa(X_u)\right]\right)^2|X_0=x\right]<\mathsf{c}_2(x)<\infty$$

for $\mathsf{c}_2(x)$ independent of t. It follows from the conditional version of the Cauchy-Schwarz Inequality that

$$\frac{1}{t}\hat{E}\left(\left[\hat{e}(X_t)-\hat{E}\left(\hat{e}(X_t)|X_0=x\right)\right]\left[\int_0^t\gamma_d(X_u)\cdot\kappa(X_u)du\right]|X_0=x\right)\to 0$$

for each x.

ii) Consider next

$$\frac{1}{t}\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)|X_0 = x\right)\right]\left[\int_0^t \gamma_d(X_u)d\hat{W}_u\right]|X_0 = x\right)$$
$$\leq \frac{1}{\sqrt{t}}\sqrt{\hat{E}\left(\left[\hat{e}(X_t) - \hat{E}\hat{e}(X_t)\right]^2|X_0 = x\right)}$$
$$\times\sqrt{\left(\hat{E}\left[\frac{1}{t}\int_0^t |\gamma_d(X_u)|^2 du|X_0 = x\right]\right)}$$

where the inequality is application of the Cauchy-Schwarz Inequality. Provided that $\gamma_d(X_u)$ has a finite second moment under the $\hat{\cdot}$ distribution, the right-hand side converges to zero for each x.

iii) Finally,

$$\hat{E}\left[\hat{e}(X_t)|X_0 = x\right] \rightarrow \int \hat{e}d\hat{Q}$$

for each x.

Given these three intermediate results, the conclusion follows from (10).

□

References

Fournia, Eric, Jean-Michel Lasry, Jrme Lebuchoux, and Pierre-Louis Lions. 1999. Applications of Malliavin Calculus to Monte-Carlo Methods in Finance. *Finance and Stochastics* 3 (4):391–412.

———. 2001. Applications of Malliavin Calculus to Monte-Carlo Methods in Finance II. *Finance and Stochastics* 5 (2):201–236.

Hansen, Lars Peter. 2008. Modeling the Long Run: Valuation in Dynamic Stochastic Economies. Fisher-Schultz Lecture at the European Meetings of the Econometric Society.

Hansen, Lars Peter and Jose Alexandre Scheinkman. 1995. Back to the Future: Generating Moment Implications for Continuous-Time Markov Processes. *Econometrica* 63 (4):767–804.

———. 2009. Long-term Risk: A Operator Approach. *Econometrica* 77 (1):177–234.

Karatzas, Ioannis and Stephen E. Shreve. 1991. *Bownian Motion and Stochastic Calculus*. New York: Springer-Verlag, second ed.

Meyn, S. P. and R. L. Tweedie. 1993. *Markov Chains and Stochastic Stability*. London: Springer-Verlag.

Rosenblatt, Murray. 1971. *Markov Processes: Structure and Asymptotic Behavior*. New York: Springer-Verlag.