Temporal Theories of Reasoning*

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ABSTRACT: In this paper we describe a general way of formalizing reasoning behaviour. Such a behaviour may be described by all the patterns which are valid for the behaviour. A pattern can be seen as a sequence of information states which describe what has been derived at each time point. A transition from an information state at a point in time to the state at the (or a) next time point is induced by one or more inference steps. We choose to model the information states by partial models and the patterns either by linear time or branching time temporal models. Using temporal logic one can define theories and look at all models of that theory. For a number of examples of reasoning behaviour we have been able to define temporal theories such that its (minimal) models correspond to the valid patterns of the behaviour. These theories prescribe that the inference steps which are possible, are "executed" in the temporal model. The examples indicate that partial temporal logic is a powerful means of describing and formalizing complex reasoning patterns, as the dynamic aspects of reasoning systems are integrated into the static ones in a clear fashion.
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1 Introduction

In practical reasoning usually there are different patterns of reasoning behaviour possible, each leading to a distinct set of conclusions. In logic one is used to express semantics in terms of models that represent descriptions of (conclusions about) the world and in terms of semantic entailment relations based on a specific class of this

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type of models. In the (sound) classical case each reasoning pattern leads to conclusions that are true in all of these models: each line of reasoning fits to each model. However, for the non-classical case the picture is quite different. For example, in default reasoning conclusion sets can be described by (Reiter) extensions. In common examples this leads to a variety of mutually contradictory extensions. It depends on the chosen line of reasoning which one of these extensions fits the pattern of reasoning.

The general idea underlying our approach is that a particular reasoning pattern can be formalized by a sequence of *information states* $\mathbf{M_0}, \mathbf{M_1}, \ldots\ldots$. Here any $\mathbf{M_t}$ is a description of the (partial) knowledge that has been deduced up to the moment in time $\mathbf{t}$. An inference step is viewed as a transition $\mathbf{M_t}$ -> $\mathbf{M_{t+1}}$ of the current information state $\mathbf{M_t}$ to the next information state $\mathbf{M_{t+1}}$. In the current paper we formalize the information states $\mathbf{M_t}$ by partial models (although other choices are possible). A particular reasoning pattern is formalized by a sequence $\mathbf{(M_t)_{t \in T}}$ of subsequent partial models labelled by elements of a flow of time $\mathbf{T}$; such a sequence is interpreted as a partial temporal model. A transition relating a next information state to the current one can be formalized by temporal formulae the partial temporal model has to satisfy. So, inference rules will be translated into temporal rules to obtain a temporal theory describing the reasoning behaviour. Each possible pattern of the reasoning process can be described by a model of this theory (in temporal partial logic).

Using our techniques the semantics of reasoning can be viewed as a set of (intended) partial temporal models. The branching character of these reasoning processes can be described by branching time partial temporal models. One (strict) line of reasoning corresponds to a linear time model: a branch in the tree of all possibilities. In one branching time model more than one line of reasoning (and the resulting conclusion sets) can be represented (even when they are mutually contradictory).

In this paper we will present temporal axiomatizations of three types of reasoning: reasoning based on a classical proof system, default reasoning and meta-level reasoning. We show that it is possible to define formal semantics where (temporal) aspects of the process of reasoning and the resulting conclusions are both integrated in an explicit manner.

In Section 2 we introduce temporal partial logic. In Section 3 it is pointed out that under some conditions a temporal theory has a unique (up to isomorphism) final branching time model that covers all possible lines of reasoning. The semantics of such a temporal theory can be defined on the basis of this unique final model: a form of final model semantics.

In Section 4 we show how proof rules in a classical proof system can be represented by temporal rules. Thus a temporal theory is provided that has a final model where all possible classical proofs are represented as branches. In Section 5 we show how default reasoning can be formalized in temporal partial logic. In Section 6 we treat the reasoning of a meta-level architecture. We show how also in this case a temporal axiomatization of the reasoning can be obtained.

The approach as worked out here can be viewed as a generalization of the manner in which modal and temporal semantics can be given to intuitionistic logic (see [GAB 82], [KRI 65]). One of the differences is that we use partial (information) states; another one is that we apply the approach to a wider class of reasoning systems.

In practical reasoning systems for complex tasks of the type analysed in [TW 93] often the dynamics of the reasoning is subject of reasoning itself (using strategic knowledge to control the reasoning). Therefore for the overall semantics of this type of reasoning system it is hard to make a distinction between static aspects and dynamic aspects. In particular, it is impossible to provide independent declarative semantics for such systems without taking into account the dynamics of the reasoning. For the overall reasoning system a formal semantical description is needed that systematically integrates both views. The lack of such (overall) semantics for complex reasoning systems (with meta-level reasoning capabilities) was one of the major open problems that were identified during the ECAI-92 workshop on Formal Specification Methods for Complex Reasoning Systems where 8 formal specification languages for reasoning systems for complex tasks where analysed and compared (see [HLM *et al.* 93], p. 280). The approach introduced in the current paper can be considered as a first step to provide semantics of formal specification languages of this type.

2 Temporal Partial Logic

In this section we introduce our temporal partial logic, based on branching time structures. Our approach is in line with what in [FG 92] is called temporalizing a given logic; in our case the given logic is partial logic with the Strong Kleene semantics. We shall start by defining the time structures, then we will define the models based on them and finally we will define temporal formulae and their interpretation.

Definition 2.1 (Flow of Time)
A *flow of time* is a pair $(\mathbf{T}, <)$ where $\mathbf{T}$ is a set of time points and $<$ is a binary relation over $\mathbf{T}$, called the *immediate successor relation*. We want to consider forward branching structures: $(\mathbf{T}, <)$ viewed as a graph has to be a *forest*, that is a disjoint union of trees. Furthermore the transitive (but not reflexive) closure $\ll$ of $<$ is introduced.

Definition 2.2 (Partial Temporal Model)
a) A (propositional) *partial temporal model* $\mathcal{M}$ of signature Σ is a triple $(\mathbf{M}, \mathbf{T}, <)$, where $(\mathbf{T}, <)$ is a flow of time and $\mathbf{M}$ is a mapping

$$\mathbf{M} : \mathbf{T} \times \mathrm{At}(\Sigma) \to \{0, 1, u\}$$

If $\mathbf{a}$ is an atom, and $\mathbf{t}$ is a time point in $\mathbf{T}$, and $\mathbf{M}(\mathbf{t}, \mathbf{a}) = 1$, then we say that in this model $\mathbf{M}$ *at time point* $\mathbf{t}$ *the atom* $\mathbf{a}$ *is true*. Similarly we say that *at time point* $\mathbf{t}$ *the atom* $\mathbf{a}$ is *false*, respectively *undefined*, if $\mathbf{M}(\mathbf{t}, \mathbf{a}) = 0$, respectively

$\mathbf{M(t, a) = u}$. We will sometimes leave out the flow of time and denote a partial temporal model by $\mathbf{M}$ only.

b) If $\mathbf{M}$ is a partial temporal model, then for any fixed time point $\mathbf{t}$ the partial model $\mathbf{M_t : At(\Sigma) \to \{0, 1, u\}}$ (the *snapshot at time point* t) is defined by

$$\mathbf{M_t:\ a \ \mapsto\ M(t, a)}$$

We sometimes will use the notation $\mathbf{(M_t)_{t \in T}}$ where each $\mathbf{M_t}$ is a partial model as an equivalent description of a partial temporal model $\mathbf{M}$.

The *ordering of truth values* is defined by $\mathbf{u \le 0}$, $\mathbf{u \le 1}$, $\mathbf{u \le u}$, $\mathbf{0 \le 0}$, $\mathbf{1 \le 1}$. We call the model $\mathbf{N}$ a *refinement* of the model $\mathbf{M}$, denoted by $\mathbf{M \le N}$, if for all atoms $\mathbf{a}$ it holds: $\mathbf{M(a) \le N(a)}$. A partial temporal model $\mathbf{M}$ is called *conservative* if for all time points $\mathbf{s}$ and $\mathbf{t}$ with $\mathbf{s \ll t}$ it holds $\mathbf{M_s \le M_t}$.

c) The refinement relation $\mathbf{\le}$ between partial temporal models based on the same flow of time is defined by: $\mathbf{M \le N}$ if for all time points $\mathbf{t}$ and atoms $\mathbf{a}$ it holds $\mathbf{M(t, a) \le N(t, a)}$.

Because our partial temporal models based on forests have a more differentiated structure towards the future than towards the past, we assume the following temporal operators. In the standard manner we build temporal formulae. In these definitions, $\mathbf{(M, t) \models^{+} \alpha}$ means that in the model $\mathbf{M}$ at time point $\mathbf{t}$ the formula $\boldsymbol{\alpha}$ is true, $\mathbf{(M, t) \models^{-} \alpha}$ that it is false and $\mathbf{(M, t) \models^{u} \alpha}$ that it is undefined. Furthermore $\mathbf{(M, t) \not\models^{+} \alpha}$ denotes that $\mathbf{(M, t) \models^{+} \alpha}$ is not the case.

Definition 2.3 (Interpretation of Temporal Formulae)

Let a (temporal) formula $\boldsymbol{\alpha}$, a partial temporal model $\mathbf{M}$, and a time point $\mathbf{t \in T}$ be given, then:

a) $\mathbf{(M, t) \models^{+} P\alpha} \quad \Leftrightarrow \quad \mathbf{\exists\, s \in T\ [\, s \ll t\ \&\ (M, s) \models^{+} \alpha\,]}$

$\mathbf{(M, t) \models^{+} C\alpha} \quad \Leftrightarrow \quad \mathbf{(M, t) \models^{+} \alpha}$

$\mathbf{(M, t) \models^{+} \exists X\alpha} \quad \Leftrightarrow \quad \mathbf{\exists\, s \in T\ [\, t < s\ \&\ (M, s) \models^{+} \alpha\,]}$

$\mathbf{(M, t) \models^{+} \exists F\alpha} \quad \Leftrightarrow \quad \mathbf{\exists\, s \in T\ [\, t \ll s\ \&\ (M, s) \models^{+} \alpha\,]}$

$\mathbf{(M, t) \models^{+} \forall F\alpha} \quad \Leftrightarrow$ **for all branches including t there exists an s in that branch such that** $\mathbf{[\, t \ll s\ \&\ (M, s) \models^{+} \alpha\,]}$

b) The temporal operators are defined in a two-valued manner, so for every $\mathbf{O \in \{\exists F, \forall F, P, C, \exists X\}}$:

$\mathbf{(M, t) \models^{-} O\alpha} \quad \Leftrightarrow \quad \mathbf{(M, t) \not\models^{+} O\alpha}$

c) The connectives are evaluated according to the strong Kleene semantics and the atoms according to Definition 2.2.

d) For a partial temporal model $\mathbf{M}$, by $\mathbf{M \models^{+} \alpha}$ we mean $\mathbf{(M, t) \models^{+} \alpha}$ for all $\mathbf{t \in T}$ and by $\mathbf{M \models^{+} K}$ we mean $\mathbf{M \models^{+} \varphi}$ for all $\boldsymbol{\varphi} \in \mathbf{K}$, where $\mathbf{K}$ is a set of formulae possibly containing any of the defined operators. We will say that $\mathbf{M}$ is a *model of* the theory $\mathbf{K}$.

e) A partial temporal model $\mathbf{M}$ of a theory $\mathbf{K}$ is called a *minimal model* of $\mathbf{K}$ if for every model $\mathbf{C}$ of $\mathbf{K}$ with $\mathbf{C \le M}$ it holds $\mathbf{C = M}$.

If in a model $\mathbf{M}$ the formula $\mathbf{P(\top)}$ is true at time point $\mathbf{t}$ then $\mathbf{t}$ must have a predecessor, and therefore $\neg \mathbf{P(\top)}$ will be true exactly in the time points which are minimal with respect to $<$.

From now on the word (temporal) formula will be used to denote a formula possibly containing any of the new operators, unless stated otherwise. If a formula contains no operators it is called *objective*. We call a subformula *guarded* if it is in the scope of an operator. A *purely temporal* formula is one in which all objective subformulae are guarded.

What is most interesting about a reasoning process, is of course its set of final conclusions. Talking about *final* conclusions, we will assume that the reasoning is *conservative*, which means that once a fact is established, it will remain true in the future of the reasoning process. In that case a fact is a final conclusion of a process if it is established at some point in time in the branch representing the process. So besides reasoning paths also the conclusions they result in are defined in a branching time model in the following manner:

Definition 2.4 (Limit Models of a Conservative Model)
Let $\mathbf{M}$ be a partial temporal model. Then $\mathbf{M}$ is *conservative* if $\mathbf{M_t \leq M_s}$ whenever $\mathbf{t \ll s}$. The *limit model of a branch* $\mathbf{B}$ of $\mathbf{M}$ based on flow of time $\mathbf{(T', <')}$, denoted by $\mathbf{lim_B\ M}$, is the partial model with for all atoms $\mathbf{p}$:

(i) $\mathbf{lim_B\ M} \models^+ \mathbf{p} \quad \Leftrightarrow \quad \exists\, \mathbf{t \in T'}\text{: } \mathbf{(M, t)} \models^+ \mathbf{p}$

(ii) $\mathbf{lim_B\ M} \models^- \mathbf{p} \quad \Leftrightarrow \quad \exists\, \mathbf{t \in T'}\text{: } \mathbf{(M, t)} \models^- \mathbf{p}$

Notice that $\mathbf{p}$ is undefined in $\mathbf{lim_B\ M}$ if and only if $\mathbf{p}$ is undefined in $\mathbf{B_t}$ for all $\mathbf{t \in T'}$.

3 Final Models

In this section $\mathbf{M}$ and $\mathbf{M'}$ denote partial temporal models based on the flows of time $\mathbf{(T, <)}$ and $\mathbf{(T', <')}$ respectively.

Definition 3.1 (Homomorphism)
A mapping $\mathbf{f : T \to T'}$ is called a *homomorphism* of $\mathbf{M}$ to $\mathbf{M'}$ if

(i) $\mathbf{s < t \Rightarrow f(s) <' f(t)}$

(ii) $\mathbf{M(s) = M'(f(s))}$

(iii) If $\mathbf{s}$ is a minimal element of $\mathbf{T}$ then $\mathbf{f(s)}$ is minimal element of $\mathbf{T'}$

Definition 3.2 (Persistency under Homomorphisms)
Let $\mathbf{f : M \to M'}$ be a homomorphism. The formula $\boldsymbol{\alpha}$ is called *forward persistent* (under $\mathbf{f}$) if for all time points $\mathbf{t}$ in $\mathbf{T}$:

$$\mathbf{(M,\ t)} \models^+ \boldsymbol{\alpha} \quad \Rightarrow \quad \mathbf{(M',\ f(t))} \models^+ \boldsymbol{\alpha}$$

The following proposition is an immediate consequence of [ET 94a].

Proposition 3.3
Let α be any formula containing at most the operators **C** and **P** and β any objective formula. Then the formulae α and $\alpha \rightarrow \exists X(\beta)$ are forward persistent under any homomorphism.

Definition 3.4 (Final Model)
The model **F** of a temporal theory **Th** is called a *final model* of **Th** if for each model **M** of **Th** there is a unique homomorphism **f : M -> F**. The model **F** is called a *final minimal model* of **Th** if **F** is a minimal model of **Th** and for each minimal model **M** of **Th** there is a unique homomorphism **f : M -> F**.

The following result shows the existence of final models for a certain class of theories (see [ET 94a]).

Theorem 3.5
If a final (minimal) model of a temporal theory **Th** exists, then it is unique (up to isomorphism). If all formulae in **Th** are forward persistent under surjective homomorphisms then there exists a (unique) final model $\mathbf{F_{Th}}$ of **Th**.

4 Temporal Axiomatization of a Classical Proof System

In this section we will apply our approach to a relatively simple type of reasoning: based on a classical proof system. We will show how proof rules can be represented by temporal formulae. As an example, consider modus ponens:

$$\frac{\mathbf{A} \quad \mathbf{A} \rightarrow \mathbf{B}}{\mathbf{B}}$$

Here **A** and **B** are meta-variables ranging over the set of formulae, and $\mathbf{A} \rightarrow \mathbf{B}$ is a term structure built from them using the logical connective $\rightarrow$. We want the partial temporal models to reflect the proof process, such that a partial model at a certain point in time reflects what has been derived up to that moment. The temporal interpretation of such a proof rule we have in mind is then the following.

if for any formulae **A** *and* **B**
in the current information state both **A** *and* $\mathbf{A} \rightarrow \mathbf{B}$ *have been derived*
then in a next information state **B** *has been derived*

This interpretation of modus ponens is formalized by the following temporal axiom scheme (for all formulae **A** and **B**):
$\mathbf{C(A) \wedge C(A \rightarrow B) \rightarrow \exists X(B)}$.
However, the truth of the formulae **A** and $\mathbf{A} \rightarrow \mathbf{B}$ in a current state already implies the truth of **B** in the same state, due to the compositional truth definition in the Strong Kleene semantics. As we want to describe the steps of reasoning by time

steps this is undesirable. A solution for this is to extend the notion of partial model to the notion of valuation of all formulae, in a manner similar to [BM 92], also see [SAN 85]. For each formula φ of the original language we define a new atom $\mathbf{at}_\varphi$, and then we take the propositional language induced by these new atoms as our new language. So if $\mathbf{FORM(\Sigma)}$ denotes the set of formulae based on the signature Σ, then we define a new signature Σ' based on the set of atoms $\mathbf{At(\Sigma')} = \{ \mathbf{at}_\varphi \mid \varphi \in \mathbf{FORM(\Sigma)} \}$. So we have a natural bijection $\varphi \text{ -> } \mathbf{at}_\varphi$ between $\mathbf{FORM(\Sigma)}$ and $\mathbf{At(\Sigma')}$. Notice that $\mathbf{At(\Sigma)}$ is embedded in $\mathbf{At(\Sigma')}$ by $\mathbf{At(\Sigma)} \ni \mathbf{p} \text{ -> } \mathbf{at_p} \in \mathbf{At(\Sigma')}$.

After this change of language has been accomplished, we can describe any instance of the proof rule modus ponens by a temporal formula as follows:

$$\mathbf{C(at_\varphi) \wedge C(at_{\varphi \to \psi}) \to \exists X(at_\psi)}$$

This allows us to give a temporal axiomatization of a proof system. In addition we need a temporal translation of the initial axioms: the theory from which conclusions are to be drawn. Suppose $\mathbf{K}$ is any set of formulae of signature Σ . Let $\mathbf{at(K)}$ be the set of atoms corresponding to the formulae in $\mathbf{K}$. We require that these atoms are true at each moment of time. Therefore for any such formulae φ we can simply add the formulae $\mathbf{C(at_\varphi)}$ to our temporal theory.

After these preparations we are ready to formalize the translation of the proof rules into temporal formulae:

Definition 4.1

a) By **Forterm** we denote the set of term structures built up from (meta-) variables, ranging over $\mathbf{FORM(\Sigma)}$, by use of the logical connectives. A *proof system* **PS** is a set of *proof rules* of type $\mathbf{(A_1 , \ldots, A_k) / B}$ where the $\mathbf{A_i , B} \in$ **Forterm**. Let a proof rule **PR**: $\mathbf{(A_1, \ldots , A_k) / B}$ be given en let $\mathbf{MV_{PR}}$ be the set of *meta-variables* occurring in $\mathbf{A_1, \ldots , A_k}$ and $\mathbf{B}$. A mapping σ: $\mathbf{MV_{PR}} \text{ -> } \mathbf{FORM(\Sigma)}$ is called a *meta-variable assignment* . Any meta-variable assignment σ can be extended in a canonical manner to a substitution mapping

$$\sigma^*: \mathbf{Forterm} \text{ -> } \mathbf{FORM(\Sigma)}$$

such that σ^* substitutes formulae for the meta-variables of $\mathbf{MV_{PR}}$ in any term structure of **Forterm**.

The *temporal translation* of a proof rule **PR** of the form $\mathbf{(A_1 , \ldots, A_k) / B}$ is the set $\mathbf{T_{PR}}$ of instances of temporal formulae defined by:

$$\{ \mathbf{C(at_{\sigma * A_1}) \wedge \ldots \wedge C(at_{\sigma * A_k}) \to \exists X(at_{\sigma * B})} \mid \sigma \textbf{ meta-variable assignment for PR} \}$$

The *temporal translation* $\mathbf{T_{PS}}$ of **PS** is defined by: $\mathbf{T_{PS}} = \bigcup_{\mathbf{PR} \in \mathbf{PS}} \mathbf{T_{PR}}$.

b) Let $\mathbf{K}$ be any set of objective formulae of signature Σ. The *temporal translation* $\mathbf{T_K}$ of $\mathbf{K}$ is defined by: $\mathbf{T_K} = \{ \mathbf{C(at_\varphi)} \mid \varphi \in \mathbf{K} \}$.

c) We have to make sure that once a fact has been established, it remains known at all later points (conservativity); this can be axiomatized by the temporal theory

$$\mathbf{C'} = \{ \mathbf{P(a) \to C(a)} \mid \mathbf{a} \in \mathbf{At(\Sigma')} \}$$

The overall translation of proof rules and theory is defined by:

$$\mathbf{Th_{PS,K}} = \mathbf{T_{PS}} \cup \mathbf{T_K} \cup \mathbf{C'}$$

Some proof systems may consist of both proof rules and axioms; these may be incorporated by adding them to the theory **K**.

The first observation about this temporal theory $\mathbf{Th_{PS,K}}$ is that there exist partial temporal models of it. Such a model could be constructed incrementally, starting with a root, adding its successor partial models in the next step, and any time the model has been constructed up until a certain level, one can construct the next level by adding successor partial models to those at the current level. This is possible since the formulae of $\mathbf{Th_{PS,K}}$ prescribe existence of successors, obeying certain properties. It is easy to see that these properties are never contradictory since only truth of certain atoms is prescribed. Taking such a model and changing the truth value of atoms which are not prescribed to be true by $\mathbf{Th_{PS,K}}$ to undefined, points out a manner to establish the existence of minimal models of $\mathbf{Th_{PS,K}}$. We have the following theorem.

Theorem 4.2

Let **PS** be any proof system and **K** any set of objective formulae of signature $\mathbf{\Sigma}$ and let $\mathbf{Th_{PS,K}}$ be the temporal theory $\mathbf{T_{PS}} \cup \mathbf{T_K} \cup \mathbf{C'}$. Let **M** be a minimal partial temporal model of $\mathbf{Th_{PS,K}}$. For any formula $\boldsymbol{\varphi}$ of signature $\mathbf{\Sigma}$ it holds

$$\mathbf{K} \vdash_{\mathbf{PS}} \boldsymbol{\varphi} \;\Leftrightarrow\; \mathbf{M} \models^{+} \neg\, \mathbf{P}(\top) \rightarrow \exists \mathbf{F}(\mathbf{at}_{\boldsymbol{\varphi}})$$

Note that for a minimal partial temporal model of $\mathbf{Th_{PS,K}}$ the partial models of time points which are minimal, are the same (atoms corresponding to formulae of **K** are true, other atoms are undefined). In this way a semantics is defined which can be seen as a generalization of the manner in which modal and temporal semantics can be given to intuitionistic logic (see [GAB 82], [KRI 65]). Apart from the use of partial models, our approach can be used for any proof system.

Proposition 4.3

The temporal theory $\mathbf{Th_{PS,K}}$ has a final model $\mathbf{F_{PS,K}}$.

If we have a proof $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$ of which (only) the first **k** formulae are axioms from **K**, then a proof trace is a sequence $(\mathbf{M_i})_{\mathbf{i=0..n-k}}$ of partial models such that $\mathbf{Lit(M_i)} = \{\mathbf{at}_{\boldsymbol{\varphi}_\mathbf{j}} \mid \mathbf{j} = 1, .., \mathbf{k+i}\}$. In such a trace the partial model $\mathbf{M_i}$ reflects exactly the formulae which have been derived up until the $\mathbf{i^{th}}$ step of the proof. It is easy to see that, although such a proof trace itself is in general not a model of $\mathbf{Th_{PS,K}}$, it can always be embedded in the final model $\mathbf{F_{PS,K}}$. Note that for a branch **B** of the final model the limit model $\mathbf{lim_B}\ \mathbf{F_{PS,K}}$ corresponds to the set of all conclusions drawn in that reasoning pattern; this is a subset of the deductive closure of **K** under **PS** (since we allow non-exhaustive reasoning patterns).

5 Temporal Theories of Default Reasoning

In this section we will show that default reasoning patterns based on normal defaults can be captured by temporal theories. The main point is how to interpret a default reasoning step

if α *and it is consistent to assume* β
then β *can be assumed*

in a temporal manner. We will view the underlying default $(\alpha : \beta)/\beta$ as a (meta-level) proof rule stating that if the formula α has already been established in the past, and there is a possible future reasoning path where the formula β remains consistent, then β can be assumed to hold in the current time point. As partial models can only be used to describe literals, instead of arbitrary formulae, we restrict our default rules to ones which are *based on literals*, which means that α and β have to be literals. As in the previous section, this is not an important hindrance, since for an arbitrary formula α we can add a new atom $\mathbf{at}_\alpha$ to our signature, adding the formula $\alpha \leftrightarrow \mathbf{at}_\alpha$ to our (non-default) knowledge in $\mathbf{W}$. The translation of the rule $(\alpha : \beta)/\beta$ in temporal partial logic will be:

$$\mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta \rightarrow \mathbf{C}\beta$$

Here $\neg \forall \mathbf{F} \neg \beta$ is true at a point in time if not for all future paths, the negation of β becomes true at some point in that branch, which is equivalent to: there is a branch, starting at the present time point, on which β is always either true or undefined. To ensure that formulae which are true at a certain point in time, will remain true in all of its future points (once a fact has been established, it remains established), we will add for each literal $\mathbf{L}$ a rule $\mathbf{P(L)} \rightarrow \mathbf{C(L)}$. Furthermore, if we have an additional (non-default) theory $\mathbf{W}$ it can be shown (see [ET 93], [ET 94b]) that there exists a temporal theory which ensures that all conclusions which can be drawn using the theory $\mathbf{W}$ and the default conclusions at a certain time point, are true in the partial model at that time point. The complete translation of a normal default theory is as follows:

Definition 5.1 (Temporal Interpretation of a Normal Default Theory)

Let $\Delta = \langle \mathbf{W}, \mathbf{D} \rangle$ be a normal default theory of signature Σ. Define

$$\mathbf{C'} = \{ \mathbf{P(L)} \rightarrow \mathbf{C(L)} \mid \mathbf{L} \in \mathbf{Lit}(\Sigma) \} \cup \{ \exists \mathbf{F}(\mathrm{T}) \}$$

$$\mathbf{D'} = \{ \mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta \rightarrow \mathbf{C}\beta \mid (\alpha : \beta) / \beta \in \mathbf{D} \}$$

$$\mathbf{W'} = \{ \mathbf{C(L)} \mid \mathbf{L} \text{ literal}, \ \mathbf{W} \models \mathbf{L} \} \cup \{ \mathbf{C(con(F))} \rightarrow \mathbf{C(L)} \mid \mathbf{L} \text{ literal}, \ \mathbf{F} \neq \varnothing \text{ a finite set of literals with } \mathbf{F} \cup \mathbf{W} \models \mathbf{L} \}$$

The *temporal interpretation* of Δ is the temporal theory

$$\mathbf{Th}_\Delta = \mathbf{C'} \cup \mathbf{D'} \cup \mathbf{W'}$$

Minimal temporal models of $\mathbf{Th}_\Delta$ describe the possible reasoning paths when reasoning with defaults from $\boldsymbol{\Delta}$. It turns out that there is a nice connection between the branches of such a minimal temporal model and Reiter extensions of the default theory. We will first give a definition of a Reiter extension of a default theory, equivalent to Reiter's original definition (in [REI 80]):

Definition 5.2 (Reiter Extension)
Let $\Delta = \langle \mathbf{W}, \mathbf{D} \rangle$ be a default theory of signature Σ, and let $\mathbf{E}$ be a consistent set of sentences for Σ. Then $\mathbf{E}$ is a Reiter extension of Δ if $\mathbf{E} = \bigcup_{i=0}^{\infty} \mathbf{E}_i$ where $\mathbf{E}_0 = \mathbf{Th(W)}$, and for all $\mathbf{i} \geq \mathbf{0}$

$$\mathbf{E}_{i+1} = \mathbf{Th}(\mathbf{E}_i \cup \{ \beta \mid (\alpha : \beta) / \beta \in \mathbf{D}, \alpha \in \mathbf{E}_i \text{ and } \neg \beta \notin \mathbf{E} \})$$

If $\mathbf{E}$ is a Reiter extension, then by $\mathbf{E}_i$ we will denote the subsets of $\mathbf{E}$ as defined in this definition. The following theorem shows that in the case of linear models there is a clear correspondence between extensions of a default theory Δ and the minimal linear time models of the temporal theory $\mathbf{Th}_\Delta$ (also see [ET 94b]). For a linear model we can always assume that it is based on the flow of time $(\mathbb{N}, <')$ with $\mathbf{s} <' \mathbf{t}$ iff $\mathbf{t} = \mathbf{s} + \mathbf{1}$. For a consistent set of literals $\mathbf{S}$ by $< \mathbf{S} >$ we denote the unique partial model $\mathbf{M}$ with $\mathbf{Lit(M)} = \mathbf{S}$.

Theorem 5.3
Let $\Delta = \langle \mathbf{W}, \mathbf{D} \rangle$ be a normal default theory.
a) If $\mathbf{M}$ is a minimal linear time temporal model of $\mathbf{Th}_\Delta$, then $\mathbf{Th}(\mathbf{Lit}(\mathbf{lim}_\mathbf{M}\, \mathbf{M}) \cup \mathbf{W})$ is a Reiter extension $\mathbf{E}$ of Δ.
Moreover, $\mathbf{E}_t = \mathbf{Th}(\mathbf{Lit}(\mathbf{M}_t) \cup \mathbf{W})$ for all $\mathbf{t} \in \mathbb{N}$.
b) If $\mathbf{W}$ is consistent and $\mathbf{E}$ a Reiter extension of Δ, then the partial temporal model $\mathbf{M}$ defined by $\mathbf{M} = (< \mathbf{Lit}(\mathbf{E}_t) >)_{t \in \mathbb{N}}$ is a minimal linear time temporal model of $\mathbf{Th}_\Delta$ with $\mathbf{Lit}(\mathbf{lim}_\mathbf{M}\, \mathbf{M}) = \mathbf{Lit(E)}$.

There is the following connection between the branches of (minimal) temporal models of $\mathbf{Th}_\Delta$ and the linear (minimal) temporal models of $\mathbf{Th}_\Delta$:

Proposition 5.4
Let Δ be a normal default theory and $\mathbf{M}$ a temporal model of $\mathbf{Th}_\Delta$.
a) Every maximal branch of $\mathbf{M}$ is a linear time model of $\mathbf{Th}_\Delta$.
b) $\mathbf{M}$ is a minimal temporal model of $\mathbf{T}_\Delta$ if and only if every maximal branch of $\mathbf{M}$ is a (linear) minimal temporal model of $\mathbf{Th}_\Delta$.

What we would now hope is that for a default theory a final model would exist which captures all of the linear partial temporal models, and thus captures all of the extensions in one model. This is however not in general the case, but there is a category of theories (which includes all the theories with a finite number of default rules) for which such a final model always exists. To identify this category we need the following notion:

Definition 5.5 (Extension Complete)
Let $\mathbf{\Delta}$ be a default theory.
a) We call a chain of sets of formulae

$$\mathbf{S_0} \subseteq \mathbf{S_1} \subseteq \mathbf{S_2} \subseteq \; \ldots .$$

approximated by a (Reiter) extension $\mathbf{E}$ of $\mathbf{\Delta}$ *up to depth* $\mathbf{n}$ if for all $\mathbf{i} \leq \mathbf{n}$ it holds $\mathbf{S_i} = \mathbf{E_i}$ (where the $\mathbf{E_i}$ are as in Definition 5.2). The chain $(\mathbf{S_k})_{\mathbf{k} \in \mathbb{N}}$ is called *approximated by a set* of Reiter extensions $\mathbf{R}$ of $\mathbf{\Delta}$ if for every $\mathbf{n} \in \mathbb{N}$ there is an extension $\mathbf{E} \in \mathbf{R}$ such that $(\mathbf{S_k})_{\mathbf{k} \in \mathbb{N}}$ is approximated by $\mathbf{E}$ up to depth $\mathbf{n}$.
b) We call $\mathbf{\Delta}$ (Reiter) *extension complete* if for any chain of sets of formulae $(\mathbf{S_k})_{\mathbf{k} \in \mathbb{N}}$ that is approximated by a set of (Reiter) extensions $\mathbf{R}$ of $\mathbf{\Delta}$, its union $\cup_{\mathbf{k} \in \mathbb{N}} \mathbf{S_k}$ is a Reiter extension $\mathbf{E}$ with (where the $\mathbf{E_i}$ are as in Definition 5.2) $\mathbf{E_i} = \mathbf{S_i}$ for all $\mathbf{i}$.

In [ET 94b] an example is given of a default theory which is not extension complete. For the default theories which are extension complete, we have the following (see [ET 94b]):

Theorem 5.6
Let $\mathbf{\Delta}$ be a normal default theory.
a) If $\mathbf{\Delta}$ is extension complete, then there exists a (unique) final minimal temporal model $\mathbf{FM_\Delta}$ of $\mathbf{Th_\Delta}$.
b) Suppose a final minimal temporal model $\mathbf{FM_\Delta}$ of $\mathbf{Th_\Delta}$ exists.
Then there is a one to one correspondence between the set $\mathbb{LT}(\mathbf{FM_\Delta})$ of maximal branches $\mathbf{B}$ of $\mathbf{FM_\Delta}$ and the set $\mathbb{E}(\mathbf{\Delta})$ of all Reiter extensions $\mathbf{E}$ of $\mathbf{\Delta}$.
Here $\mathbf{B}$ and $\mathbf{E}$ correspond to each other if and only if $\mathbf{B} = (< \mathbf{Lit(E_t)} >)_{\mathbf{t} \in \mathbb{N}}$ and $\mathbf{E} = \mathbf{Th(Lit(lim_B\ M)} \cup \mathbf{W)}$.

6 Temporal Axiomatization of a Meta-level Architecture

In this section we apply our approach to a third kind of reasoning patterns: generated by a meta-level architecture reasoning system. Meta-level architectures form the basis of quite powerful reasoning systems: they have been applied for example to non-monotonic reasoning and reasoning about control (e.g., [BK 82], [CB 88], [DAV 80], [GTG 93], [MN 88], [TT 91], [TT 92], [WEY 80]). A meta-level architecture consists of two separate reasoning levels or components: the object level component and the meta-level component. The connections between the components are defined by so called upward and downward reflections.

As an example, suppose the meta-level reasoning component has (meta-) knowledge by which it can be deduced in which state what goal is adequate for the reasoning of the object level component:

if the atom **a** *is unknown, then the atom* **b** *is proposed as a goal*

If we assume that after downward reflection indeed the proposed goal has been chosen (in the literature this is called the causal connection assumption; [MN 88]), this meta-knowledge can be interpreted in a temporal manner:

if in the current state the atom **a** *is unknown* *(in the object level reasoning component)*
then in a next state the atom **b** *is a goal* *(for the object level reasoning process)*

Thus meta-level reasoning implies a shift in time, replacing the goals at the object-level by new goals (the ones proposed by the meta-level). We will formalize these notions in subsequent (sub)sections.

6.1 Formalizing the Object Level Component

In the sequel by $\vdash$ we will denote any sound inference relation that is not necessarily complete (e.g., one of: natural deduction, chaining, full resolution, SLD resolution, unit resolution, etc.). A partial model is complete if it does not assign the truth value undefined to any atom. For a consistent set of formulae $\mathbf{K}$, of signature $\mathbf{\Sigma}$, by $\mathbf{IS_K(\Sigma)}$ we denote the set of partial models which have a complete refinement (with respect to $\leq$) which is a model of $\mathbf{K}$. This set can be seen as the set of partial models which are "consistent" with $\mathbf{K}$.

Definition 6.1 (Deductive Closure)
Let $\mathbf{K}$ be a consistent set of objective formulae of signature $\mathbf{\Sigma}$.
For $\mathbf{M} \in \mathbf{IS_K(\Sigma)}$ we define the partial model $\mathbf{dc_K{}^{\vdash}(M)}$ by

$$\mathbf{dc_K{}^{\vdash}(M) \models^+ L} \quad \Leftrightarrow \quad \mathbf{K \cup Lit(M)} \ \vdash \ \mathbf{L}$$

for any literal $\mathbf{L}$. This model is called the *deductive closure* of $\mathbf{M}$ under $\mathbf{K}$.
We call $\mathbf{M}$ *deductively closed* under $\mathbf{K}$ if $\mathbf{M = dc_K{}^{\vdash}(M)}$.

Definition 6.2 (Conservation, Monotonicity, Idempotency)
Let $\mathbf{K}$ be a consistent set of objective formulae of signature $\mathbf{\Sigma}$.
The mapping $\alpha : \mathbf{IS_K(\Sigma)} \to \mathbf{IS_K(\Sigma)}$ is called:

(i) *conservative* if $\mathbf{M} \leq \alpha(\mathbf{M})$ for all $\mathbf{M} \in \mathbf{IS_K(\Sigma)}$
(ii) *monotonic* if $\alpha(\mathbf{M}) \leq \alpha(\mathbf{N})$ for all $\mathbf{M, N} \in \mathbf{IS_K(\Sigma)}$ with $\mathbf{M} \leq \mathbf{N}$
(iii) *idempotent* if $\alpha(\alpha(\mathbf{M})) = \alpha(\mathbf{M})$ for all $\mathbf{M} \in \mathbf{IS_K(\Sigma)}$

Proposition 6.3
Let $\mathbf{K}$ be a consistent set of objective formulae of signature $\mathbf{\Sigma}$. Then the mapping $\mathbf{dc_K{}^{\vdash}} : \mathbf{IS_K(\Sigma)} \to \mathbf{IS_K(\Sigma)}$ is conservative, monotonic and idempotent.
Moreover, for any $\mathbf{M} \in \mathbf{IS_K(\Sigma)}$ and any complete model $\mathbf{N}$ of $\mathbf{K}$ with $\mathbf{M} \leq \mathbf{N}$ it holds $\mathbf{dc_K{}^{\vdash}(M)} \leq \mathbf{N}$.

In case of controlled non-exhaustive reasoning, mappings are involved (depending on certain control settings) that in principle are not idempotent. However, we still require these mappings (called controlled inference functions) to be conservative and monotonic. Now we can formalize the object level component as follows.

Definition 6.4 (Object Level Component)
The *object-level reasoning component* **OC** is defined by a tuple

$$\mathbf{OC} = \langle \langle \Sigma_o, \mathrm{OT}, \vdash \rangle, \langle \Sigma_c, \mu_{OT}{}^{\vdash}, \upsilon_{OT}{}^{\vdash} \rangle \rangle$$

with
Σ_o a signature, called the *object-signature*
OT a set of ground formulae expressed in terms of Σ_o : the *object theory*
$\vdash$ a classical inference relation (assumed sound but not necessarily complete)
Σ_c a signature, called the *control signature* and
$\mu_{OT}{}^{\vdash}$ $: IS_{OT}(\Sigma_o) \times IS(\Sigma_c) \rightarrow IS_{OT}(\Sigma_o)$
$\upsilon_{OT}{}^{\vdash}$ $: IS_{OT}(\Sigma_o) \times IS(\Sigma_c) \rightarrow IS(\Sigma_c)$

We call $\mu_{OT}{}^{\vdash}$ the *(controlled) inference function* for the object-level, and $\upsilon_{OT}{}^{\vdash}$ the *process state update function*. For any $N \in IS(\Sigma_c)$ the mapping

$$\mu_{OT}{}^{N} : IS_{OT}(\Sigma_o) \rightarrow IS_{OT}(\Sigma_o)$$

is defined by $\mu_{OT}{}^{N}(M) = \mu_{OT}{}^{\vdash}(M, N)$. We assume that for any $N \in IS(\Sigma_c)$ this $\mu_{OT}{}^{N}$ is conservative and monotonic and satisfies $\mu_{OT}{}^{N}(M) \leq dc_{OT}{}^{\vdash}(M)$ for all $M \in IS_{OT}(\Sigma_o)$. When no confusion is expected, we will leave out the subscript and superscript of $\mu_{OT}{}^{\vdash}$ and $\upsilon_{OT}{}^{\vdash}$ and write shortly μ and υ.

The control-information state **N** specifies at a high level of abstraction all information relevant to the control of the (future) reasoning behaviour; i.e., the object-information state **M** and the control-information state **N** together determine in a deterministic manner the behaviour of the object-level reasoning component during its next activation. The process state update function expresses what the process brings about with respect to the descriptors in the control-information state. Examples: an object-atom was unknown, but becomes known during the reasoning; an object atom that was a goal has failed to be found.

6.2 Formalizing the Meta-level Component

In the meta-reasoning we distinguish two special types of (meta-)information: a) information on relevant aspects of the *current* (control-)state of the object-level reasoning process (possibly also including facts inherited from the past), and b) information on proposals for control parameters that are meant to guide the object-level reasoning process in the next activation. Therefore we assume that in the meta-signature two copies of the control signature of the object-level component are included as a subsignature: one that refers to the current state and another one referring to the proposed truth values for the next state of the object-level reasoning process. For example, if **goal(h)** is an atom of the control signature, then there are copies **current_goal(h)** and **proposed_goal(h)** in the set of atoms for the meta-signature. Two syntactic functions **c** and **p** transforming a meta-atom into a current variant and a proposed variant of it can simply be defined; e.g.,

c(goal(h)) = current_goal(h), p(goal(h)) = proposed_goal(h)

We assume that the reasoning of the meta-level itself has no sophisticated control: for simplicity we assume that it concerns taking deductive closures with respect to the inference relation used at the meta-level.

Definition 6.5 (Meta-level Architecture)
a) The *meta-level component* **MC** related to $\boldsymbol{\Sigma_c}$ is defined as a tuple

$$\mathbf{MC} = \langle\langle \Sigma_m, \mathbf{MT}, \vdash_m \rangle, \langle \mathbf{c}, \mathbf{p} \rangle\rangle$$

with

$\boldsymbol{\Sigma_m}$ a signature, called the *meta-signature* related to $\boldsymbol{\Sigma_c}$

MT a set of objective ground formulae of signature $\boldsymbol{\Sigma_m}$: the *meta-theory*

$\vdash_m$ a classical inference relation (assumed sound but not necessarily complete)

c, p $: \mathbf{At}(\Sigma_c) \to \mathbf{At}(\Sigma_m)$ injective mappings

It is assumed that every information state in $\mathbf{M} \in \mathbf{IS}(\Sigma_m)$ with $\mathbf{M(a)} = \mathbf{u}$ for all $\mathbf{a} \in \mathbf{At}(\Sigma_m)\backslash \mathbf{c}(\mathbf{At}(\Sigma_c))$ is consistent with **MT**, i.e. $\mathbf{M} \in \mathbf{IS}_{\mathbf{MT}}(\Sigma_m)$.

The *inference function of the meta-level* $\mu_{\mathbf{MT}}{}^{\vdash_m}$ (or shortly μ^*)

$$\mu_{\mathbf{MT}}{}^{\vdash_m}: \mathbf{IS}_{\mathbf{MT}}(\Sigma_m) \to \mathbf{IS}_{\mathbf{MT}}(\Sigma_m)$$

is defined by $\mu_{\mathbf{MT}}{}^{\vdash_m}(\mathbf{N}) = \mathbf{dc}_{\mathbf{MT}}{}^{\vdash_m}(\mathbf{N})$.

b) A *meta-level architecture* is defined as a pair

$$\mathbf{MA} = \langle \mathbf{OC}; \mathbf{MC} \rangle$$

with **OC** an object level component and **MC** a related meta-level component.

c) For a meta-level architecture **MA** the *upward reflection function*

$\alpha_u: \mathbf{IS}(\Sigma_c) \to \mathbf{IS}(\Sigma_m)$ is defined for $\mathbf{N} \in \mathbf{IS}(\Sigma_c)$ and $\mathbf{b} \in \mathbf{At}(\Sigma_m)$ by

$$\alpha_u(\mathbf{N})(\mathbf{b}) = \begin{cases} \mathbf{N(a)} & \text{if } \mathbf{b} = \mathbf{c(a)} \text{ for some } \mathbf{a} \in \mathbf{At}(\Sigma_c) \\ \mathbf{u} & \text{otherwise} \end{cases}$$

The *downward reflection function* $\alpha_d: \mathbf{IS}(\Sigma_m) \to \mathbf{IS}(\Sigma_c)$ is defined for $\mathbf{N} \in \mathbf{IS}(\Sigma_m)$ and $\mathbf{a} \in \mathbf{At}(\Sigma_c)$ by

$$\alpha_d(\mathbf{N})(\mathbf{a}) = \begin{cases} \mathbf{1} & \text{if } \mathbf{N(p(a))} = \mathbf{1} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

6.3 Formalizing the Overall Reasoning Behaviour

Four types of actions take place as depicted in Fig. 1. For a formal description, see the following definition.

For a meta-level architecture with propositional signatures $\boldsymbol{\Sigma_o}$ and $\boldsymbol{\Sigma_m}$ we denote the combined signature (based on the disjoint union of their sets of atoms) by $\boldsymbol{\Sigma_o} \oplus \boldsymbol{\Sigma_m}$. We denote partial models for this combined signature as a pair of partial models $\mathbf{M} \oplus \mathbf{N}$ for $\boldsymbol{\Sigma_o}$ and $\boldsymbol{\Sigma_m}$.

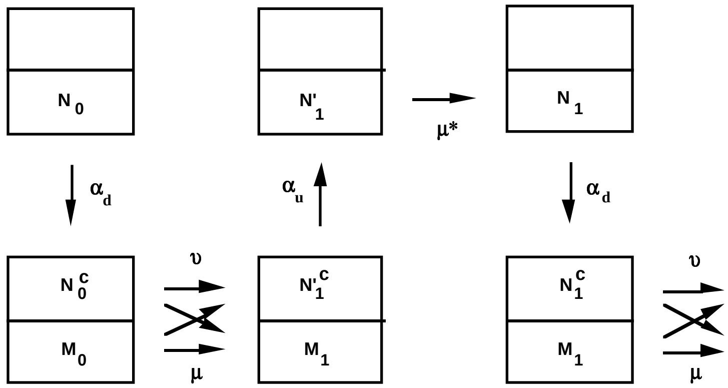

Fig 1 Reasoning pattern in a meta-level architecture

Definition 6.6 (Semantics Based on Overall Traces)
a) An *overall trace* for the meta-level architecture **MA** is a linear partial temporal model $(\mathbf{M_t} \oplus \mathbf{N_t})_{\mathbf{t} \in \mathbb{N}}$ of signature $\Sigma_o \oplus \Sigma_m$ satisfying for each $\mathbf{t} \in \mathbb{N}$:

$$\mathbf{M_{t+1}} = \mu(\mathbf{M_t}, \alpha_d(\mathbf{N_t}))$$
$$\mathbf{N_{t+1}} = \mu^*(\alpha_u(\upsilon(\mathbf{M_t}, \alpha_d(\mathbf{N_t}))))$$

b) The *(intended) trace semantics* of **MA** is the set of overall traces.

The following theorem (also see [TRE 94]) shows that for any meta-level architecture **MA** with finite sets of atoms there exists a temporal theory such that a specific class of its models are precisely the overall traces of **MA**. For any $\langle \mathbf{M'}, \mathbf{N'} \rangle \in \mathbf{IS}(\Sigma_o) \times \mathbf{IS}(\Sigma_m)$ let the temporal theory $\mathbf{T}_{\langle \mathbf{M'}, \mathbf{N'} \rangle}$ be given that defines all partial temporal models with initial state $\langle \mathbf{M'}, \mathbf{N'} \rangle$, i.e. with $(\mathbf{M_t} \oplus \mathbf{N_t})_{\mathbf{t} \in \mathbb{N}}$ is a minimal model of $\mathbf{T}_{\langle \mathbf{M'}, \mathbf{N'} \rangle}$ iff $\mathbf{M_0} = \mathbf{M'}$ and $\mathbf{N_0} = \mathbf{N'}$.

Theorem 6.7 (Temporal Theory of a Meta-level Architecture)
Let **MA** be a meta-level architecture with finite sets of atoms.
There exists a temporal theory $\mathbf{Th_{MA}}$ (consisting of formulae of the form **A** or $\mathbf{A} \rightarrow \exists\mathbf{X}(\mathbf{L})$ where **L** is some objective literal and **A** a formula only referring to the past and current state) such that:
a) For any linear partial temporal model $(\mathbf{M_t} \oplus \mathbf{N_t})_{\mathbf{t} \in \mathbb{N}}$ of signature $\Sigma_o \oplus \Sigma_m$ the following are equivalent:
(i) $(\mathbf{M_t} \oplus \mathbf{N_t})_{\mathbf{t} \in \mathbb{N}}$ is a minimal model of $\mathbf{Th_{MA}} \cup \mathbf{T}_{\langle \mathbf{M'}, \mathbf{N'} \rangle}$.
(ii) $(\mathbf{M_t} \oplus \mathbf{N_t})_{\mathbf{t} \in \mathbb{N}}$ is an overall trace for **MA** with initial state $\langle \mathbf{M'}, \mathbf{N'} \rangle$.
In other words: the intended semantics of **MA** is described by the linear models of $\mathbf{Th_{MA}}$ that are minimal in the set of models with fixed initial state.
b) The theory $\mathbf{Th_{MA}}$ has a final model $\mathbf{F_{MA}}$.

To get an idea how such a temporal theory can be defined, the example given in the beginning of this section can be formalized by the rules

C(¬ known(a)) $\rightarrow$ **C(proposed_goal(b))** (meta-knowledge)

C(proposed_goal(b)) $\rightarrow$ **∃X(goal(b))** (downward reflection)

The final model $\mathbf{F_{MA}}$ contains all overall traces as linear submodels (branches), but in general it will contain other, less useful branches as well. Another variant can be obtained by using the temporal operator **∀X** instead of **∃X** in the formulae of $\mathbf{Th_{MA}}$; this leads to a temporal theory with the same linear time models but with a more restricted final model.

7 Conclusions

Partial temporal models can be used to describe the behaviour of dynamic reasoning processes, such as those performed by reasoning agents. The linear models usually describe a particular reasoning pattern, and a set of such models can be used to describe all possible patterns. These models may be described by a temporal theory. Another way of describing possible behaviour is by a branching time process which is branching at any time a pattern can continue in more than one way. These models can also often be axiomatized by a temporal theory. In this fashion we can use branching time temporal partial logic to obtain semantics for a variety of reasoning patterns including regular monotonic logics, default logic and for all patterns a meta-level architecture can perform. In these patterns one can often identify object level reasoning (by means of classical logic) and meta-level reasoning which complements it. The inferences on the object level can be axiomatized by a theory which restricts the (current) partial models at each time point, whereas the meta-level inferences (potentially introducing non-monotonicity) can be axiomatized by a theory which restricts the successor partial models at any point in time. The theory axiomatizing object level inference consists of formulae containing no operators but the **C** operator, whereas the theory for meta-level inference will consist of formulae containing at least one of the other operators.

In default logic the classical propositional logic is used for the object level inferences (axiomatized by the theory **W'**), whereas the meta-level inferences are performed by the default rules (axiomatized by the theory **D'**). The truth of formulae in the theory **W'** at a certain point depends only on the partial model a that point, whereas the theory **D'** restricts successor partial models.

In the example of the classical proof system, the proof rules have been lifted to the meta-level so that "proving" a formula is an explicit temporal process. Note that the partiality in this case is not explicitly needed: in a minimal model the truth value false will never occur. In the meta-level architecture some aspects of the object inferences have been lifted to the meta-level to allow for explicit control of the reasoning process.

In a number of cases it is possible to identify a final model, that is a "biggest" branching time model in which all possible patterns are incorporated in the most compact manner. Not only do we then have one structure which holds all information about a reasoning process, also the points in a process where a choice has to be made are explicitly identified. Based on this final model one can define a number of entailment relations, depending for instance on whether conclusions have to be established in all possible patterns, or if it is enough if there is at least one possibility to establish the conclusion.

We feel that temporal partial logic and other temporalized logics are a powerful way of describing complex reasoning patterns as they can be used to model a variety of reasoning patterns in a clear fashion. This approach contributes to a better integration of dynamic aspects in logical systems, as, for instance, advocated in [BEN 91]. It can be used as a basis for providing formal semantics of (specification languages for) complex reasoning systems, where often control of reasoning itself is a subject of reasoning: one of the open problems formulated in [HLM *et al.* 93].

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Appendix A to Chapter 2

Flows of Time

Definition 2.1' (Flow of Time)
A *flow of time* **(T, <)** is a pair consisting of a non empty set **T** of time points, and a binary relation < on **T x T**, called the *immediate successor relation* which has to be:
(i) *Antitransitive*: for no **a**, **b** and **c** in **T** it holds **a** < **b**, **b** < **c** and **a** < **c**.
(ii) *A-cyclic*: there is no $\mathbf{n \geq 0}$ and $\mathbf{a_0}, \ldots, \mathbf{a_n}$ in **T** with for $\mathbf{0 \leq i \leq n - 1}$: $\mathbf{a_i} < \mathbf{a_{i+1}}$ and $\mathbf{a_n} < \mathbf{a_0}$. (This implies < is irreflexive and antisymmetrical).
Here for **s, t** in **T** the expression **s < t** denotes that **t** is an *immediate successor* of **s**, and that **s** is an *immediate predecessor* of **t**.
We also introduce the transitive (but not reflexive) closure **«** of this binary relation: **«** = $<^+$. A flow of time is called *linear* if **«** is a total ordering.

Note that with this definition **T** together with **«** is a discrete time structure.
We will further limit our flows of time to be forests or trees:

Definition 2.2' (Tree and Forest)
a) The following properties are defined:
(i) *Successor existence*
Every time point has at least one successor:
for all $\mathbf{s} \in \mathbf{T}$ there exists a $\mathbf{t} \in \mathbf{T}$ such that **s < t**.
(ii) *Rooted*
A flow of time is rooted with root **r** if **r** is a (unique) smallest element:
for all **t** it holds **r = t** or **r « t**.
(iii) *Left linear*
For all **t** the set of **s** with **s « t** is totally ordered by «.
(iv) *Well-founded*

There are no infinite descending chains of elements $s_i < s_{i-1}$

b) A flow of time is called a *tree* if it is rooted and left linear.

c) A flow of time is called a *forest* if it is well-founded, left linear.

Note that a forest is just a disjoint union of trees. We will assume all flows of time to be forests.

Definition 2.3' (Sub-ft and Branch)

a) A flow of time $(T', <')$ is called a *sub-ft (sub-flow of time)* of a flow of time $(T, <)$ if $T' \subseteq T$ and $<' = < \cap T' \times T'$. It is also called the sub-ft of $(T, <)$ *defined by* T', or the *restriction of* $(T, <)$ to T'.

b) A *branch* in a flow of time T is a sub-ft $B = (T', <')$ of T such that:

(i) $\ll'$ (the transitive closure of $<'$) is a total ordering on $T' \times T'$

(ii) Every $t \in T'$ with a successor in T also has a successor in T':

for all $s \in T', t \in T : s < t \Rightarrow$ there is a $t' \in T' : s < t'$

(iii) Every element of T that is in between elements of T' is itself in T':

for all $s \in T', t \in T, u \in T' : s \ll t \ll u \Rightarrow t \in T'$

A branch is called *maximal* if every t in T' with a predecessor in T also has a predecessor in T': for all $s \in T, t \in T' : s < t \Rightarrow$ there is an $s' \in T' : s' < t$.

Partial Temporal Models

By a signature Σ for convenience we mean a sequence of proposition symbols (propositional atom names). What counts is the set of atoms $At(\Sigma)$ and the set of literals $Lit(\Sigma)$ based on this signature.

Definition 2.4' (Partial Model)

Let Σ be a signature.

a) A *partial model* M for the signature Σ is an assignment of a truth value from $\{0, 1, u\}$ to each of the atoms of Σ, i.e. $M: At(\Sigma) \rightarrow \{0, 1, u\}$.

We say an atom a is *true* in M if 1 is assigned to it, and *false* if 0 is assigned; else it is called *undefined* (or *unknown*). In the same way we say a literal $\neg p$ is *true* in M if M assigns 0 to p and it is *false* if M assigns 1 to p. Otherwise it is *undefined*.

By $Lit(M)$ we denote the set of literals (atoms or negations of atoms) with truth value *true* in M.

b) The truth, falsity or undefinedness of any formulae in a partial model is evaluated according to the Strong Kleene semantics (e.g., [BLA 86], [LAN 88]).

c) The *ordering of truth values* is defined by $u \leq 0$, $u \leq 1$, $u \leq u$, $0 \leq 0$, $1 \leq 1$. We call the model N a *refinement* of the model M, denoted by $M \leq N$, if for all atoms a it holds: $M(a) \leq N(a)$.

d) For a consistent set of literals S the unique partial model M with $Lit(M) = S$ is denoted by $< S >$.

Definition 2.5'

The partial temporal model **M'** is *sub-model* of **M** if **(T', <')** is a sub-flow of time of **(T, <)** with $M(t) = M'(t)$ for all **t** in **T'**. We also call **M'** the *restriction* of **M** to **T'**, denoted by **M|T'**.

Also the other notions defined in the above subsection for flows of time inherit to models.

Definition 2.6' (Temporal Operators and Their Semantics)

Let a formula α, a partial temporal model **M**, and a time point $t \in T$ be given, then:

a) $(M, t) \vDash^{+} \exists F\alpha \;\Leftrightarrow\; \exists\, s \in T\ [\, t \ll s \ \&\ (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \exists F\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \exists F\alpha$

b) $(M, t) \vDash^{+} \forall F\alpha \;\Leftrightarrow\;$ **for all branches including t there exists an s in that branch such that** $[\, t \ll s \ \&\ (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \forall F\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \forall F\alpha$

c) $(M, t) \vDash^{+} \forall G\alpha \;\Leftrightarrow\; \forall\, s \in T\ [\, t \ll s \Rightarrow (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \forall G\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \forall G\alpha$

d) $(M, t) \vDash^{+} \exists G\alpha \;\Leftrightarrow\;$ **there exists a branch including t such that for all s in that branch** $[\, t \ll s \Rightarrow (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \exists G\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \exists G\alpha$

e) $(M, t) \vDash^{+} P\alpha \;\Leftrightarrow\; \exists\, s \in T\ [\, s \ll t \ \&\ (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} P\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} P\alpha$

f) $(M, t) \vDash^{+} H\alpha \;\Leftrightarrow\; \forall\, s \in T\ [\, s \ll t \Rightarrow (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} H\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} H\alpha$

g) $(M, t) \vDash^{+} C\alpha \;\Leftrightarrow\; (M, t) \vDash^{+} \alpha$

$(M, t) \vDash^{-} C\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} C\alpha$

h) $(M, t) \vDash^{+} \exists X\alpha \;\Leftrightarrow\; \exists\, s \in T\ [\, t < s \ \&\ (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \exists X\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \exists X\alpha$

i) $(M, t) \vDash^{+} \forall X\alpha \;\Leftrightarrow\; \forall\, s \in T\ [\, t < s \Rightarrow (M, s) \vDash^{+} \alpha \,]$

$(M, t) \vDash^{-} \forall X\alpha \;\Leftrightarrow\; (M, t) \nvDash^{+} \forall X\alpha$

Definition 2.7' (Interpretation of Temporal Formulae)

Let Σ be a signature, let **M** be a partial temporal model for Σ, and $t \in T$ a time point.

a) For any propositional atom $p \in At(\Sigma)$:

$$(M, t) \vDash^{+} p \;\Leftrightarrow\; M(t, p) = 1$$

$$(M, t) \vDash^{-} p \;\Leftrightarrow\; M(t, p) = 0$$

b) For a formula of the form $\exists F\alpha$, $\forall F\alpha$, etcetera, see Definition 2.6'

c) For any two temporal formulae φ and ψ:

(i) $(M, t) \vDash^{+} \varphi \wedge \psi \Leftrightarrow (M, t) \vDash^{+} \varphi$ and $(M, t) \vDash^{+} \psi$

$(M, t) \vDash^{-} \varphi \wedge \psi \Leftrightarrow (M, t) \vDash^{-} \varphi$ or $(M, t) \vDash^{-} \psi$

(ii) $\quad (\mathbf{M}, \mathbf{t}) \vDash^{+} \varphi \rightarrow \psi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{-} \varphi$ or $(\mathbf{M}, \mathbf{t}) \vDash^{+} \psi$

$\quad (\mathbf{M}, \mathbf{t}) \vDash^{-} \varphi \rightarrow \psi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{+} \varphi$ and $(\mathbf{M}, \mathbf{t}) \vDash^{-} \psi$

(iii) $\quad (\mathbf{M}, \mathbf{t}) \vDash^{+} \neg \varphi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{-} \varphi$

$\quad (\mathbf{M}, \mathbf{t}) \vDash^{-} \neg \varphi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{+} \varphi$

d) For any temporal formula φ:

$\quad (\mathbf{M}, \mathbf{t}) \nvDash^{+} \varphi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{+} \varphi$ does not hold

$\quad (\mathbf{M}, \mathbf{t}) \nvDash^{-} \varphi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \vDash^{-} \varphi$ does not hold

$\quad (\mathbf{M}, \mathbf{t}) \vDash^{u} \varphi \quad \Leftrightarrow \quad (\mathbf{M}, \mathbf{t}) \nvDash^{+} \varphi$ and $(\mathbf{M}, \mathbf{t}) \nvDash^{-} \varphi$

e) For a partial temporal model **M**, by $\mathbf{M} \vDash^{+} \varphi$ we mean $(\mathbf{M}, \mathbf{t}) \vDash^{+} \varphi$ for all $\mathbf{t} \in \mathbf{T}$ and by $\mathbf{M} \vDash^{+} \mathbf{K}$ we mean $\mathbf{M} \vDash^{+} \varphi$ for all $\varphi \in \mathbf{K}$, where **K** is a set of formulae possibly containing any of the defined operators. We will say that **M** is a model of the theory **K**.

f) A partial temporal model **M** of a theory **K** is called a *minimal* model of **K** if for every model **C** of **K** with $\mathbf{C} \leq \mathbf{M}$ it holds $\mathbf{C} = \mathbf{M}$.

Appendix B Proofs

In this Appendix we give proofs of results in Section 4. Proofs of results in the other sections can be found in [ET 93], [ET 94a], [ET 94b], [TRE 94].

Theorem 4.2

Let **PS** be any proof system and **K** any set of formulae of signature Σ and let $\mathbf{Th_{PS,K}}$ be the temporal theory $\mathbf{T_{PS}} \cup \mathbf{T_K} \cup \mathbf{C}'$. Let **M** be a minimal partial temporal model of $\mathbf{Th_{PS,K}}$. For any formula φ of signature Σ it holds

$$\mathbf{K} \vdash_{\mathbf{PS}} \varphi \iff \mathbf{M} \vDash^{+} \neg \mathbf{P}(\mathrm{T}) \rightarrow \exists \mathbf{F}(\mathbf{at}_{\varphi})$$

Proof

"$\Rightarrow$" Suppose $\mathbf{K} \vdash_{\mathbf{PS}} \varphi$ and suppose that $\psi_1, \ldots, \psi_{n-1}, \psi_n$, with $\psi_n = \varphi$, is a proof for φ. For a non-minimal element **t** in **T** it holds trivially that $(\mathbf{M}, \mathbf{t}) \vDash^{+} \neg \mathbf{P}(\mathbf{T}) \rightarrow \exists \mathbf{F}(\mathbf{at}_{\varphi})$, so let **r** be a minimal element in **T**. We shall prove the following by induction:

For every $1 \leq i \leq n$ there is a time point **s** reachable from **r** such that $\mathbf{at}_{\psi_1}, \ldots, \mathbf{at}_{\psi_i}$ are true in **M** at time point **s**.

$\mathbf{i = 1}$: ψ_1 has to be an element of **K** and as **M** is a model of $\mathbf{T_K}$, $\mathbf{at}_{\varphi}$ has to be true in **M** at time point **r**.

$\mathbf{i \rightarrow i + 1}$: suppose that **s** is a time point reachable from **r** and that $\mathbf{at}_{\psi_1}, \ldots, \mathbf{at}_{\psi_i}$ are true in **M** at time point **s**. If ψ_{i+1} is an element of **K** then the same argument as above yields that $\mathbf{at}_{\psi_{i+1}}$ must be true in **M** at point **s**, so assume that ψ_{i+1} is the result of applying a proof rule **PR** to a subset of the formulae $\psi_1, \ldots, \psi_i$ (say $\alpha_1, \ldots, \alpha_k$). Then there is a rule $\mathbf{C}(\mathbf{at}_{\alpha_1}) \wedge \ldots \wedge \mathbf{C}(\mathbf{at}_{\alpha_k}) \rightarrow \exists \mathbf{X}(\mathbf{at}_{\psi_{i+1}})$ in $\mathbf{T_{PS}}$ which has to be true in **M** at point s. As $at_{\alpha 1}, .., at_{\alpha k}$ are true in M at point s, there has to be a successor t to s in which $at_{\psi i+1}$ is true. The rules in C' ensure that $at_{\psi 1}, .., at_{\psi i}$ have to be true in M at point t too.

Taking n for i we have that there must be a point s reachable from r such that $at_{\psi n}$ is true in M at point s. It follows that $(M, s) \models^+ \neg P(T) \rightarrow \exists F(at_\varphi)$.

"$\Leftarrow$" Suppose there is a formula φ and a minimal element r such that $(M, r) \models^+ \exists F(at_\varphi)$ although $K \nvdash_{PS} \varphi$. Take the formulae φ at minimal depth, i.e. if s is a point at minimal depth for which $(M, s) \models^+ at_\varphi$, then there is no formula α such that there is a point t at smaller depth than s with $(M, t) \models^+ at_\alpha$ but $K \nvdash_{PS} \alpha$. As M is a minimal model of Th, if at_φ were undefined in M at point s, a formula in Th would become false. If this is a formula from T_K then it has to be the formula $C(at_\varphi)$, but then φ is in K and therefore $K \vdash_{PS} \varphi$. If it is a formula in C' then it must be the rule $P(at_\varphi) \rightarrow C(at_\varphi)$ at time point s. This means that at_φ is true in a point at smaller depth, which was not the case. Therefore it must be a rule of T_{PS}, say $C(at_{\alpha 1}) \wedge ... \wedge C(at_{\alpha k}) \rightarrow \exists X(at_\varphi)$ which will become false in a point t with $t < s$. But as $at_{\alpha 1}, .., at_{\alpha k}$ have to be true in M at point t and t is at smaller depth than s, we must have that $K \vdash_{PS} \alpha_1, .., K \vdash_{PS} \alpha_k$. But there is a proof rule in PS which can be applied to $\alpha_1, .., \alpha_k$ yielding φ, and therefore $K \vdash_{PS} \varphi$. This shows that such a formula can not exist.

Proposition 4.3

The temporal theory $Th_{PS,K}$ has a final model $F_{PS,K}$.

Proof

The theory $Th_{PS,K}$ consists of formulae which are forward persistent under any homomorphism (Proposition 3.3) and therefore by Theorem 3.5 a final model exists.