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# On Weil-Petersson Volumes and Geometry of Random Hyperbolic Surfaces

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**Abstract**

This paper investigates the geometric properties of random hyperbolic surfaces with respect to the Weil-Petersson measure. We describe the relationship between the behavior of lengths of simple closed geodesics on a hyperbolic surface and properties of the moduli space of such surfaces. First, we study the asymptotic behavior of Weil-Petersson volumes of the moduli spaces of hyperbolic surfaces of genus $g$ as $g \to \infty$. Then we apply these asymptotic estimates to study the geometric properties of random hyperbolic surfaces, such as the length of the shortest simple closed geodesic of a given combinatorial type.

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**Keywords.** Moduli space, Weil-Petersson volume form, simple closed geodesic, hyperbolic surface

## 1. Introduction

The space of hyperbolic surfaces of a given genus is equipped with a natural notion of measure, which is induced by the *Weil-Petersson* symplectic form. We are interested in geometric properties of a random hyperbolic surface with respect to this measure. In particular, we are interested in the behavior of the length of the shortest separating/non-separating simple closed geodesic on a random surface of genus $g$ as $g \to \infty$.

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**Notation.** Let $\mathcal{M}_{g,n}$ be the moduli space of complete hyperbolic surfaces of genus $g$ with $n$ punctures. The universal cover of $\mathcal{M}_{g,n}$ is the Teichmüller space $\mathcal{T}_{g,n}$; every $X \in \mathcal{T}_{g,n}$ represents a *marked* hyperbolic structure on a surface of genus $g$ with $n$ punctures. The space $\mathcal{M}_{g,n}$ is a connected orbifold of dimension $6g-6+2n$, while $\mathcal{T}_{g,n}$ is homeomorphic to $\mathbb{R}^{3g-3+n} \times \mathbb{R}_+^{3g-3+n}$.

Every isotopy class of a closed curve on a hyperbolic surface contains a unique closed geodesic. Given a homotopy class of a closed curve $\alpha$ on a topological surface $S_{g,n}$ of genus $g$ with $n$ marked points and $X \in \mathcal{T}_{g,n}$, let $\ell_\alpha(X)$ be the length of the unique geodesic in the homotopy class of $\alpha$ on $X$. This defines a length function $\ell_\alpha$ on the Teichmüller space $\mathcal{T}_{g,n}$.

When studying the behavior of these length functions, it proves fruitful to consider more generally bordered hyperbolic surfaces with geodesic boundary components. Given $L = (L_1, \ldots, L_n) \in \mathbb{R}_+^n$, we consider the Teichmüller space $\mathcal{T}_{g,n}(L)$ of hyperbolic structures with geodesic boundary components of length $L_1, \ldots, L_n$. Note that a geodesic of length zero is the same as a puncture. The space $\mathcal{T}_{g,n}(L)$ is naturally equipped with a symplectic structure; this symplectic form $\omega = \omega_{wp}$ is called the Weil-Petersson symplectic form. When $L = 0$, this form is the symplectic form of a Kähler noncomplete metric on the moduli space $\mathcal{M}_{g,n}$ introduced by Weil [IT]. Wolpert showed that the Weil-Petersson symplectic form has a simple expression in terms of the Fenchel-Nielsen twist-length coordinates on the Teichmüller space (§2). As a result, there is a close relationship between the Weil-Petersson geometry and the lengths of simple closed geodesics on surfaces in $\mathcal{M}_g$.

**Our results.** In this paper, we present the following results, old and new:

1. In §2, following [M2] and [M1], we discuss a method to integrate *geometric* functions given in terms of the hyperbolic length functions over $\mathcal{M}_{g,n}$. This implies that the Weil-Petersson volume $V_{g,n}(L)$ of $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ is a polynomial in $L_1^2, \ldots, L_n^2$. The constant term of this polynomial, $V_{g,n} = V_{g,n}(0, \ldots, 0)$, is the Weil-Petersson volume of the moduli space of complete hyperbolic surfaces of genus $g$ with $n$ punctures. More generally, the coefficients of $V_{g,n}(L)$ can be written in terms of the intersection pairings of tautological line bundles over Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space.

2. Next, in §3, we study the asymptotic behavior of Weil-Petersson volumes, and the other coefficients of volume polynomials. In particular, we show that for $n \geq 0$
$$\lim_{g\to\infty} \frac{V_{g,n}}{V_{g-1,n+2}} = 1,$$
and
$$\lim_{g\to\infty} \frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2.$$

These results were predicted by Zograf [Z2]. We obtain several related estimates for the growth of the volumes of moduli spaces.

3. Finally, in §4, we describe the relationship between the asymptotic behavior of the Weil-Petersson volumes and the geometry of a random hyperbolic surface. In particular, we will see that in a typical hyperbolic surface of large genus, the shortest non-separating simple closed geodesic tends to be shorter than any separating simple closed geodesic. Further, we get lower bounds on the expected length of the shortest closed geodesic of a given type. For example, the shortest simple closed geodesic separating the surface into two roughly equal areas has expected length at least linear in $g$.

**Notation.** In this paper, $A \asymp B$ means that $A/C < B < AC$ for some universal constant $C$. Also, $A = O(B)$ means that $A < BC$, for some universal constant $C$.

**Notes and remarks.**

1. In [BM] Brooks and Makover developed a method for the study of *typical* Riemann surfaces with large genus by using trivalent graphs. In this model the expected value of the systole of a random Riemann surface turns out to be bounded (independent of the genus) [MM]. See also [Ga]. It seems that a random Riemann surface with respect to the Weil-Petersson volume form has some similar features. However, it is not clear how the measure induced by their model is related to the measure induced by the Weil-Petersson volume.

2. The distribution of hyperbolic surfaces of genus $g$ produced randomly by gluing Riemann surfaces with long geodesic boundary components is closely related to the measure induced by $\omega$ on $\mathcal{M}_{g,n}$. See [M4] for details.

3. The following exact asymptotic formula was proved in [MZ]. There exists $C > 0$ such that for any fixed $g \geq 0$

$$V_{g,n} = n! C^n n^{(5g-7)/2}(a_g + O(1/n)), \tag{1}$$

as $n \to \infty$.

Moreover, Zograf developed a fast algorithm for calculating the volume polynomials, and made the following conjecture on the basis of the numerical data obtained by his algorithm [Z2]:

**Conjecture 1.1** (Zograf). *For any fixed* $n \geq 0$

$$V_{g,n} = (4\pi^2)^{2g+n-3}(2g-3+n)! \frac{1}{\sqrt{g\pi}} \left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right)\right)$$

*as* $g \to \infty$.

4. We warn the reader that there are some small differences in the normalization of the Weil-Petersson volume form in the literature; in this paper,
$$V_{g,n} = V_{g,n}(0,\dots,0) = \frac{1}{(3g-3+n)!}\int_{\mathcal{M}_{g,n}} \omega^{3g-3+n}$$
which is slightly different from the notation used in [Z2] and [ST]. Also, in [Z1] the Weil-Petersson Kähler form is 1/2 the imaginary part of the Weil-Petersson pairing, while here the factor 1/2 does not appear. So our answers are different by a power of 2.

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## 2. Weil-Petersson Measure on $\mathcal{M}_{g,n}$

First, we briefly recall some background material and constructions in Teichmüller theory of Reimann surfaces with geodesic boundary components. For further details see [IT], [M2] and [Bu].

**Teichmüller Space.** A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface $X$ equipped with a diffeomorphism $f : S \to X$. The map $f$ provides a *marking* on $X$ by $S$. Two marked surfaces $f : S \to X$ and $g : S \to Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When $\partial S$ is nonempty, consider hyperbolic Riemann surfaces homeomorphic to $S$ with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha\in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}(S, L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have
$$\ell_\beta(X) = L_\beta.$$
Let $S_{g,n}$ be an oriented connected surface of genus $g$ with $n$ boundary components $(\beta_1,\dots,\beta_n)$. Then
$$\mathcal{T}_{g,n}(L_1,\dots,L_n) = \mathcal{T}(S_{g,n}, L_1,\dots,L_n),$$
denote the Teichmüller space of hyperbolic structures on $S_{g,n}$ with geodesic boundary components of length $L_1,\dots,L_n$. By convention, a geodesic of length zero is a cusp and we have
$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0,\dots,0).$$

Let $\mathrm{Mod}(S)$ denote the mapping class group of $S$, or the group of isotopy classes of orientation preserving self homeomorphisms of $S$ leaving each boundary component setwise fixed. The mapping class group $\mathrm{Mod}_{g,n} = \mathrm{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \ldots, L_n)/\operatorname{Mod}_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with $n$ boundary components of length $\ell_{\beta_i} = L_i$. Also, we have

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0).$$

For a disconnected surface $S = \bigcup_{i=1}^{k} S_i$ such that $A_i = \partial S_i \subset \partial S$, we have

$$\mathcal{M}(S, L) = \prod_{i=1}^{k} \mathcal{M}(S_i, L_{A_i}),$$

where $L_{A_i} = (L_s)_{s \in A_i}$.

**The Weil-Petersson symplectic form.** By work of Goldman [Go], the space $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called the *Weil-Petersson symplectic form,* and denoted by $\omega$ or $\omega_{wp}$. We investigate the volume of the moduli space with respect to the volume form induced by the Weil-Petersson symplectic form. Also, when $S$ is disconnected, we have

$$\mathrm{Vol}(\mathcal{M}(S, L)) = \prod_{i=1}^{k} \mathrm{Vol}(\mathcal{M}(S_i, L_{A_i})).$$

When $L = 0$, there is a natural complex structure on $\mathcal{T}_{g,n}$, and this symplectic form is in fact the Kähler form of a Kähler metric [IT].

**The Fenchel-Nielsen coordinates.** A *pants decomposition* of $S$ is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^{k}$, where $k = 6g-6+2n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with $\mathcal{P}$, $\{\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_k}(X)\}$, consists of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism

$$\mathcal{T}_{g,n}(L) \cong \mathbb{R}_{+}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \to (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

See [Bu] for more details.

By work of Wolpert, over Teichmüller space the Weil-Petersson symplectic structure has a simple form in Fenchel-Nielsen coordinates [W1].

**Theorem 2.1** (Wolpert). *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

Given a simple closed geodesic $\alpha$ on $X$ and $t \in \mathbb{R}$, we can deform the hyperbolic structure of $X$ by a right twist along $\alpha$ as follows. First, cut $X$ along $\alpha$ and then reglue back after twisting distance t to the right. We observe that the hyperbolic structure of the complement of the cut extends to a new hyperbolic structure on $S$. The resulting continuous path in Teichmüller space is the Fenchel-Nielsen deformation of $X$ along $\alpha$. By Theorem 2.1 the vector field generated by twisting around is symplectically dual to the exact one-form $d\ell_\alpha$. In other words, the natural twisting around $\alpha$ is the Hamiltonian flow of the length function of $\alpha$.

**Integrating geometric functions over moduli spaces.** Here, we develop a method for integrating certain geometric functions over $\mathcal{M}_{g,n}(L)$. Working with bordered Riemann surfaces allows us to exploit the existence of commuting Hamiltonian $S^1$-actions on certain coverings of the moduli space in order to integrate certain geometric functions over the moduli space of curves.

Let $S_{g,n}$ be a closed surface of genus $g$ with $n$ boundary components and let $Y \in \mathcal{T}_{g,n}$. For a simple closed curve $\gamma$ on $S_{g,n}$, let $[\gamma]$ denote the homotopy class of $\gamma$ and let $\ell_\gamma(Y)$ denote the hyperbolic length of the geodesic representative of $[\gamma]$ on $Y$. To each simple closed curve $\gamma$ on $S_{g,n}$, we associate the set

$$\mathcal{O}_\gamma = \{[\alpha] \mid \alpha \in \mathrm{Mod}_{g,n} \cdot \gamma\}$$

of homotopy classes of simple closed curves in the $\mathrm{Mod}_{g,n}$-orbit of $\gamma$ on $X \in \mathcal{M}_{g,n}$. Given a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, and a multicurve $\gamma$ on $S_{g,n}$ define

$$f_\gamma : \mathcal{M}_{g,n} \to \mathbb{R}$$

by

$$f_\gamma(X) = \sum_{[\alpha] \in \mathcal{O}_\gamma} f(\ell_\alpha(X)). \tag{2}$$

The main idea for integrating over $\mathcal{M}_{g,n}^\gamma$ is that the decomposition of the surface along $\gamma$ gives rise to a description of $\mathcal{M}_{g,n}^\gamma$ in terms of moduli spaces corresponding to simpler surfaces. This leads to formulas for the integral of $f_\gamma$ in terms of the Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces and the function $f$.

We sketch the proof for the case where $\gamma$ is a connected simple closed curve. See Theorem 2.2 for the general case.

First, consider the covering space of $\mathcal{M}_{g,n}$

$$\pi^\gamma : \mathcal{M}_{g,n}^\gamma = \{(X, \alpha) \mid X \in \mathcal{M}_{g,n}, \text{and } \alpha \in \mathcal{O}_\gamma \text{ is a geodesic on } X\} \to \mathcal{M}_{g,n},$$

where $\pi^\gamma(X,\alpha) = X$. The hyperbolic length function descends to the function,

$$\ell : \mathcal{M}_{g,n}^{\gamma} \to \mathbb{R}$$

defined by $\ell(X,\eta) = \ell_\eta(X)$. Therefore, we have

$$\int\limits_{\mathcal{M}_{g,n}} f_\gamma(X)\, dX = \int\limits_{\mathcal{M}_{g,n}^{\gamma}} f \circ \ell(Y)\, dY.$$

On the other hand, the function $f$ is constant on each level set of $\ell$ and we have

$$\int\limits_{\mathcal{M}_{g,n}^{\gamma}} f \circ \ell(Y)\, dY = \int\limits_{0}^{\infty} f(t)\,\mathrm{Vol}(\ell^{-1}(t))\, dt,$$

where the volume is taken with respect to the volume form $- * d\ell$ on $\ell^{-1}(t)$.

Let $S_{g,n}(\gamma)$ be the result of cutting the surface $S_{g,n}$ along $\gamma$; that is $S_{g,n}(\gamma) \cong S_{g,n} - U_\gamma$, where $U_\gamma$ is an open neighborhood of $\gamma$ homeomorphic to $\gamma \times (0,1)$. Thus $S_{g,n}(\gamma)$ is a possibly disconnected compact surface with $n+2$ boundary components. We define $\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$ to be the moduli space of Riemann surfaces homeomorphic to $S_{g,n}(\gamma)$ such that the lengths of the 2 boundary components corresponding to $\gamma$ are equal to $t$. We have a natural circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & \ell^{-1}(t) \subset \mathcal{M}_{g,n}^{\gamma} \\ & & \big\downarrow \\ & & \mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t) \end{array}$$

We will study the $S^1$-action on the level set $\ell^{-1}(t) \subset \mathcal{M}_{g,n}^{\gamma}$ induced by twisting the surface along $\gamma$. The quotient space $\ell^{-1}(t)/S^1$ inherits a symplectic form from the Weil-Petersson symplectic form. On the other hand, $\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$ is equipped with the Weil-Petersson symplectic form. By investigating these $S^1$-actions in more detail, one can show that

$$\ell^{-1}(t)/S^1 \cong \mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)$$

as symplectic manifolds. Therefore, we have

$$\mathrm{Vol}(\ell^{-1}(t)) = t\ \mathrm{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t)).$$

For any connected simple closed curve $\gamma$ on $S_{g,n}$, we have

$$\int_{\mathcal{M}_{g,n}} f_\gamma(X)\, dX = \int\limits_{0}^{\infty} f(t)\, t\ \mathrm{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\gamma = t))\, dt. \tag{3}$$

In general, we have ([M2]):

**Theorem 2.2.** *For any multicurve* $\gamma = \sum_{i=1}^{k} c_i\gamma_i$, *the integral of* $f_\gamma$ *over* $\mathcal{M}_{g,n}(L)$ *with respect to the Weil-Petersson volume form is given by*

$$\int_{\mathcal{M}_{g,n}(L)} f_\gamma(X)\, dX = \frac{2^{-M(\gamma)}}{|\operatorname{Sym}(\gamma)|} \int_{\mathbf{x}\in\mathbb{R}_+^k} f(|\mathbf{x}|)\, V_{g,n}(\Gamma, \mathbf{x}, \beta, L)\, \mathbf{x}\cdot d\mathbf{x},$$

*where* $\Gamma = (\gamma_1, \ldots, \gamma_k)$, $|\mathbf{x}| = \sum_{i=1}^{k} c_i\, x_i$, $\mathbf{x}\cdot d\mathbf{x} = x_1 \cdots x_k \cdot dx_1 \wedge \cdots \wedge dx_k$, *and*

$$M(\gamma) = |\{i | \gamma_i \quad \textit{separates off a one-handle from } \; S_{g,n}\}|.$$

Given a multicurve $\gamma = \sum_{i=1}^{k} c_i\gamma_i$, the symmetry group of $\gamma$, $\operatorname{Sym}(\gamma)$, is defined by

$$\operatorname{Sym}(\gamma) = \operatorname{Stab}(\gamma)/\cap_{i=1}^{k} \operatorname{Stab}(\gamma_i).$$

Recall that given $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$, $V_{g,n}(\Gamma, \mathbf{x}, \beta, L)$ is defined by

$$V_{g,n}(\Gamma, \mathbf{x}, \beta, L) = \operatorname{Vol}(\mathcal{M}(S_{g,n}(\gamma), \ell_\Gamma = \mathbf{x}, \ell_\beta = L)).$$

Also,

$$V_{g,n}(\Gamma, \mathbf{x}, \beta, L) = \prod_{i=1}^{s} V_{g_i,n_i}(\ell_{A_i}),$$

where

$$S_{g,n}(\gamma) = \bigcup_{i=1}^{s} S_i\ , \tag{4}$$

$S_i \cong S_{g_i,n_i}$, and $A_i = \partial S_i$.

By Theorem 2.2 integrating $f_\gamma$, even for a compact Riemann surface, reduces to the calculation of volumes of moduli spaces of bordered Riemann surfaces. This formula can be used to relate the growth of the number of simple closed geodesics on $X \in \mathcal{M}_g$ to the volume polynomials [M3].

**Remark.** Let $g \in \operatorname{Sym}(\gamma)$, where $\gamma = \sum_{i=1}^{k} c_i\gamma_i$. Then $g(\gamma_i) = \gamma_j$ implies that $c_i = c_j$.

**Connection with the intersection pairings of tautological line bundles.** The moduli space $\mathcal{M}_{g,n}$ is endowed with natural cohomology classes. An example of such a class is the Chern class of a vector bundle on the moduli space. When $n > 0$, there are $n$ tautological line bundles defined on $\overline{\mathcal{M}}_{g,n}$ as follows. For each marked point $i$, there exists a canonical line bundle $\mathcal{L}_i$ in the orbifold sense whose fiber at the point $(C, x_1, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of $C$ at $x_i$. The first Chern class of this bundle is denoted by $\psi_i = c_1(\mathcal{L}_i)$. Note that although the complex curve $C$ may have nodes, $x_i$ never coincides with the singular points. For any set $\{d_1, \ldots, d_n\}$ of integers, define the top intersection number of $\psi$ classes by

$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \quad .$$

Such products are well defined when the $d_i'$s are non-negative integers and $\sum_{i=1}^{n} d_i = 3g - 3 + n$. In other cases $\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g$ is defined to be zero. Since we are in the orbifold setting, these intersection numbers are rational numbers. See [HM] and [AC] for more details. In [M1], we use the symplectic geometry of moduli spaces of bordered Riemann surfaces to relate these intersection pairings to the volume polynomials. This method allows us to read off the intersection numbers of tautological line bundles from the volume polynomials:

**Theorem 2.3.** *In terms of the above notation,*

$$\mathrm{Vol}(\mathcal{M}_{g,n}(L_1, \ldots, L_n)) = \sum_{|\mathbf{d}| \leq 3g-3+n} C_g(\mathbf{d}) \;\; L_1^{2d_1} \ldots L_n^{2d_n},$$

*where* $\mathbf{d} = (d_1, \ldots, d_n)$*, and* $C_g(\mathbf{d})$ *is equal to*

$$\frac{2^{m(g,n)|\mathbf{d}|}}{2^{|\mathbf{d}|} \; |\mathbf{d}|! \; (3g-3+n-|\mathbf{d}|)!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \cdot \omega^{3g-3+n-|\mathbf{d}|}.$$

*Here* $m(g,n) = \delta(g-1) \times \delta(n-1)$*,* $\mathbf{d}! = \prod_{i=1}^{n} d_i!$*, and* $|\mathbf{d}| = \sum_{i=1}^{n} d_i$*.*

**Recursive formulas for volume polynomials.** We approach the study of the volumes of $\mathcal{M}_{g,n}(L)$ via the length functions of simple closed geodesics on a hyperbolic surface in $\mathcal{T}_{g,n}(L)$. Our point of departure for calculating these volume polynomials is a result due to McShane [Mc] which gives an identity for the lengths of certain types of simple closed geodesics on a surface $X \in \mathcal{M}_{g,n}$ when $n > 0$. Here we just cite the simplest case of this identity for $g = n = 1$.

Let $X \in \mathcal{T}_{1,1}$ be a hyperbolic once-punctured torus. Then we have

$$\sum_{\gamma} (1 + e^{\ell_\gamma(X)})^{-1} = \frac{1}{2},$$

where the sum is over all simple closed geodesics $\gamma$ on $X$. Note that the left hand side of this identity is a *geometric function* for $f(t) = 1/(1+e^t)$ in the sense of (2), and the right hand side is independent of $X$.

This identity can be generalized to hyperbolic surfaces with finitely many geodesic boundary components or cone singularities [TWZ2]. In [LM], Labourie and McShane generalize the length identities to arbitrary cross ratios; as a result they obtain new identities for the Hitchin representations of surface groups in $SL(n, \mathbb{R})$. For further generalization of these identities see [TWZ1] and [Bo].

**Remark.** A recursive formula for the Weil-Petersson volume of the moduli space of punctured spheres was obtained by Zograf [Z1]. Moreover, Zograf and Manin have obtained generating functions for the Weil-Petersson volume of $\mathcal{M}_{g,n}$[MZ]. See also [KMZ]. Penner has developed a different method for calculating the Weil-Petersson volume of the moduli spaces of curves with marked points by using decorated Teichmüller theory [Pe].

# 3. Asymptotic Behavior of Weil-Petersson Volumes and Tautological Intersection Pairings

In this section, we study the asymptotics behavior of the Weil-Petersson volume of $\mathcal{M}_{g,n}$ as $g \to \infty$.

It is known [Gr] that for a fixed $n > 0$ there are $c_1, c_2 > 0$ such that

$$c_2^g (2g)! < \mathrm{Vol}(\mathcal{M}_{g,n}) < c_1^g (2g)!.$$

This result was extended to the case of $n = 0$ in [ST]. However, these estimates do not give much information about the growth of

$$B_{g,n} = V_{g,n}/V_{g-1,n+2}$$

and

$$C_{g,n} = V_{g,n+1}/(2gV_{g,n})$$

when $g \to \infty$.

**Notation.** For $\mathbf{d} = (d_1, \dots, d_n)$ with $d_i \in \mathbb{N} \cup \{0\}$ and $|\mathbf{d}| = d_1 + \ldots + d_n \leq 3g - 3 + n$, let $d_0 = 3g - 3 - |\mathbf{d}|$ and define

$$\left[\prod_{i=1}^{n} \tau_{d_i}\right]_{g,n} = \frac{\prod_{i=1}^{n}(2d_i+1)!2^{|\mathbf{d}|}}{\prod_{i=0}^{n} d_i!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \omega^{d_0} =$$

$$= \frac{(2\pi^2)^{d_0} \prod_{i=1}^{n}(2d_i+1)!!2^{2|d|}}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_1^{d_0},$$

where $\kappa_1 = \frac{\omega}{2\pi^2}$ is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$ [AC]. By Theorem 2.3 for $L = (L_1, \dots, L_n)$ we have:

$$V_{g,n}(2L) = \sum_{|\mathbf{d}| \leq 3g-3+n} [\tau_{d_1}, \dots \tau_{d_n}]_{g,n} \frac{L_1^{2d_1}}{(2d_1+1)!} \cdots \frac{L_n^{2d_n}}{(2d_n+1)!}. \tag{5}$$

**Some useful recursive formulas for the intersection pairings.** Given $\mathbf{d} = (d_1, \ldots, d_n)$ with $|\mathbf{d}| \leq 3g - 3 + n$, the following recursive formulas hold:

**I.**

$$\left[\tau_0 \tau_1 \prod_{i=1}^{n} \tau_{d_i}\right]_{g,n+2} = \left[\tau_0^4 \prod_{i=1}^{n} \tau_{d_i}\right]_{g-1,n+4} +$$

$$+\frac{1}{2} \sum_{\substack{g_1+g_2=g \\ \{1,\ldots,n\}=I \amalg J}} \left[\tau_0^2 \prod_{i\in I} \tau_{d_i}\right]_{g_1,|I|+2} \cdot \left[\tau_0^2 \prod_{i\in J} \tau_{d_i}\right]_{g_2,|J|+2},$$

**II.**

$$(2g-2+n)\left[\prod_{i=1}^{n} \tau_{d_i}\right]_{g,n} = \frac{1}{2} \sum_{L=0}^{3g-3+n} (-1)^L (L+1) \frac{\pi^{2L}}{(2L+3)!} \left[\tau_{L+1} \prod_{i=1}^{n} \tau_{d_i}\right]_{g,n+1}.$$

**III.** Let $a_0 = 1/2$, and for $n \geq 1$,

$$a_n = \zeta(2n)(1 - 2^{1-2n}).$$

Then we have

$$[\tau_{d_1}, \ldots, \tau_{d_n}]_{g,n} = \sum_{j=2}^{n} \mathcal{A}_{\mathbf{d}}^j + \frac{1}{2}\mathcal{B}_{\mathbf{d}} + \frac{1}{2}\mathcal{C}_{\mathbf{d}},$$

where

$$\mathcal{A}_{\mathbf{d}}^j = \sum_{L=0}^{d_0} (2d_j + 1)\; a_L \left[\tau_{d_1+d_j+L-1}, \prod_{i\neq 1,j} \tau_{d_i}\right]_{g,n-1},$$

$$\mathcal{B}_{\mathbf{d}} = \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L \left[\tau_{k_1}\tau_{k_2} \prod_{i\neq 1} \tau_{d_i}\right]_{g-1,n+1},$$

and

$$\mathcal{C}_{\mathbf{d}} = \sum_{\substack{I \amalg J=\{2,\ldots,n\} \\ 0\leq g'\leq g}} \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L \left[\tau_{k_1} \prod_{i\in I} \tau_{d_i}\right]_{g',|I|+1} \times [\tau_{k_2} \prod_{i\in J} \tau_{d_i}]_{g-g',|J|+1}.$$

**Remarks**.

- Formula (**I**) is a special case of Proposition 3.3 in [LX1]. For different proofs of (**II**) see [DN] and [LX1]. The proof presented in [DN] uses the properties of moduli spaces of of hyperbolic surfaces with cone points. See also [KMZ] and [AC].

- In terms of the volume polynomials (**II**) can be written as ([DN]):

$$\frac{\partial V_{g,n+1}}{\partial L}(L, 2\pi i) = 2\pi i(2g-2+n)V_{g,n}(L).$$

When $n = 0$,

$$V_{g,1}(2\pi i) = 0,$$

and

$$\frac{\partial V_{g,1}}{\partial L}(2\pi i) = 2\pi i(2g-2)V_g. \tag{6}$$

Note that (**III**) applies only when $n > 0$. In the case of $n = 0$, (6) allows us to prove necessary estimates for the growth of $V_{g,0}$.

- Although (**III**) has been described in purely combinatorial terms, it is closely related to the topology of different types of pairs of pants in a surface. In fact, this formula gives us the volume of $\mathcal{M}_{g,n}(L)$ in terms of volumes of moduli spaces of Riemann surfaces that we get by removing a pair of pants containing at least one boundary component of $S_{g,n}$.

- If $d_1 + \ldots + d_n = 3g - 3 + n$, (**III**) gives rise to a recursive formula for the intersection pairings of $\psi_i$ classes which is the same as the Virasoro constraints for a point. This result is equivalent to the Witten-Kontsevich formula [M2], [LX2]. See also [MS]. For different proofs and discussions related to these relations see [Wi], [Ko], [OP], [M1], [KL], and [EO]. In this paper, we are mainly interested in the intersection parings only containing $\kappa_1$ and $\psi_i$ classes. For generalizations of (**III**) to the case of higher Mumford's $\kappa$ classes see [LX1] and [E].

**Basic general estimates.** The main advantage of using (**III**) is that all the coefficients are positive. Moreover, it is easy to check that

$$\zeta(2n)(1-2^{1-2n}) = \frac{1}{(2n-1)!}\int_0^\infty \frac{t^{2n-1}}{1+e^t}\, dt.$$

Hence,

$$a_{n+1} - a_n = \int_0^\infty \frac{1}{(1+e^t)^2}\left(\frac{t^{2n+1}}{(2n+1)!} + \frac{t^{2n}}{2n!}\right) dt.$$

As a result, we have:

1. $\{a_n\}_{n=1}^\infty$ is an increasing sequence, and $\lim_{n\to\infty} a_n = 1$;

2. $$a_{n+1} - a_n \asymp 1/2^{2n}. \tag{7}$$

Using this observation one can prove the following general estimates:

- For any $\mathbf{d} = (d_1, \ldots, d_n)$

$$[\tau_{d_1}, \ldots, \tau_{d_n}]_{g,n} \leq [\tau_0, \ldots, \tau_0]_{g,n} = V_{g,n}.$$

- Then (5) implies that

$$V_{g,n}(2L_1, \ldots, 2L_n) \leq e^L V_{g,n}, \tag{8}$$

where $L = L_1 + \ldots + L_n$.

- Moreover, since

$$[\tau_1, \tau_0, \ldots, \tau_0] \leq V_{g,n}$$

(**I**) and (**II**) for $\mathbf{d} = 0$ imply that for any $g, n \geq 0$,

$$V_{g,n+2} \geq V_{g-1,n+4}, \quad \text{and} \quad b \cdot V_{g,n+1} > (2g-2+n)V_{g,n}, \tag{9}$$

where $b = \sum_{L=0}^{\infty} \pi^{2L}(L+1)/(2L+3)!$.

**Remark.** We will show that as $g \to \infty$ the first inequality of (9) is asymptotically sharp. However, (1) implies that when $g$ is fixed and $n$ is large this inequality is far from being sharp; in fact, given $g \geq 1$ as $n \to \infty$

$$V_{g,n+2} \asymp \sqrt{n}\, V_{g-1,n+4}.$$

**Asymptotic behavior of the coefficients of volume polynomials.** Let $n \geq 0$. The following estimates hold:

- Combining (9) and (7) with a more careful analysis of (**III**) implies that for any $k \in \mathbb{N}$

$$\frac{[\tau_k, \tau_0, \ldots, \tau_0]_{g,n}}{V_{g,n}} = 1 + O(1/g), \tag{10}$$

as $g \to \infty$.

This is a special case of Conjecture 2 in [Z2]. However, (10) fails if $k$ is not very small compare to $g$.

- Also, (**III**) implies that:

$$\sum_{i=1}^{g-1} i(g-i) V_{i,1+n} V_{g-i,1+n} = O(V_{g,1+n}),$$

and hence by (9), we get

$$\sum_{i=1}^{g-1} V_{i,2+n} V_{g-i,2+n} = O(V_{g,2+n}/g), \tag{11}$$

as $g \to \infty$.

In fact, one can prove a stronger version of (11) by replacing the right-hand side with $O(V_{g,2+n}/g^2)$.

**Asymptotic behavior of ratios $B_{g,n}$ and $C_{g,n}$.** We use (**II**) to show that when $n$ is fixed

$$V_{g,n} \asymp V_{g-1,n+2}, \quad \text{and} \quad V_{g,n+1} \asymp g V_{g,n}. \tag{12}$$

More generally, we show:

**Theorem 3.1.** *Let $n \geq 0$. As $g \to \infty$*

**a):**

$$\frac{V_{g,n+1}}{2g V_{g,n}} \to 4\pi^2,$$

**b):**

$$\frac{V_{g,n}}{V_{g-1,n+2}} \to 1.$$

**Sketch of Proof.** We use the following elementary observation to prove (a):

**Elementary fact**. *Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of real numbers and $\{k_g\}_{g=1}^{\infty}$ be an increasing sequence of positive integers. Assume that for $g \geq 1$, and $i \in \mathbb{N}$, $0 \leq c_{g,i} \leq c_i$, and $\lim_{g\to\infty} c_{g,i} = c_i$. If $\sum_{i=1}^{\infty} |c_i r_i| < \infty$, then*

$$\lim_{g\to\infty} \sum_{i=1}^{k_g} r_i c_{g,i} = \sum_{i=1}^{\infty} r_i c_i. \tag{13}$$

Now, let

$$r_i = (-1)^i \frac{\pi^{2i}(i+1)}{(2i+3)!}, \; k_g = 3g-3+n \,, c_i = 1 \;\text{ and }\; c_{g,i} = \frac{[\tau_{i+1}\tau_0 \ldots \tau_0]_{g,n}}{V_{g,n+1}}.$$

By (13), and (**II**) for $\mathbf{d} = 0$ we get

$$\lim_{g\to\infty} \frac{2(2g-2+n)V_{g,n}}{V_{g,n+1}} = \frac{1}{3!} - \frac{2\pi^2}{5!} + \ldots + (-1)^L (L+1) \frac{\pi^{2L}}{(2L+3)!} + \ldots = \frac{1}{2\pi^2}.$$

On the other hand, from (**I**) and (11) we get that for $n \geq 2$ :

$$\lim_{g\to\infty} \frac{V_{g,n}}{V_{g-1,n+2}} = 1.$$

Finally, we can use part (a) to get the same result for $n = 0, 1$. □

In fact, we can prove that as $g \to \infty$:

$$\frac{V_{g,n+2}}{2g V_{g,n+1}} = 4\pi^2 + O(1/g), \;\text{ and }\; \frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g). \tag{14}$$

These stronger results imply that:

$$\sum_{i=1}^{g-1} V_{i,1} \times V_{g-i,1} \asymp V_{g,1}/g^2 \asymp V_g/g. \tag{15}$$

**Remark.** These estimates are all consistent with the conjectures of Zograf [Z2] on the growth of Weil-Petersson volumes as $g \to \infty$.

## 4. Random Riemann Surfaces of High Genus

In this section, we will discuss the typical behavior of a Riemann surfaces of large genus with respect to the Weil-Petersson measure.

**Notation.** The mapping class group $\mathrm{Mod}_{g,n}$ acts naturally on the set of isotopy classes of simple closed curves on $S_{g,n}$: Two simple closed curves $\alpha_1$ and $\alpha_2$ are of the same *type* if and only if there exists $g \in \mathrm{Mod}_{g,n}$ such that $g \cdot \alpha_1 = \alpha_2$. The type of a simple closed curve is determined by the topology of $S_{g,n} - \alpha$, the surface that we get by cutting $S_{g,n}$ along $\alpha$.

To simplify the notation, let $\gamma_0$ be a non-separating simple closed curve on $S_g$, and $\gamma_i$ be a separating simple closed curve on $S_g$ such that

$$S_g - \gamma_i = S_{i,1} \cup S_{g-i,1}.$$

**Thin part of $\mathcal{M}_g$.** First, we discuss the probability of appearance of a short closed geodesic in a random surface. Recall that every hyperbolic surface has a thick-thin decomposition; the thin part is the region of injectivity radius is less than a fixed small number. The thin components of a hyperbolic surface are neighborhoods of cusps or tubular neighborhoods of short geodesics.

The set of hyperbolic surfaces with lengths of closed geodesics bounded below by a constant $\epsilon > 0$ is a compact subset $\mathcal{C}^{\epsilon}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$. Some geometric properties of the moduli space can be controlled more easily in $\mathcal{C}^{\epsilon}_{g,n}$. See [Hu] and [Te]. Let $\mathcal{M}^{\epsilon}_{g,n} = \mathcal{M}_{g,n} - \mathcal{C}^{\epsilon}_{g,n}$.

**Theorem 4.1.** *Given $c > 0$, and $n \geq 0$, there exists $\epsilon > 0$ such that for any $g \geq 2$*

$$\mathrm{Vol}_{wp}(\mathcal{M}^{\epsilon}_{g,n}) < c\,\mathrm{Vol}_{wp}(\mathcal{M}_{g,n}).$$

Here we sketch the proof for the case of $n = 0$. Consider the function

$$F^{\epsilon} : \mathcal{M}_g \to \mathbb{R}_+$$

defined by

$$F^{\epsilon}(X) = |\{\gamma | \ell_{\gamma}(X) \leq \epsilon\}| = F^{\epsilon}_0(X) + \ldots F^{\epsilon}_{g/2}(X),$$

where $F_i^\epsilon(X) = |\{\gamma | \gamma \in \mathcal{O}_{\gamma_i}, \ell_\gamma(X) \leq \epsilon\}|$. Then by Theorem 2.2, we have

$$\mathrm{Vol}_{wp}(\mathcal{M}_g^\epsilon) \leq \int_{\mathcal{M}_g} F^\epsilon(X)\, dX \leq$$

$$\leq \sum_{i=1}^{g/2} \int_0^\epsilon t\, \mathrm{Vol}_{wp}(\mathcal{M}(S_g - \gamma_i, t, t))\, dt \ + \int_0^\epsilon t\, \mathrm{Vol}_{wp}(\mathcal{M}_{g-1,2}(t,t))\, dt$$

On the other hand, by (8) we know that if $t$ is small enough for $i \geq 1$,

$$\mathrm{Vol}_{wp}(\mathcal{M}(S_g - \gamma_i, t, t)) \leq 2V_{i,1} \times V_{g-i,1},$$

and

$$\mathrm{Vol}_{wp}(\mathcal{M}_{g-1,2}(t,t)) \leq 2V_{g-1,2}.$$

Hence, when $\epsilon$ is small (independent of $g$), from (12) and (15) we get

$$\mathrm{Vol}_{wp}(\mathcal{M}_g^\epsilon) = O\left(\epsilon^2 \left(\sum_{i=1}^{g/2} V_{i,1} V_{g-i,1} + V_{g-1,2}\right)\right) = O(\epsilon^2 V_g).$$

**Remark.** Even though we can make the ratio $T_{g,n}^\epsilon = \mathrm{Vol}(\mathcal{M}_{g,n}^\epsilon)/\,\mathrm{Vol}(\mathcal{M}_{g,n})$ small, for any fixed $\epsilon > 0$, $T_{g,n}^\epsilon$ does not tend to zero as $g \to \infty$.

**Behavior of the systoles.** Next, we would like to know how the length of the shortest closed geodesic on a random Riemann surface grows with the genus. In general, the systole of a compact metric space $X$ is defined to be the least length of a noncontractible loop in $X$. It is known that there are Riemann surfaces of large genus whose systole behaves logarithmically in the the genus [BP]. In fact, by [KSV] there is a principal congruence tower of Hurwitz surfaces (PCH), such that

$$\ell_{syst}(X_{PCH}) \geq \frac{4}{3} \log(g(X_{PCH})),$$

where $\ell_{syst}(X)$ is the length of a shortest simple closed geodesic on $X$. However, such a closed geodesic could be separating or non-separating. For more on properties of the function $\ell_{syst} : \mathcal{T}_{g,n} \to \mathbb{R}_+$ see [S1] and [S2].

First, consider the set of separating simple closed geodesics of typer $\gamma_1$. Since

$$\frac{V_g}{V_{1,1} \times V_{g-1,1}} \asymp g,$$

Theorem 2.2 and (8) imply that we can not cover $\mathcal{M}_g$ with surfaces which have a short separating curve $\gamma \in \mathcal{O}_{\gamma_1}$. More precisely, let

$$\mathcal{C}_g(L) = \{X \in \mathcal{M}_g |\ \exists \ \text{ separating curve } \alpha,\ \ell_\alpha(X) \leq L\} \subset \mathcal{M}_g.$$

Then

$$\mathcal{C}_g(L) = \bigcup_{i=1}^{g/2} \mathcal{C}_g(\gamma_i, L),$$

where

$$\mathcal{C}_g(\gamma, L) = \{X \in \mathcal{M}_g | \ \exists \alpha \in \mathcal{O}_\gamma \ , \ell_\alpha(X) \le L\} \subset \mathcal{M}_g.$$

Then by Theorem 2.2 and (8), for $1 \le i \le g/2$

$$\mathrm{Vol}_{wp}(\mathcal{C}_g(\gamma_i, L)) \le V_{i,1} \times V_{g-i,1} e^L L^2.$$

Hence (15) implies that

$$\frac{\mathrm{Vol}_{wp}(\mathcal{C}_g(L))}{V_g} \le \frac{\sum_{i=1}^{g/2} \mathrm{Vol}_{wp}(\mathcal{C}_g(\gamma_i, L))}{V_g} \le L^2 e^L \sum_{i=1}^{g/2} \frac{V_{i,1} \times V_{g-i,1}}{V_g} = O\left(\frac{L^2 e^L}{g}\right).$$

This implies:

**Theorem 4.2.** *The probability that a Riemann surface in $\mathcal{M}_g$ has a separating simple closed geodesic of length $\le \frac{1}{3}\log(g)$ tends to zero as $g \to \infty$.*

**Remark.** On the other hand, because $\frac{V_g}{V_{g-1,2}}$ is bounded, the situation is very different for a non-separating simple closed curve. In fact, the probability that a random Riemann surface has a short non-separating simple closed geodesic is asymptotically positive.

Finally, we consider the following quantity similar to the Cheeger constant [Bu] of a Riemann surface. Given $X \in \mathcal{T}_g$, let

$$L(X) = \inf_C \frac{\ell_C(X)}{\min[\mathrm{area}(A), \mathrm{area}(B)]},$$

where $C$ runs over (possibly disconnected) simple closed *geodesics* on $X$, with $X - C = A \cup B$. Then there exists $c > 0$ such that

$$\frac{\mathrm{Vol}_{wp}\{X \ |X \in \mathcal{M}_g, L(X) < c\}}{V_g} \to 0$$

as $g \to \infty$.

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