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Geometric Construction of Representations of Affine Algebras

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Abstract

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$. We consider Γ-fixed point sets in Hilbert schemes of points on the affine plane $\mathbf{C}^2$. The direct sum of homology groups of components has a structure of a representation of the affine Lie algebra $\widehat{\mathfrak{g}}$ corresponding to Γ. If we replace homology groups by equivariant K-homology groups, we get a representation of the quantum toroidal algebra $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$. We also discuss a higher rank generalization and character formulas in terms of intersection homology groups.

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1. Finite subgroups of $\mathrm{SL}_2(\mathrm{C})$ and simple Lie algebras

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$. The classification of such subgroups has been well-known to us, since they are essentially symmetry groups of regular polytopes. They are cyclic groups, binary dihedral groups, and binary polyhedral groups (Klein (1884)).

It has been also known that we can associate a complex simple Lie algebra $\mathfrak{g}$ to Γ. This can be done in two ways. The first one is geometric and due to DuVal (1934). The second one is algebraic, and is due to McKay (1979).

Let us explain the two constructions and subsequent developments briefly. More detailed account can be found in [17].

1.1. Minimal resolution of C^2/Γ

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Let us consider the quotient space $\mathbf{C}^2/\Gamma$. This space has an isolated singularity at the origin. We have a unique *minimal* resolution $\pi\colon M \to \mathbf{C}^2/\Gamma$, in the sense that all other resolutions factor through π. (For general singularities, we have many resolutions. This speciality occurs in 2-dimensional case.) This singularity is called a *simple singularity*, and has been intensively studied from various points of view. In particular, the following are known (see e.g., [2]):

1. The exceptional set $\pi^{-1}(0)$ consists of the union of projective lines.
2. We draw a diagram so that vertices correspond to projective lines (irreducible components) and two vertices are connected by an edge if they intersect. Then we obtain a Dynkin diagram of type ADE.

We thus have bijections

$$\left\{\text{irreducible components of } \pi^{-1}(0)\right\} \longleftrightarrow \{\text{vertices of the Dynkin diagram}\}.$$

The Dynkin diagram appears in the classification of simple Lie algebras. Thus we have a complex simple Lie algebra $\mathfrak{g}$ corresponding to Γ. Since vertices of the Dynkin diagram correspond to simple coroots of $\mathfrak{g}$, the above bijection gives an isomorphism (of vector spaces)

$$\mathfrak{h} \xrightarrow{\sim} H_2(\pi^{-1}(0), \mathbf{C}), \tag{1.1}$$

where $\mathfrak{h}$ is the complex Cartan subalgebra of $\mathfrak{g}$.

This correspondence $\Gamma \to \mathfrak{g}$ is based on the classification of simple Lie algebras since they attach a Dynkin diagram to Γ. So the reason why such a result holds remained misterious. A deeper connection between two objects were conjectured by Grothendieck, and obtained by Brieskorn (1970) and Slodowy (1980). They constructed the simple singularity $\mathbf{C}^2/\Gamma$ in $\mathfrak{g}$. Moreover, its semi-universal deformation and a simultaneous resolution were also constructed using geometry related to $\mathfrak{g}$. We do not recall their results here, so the interested reader should consult [40].

1.2. McKay correspondence

Let $\{\rho_i\}_{i\in I}$ be the set of (isomorphism classes of) irreducible representations of Γ. It has a special element ρ_0, the class of trivial representation. Let Q be the 2-dimensional representation given by the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$. Let us decompose $Q\otimes\rho_i$ into irreducibles, $Q\otimes\rho_i = \bigoplus_j a_{ij}\rho_j$, where a_{ij} is the multiplicity. We draw a diagram so that vertices correspond to ρ_i's, and there are a_{ij} edges between ρ_i and ρ_j. (Note that $a_{ij} = a_{ji}$ thanks to the self-duality of Q). Then McKay [26] observed that the graph is an affine Dynkin diagram of $\tilde{A}_n^{(1)}, \tilde{D}_n^{(1)}, \tilde{E}_6^{(1)}, \tilde{E}_7^{(1)}$ or $\tilde{E}_8^{(1)}$, i.e., the Dynkin diagram of an untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ attached to a simple Lie algebra $\mathfrak{g}$ of type ADE. Furthermore it is also known that the Dynkin diagram given in the previous subsection is obtained by the affine Dynkin diagram by removing the vertex corresponding to the trivial representation ρ_0. We thus have bijections

$$\{\text{irreducible representations of } \Gamma\} \longleftrightarrow \{\text{vertices of the affine Dynkin diagram}\}.$$

The original McKay's proof was based on the explicit calculation of characters. The reason why such a result holds remained misterious also in this case. A geometric explanation via the K-theory of the minimal resolution M of $\mathbf{C}^2/\Gamma$ was subsequently given by Gonzalez-Sprinberg and Verdier [14]. In particular, they proved that there exists a natural geometric construction of an isomorphism (of abelian groups)

$$R(\Gamma) \xrightarrow{\sim} K(M),$$

where $R(\Gamma)$ is the representation ring of Γ, and $K(M)$ is the Grothendieck group of the abelian category of algebraic vector bundles over M. This result is strengthened and generalized to the higher dimensional case $\Gamma \subset \mathrm{SL}_3(\mathbf{C})$ [5].

Note that the above isomorphism together with the Chern character homomorphism leads to an isomorphism $R(\Gamma) \otimes_{\mathbf{Z}} \mathbf{C} \xrightarrow{\sim} H^*(M, \mathbf{C})$, which gives an isomorphism

$$R(\Gamma) \otimes_{\mathbf{Z}} \mathbf{C} \xrightarrow{\sim} (\mathfrak{h} \oplus \mathbf{C}h_0)^*, \tag{1.2}$$

combined with (1.1). Here h_0 is the 0th simple coroot of the affine Lie algebra $\widehat{\mathfrak{g}}$, and corresponds to the dual of the trivial representation ρ_0. It corresponds to $H_0(M, \mathbf{C}) \cong H_0(\pi^{-1}(0), \mathbf{C})$.

Compared with correspondence in §1.1, our situation is less satisfactory: we only get $\mathfrak{h}$ and the role of $\mathfrak{g}$ or $\widehat{\mathfrak{g}}$ is less clear. This is the starting point of our whole construction. We construct $\widehat{\mathfrak{g}}$ entirely from Γ in some sense. For another approach, see [13].

2. Hilbert schemes of points and their Γ-fixed point components — quiver varieties

In 1986, Kronheimer [20] constructed a simple singularity $\mathbf{C}^2/\Gamma$, its deformation and simultaneous resolution, i.e., those spaces constructed by Brieskorn-Slodowy by a totally different method. His construction is based on the theory of 'quivers', which is a subject in noncommutative algebras. (See also [6] for a different approach.) Subsequently in 1989, Kronheimer and the author [21] gave a description of moduli spaces of instantons (and coherent sheaves) on those spaces in terms of a quiver. It is an analog of the celebrated ADHM description of instantons on S^4. In 1994, this description was further generalized under the name of 'quiver varieties' by the author [27]. The purpose of this and next sections is to define quiver varieties from a slightly different point of view. This is a most economical approach to introduce quiver varieties, while it does not explain why it is something to do with quivers.

Let $\mathrm{Hilb}^n(\mathbf{C}^2)$ be the Hilbert scheme of n points in the affine plane $\mathbf{C}^2$. As a set, it consists of ideals I of the polynomial ring $\mathbf{C}[x, y]$ such that the quotient $\mathbf{C}[x, y]/I$ has dimension n as a vector space. Grothendieck constructed $\mathrm{Hilb}^n(\mathbf{C}^2)$ as a quasi-projective scheme (for more general setting), but we do not go to this direction in detail. A typical point of $\mathrm{Hilb}^n(\mathbf{C}^2)$ is an ideal of functions vanishing at n distinct points in $\mathbf{C}^2$. The space parametrizing (unordered) n distinct points is an open subset of the nth symmetric product $S^n(\mathbf{C}^2) = (\mathbf{C}^2)^n/S_n$ of $\mathbf{C}^2$, where

S_n is the symmetric group of n letters acting on $(\mathbf{C}^2)^n$ by permutation of factors. The symmetric product parametrises unordered n points with multiplicities. The Hilbert scheme $\mathrm{Hilb}^n(\mathbf{C}^2)$ is a different completion of the open set. Two completions are related: Mapping I to its support counted with multiplicities, we get a morphism $\pi\colon \mathrm{Hilb}^n(\mathbf{C}^2) \to S^n(\mathbf{C}^2)$ which is called a *Hilbert-Chow morphism.* We have following important geometric results on $\mathrm{Hilb}^n(\mathbf{C}^2)$:

1. $\mathrm{Hilb}^n(\mathbf{C}^2)$ is a resolution of singularities of $S^n(\mathbf{C}^2)$ (Fogarty).
2. $\mathrm{Hilb}^n(\mathbf{C}^2)$ has a holomorphic symplectic structure (Beauville, Mukai).

In fact, the author constructed a hyper-Kähler structure on $\mathrm{Hilb}^n(\mathbf{C}^2)$, which induces Beauville-Mukai's symplectic form, by describing it as a hyper-Kähler quotient. See [29] and Göttsche's article in this ICM proceeding for more recent results on $\mathrm{Hilb}^n(\mathbf{C}^2)$.

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$ as above. Its natural action on $\mathbf{C}^2$ induces an action on $\mathrm{Hilb}^n(\mathbf{C}^2)$ and $S^n(\mathbf{C}^2)$ such that the Hilbert-Chow morphism π is Γ-equivariant. Let us consider the fixed point set $\mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma$, $(S^n(\mathbf{C}^2))^\Gamma$. The latter is easy to describe:

$$(S^n(\mathbf{C}^2))^\Gamma = S^m(\mathbf{C}^2/\Gamma),$$

where m is the largest integer less than or equal to $n/\#\Gamma$. The difference $n - m\#\Gamma$ is the multiplicity of the origin. The former space $\mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma$ is a union of nonsingular submanifolds of $\mathrm{Hilb}^n(\mathbf{C}^2)$. If $I \in \mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma$, the quotient $\mathbf{C}[x,y]/I$ has a structure of a representation of Γ. For an isomorphism class $\mathbf{v}$ of a representation of Γ, we define $M(\mathbf{v})$ as

$$M(\mathbf{v}) = \left\{ I \in \mathrm{Hilb}^n(\mathbf{C}^2)^\Gamma \;\middle|\; [\mathbf{C}[x,y]/I] = \mathbf{v} \right\},$$

where $[\mathbf{C}[x,y]/I]$ is the isomorphism class of $\mathbf{C}[x,y]/I$. Since isomorphism classes are parametrized by discrete data, i.e., dimensions of isotropic components, the isomorphism class of $[\mathbf{C}[x,y]/I]$ is constant on each connected component. Therefore $M(\mathbf{v})$ is a union of connected component. In fact, Crawley-Boevey recently proves that $M(\mathbf{v})$ is connected (in fact, he proved it for more general case including varieties discussed in next section) [9]. Moreover, $M(\mathbf{v})$ has induced holomorphic symplectic and hyper-Kähelr structures. It is an example of quiver varities of affine type. (See remark at the end of the next section.)

The simplest but nontrivial example is the case when $\mathbf{v}$ is the class of the regular representation of Γ. Under (1.2), the regular representation corresponds to the imaginary root δ, which is the positive generator of the kernel of the affine Cartan matrix, is identified with the dimension vector of the regular representation of Γ. The dimension of the regular representation is equal to $\#\Gamma$, and thus the fixed point set in the symmetric product is $(S^{\#\Gamma}(\mathbf{C}^2))^\Gamma = \mathbf{C}^2/\Gamma$. We can consider this as the space of Γ-orbits. A typical point is a free Γ-orbit, and is also a point in $\mathrm{Hilb}^{\#\Gamma}(\mathbf{C}^2)^\Gamma$ as the ideal vanishing at the orbit. In fact, it is not difficult to see that $M(\mathbf{v})$ is isomorphic to the minimal resolution M of $\mathbf{C}^2/\Gamma$. The resolution map $\pi\colon M \to \mathbf{C}^2/\Gamma$ is given by the restriction of the Hilbert-Chow morphism. This result was obtained by Ginzburg-Kapranov (unpublished) and Ito-Nakamura [17] independently, but is also a re-interpretation of Kronheimer's construction [20]. The precise explanation was given in [29, Chapter 4].

Recently higher dimensional $M(\mathbf{v})$ attract attention in connection with the McKay correspondence for wreath products $\Gamma \wr S_n$ [44, 22, 15]. These $M(\mathbf{v})$ are diffeomorphic to the Hilbert schemes of points on the minimal resolution $\operatorname{Hilb}^n M$.

3. A higher rank generalization of Hilbert schemes

We give a higher rank generalization of Hilbert schemes in this section. But geometric structures remain unchanged for general cases. So a reader, who wants to catch only a *rough* picture, could safely skip this section.

Let $\mathbf{P}^2$ be the projective plane with a fixed line ℓ_∞. So $\mathbf{P}^2 = \mathbf{C}^2 \sqcup \ell_\infty$. Let $\mathfrak{M}(n,r)$ be the framed moduli space of torsion free sheaves on $\mathbf{P}^2$ with rank r and $c_2 = n$, i.e. the set of isomorphism classes of pairs (E,φ), where E is a torsion free sheaf of $\operatorname{rank} E = r$, $c_2(E) = n$, which is locally free in a neighbourhood of ℓ_∞, and φ is an isomorphism $\varphi\colon E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$ (framing at infinity). It is known that this space has a structure of a quasi-projective variety [16]. This is a higher rank generalization of the Hilbert scheme $\operatorname{Hilb}^n(\mathbf{C}^2)$. The analog of $\mathbf{C}[x,y]/I$ is $H^1(\mathbf{P}^2, E(-1))$ and it is known that $H^0(\mathbf{P}^2, E(-1)) = H^2(\mathbf{P}^2, E(-1)) = 0$ [29, Chapter 2]. It is also known that $\mathfrak{M}(n,r)$ has a holomorphic symplectic (in fact, hyper-Kähler) structure [29, Chapter 3].

The higher rank generalization of the symmetric product $S^n(\mathbf{C}^2)$ is the so-called Uhlenbeck compactification of the framed moduli space of locally free sheaves. (On the other hand, $\mathfrak{M}(n,r)$ is called Gieseker-Maruyama compactification.) It is

$$\mathfrak{M}_0(n,r) = \bigsqcup_{n'+n''=n} \mathfrak{M}_0^{\mathrm{reg}}(n',r) \times S^{n''}\mathbf{C}^2,$$

where $\mathfrak{M}_0^{\mathrm{reg}}(n',r)$ is the open subset of $\mathfrak{M}(n',r)$ consisting of framed *locally free* sheaves (E,φ). It is known that $\mathfrak{M}_0(n,r)$ has a structure of an affine algebraic variety [10, Chapter 3]. Moreover, the map

$$(E,\varphi) \mapsto (E^{\vee\vee}, \varphi, \operatorname{Supp}(E^{\vee\vee}/E))$$

gives a projective morphism $\pi\colon \mathfrak{M}(n,r) \to \mathfrak{M}_0(n,r)$ [29, Chapter 3], where $E^{\vee\vee}$ is the double dual of E, which is locally free on surfaces, and $\operatorname{Supp}(E^{\vee\vee}/E)$ is the support of $E^{\vee\vee}/E$, counted with multiplicities.

When $r = 1$, there exists only one locally free sheaf which is trivial at ℓ_∞, i.e., the trivial line bundle $\mathcal{O}_{\mathbf{P}^2}$. So the first factor of the above disappears : $\mathfrak{M}_0(n,1) = S^n\mathbf{C}^2$. Moreover, for $E \in \mathfrak{M}(n,1)$, the double dual $E^{\vee\vee}$ must be the trivial line bundle by the same reason. It means that E is an ideal sheaf of the structure sheaf $\mathcal{O}_{\mathbf{P}^2}$, so is a point in the Hilbert scheme $\operatorname{Hilb}^n(\mathbf{C}^2)$. Thus we recover the situation studied in §2.

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbf{C})$ as before. We take and fix a lift of the Γ-action to $\mathcal{O}_{\ell_\infty}^{\oplus r}$. It is written as $W \otimes_{\mathbf{C}} \mathcal{O}_{\ell_\infty}$, where W is a representation

W of Γ. We denote by $\mathbf{w}$ the isomorphism class of W as before. Now Γ acts on $\mathfrak{M}(n,r)$, $\mathfrak{M}_0(n,r)$ and we can consider the fixed point sets $\mathfrak{M}(n,r)^\Gamma$, $\mathfrak{M}_0(n,r)^\Gamma$. We decompose the former as

$$\mathfrak{M}(n,r)^\Gamma = \bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v},\mathbf{w}),$$

where $\mathfrak{M}(\mathbf{v},\mathbf{w})$ consists of the framed torsion free sheaves (E,φ) such that the isomorphism class of $H^1(\mathbf{P}^2, E(-1))$, as a representation of Γ, is $\mathbf{v}$. Each $\mathfrak{M}(\mathbf{v},\mathbf{w})$, if it is nonempty, inherits a holomorphic symplectic and hyper-Kähler structures from $\mathfrak{M}(n,r)$.

Arbitrary quiver varieties of affine types with complex parameter equal to 0 are some $\mathfrak{M}(\mathbf{v},\mathbf{w})$. The identification with original definition was implicitly given in [29]. It was independently rediscovered by Lusztig [24]. See also [42]. Arbitrary quiver varieties of affine types with *nonzero* complex parameter are also important in representation theory [12], though we do not discuss here. Original definition of the varieties was given in terms of quivers. Later these were identified with framed moduli spaces of instantons on a noncommutative deformation $\mathbf{R}^4$ [36] or those of torsion free sheaves on a noncommutative deformation of $\mathbf{P}^2$ [18, 1].

4. Stratification and fibers of π

This technical section will be used to state character formulas later. A reader who only want to know only a *rough* picture can be skip this section.

We have the following stratification of $(S^n(\mathbf{C}^2))^\Gamma$ and its higher rank analog $\mathfrak{M}_0(n,r)^\Gamma$. The space $\mathfrak{M}_0(n,r)^\Gamma$ also decompose as

$$\begin{aligned}(S^n(\mathbf{C}^2))^\Gamma &= \bigsqcup_{m\le n} S^m_\lambda(\mathbf{C}^2/\Gamma),\\ \mathfrak{M}_0(n,r)^\Gamma &= \bigsqcup_{\substack{\mathbf{v}^0,\lambda\\ m+|\mathbf{v}^0|\le n}} \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w})\times S^m_\lambda(\mathbf{C}^2/\Gamma),\end{aligned} \tag{4.1}$$

where $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w})$ is defined exactly as above (it is possibly an empty set), $|\mathbf{v}^0|$ is the dimension of $\mathbf{v}^0$ as a complex vector space, $\lambda = (\lambda_1,\dots,\lambda_r)$ is a partition of m and

$$S^m_\lambda(\mathbf{C}^2/\Gamma) = \left\{ \sum_{i=1}^r \lambda_i[x_i] \in S^m(\mathbf{C}^2/\Gamma) \,\middle|\, x_i \ne 0 \text{ and } x_i \ne x_j \text{ for } i \ne j \right\}.$$

The differences $n-m$ and $n-(m+|\mathbf{v}^0|)$ are the multiplicity of the cycle at the origin 0.

Now it becomes clear that the case $(S^n(\mathbf{C}^2))^\Gamma$ is the special case of $\mathfrak{M}_0(n,r)^\Gamma$ with $\mathbf{w}=\rho_0$, $\mathbf{v}^0=0$. So from now, we only consider the second case.

For $x\in\mathfrak{M}_0(n,r)^\Gamma$, let $\mathfrak{M}(\mathbf{v},\mathbf{w})_x$ be the inverse image $\pi^{-1}(x)$ in $\mathfrak{M}(\mathbf{v},\mathbf{w})$. The most important one is the central fiber, i.e., the fiber over

$$x = (W\otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^2}, \varphi, 0).$$

In this case, we use the special notation $\mathfrak{L}(\mathbf{v},\mathbf{w})$. It is known that this is a Lagrangian subvariety of $\mathfrak{M}(\mathbf{v},\mathbf{w})$. Suppose that $x = (E_0, \varphi, C)$ is contained in the stratum $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v}^0,\mathbf{w}) \times S^m_\lambda(\mathbf{C}^2/\Gamma)$. Then the fiber $\mathfrak{M}(\mathbf{v},\mathbf{w})_x$ is a pure dimensional subvariety in $\mathfrak{M}(\mathbf{v},\mathbf{w})$, which is a product of $\mathfrak{L}(\mathbf{v}_s,\mathbf{w}_s)$ and copies of punctual Hilbert schemes $\mathrm{Hilb}_0^{\lambda_i}(\mathbf{C}^2)$ for some $\mathbf{v}_s$, $\mathbf{w}_s$. The proof of this statement in [27, §6], [31, §3] was given only when $m = 0$ and explained in terms of quivers, so we give more direct argument in our situation. The fiber $\mathfrak{M}(\mathbf{v},\mathbf{w})_x$ parametrises Γ-invariant subsheaves E of E_0 such that $\left[H^1(\mathbf{P}^2, E(-1))\right] = \mathbf{v}$ and $\operatorname{Supp} E_0/E = C$. Equivalently, it parametrises Γ-equivariant 0-dimensional quotients $E_0 \to Q$ such that $\left[H^0(\mathbf{P}^2, Q)\right] = \mathbf{v} - \mathbf{v}^0$ and $\operatorname{Supp} Q = C$. Such quotients depend only on a local structure on E_0, so we can replace E_0 by $W_s \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{P}^2}$, where W_s is the fiber of E_0 at the origin considered as a representation of Γ. The isomorphism class $\mathbf{w}_s$ of W_s is given by $\mathbf{w}_s = \mathbf{w} - \mathbf{C}\mathbf{v}^0$, where $\mathbf{C}$ is the class of the virtual representation $\bigwedge^0 Q - \bigwedge^1 Q + \bigwedge^2 Q = 2\rho_0 - Q$, and $\mathbf{C}\mathbf{v}^0$ means the tensor product $\mathbf{C} \otimes \mathbf{v}^0$. Therefore it becomes clear now that we have

$$\mathfrak{M}(\mathbf{v},\mathbf{w})_x \cong \mathfrak{L}(\mathbf{v} - \mathbf{v}^0 - m\delta, \mathbf{w}_s) \times \prod_i \mathrm{Hilb}_0^{\lambda_i}(\mathbf{C}^2),$$

where δ is considered as the class of the regular representation, and $\mathrm{Hilb}_0^{\lambda_i}(\mathbf{C}^2)$ is the punctural Hilbert scheme, i.e., the inverse image of $\lambda_i[0]$ by the Hilbert-Chow morphism $\pi\colon \mathrm{Hilb}^{\lambda_i}(\mathbf{C}^2) \to S^{\lambda_i}(\mathbf{C}^2)$. The punctural Hilbert schemes are known to be irreducible, thus $\mathfrak{M}(\mathbf{v},\mathbf{w})_x$ is pure-dimensional if and only if $\mathfrak{L}(\mathbf{v} - \mathbf{v}^0 - m\delta, \mathbf{w}_s)$ is so. But the latter statement is known [27, §5].

5. A geometric construction of the affine Lie algebra

After writing [21], the author tried to use this generalized ADHM description to study these varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$. But it turned out to be not so easy as he had originally hoped. When he struggled the problem, he heard a talk by Lusztig in ICM 90 Kyoto on a construction of canonical bases by using quivers. Lusztig's construction [23] was motivated by Ringel's construction [37] of the upper half part of the quantized enveloping algebra via the Hall algebra. The author thought that this construction should be useful to attack the problem. Two years later, he began to understand the picture. Quiver varieties $\mathfrak{M}(\mathbf{v},\mathbf{w})$ are, very *roughly*, cotangent bundles of varieties used by Ringel and Lusztig, and similar construction is possible [27]. A little later, he graduately realized that quiver varieties are also similar to cotangent bundles of flag varieties and the map π is an analog of Springer resolution. These varieties had been used to give geometric constructions of Weyl groups (Springer representations) and affine Hecke algebras (Deligne-Langlands conjecture). (See a beautifully written text book by N. Chriss and V. Ginzburg [8] and the references therein for these matereial.) The technique is the convolution product (see below) and works quite general. So he (and some others) conjectured that these construction should be adapted to quiver varieties. This conjecture turned out to be true [28, 31]. We explain the constructions in this and next sections. The relation between our constructions and Ringel-Lusztig construction was explained in [30] and will not be reproduced here.

5.1. Convolution algebra

We apply the theory of the convolution algebra to varieties introduced in the previous sections to obtain the universal enveloping algebra $\mathbf{U}(\widehat{\mathfrak{g}})$ of the affine algebra $\widehat{\mathfrak{g}}$.

We continue to fix a representation W of Γ and denote by $\mathbf{w}$ its isomorphism class. For the Γ-fixed point set of Hilbert schemes, studied in §2, W is the trivial representation.

We introduce the following notation:

$$\mathfrak{M}(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_n \mathfrak{M}(n,r)^\Gamma = \bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v},\mathbf{w}), \qquad \mathfrak{L}(\mathbf{w}) \overset{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v},\mathbf{w}),$$
$$\mathfrak{M}_0(\infty,\mathbf{w}) \overset{\text{def.}}{=} \bigcup_n \mathfrak{M}_0(n,r)^\Gamma.$$

The first and second are disjoint union. For the last, we use the inclusion $\mathfrak{M}_0(n,r)^\Gamma \subset \mathfrak{M}_0(n',r)^\Gamma$ for $n \le n'$ given by

$$(E,\varphi,C) \mapsto (E,\varphi,C+(n'-n)0)\,.$$

For $r=1$, these are

$$\bigsqcup_n \left(\operatorname{Hilb}^n(\mathbf{C}^2)\right)^\Gamma, \quad \bigcup_n (S^n\mathbf{C}^2)^\Gamma = \bigcup_n S^n(\mathbf{C}^2/\Gamma),$$

where the inclusion $S^n(\mathbf{C}^2/\Gamma) \subset S^{n'}(\mathbf{C}^2/\Gamma)$ is given by adding $(n'-n)0$ as above.

Rigorously speaking, we cannot study $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty,\mathbf{w})$ directly since they are infinite dimensional. We need to work individual spaces $\mathfrak{M}(\mathbf{v},\mathbf{w})$, $\mathfrak{M}_0(n,r)^\Gamma$. But we use those spaces as if they are finite dimensional spaces for a notational convenience.

We consider the fiber product

$$Z(\mathbf{w}) \overset{\text{def.}}{=} \mathfrak{M}(\mathbf{w}) \times_{\mathfrak{M}_0(\infty,\mathbf{w})} \mathfrak{M}(\mathbf{w}).$$

It consists of pairs (E,φ), (E',φ') such that

1. $E^{\vee\vee} \cong E'^{\vee\vee}$
2. $\operatorname{Supp} E^{\vee\vee}$ and $\operatorname{Supp} E'^{\vee\vee}$ are equal in the complement of the origin.

The multiplicities of $\operatorname{Supp} E^{\vee\vee}$ and $\operatorname{Supp} E'^{\vee\vee}$ at the origin may be different since we consider the inclusion above.

One can show that this is a lagrangian subvariety in $\mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$. (The same remark as $\mathfrak{M}(\mathbf{v},\mathbf{w})_x$ in §4 applies here also.) Let us consider its top degree Borel-Moore homology group

$$H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C}).$$

More precisely, it is the subspace of

$$\prod_{\mathbf{v}^1,\mathbf{v}^2} H_{\mathrm{top}}(Z(\mathbf{w}) \cap \left(\mathfrak{M}(\mathbf{v}^1,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w})\right), \mathbf{C})$$

consisting of elements $(F_{\mathbf{v}^1,\mathbf{v}^2})$ such that

1. for fixed $\mathbf{v}^1$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ for all but finitely many choices of $\mathbf{v}^2$,
2. for fixed $\mathbf{v}^2$, $F_{\mathbf{v}^1,\mathbf{v}^2} = 0$ for all but finitely many choices of $\mathbf{v}^1$.

The degree top depends on $\mathbf{v}^1$, $\mathbf{v}^2$, but we supress the dependency for brevity.

Let us consider the convolution product

$$*\colon H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C}) \otimes H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C}) \to H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C})$$

given by

$$c * c' = p_{13*}\left(p_{12}^*(c) \cap p_{23}^*(c')\right),$$

where p_{ij} is the projection from the triple product $\mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$ to the double product $\mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$ of the ith and jth factors. More detail for the definition of the convolution product, say p_{12}^*, $\cap$, is explained in [8], but we want to emphasize one point. The statement that the result $c * c'$ has top degree is the consequence of $\dim Z(\mathbf{w}) = \frac{1}{2}\dim \mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$. Although we are considering $Z(\mathbf{w})$ having infinitely many connected components, the convolution is well-defined and $H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C})$ is an associative algebra with unit, thanks to the above definition of the subspace in the direct product.

For $x \in \mathfrak{M}_0(\infty,\mathbf{w})$, let $\mathfrak{M}(\mathbf{w})_x$ be the inverse image $\pi^{-1}(x)$ in $\mathfrak{M}(\mathbf{w})$. We consider the top degree homology group

$$H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x,\mathbf{C})$$

which is the usual direct sum of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x \cap \mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbf{C})$ (unlike the case of $Z(\mathbf{w})$). The convolution product makes this space into a module of $H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C})$.

Theorem 5.1. *Let $\mathbf{U}(\widehat{\mathfrak{g}})$ be the universal enveloping algebra of the untwisted affine algebra $\widehat{\mathfrak{g}}$ corresponding to Γ.* (**NB**: *not a 'quantum' version*). *There exists an algebra homomorphism*

$$\mathbf{U}(\widehat{\mathfrak{g}}) \to H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C}).$$

Furthermore, if we consider $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x,\mathbf{C})$ as a $\mathbf{U}(\widehat{\mathfrak{g}})$-module via the homomorphism, it is an irreducible integrable highest weight representation and the direct summands $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x \cap \mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbf{C})$ are weight spaces.

This theorem was essentially proved in [27] with a modification for general x mentioned above.

The highest weight of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x,\mathbf{C})$ and weights of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x \cap \mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbf{C})$ are determined explicitly in terms of $\mathbf{v}$, $\mathbf{w}$ and the stratum to which x belongs. For example, in the case of the central fiber $\mathfrak{L}(\mathbf{v},\mathbf{w})$, the highest weight is $\mathbf{w}$, considered as a dominant integral weight as $\mathbf{w} = \sum_i w_i \Lambda_i$, where w_i is the ρ_i component of

$\mathbf{w}$, and Λ_i is the ith fundamental weight. Here we use the identification of the irreducible representation ρ_i and a vertex of the affine Dynkin diagram given by McKay correspondence. The weight of $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v},\mathbf{w}),\mathbf{C})$ is $\mathbf{w}-\mathbf{v}$, where $\mathbf{v}=\sum_i v_i\alpha_i$ with the ρ_i-component v_i of $\mathbf{v}$ and ith simple root α_i. The highest weight vector is the fundamental class $[\mathfrak{L}(0,\mathbf{w})]$, where $\mathfrak{M}(0,\mathbf{w})=\mathfrak{L}(0,\mathbf{w})$ consists of a single point $E=W\otimes_{\mathbf{C}}\mathcal{O}_{\mathbf{P}^2}$.

For the case studied in §2, $\mathbf{w}$ is the 0th fundamental weight Λ_0. The corresponding integrable highest weight representation is called the *basic* representation in literature. If we vary $\mathbf{w}$, we get all integrable highest weight representations as $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{L}(\mathbf{v},\mathbf{w}),\mathbf{C})$. It is worth while remarking that this is an extension of (1.1) since the Cartan subalgebra $\mathfrak{h}$ is naturally contained in the Cartan subalgebra. Furthermore, the finite dimensional Lie algebra $\mathfrak{g}$ is embedded in the basic representation, and we get

$$\mathfrak{g}\cong\bigoplus_{v_0=1} H_{\mathrm{top}}(M(\mathbf{v}),\mathbf{C}),$$

where v_0 is the ρ_0-isotropic component of $\mathbf{v}$. This is an extension of (1.1), mentioned before. In fact, it is easy to see that if $\mathbf{v}$ is not δ, then $M(\mathbf{v})$ is either empty, or a single point. The latter holds if and only if $\mathbf{v}$, considered as an element of $\mathfrak{h}^*$ by removing v_0, is a root of $\mathfrak{g}$.

If we fix $\mathbf{w}$ and vary x, we still obtain various integrable highest weight representations. The highest weight of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x,\mathbf{C})$ is $\mathbf{w}-\mathbf{v}^0-m\delta$, where $\mathbf{v}^0$, m are determined by x as in §4, and $\mathbf{w}$, $\mathbf{v}^0$ are considered as weights as above. The weight of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x\cap\mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbf{C})$ is equal to $\mathbf{w}-\mathbf{v}$. All of their highest weights are less than or equal to $\mathbf{w}$ with respect to the dominance order. In particular, when $\mathbf{w}=\Lambda_0$, those have highest weights $\Lambda_0-n\delta$ for some $n\in\mathbf{Z}_{\geq 0}$. They are essentially isomorphic to the basic representation.

We explain how the algebra homomorphism $\mathbf{U}(\widehat{\mathfrak{g}})\to H_{\mathrm{top}}(Z(\mathbf{w}),\mathbf{C})$ is defined. It is enough to define the image of Chevalley generators e_i, f_i, h_i ($i\in I$), d of $\mathbf{U}(\widehat{\mathfrak{g}})$ (and check the defining relations). The images of h_i and d are multiples of fundamental classes of diagonales in $\mathfrak{M}(\mathbf{w})\times\mathfrak{M}(\mathbf{w})$. More precisely, the multiple is determined so that the weight of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x\cap\mathfrak{M}(\mathbf{v},\mathbf{w}),\mathbf{C})$ is equal to $\mathbf{w}-\mathbf{v}$. The image of e_i is the fundamental classe of the so-called 'Hecke correspondence':

$$\bigsqcup_{\mathbf{v}}\{((E,\varphi),(E',\varphi'))\in\mathfrak{M}(\mathbf{v},\mathbf{w})\times\mathfrak{M}(\mathbf{v}+\rho_i,\mathbf{w})\mid E\subset E'\}. \tag{5.1}$$

It is known that each component is a nonsingular lagrangian subvariety of $\mathfrak{M}(\mathbf{v},\mathbf{w})\times\mathfrak{M}(\mathbf{v}+\rho_i,\mathbf{w})$. Hence it is an irreducible component of $Z(\mathbf{w})$. The image of f_i is given by swapping the first and second factors, up to sign.

As an application of the above construction, we get a base of $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_x,\mathbf{C})$ indexed by the irreducible components of $\mathfrak{M}(\mathbf{w})_x$. It has a structure of the crystal in the sense of Kashiwara, and is isomorphic to the crystal of the corresponding integrable highest weight module of the quantum affine algebra by Kashiwara-Saito [19, 38]. (See also [32] for a different proof.) However, the base itself is different from the specialization of the canonical (= global crystal) base of the quantum affine algebra module at $q = 1$. A counter example was found in [19]. The base given by irreducible components is named *semicanonical base* by Lusztig [25].

5.2. Lower degree homology groups

The construction in the previous subsection, in fact, gives us a structure of a representation of $\widehat{\mathfrak{g}}$ on

$$H_{\mathrm{top}\,-d}(\mathfrak{M}(\mathbf{w})_x, \mathbf{C})$$

for each fixed integer d. It is an integrable representation, and decomposes into irreducible representations. The multiplicity formula can be expressed in terms of the intersection cohomology thanks to Beilinson-Bernstein-Deligne-Gabber's decomposition theorem [3] applied to the morphism $\pi\colon \mathfrak{M}(\mathbf{w}) \to \mathfrak{M}_0(\infty, \mathbf{w})$. In our situation, π is a *semi-small* morphism, i.e., the restriction of π to the inverse image of the stratum (4.1) is a topological fiber bundle, and

$$2 \dim \mathfrak{M}(\mathbf{w})_x \le \operatorname{codim} \mathcal{O}_x,$$

where $\mathcal{O}_x$ is the stratum containing x. Then as observed by Borho-MacPherson [4], the decomposition theorem is simplified. We introduce several notation to describe the formula. We choose a point y from each stratum in (4.1). We denote the stratum containing y by $\mathcal{O}_y$. Let $IC(\mathcal{O}_y)$ is the intersection homology complex of $\mathcal{O}_y$ with respect to the trivial local system.

Theorem 5.2. *We have the following decomposition as a representation of* $\widehat{\mathfrak{g}}$:

$$H_{\mathrm{top}\,-d}(\mathfrak{M}(\mathbf{w})_x, \mathbf{C}) = \bigoplus_y H^{d+\dim \mathcal{O}_x}(i_x^! IC(\mathcal{O}_y)) \otimes H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_y, \mathbf{C}),$$

where $i_x\colon \{x\} \to \mathfrak{M}_0(\infty, \mathbf{w})$ *is the inclusion. Here* $\widehat{\mathfrak{g}}$ *acts trivially on the first factor of the right hand side.*

For a general semi-small morphism, we may have an intersection homology complex with respect to a *nontrivial* local system in the decomposition. In order to show that such a summand does not appear, the fact that $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w})_y, \mathbf{C})$ is a highest weight module plays a crucial role (see [31] for detail).

Note also that when $\mathbf{w} = \rho_0$, the closure of each stratum is a symmetric product of $\mathbf{C}^2/\Gamma$ (4.1). In particular, they only have finite quotient singularities, and their intersection homology complex are equal to the constant sheaf. Therefore our formula are simplified. One finds that $H_*(\mathfrak{L}(\mathbf{w}), \mathbf{C})$ is isomorphic to the so-called 'Fock space'. Later we will show that the total homology $H_*(\mathfrak{L}(\mathbf{w}), \mathbf{C})$ has a structure of a representation of a specialized quantum toroidal algebra in the next section. Then this observation generalizes a result [41, 39] for type $A_n^{(1)}$ to untwisted affine Lie algebras of type ADE.

6. Equivariant K-theory and quantum toroidal algebras

In this section, we replace the top degree homology group H_{top} in the previous section by equivariant K-groups. Then we obtain a geometric construction of a quantum toroidal algebra $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$. It is a q–analog of the loop algebra $\mathbf{L}\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}} \otimes_{\mathbf{C}} \mathbf{C}[z, z^{-1}]$ of the affine algebra $\widehat{\mathfrak{g}}$. Since $\widehat{\mathfrak{g}}$ is already a (central extension of) loop alebra of $\mathfrak{g}$, $\mathbf{L}\widehat{\mathfrak{g}}$ is a 'double-loop' algebra of $\mathfrak{g}$. The quantum toroidal algebra is defined by replacing $\mathfrak{g}$ by $\widehat{\mathfrak{g}}$ in the so-called Drinfeld realization of the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, which is a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, defined by Drinfeld, Jimbo.

Let $G_{\mathbf{w}} = \mathrm{Aut}_\Gamma(W)$ be the group of automorphisms of the Γ-module W. If w_i is the multiplicity of ρ_i in W, we have $G_{\mathbf{w}} \cong \prod_i \mathrm{GL}_{w_i}(\mathbf{C})$. We have a natural action of $G_{\mathbf{w}}$ on $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty, \mathbf{w})$ by the change of the framing:

$$\varphi \mapsto g \circ \varphi, \qquad g \in G_{\mathbf{w}}.$$

The projective morphism $\mathfrak{M}(\mathbf{w}) \to \mathfrak{M}_0(\infty, \mathbf{w})$ is equivariant.

Let $\mathbf{C}^*$ act on $\mathbf{C}^2$ by $t \cdot (x, y) = (tx, ty)$. It extends to an action on $\mathbf{P}^2$, where it acts trivially on ℓ_∞. Note that this action commutes with the Γ-action. Then we have a natural induced $\mathbf{C}^*$-action on $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty, \mathbf{w})$ so that the projection π is equivariant. Combining two actions we have an action of $G_{\mathbf{w}} \times \mathbf{C}^*$ on $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty, \mathbf{w})$. (This action is different from the action studied in [31] for type $A_1^{(1)}$. We need to change the definition of $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$ in that paper to apply the result in this section. This comes from the umbiguity of the definition of a q-analog of the Cartan matrix.)

Let $K^{G_{\mathbf{w}} \times \mathbf{C}^*}(Z(\mathbf{w}))$ be the equivariant K-homology group of $Z(\mathbf{w})$ with respect to the above $G_{\mathbf{w}} \times \mathbf{C}^*$-action. (More precisely, it should be defined as a subspace of the direct product as in the case of homology groups.) It is a module over the representation ring $R(G_{\mathbf{w}} \times \mathbf{C}^*) = \mathbf{Z}[q, q^{-1}] \otimes_{\mathbf{Z}} R(G_{\mathbf{w}})$, where q is the natural 1-dimensional representation of $\mathbf{C}^*$. The convolution product makes $K^{G_{\mathbf{w}} \times \mathbf{C}^*}(Z(\mathbf{w}))$ into a $R(G_{\mathbf{w}} \times \mathbf{C}^*)$-algebra. We divide its torsion part over $\mathbf{Z}[q, q^{-1}]$ and denote it by $K^{G_{\mathbf{w}} \times \mathbf{C}^*}(Z(\mathbf{w}))/\,\mathrm{torsion}$. (It is conjectured that the torsion is, in fact, 0.)

Theorem 6.1. *There exists a* $\mathbf{Z}[q, q^{-1}]$*-algebra homomorphism*

$$\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}}) \to K^{G_{\mathbf{w}} \times \mathbf{C}^*}(Z(\mathbf{w}))/\,\mathrm{torsion},$$

where $\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}})$ *is a certain* $\mathbf{Z}[q, q^{-1}]$*-subalgebra* (*conjecturally an integral form*) *of* $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$.

The definition of the homomorphism is similar to the case of homology groups. The image of the q-analog of $e_i \otimes z^r$ is given by a natural line bundles on the Hecke correspondence (5.1) whose fiber at $((E, \varphi), (E', \varphi'))$ is $H^0(E'/E)^{\otimes r}$.

Let us explain how we can use this algebra homomorphism to study representations of specialized quantum toroidal algebra $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}}) = \mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}})|_{q=\varepsilon}$, where ε is a nonzero complex number which may or may not be a root of unity. A natural generalization of finite dimensional representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ are l–integrable representations. (See [31] for the definition.) The Drinfeld-Chari-Pressley classification [11, 7] of irreducible finite dimensional reprentation of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ has a natural analog in $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}})$: Irreducible l–integrable representations of $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}})$ are parametrized by I–tuple of polynomials $P_i(u)$ with $P_i(0) = 1$, where I is the set of vertices of the affine Dynkin diagram.

Irreducible representations are obtained in the following way. Let us consider the equivariant homology $K^{G_{\mathbf{w}}\times\mathbf{C}^*}(\mathfrak{L}(\mathbf{w}))$, which is without torsion. It is a module of $\mathbf{U}_q^{\mathbf{Z}}(\mathbf{L}\widehat{\mathfrak{g}})$ and called a *universal standard module.* For a semisimple element $(s,\varepsilon) \in G_{\mathbf{w}} \times \mathbf{C}^*$, we consider the evaluation homomorphism $R(G_{\mathbf{w}} \times \mathbf{C}^*) \to \mathbf{C}$. Then the specialization

$$K^{G_{\mathbf{w}}\times\mathbf{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbf{C}^*)} \mathbf{C}$$

is a representation of $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}})$. This is called a *standard module.* It has a unique irreducible quotient, and the associated polynomials are the characteristic polynomials of components of s. (Recall $G_{\mathbf{w}} = \prod_{i\in I} \mathrm{GL}_{w_i}(\mathbf{C})$.)

In order to state character formulas, which is very similar to Theorem 5.2, we need a little more notation. Let A be the Zariski closure of powers of (s,ε) in $G_{\mathbf{w}} \times \mathbf{C}^*$. Let $\mathfrak{M}(\mathbf{w})^A$, $\mathfrak{M}_0(\infty,\mathbf{w})^A$, $Z(\mathbf{w})^A$ be the fixed point sets. We have a chain of natural algebra homomorphisms

$$\begin{aligned} K^{G_{\mathbf{w}}\times\mathbf{C}^*}(Z(\mathbf{w}))_{R(G_{\mathbf{w}}\times\mathbf{C}^*)}\mathbf{C} &\to K^A(Z(\mathbf{w}))_{R(A)}\mathbf{C} \\ &\to K(Z(\mathbf{w})^A) \otimes_{\mathbf{Z}} \mathbf{C} \to H_*(Z(\mathbf{w})^A, \mathbf{C}), \end{aligned}$$

where the first one is induced by the inclusion $A \subset G_{\mathbf{w}} \times \mathbf{C}^*$, the second one is given by the localization theorem in the equivariant K-theory, and the last one is the Chern character homomorphism. (In fact, we need ‘twists’ for the last two. See [8] for detail.)

There exists a natural stratification of $\mathfrak{M}_0(\infty,\mathbf{w})^A$ similar to (4.1). We choose a point y in each stratum and denote by $\mathcal{O}_y$ the stratum containing y. If $\mathfrak{M}(\mathbf{w})^A_y$ denotes the inverse image of y in $\mathfrak{M}(\mathbf{w})^A$ under π, the homology group $H_*(\mathfrak{M}(\mathbf{w})^A_y, \mathbf{C})$ is a representation of $H_*(Z(\mathbf{w})^A, \mathbf{C})$, and hence that of $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}})$. (When $y = 0$, it is the standard module.) Analog of Theorem 5.2 is the following:

Theorem 6.2. *We have the following in the Grothendieck group of the abelian category of l–integrable representations of* $\mathbf{U}_\varepsilon(\mathbf{L}\widehat{\mathfrak{g}})$

$$H_*(\mathfrak{M}(\mathbf{w})^A_x, \mathbf{C}) = \sum_y H^*(i_x^! IC(\mathcal{O}_y)) \otimes L_y,$$

where L_y is the unique irreducible quotient of $H_*(\mathfrak{M}(\mathbf{w})^A_y, \mathbf{C})$.

(The right hand side is an infinite sum, so we must understand it in an appropriate way. But it should be clear how it can be done.)

In this case, we apply the decomposition theorem to $\mathfrak{M}(\mathbf{w})^A \to \mathfrak{M}_0(\infty,\mathbf{w})^A$. It is *not* semi-small any more. So the degrees in the left and righ hand sides do not have clear relations.

Remarks 6.3. (1) We stated our results for the affine Lie algebra $\widehat{\mathfrak{g}}$ and the quantum toroidal algebra $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$. But they also hold for the finite dimensional Lie algebra $\widehat{\mathfrak{g}}$ and the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, if we impose the condition $v_0 = w_0 = 0$. It is known that $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is a Hopf algebra (since it is a subquotient of the quantum affine algebra), and the standard modules are isomorphic to tensor products of l–fundamental representations when ε is not a root of unity [43]. Here l–fundamental representations are irreducible representations corresponding to $\mathbf{w} = \rho_i$. In particular, the tensor product decomposition in the representation ring can be expressed in terms of intersection homology groups.

(2) Our remaining tasks are computing dimensions of $H^*(i_x^! IC(\mathcal{O}_y))$ appearing in Theorems 5.2, 6.2. In [33, 34] we gave a purely combinatorial algorithm to compute them. This algorithm can be made into a computer program. The algorithm was stated for the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$, but works also for $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$. This means that for any given stratum $\mathcal{O}_y$, $\dim H^*(i_x^! IC(\mathcal{O}_y))$ is, in principle, computable. However, it is practically difficult to compute because we need lots of memory. And, for $\mathbf{U}_q(\mathbf{L}\widehat{\mathfrak{g}})$, the summation is infinite. So having an algorithm to compute each term is not a strong statement. It is desirable to have an alternative method to compute them. That has be done in some special classes of representations [35].

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