Stable determination of corrosion by a single electrostatic boundary measurement

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Abstract

We prove an optimal stability estimate for an inverse Robin boundary value problem arising in corrosion detection by electrostatic boundary measurements.

1 Introduction

Consider the following boundary value problem

$$\Delta u = 0 \qquad \text{in } \Omega \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \phi \qquad \text{on } \Gamma_N \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \gamma u = 0 \qquad \text{on } \Gamma \tag{1.3}$$

$$u = 0 \qquad \text{on } \Gamma_D \tag{1.4}$$

where Γ_N, Γ are two open, disjoint portions of $\partial\Omega$, $\Gamma_D = \partial\Omega \backslash (\Gamma_N \cup \Gamma)$ and γ is a nonnegative coefficient.

This problem has been introduced as a model of an electrostatic conductor, Ω, in which a part of the boundary Γ is subject to corrosion, [K-S, K-S-V, F-I, I, C-J], see also [V-X, K-V] for more accurate nonlinear models.

In this model, u represents the electrostatic potential, ϕ the prescribed current density on the accessible part of the boundary Γ_N, and the possible presence of corrosion damage on the inaccessible portion Γ of the boundary is modelled by the Robin boundary condition (1.3) where γ represents the reciprocal of the surface impedance.

The inverse problem of corrosion detection consists of the determination of γ when the available data are: a fixed choice of the current density ϕ and the measurement of the corresponding boundary voltage $u|_{\Gamma_N}$ on the accessible part of $\partial\Omega$.

In this paper we deal with the stability issue and concentrate on the case when the space dimension is $n = 2$. Let us recall that a stability estimate for

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the same inverse problem was obtained by Chaabane and Jaoua, [C-J], under the assumption that $\phi \geq 0$. In fact, in [C-J] the same boundary value problem (1.1)–(1.4) was introduced as a model of stationary heat conduction, and, in such a context, the assumption of a nonnegative prescribed boundary heat flux ϕ is well justified. However, this is not the case in the electrostatic setting where the simultaneous presence of positive and negative electrodes must be admitted.

In the case when ϕ has variable sign, additional difficulties occur. In order to explain this, let us illustrate briefly the underlying ideas for the identification of γ. If for the harmonic function u in Ω the Cauchy data $\frac{\partial u}{\partial \nu}|_{\Gamma_N} = \phi$ and $u|_{\Gamma_N}$ are known, then u is uniquely determined in all of $\overline{\Omega}$. Hence we can use (1.3) to compute

$$\gamma = -\frac{1}{u}\frac{\partial u}{\partial \nu} \quad \text{on } \Gamma. \tag{1.5}$$

This operation is well justified if $u > 0$ on Γ, and in fact (by the maximum principle) this is the case when $\phi \geq 0$, see [C-J]. If instead ϕ has variable sign, then u may vanish somewhere in Ω and on Γ and thus formula (1.5) may be undetermined or highly unstable. Thus, it is required to have a quantitative control on the possible vanishing rate of u. This is achieved in Proposition 2.3 below, by assuming that a control on the oscillation character of ϕ is a priori given. More precisely, letting $\Phi = \int \phi$ be the antiderivative of ϕ on Γ, such that $\int_{\Gamma_N} \Phi = 0$, and assuming that for a given $F > 0$ we have that

$$\frac{\|\phi\|_{L^2(\Gamma_N)}}{\|\Phi\|_{L^2(\Gamma_N)}} \leq F,$$

we obtain an upper bound, (2.11) and (2.12), on the number and on the order of the zeroes of u on Γ.

By combining this result with stability estimates for a Cauchy problem, Propositions 2.4 and 2.5, we obtain a stability estimate of logarithmic type for the determination of γ, Theorem 2.2.

Some remarks are in order:

1) The method introduced by Mandache, [M], and developed in [DC-R] could be adapted to the present setting to show by examples that indeed logarithmic stability is the best possible, and thus the result of Theorem 2.2 is essentially optimal.

2) The results obtained here, and especially those of Proposition 2.3, are based on methods of complex analytic function theory and thus are limited to the two-dimensional setting. A generalization of these results to the higher dimensional case requires the introduction of different tools. This will be the object of future research.

3) Results completely analogous to those presented here might also be obtained for some variations of the boundary value problem (1.1)–(1.4). For instance, the Dirichlet boundary condition (1.4) might be replaced by a homogeneous Neumann condition and the (known) conductivity in Ω might be variable and anisotropic.

The plan of the paper is as follows. In Section 2 we state and prove the stability result, Theorem 2.2. The main steps of the proof of Theorem 2.2 are described in a series of propositions. Proposition 2.3 contains the lower bound estimate for the solution to the direct problem, whereas Propositions 2.4 and 2.5 concern stability estimates for a Cauchy type problem. In Section 3 the proof of Proposition 2.3 is developed and in Section 4 we treat the Cauchy type problem and prove Propositions 2.4 and 2.5.

2 The stability theorem

Given $z \in \mathbb{C}$ and $r > 0$, we denote by $B_r(z)$ the open disc with centre z and radius r. We shall identify complex numbers $z = x + iy \in \mathbb{C}$ with points $(x, y) \in \mathbb{R}^2$, and we shall denote complex derivatives as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Definition 2.1 Given an integer $k = 0, 1, 2, \dots$, a number α, $0 < \alpha \leq 1$, and a simple closed curve γ, we shall say that γ is $C^{k,\alpha}$ with constants $\delta, M > 0$ if, for any $z \in \gamma$, $\gamma \cap B_\delta(z)$ is given, up to a rigid transformation, by the graph $\{y = g(x),\ x^2 + y^2 < \delta^2\}$ of a function $g \in C^{k,\alpha}[-\delta, \delta]$ such that $\|g\|_{C^{k,\alpha}[-\delta,\delta]} \leq M$. In the special case when $k = 0$, $\alpha = 1$, we shall say that γ is a *Lipschitz* curve.

Prior information on the domain. Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$, whose boundary $\partial\Omega$ is a simple closed curve which is Lipschitz with positive constants δ, M, and whose diameter is bounded by L. Furthermore we decompose $\partial\Omega$ into four closed subarcs, each of length at least δ, which are pairwise internally disjoint. Following the counterclockwise orientation of $\partial\Omega$, we call these subarcs Γ, Γ_D^1, Γ_N, Γ_D^2 respectively. We shall denote $\Gamma_D = \Gamma_D^1 \cup \Gamma_D^2$. About Γ we further assume that for a given constant α, $0 < \alpha \leq 1$, the following holds:

$$\begin{array}{l} \text{for any } z_0 \in \Gamma, \text{ we have that, up to a rigid change of coordinates,} \\ \Gamma \cap B_\delta(z_0) \subseteq \{z = x + iy :\ y = g(x)\} \\ \text{and } \Omega \cap B_\delta(z_0) \subseteq \{z = x + iy :\ y < g(x)\}, \text{ where} \\ g : [-\delta, \delta] \to \mathbb{R} \text{ is a } C^{1,\alpha} \text{ function satisfying } \|g\|_{C^{1,\alpha}[-\delta,\delta]} \leq M. \end{array} \tag{2.1}$$

Prior information on the boundary data. The current flux ϕ is a prescribed function in $L^2(\Gamma_N)$ such that, for a given constant $H_1 > 0$, we have

$$\|\phi\|_{L^2(\Gamma_N)} \leq H_1. \tag{2.2}$$

We define the antiderivative along Γ_N of ϕ as

$$\Phi = \int \phi(s)\mathrm{d}s,$$

where the indefinite integral is taken with respect to the arclength on Γ_N oriented in the counterclockwise direction. We suppose that, for a given constant $H_2 > 0$, we have

$$\|\Phi - \Phi_{\Gamma_N}\|_{L^2(\Gamma_N)} \geq H_2, \tag{2.3}$$

where Φ_{Γ_N} denotes the average of Φ over Γ_N.

Prior information on the Robin coefficient. We assume that the Robin coefficient γ belongs to $C^{\beta}(\Gamma)$, $0 < \beta \leq 1$, and that $\gamma \geq 0$ on Γ. More precisely, we assume that, for a given constant $H_3 > 0$, we have

$$\|\gamma\|_{C^{\beta}(\Gamma)} = \sup_{z\in\Gamma} |\gamma(z)| + |\gamma|_{C^{\beta}(\Gamma)} \leq H_3, \tag{2.4}$$

where

$$|\gamma|_{C^{\beta}(\Gamma)} = \sup_{\substack{z_0,z_1\in\Gamma \\ z_0\neq z_1}} \frac{|\gamma(z_0)-\gamma(z_1)|}{|z_0-z_1|^{\beta}}.$$

The set of constants δ, M, L, α, H_1, H_2, H_3 and β will be referred to as the *a priori* data.

We consider the boundary value problem (1.1)–(1.4). A weak solution to this problem is a function $u \in H^1(\Omega)$, such that $u|_{\Gamma_D} = 0$, which satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \rho = \int_{\Gamma_N} \phi\rho - \int_{\Gamma} \gamma u \rho, \qquad \text{for every } \rho \in H^1(\Omega)\text{: } \rho|_{\Gamma_D} = 0. \tag{2.5}$$

Here and in the following we denote by $u|_{\Gamma_D}$ the trace of a function $u \in H^1(\Omega)$ on Γ_D.

Existence and uniqueness of a weak solution to (1.1)–(1.4) derive from standard theory of boundary value problems for Laplace's equation. Moreover, from the weak formulation of problem (1.1)–(1.4) and a Poincaré type inequality we infer that

$$\|u\|_{H^1(\Omega)} + \|u\|_{L^2(\partial\Omega)} \leq \tilde{K}\|\phi\|_{L^2(\Gamma_N)}, \tag{2.6}$$

where $\tilde{K} > 0$ depends on the *a priori* data only.

We shall denote, for any $d > 0$,

$$(\Omega_\Gamma)_d = \{z \in \Omega : \ \mathrm{dist}(z, \partial\Omega\backslash\Gamma) > d\}, \tag{2.7}$$

and

$$\Gamma_d = \partial(\Omega_\Gamma)_d \cap \Gamma. \tag{2.8}$$

The main result is the following theorem.

Theorem 2.2 *Let Ω, $\phi \in L^2(\Gamma_N)$ and $\gamma_i \in C^{\beta}(\Gamma)$, $i = 1, 2$, satisfy the prior assumptions described above. Let $u_i \in H^1(\Omega)$, $i = 1, 2$, be the weak solution to the problem (1.1)–(1.4) when γ is replaced by γ_i, respectively.*

Moreover, let $\psi_i = u_i|_{\Gamma_N}$, $i = 1, 2$. Suppose that, given $\varepsilon > 0$, we have

$$\|\psi_1 - \psi_2\|_{L^\infty(\Gamma_N)} \leq \varepsilon. \tag{2.9}$$

Then for any $d > 0$ we have

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_d)} \leq \omega(\varepsilon),$$

where $\omega(\varepsilon)$ is a positive increasing function defined on $(0, +\infty)$ that satisfies

$$\omega(\varepsilon) \leq K|\log\varepsilon|^{-\sigma}, \qquad \textit{for every } \varepsilon,\ 0 < \varepsilon \leq 1/e, \tag{2.10}$$

where $K > 0$ and σ, $0 < \sigma < 1$, depend on the a priori *data and on d only.*

The proof of this theorem is obtained from the following propositions.

Proposition 2.3 *Let Ω, $\phi \in L^2(\Gamma_N)$ and $\gamma \in C^\beta(\Gamma)$ satisfy the prior assumptions above. Let $u \in H^1(\Omega)$ be the weak solution to* (1.1)–(1.4). *There exists $d_0 > 0$, depending on the* a priori *data only, such that for any d, $0 < d \le d_0$, there exists a finite number K of points $z_1, \ldots, z_K \in \Omega \cup \Gamma$ such that*

$$|u(z)| \ge c(d) \prod_{k=1}^{K} \left(\frac{|z - z_k|}{\tilde{C}} \right)^{b_k/\tilde{\alpha}}, \qquad \textit{for any } z \in \Gamma_d, \tag{2.11}$$

where $b_1, \ldots, b_K$ are positive integers satisfying

$$\sum_{k=1}^{K} b_k \le C(d), \tag{2.12}$$

where $\tilde{C} > 0$ and $\tilde{\alpha}$, $0 < \tilde{\alpha} < 1$, depend on the a priori *data only, and $c(d) > 0$ and $C(d)$ depend on the* a priori *data and on d only.*

Proposition 2.4 *Under the same assumptions as in Theorem* 2.2, *we have*

$$\|u_1 - u_2\|_{L^\infty(\Gamma)} \le \eta(\varepsilon),$$

where $\eta(\varepsilon)$ is a positive function satisfying (2.10) *with constants $R > 0$ and θ, $0 < \theta < 1$, depending on the* a priori *data only.*

Proposition 2.5 *Under the same assumptions as in Theorem* 2.2, *we have*

$$\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{L^\infty(\Gamma_d)} \le \tilde{\eta}(\varepsilon),$$

where $\tilde{\eta}(\varepsilon)$ is a positive function satisfying (2.10) *with constants $\tilde{R} > 0$ and $\tilde{\theta}$, $0 < \tilde{\theta} < \theta$, depending on the* a priori *data and on d only.*

We postpone the proof of Proposition 2.3 to Section 3 and those of Propositions 2.4 and 2.5 to Section 4. Using these results we conclude the proof of the stability theorem.

Proof of Theorem 2.2. Since $\frac{\partial u_i}{\partial \nu} + \gamma_i u_i = 0$ on Γ, $i = 1, 2$, we obtain

$$\gamma_1 - \gamma_2 = \frac{-\dfrac{\partial u_1}{\partial \nu}}{u_1} - \frac{-\dfrac{\partial u_2}{\partial \nu}}{u_2} = \frac{\partial u_1}{\partial \nu} \frac{(u_1 - u_2)}{u_1 u_2} + \frac{1}{u_2} \left(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu} \right).$$

Using again that $\frac{\partial u_1}{\partial \nu} + \gamma_1 u_1 = 0$ on Γ

$$\gamma_1 - \gamma_2 = -\gamma_1 \frac{(u_1 - u_2)}{u_2} + \frac{1}{u_2} \left(\frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu} \right). \tag{2.13}$$

We note that there exists $\tilde{d} > 0$, depending on δ, M, L and α only, so that for any d, $0 < d \le \tilde{d}$, we can find positive constants $\tilde{\delta}$, $\tilde{M}$ and M_1, $0 < M_1 < 1$, depending on δ, M, L, α and d only, so that the following conditions are satisfied. Let $(\Omega_\Gamma)_d$ be defined as in (2.7) and Γ_d be defined as in (2.8). Then there exists

$\hat{\Omega}_d$ so that $\partial\hat{\Omega}_d$ is a simple closed curve which is $C^{1,\alpha}$ with constants $\tilde{\delta}$, $\tilde{M}$ and $(\Omega_\Gamma)_d \subset \hat{\Omega}_d \subset (\Omega_\Gamma)_{M_1 d}$. Furthermore $\partial\hat{\Omega}_d \cap \Gamma = \hat{\Gamma}_d$ is a subarc of Γ of length at least $\delta/2$ so that $\Gamma_d \subset \hat{\Gamma}_d \subset \{z \in \Gamma : \ \mathrm{dist}(z, \partial\Omega\backslash\Gamma) \geq M_1 d\}$.

Let us fix $d > 0$ so that $d \leq \min\{\tilde{d}, d_0\}$, and let $\hat{\Gamma}_d$ be the subarc of Γ of length at least $\delta/2$ so that $\Gamma_d \subset \hat{\Gamma}_d \subset \Gamma_{M_1 d}$.

By (2.13), from (2.4) and Propositions 2.4 and 2.5, we infer that for any $z \in \hat{\Gamma}_d$ and every ε, $0 < \varepsilon \leq 1/e$,

$$|\gamma_1(z) - \gamma_2(z)| \leq \frac{1}{|u_2(z)|}\left(H_3 R|\log\varepsilon|^{-\theta} + \tilde{R}|\log\varepsilon|^{-\tilde{\theta}}\right) \leq \hat{R}\frac{|\log\varepsilon|^{-\tilde{\theta}}}{|u_2(z)|}, \quad (2.14)$$

where $\hat{R} = H_3 R + \tilde{R} > 0$, and $\tilde{\theta}$, $0 < \tilde{\theta} < \theta$, depend on the *a priori* data and on d only.

In order to estimate $\gamma_1 - \gamma_2$ also on the points where u_2 vanishes or is close to zero, we apply Proposition 2.3 to $u = u_2$, and we use the notation introduced there.

Then, we consider the following subset of $\hat{\Gamma}_d$, obtained by possibly removing from $\hat{\Gamma}_d$ a finite number of discs where u_2 is close to zero. We denote, for any p, $0 < p \leq 2\tilde{C}$,

$$\hat{\Gamma}^p_d = \{z \in \hat{\Gamma}_d : \ |z - z_k| \geq p/2, \text{ for any } z_k, \ k = 1, \ldots, K\}.$$

So we obtain

$$|u_2(z)| \geq c(d)\prod_{k=1}^{K}\left(\frac{p}{2\tilde{C}}\right)^{b_k/\tilde{\alpha}} \geq c(d)\left(\frac{p}{2\tilde{C}}\right)^{C(d)/\tilde{\alpha}}, \quad \text{for any } z \in \hat{\Gamma}^p_d.$$

Then from (2.14), for any p, $0 < p \leq 2\tilde{C}$, we have

$$|\gamma_1(z) - \gamma_2(z)| \leq \hat{R}\frac{|\log\varepsilon|^{-\tilde{\theta}}}{c(d)}\left(\frac{2\tilde{C}}{p}\right)^{C(d)/\tilde{\alpha}}, \quad \text{for any } z \in \hat{\Gamma}^p_d. \quad (2.15)$$

We have that there exist p_0, $0 < p_0 \leq 2\tilde{C}$, and $C_1 > 0$, depending on the *a priori* data and on d only, so that for every p, $0 < p \leq p_0$, the following property holds true. For every $z \in B_{p/2}(z_k) \cap \hat{\Gamma}_d$ there exists $z' \in \hat{\Gamma}^p_d$ so that $|z - z'| \leq C_1 p$. Since we have supposed $\|\gamma_i\|_{C^\beta(\Gamma)} \leq H_3$, $i = 1, 2$, we have

$$|\gamma_i(z) - \gamma_i(z')| \leq H_3|z - z'|^\beta \leq H_3 C_1^\beta p^\beta, \quad i = 1, 2,$$

and then

$$|\gamma_1(z) - \gamma_2(z)| \leq |\gamma_1(z) - \gamma_1(z')| + |\gamma_1(z') - \gamma_2(z')| + |\gamma_2(z') - \gamma_2(z)| \leq$$
$$\leq 2H_3 C_1^\beta p^\beta + |\gamma_1(z') - \gamma_2(z')|.$$

Since $z' \in \hat{\Gamma}^p_d$, by (2.15), the following estimate holds for every ε, $0 < \varepsilon \leq 1/e$,

$$|\gamma_1(z) - \gamma_2(z)| \leq$$
$$\leq 2H_3 C_1^\beta p^\beta + \hat{R}\frac{|\log\varepsilon|^{-\tilde{\theta}}}{c(d)}\left(\frac{2\tilde{C}}{p}\right)^{C(d)/\tilde{\alpha}}, \quad \text{for any } z \in \hat{\Gamma}_d.$$

It follows that, for every p, $0 < p \leq p_0$, and every ε, $0 < \varepsilon \leq 1/e$,

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_d)} \leq T_1 p^\beta + T_2 |\log\varepsilon|^{-\tilde{\theta}} p^{-\theta_1},$$

where T_1, T_2 and $\theta_1 > 0$ depend on the *a priori* data and on d only. By a standard minimization argument the conclusion follows. $\square$

3 Proof of Proposition 2.3

We remark that the prior assumptions on the domain Ω imply that Γ is a $C^{1,\alpha}$ curve with constants δ, M, and we can find a domain Ω' so that $\Omega \subset \Omega'$ and $\Gamma \subset \partial\Omega'$ satisfying the following properties. First, the diameter of Ω' is bounded by L'; second its boundary $\partial\Omega'$ is a simple closed curve which is $C^{1,\alpha}$ with constants δ', M', where δ', M' and L' depend on δ, M, L and α only.

Proposition 3.1 *Let $U \in H^1(\Omega')$ be the weak solution to the following problem:*

$$\begin{cases} \Delta U = 0 & \text{in } \Omega', \\ U = 1 & \text{on } \partial\Omega' \backslash \Gamma, \\ \dfrac{\partial U}{\partial \nu} + \gamma U = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

We have

$$0 < h \leq U(z) \leq 1, \qquad \textit{almost everywhere in } \Omega', \tag{3.2}$$

where $h > 0$ depends on the a priori *data only.*

Proof. First of all we recall that a weak solution to (3.1) is a function $U \in H^1(\Omega')$, such that $U|_{\partial\Omega'\backslash\Gamma} = 1$, which satisfies

$$\int_{\Omega'} \nabla U \cdot \nabla\rho = -\int_\Gamma \gamma U \rho, \qquad \text{for every } \rho \in H^1(\Omega') \colon \rho|_{\partial\Omega'\backslash\Gamma} = 0.$$

Then we observe that $0 < U < 1$ almost everywhere in Ω'. In fact, let $\rho = [U]^-$, where $[\cdot]^-$ denotes the negative part; then $\rho \in H^1(\Omega')$ and $\rho|_{\partial\Omega'\backslash\Gamma} = 0$. Hence

$$0 \geq -\int_{\{U<0\}} |\nabla U|^2 = -\int_\Gamma \gamma U [U]^- = \int_\Gamma \gamma \left([U]^-\right)^2 \geq 0.$$

Therefore $[U]^-|_\Gamma = 0$, then $U \geq 0$ almost everywhere in Ω', and, by the strong maximum principle, $U > 0$ almost everywhere in Ω'. In an analogous way, we can show that $U < 1$ almost everywhere in Ω'.

Let us also fix a domain $\Omega_0 \subset \{z = x + iy \in \mathbb{C} : \ -1/2 < x \leq 0, \ \pi/4 < y < 3\pi/4\}$, so that $\partial\Omega_0$ is a closed curve which is $C^{1,\alpha}$ with constants δ and M. We further assume that $\Gamma_0 = \{z = x + iy \in \mathbb{C} : \ x = 0, \ \pi/3 < y < 2\pi/3\}$ is contained in $\partial\Omega_0$. Let $F : \Omega' \longrightarrow \Omega_0$ be a conformal mapping so that $F(\Gamma) = \Gamma_0$. By boundary regularity estimates for conformal mappings, see for instance [P, Chapter 3], we infer that F and its inverse F^{-1} are C^1 up to the boundary and there exists a constant K_1, depending on the *a priori* data only, so that

$$\|F\|_{C^1(\overline{\Omega'})} \leq K_1; \quad \|F^{-1}\|_{C^1(\overline{\Omega_0})} \leq K_1. \tag{3.3}$$

Let us define $\tilde{U} = U \circ F^{-1}$. We know that $\tilde{U} \in H^1(\Omega_0)$ and $\tilde{U}|_{\partial\Omega_0\setminus\Gamma_0} = 1$, and that $\tilde{U}$ is a weak solution to the following problem:

$$\begin{cases} \Delta\tilde{U} = 0 & \text{in } \Omega_0, \\ \tilde{U} = 1 & \text{on } \partial\Omega_0\setminus\Gamma_0, \\ \dfrac{\partial\tilde{U}}{\partial\nu} + \tilde{\gamma}\tilde{U} = 0 & \text{on } \Gamma_0, \end{cases}$$

where

$$\tilde{\gamma} = \frac{\gamma \circ F^{-1}}{|\det J_F|},$$

and J_F denotes the Jacobian matrix of F. By (3.3) and the prior information on γ, we have

$$0 \leq \tilde{\gamma} \leq k,$$

where k depends on the *a priori* data only.

We introduce the following barrier function

$$U_1(z) = \left(\frac{1-k}{1+k}e^x + e^{-x}\right)\sin y.$$

We have that $\Delta U_1 = 0$ on the whole plane, $\frac{\partial U_1}{\partial x} + kU_1 = 0$ on $x = 0$, and $U_1 > 0$ for $0 < y < \pi$ and $x < 0$. Moreover, we notice that $\max_{\partial\Omega_0\setminus\Gamma_0} U_1 < Q$, where $Q = \frac{1-k}{1+k} + e^{1/2} > 0$, and $\min_{\partial\Omega_0} U_1 > q$, where $q > 0$ is a constant depending on k only.

Let us denote $W = \tilde{U} - \frac{1}{Q}U_1$. Hence $W \in H^1(\Omega_0)$, and it satisfies

$$\begin{cases} \Delta W = 0 & \text{in } \Omega_0, \\ W > 0 & \text{on } \partial\Omega_0\setminus\Gamma_0, \\ \dfrac{\partial W}{\partial\nu} + \tilde{\gamma}W = (k - \tilde{\gamma})\dfrac{1}{Q}U_1 \geq 0 & \text{on } \Gamma_0, \end{cases}$$

and consequently, from the maximum principle, with an argument analogous to the one used at the beginning of this proof, we obtain that $W > 0$ in Ω_0. Thus $\tilde{U} > \frac{1}{Q}U_1$, and, choosing $h = q/Q$, (3.2) follows. $\square$

Now, we write the weak solution to (1.1)–(1.4) as $u = Uv$, where U is the weak solution to (3.1). As a consequence the function $v \in H^1(\Omega)$ is a weak solution to the following problem:

$$\begin{cases} \operatorname{div}(U^2\nabla v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_D, \\ U^2\dfrac{\partial v}{\partial\nu} = U\phi - u\dfrac{\partial U}{\partial\nu} & \text{on } \Gamma_N, \\ U^2\dfrac{\partial v}{\partial\nu} = 0 & \text{on } \Gamma. \end{cases} \tag{3.4}$$

We observe that the function $\tilde{\phi}$ given by $\tilde{\phi} = U\phi - u\frac{\partial U}{\partial\nu}$ belongs to $L^2(\Gamma_N)$ and

$$\|\tilde{\phi}\|_{L^2(\Gamma_N)} \leq K_2 H_1, \tag{3.5}$$

where $K_2 > 0$ depends on the *a priori* data only. This follows from (2.6) and from the fact that if $U \in H^1(\Omega')$ is a weak solution to the problem (3.1), we have that $|U| \leq 1$ on Γ_N and, by standard regularity estimates up to the boundary, $|\frac{\partial U}{\partial \nu}|$ is uniformly bounded on Γ_N by a constant depending on the *a priori* data only.

Since Ω is simply connected, there exists a function $w \in H^1(\Omega)$ that satisfies

$$\nabla w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} U^2 \nabla v, \qquad \text{almost everywhere in } \Omega, \tag{3.6}$$

see [A-M, Theorem 2.1]. The function w is called the *stream function* associated with v, a notion which generalizes the one of harmonic conjugate. We have that w is a weak solution to the following problem, for some constant c,

$$\begin{cases} \mathrm{div}(\frac{1}{U^2}\nabla w) = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_D, \\ w = \tilde{\Phi} & \text{on } \Gamma_N, \\ w = c & \text{on } \Gamma, \end{cases} \tag{3.7}$$

where $\tilde{\Phi} = \int \tilde{\phi}(s)\mathrm{d}s$.

Lemma 3.2 *The function $\tilde{\Phi}$ satisfies*

$$\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|_{L^2(\Gamma_N)} \geq K_3,$$

where $K_3 > 0$ depends on the a priori *data only.*

Proof. By an interpolation inequality and Poincaré inequality, we have that

$$\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|_{H^{\frac{1}{2}}(\Gamma_N)} \leq N_1 \|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|^{\frac{1}{2}}_{L^2(\Gamma_N)} \|\tilde{\phi}\|^{\frac{1}{2}}_{L^2(\Gamma_N)},$$

where $N_1 > 0$ depends on the *a priori* data only. By (3.5), we obtain

$$\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|_{L^2(\Gamma_N)} \geq \frac{\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|^2_{H^{\frac{1}{2}}(\Gamma_N)}}{N_1^2 \|\tilde{\phi}\|_{L^2(\Gamma_N)}} \geq \frac{\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|^2_{H^{\frac{1}{2}}(\Gamma_N)}}{N_1^2 K_2 H_1}. \tag{3.8}$$

We have that there exists a positive constant N_2, depending on the *a priori* data only, so that

$$\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|^2_{H^{\frac{1}{2}}(\Gamma_N)} \geq N_2 \min_{\substack{\rho \in H^1(\Omega) \\ \rho|_{\Gamma_N} = \tilde{\Phi} - \tilde{\Phi}_{\Gamma_N} \\ \rho|_{\Gamma} = const.}} \int_\Omega \frac{1}{U^2} \nabla\rho \cdot \nabla\rho = N_2 \int_\Omega \frac{1}{U^2} \nabla w \cdot \nabla w$$

and from (3.6), (3.2) and a Poincaré type inequality

$$\int_\Omega \frac{1}{U^2} \nabla w \cdot \nabla w = \int_\Omega U^2 \nabla v \cdot \nabla v \geq h^2 N_3 \|v\|^2_{L^2(\Gamma_N)},$$

where $N_3 > 0$ depends on the *a priori* data only. Therefore from $v = u/U$ and (3.2) again, we obtain

$$\|v\|_{L^2(\Gamma_N)} \geq \|u\|_{L^2(\Gamma_N)}.$$

The solution u to (2.5) trivially satisfies

$$\int_\Omega |\nabla u|^2 \le \|\phi\|_{L^2(\Gamma_N)} \|u\|_{L^2(\Gamma_N)},$$

and hence

$$\|\tilde{\Phi} - \tilde{\Phi}_{\Gamma_N}\|_{H^{\frac{1}{2}}(\Gamma_N)} \ge \frac{h\sqrt{N_2 N_3}}{H_1} \int_\Omega |\nabla u|^2 = \frac{h\sqrt{N_2 N_3}}{H_1} \int_\Omega |\nabla \tilde{u}|^2, \tag{3.9}$$

where $\tilde{u}$ is a harmonic conjugate of u; by a Poincaré type inequality and a trace inequality we infer that:

$$\|\Phi - \Phi_{\Gamma_N}\|_{L^2(\Gamma_N)} = \|\tilde{u} - \tilde{u}_{\Gamma_N}\|_{L^2(\Gamma_N)} \le N_4 \|\nabla \tilde{u}\|_{L^2(\Omega)},$$

where $N_4 > 0$ depends on the *a priori* data only. Then the conclusion follows by using (2.3) and coupling this last equation with (3.8) and (3.9). $\square$

Let us denote $f = v + iw$; f is a complex valued function, defined in Ω, that satisfies

$$f_{\overline{z}} = \mu f_z + \nu \overline{f_z} \quad \text{almost everywhere in } \Omega, \tag{3.10}$$

where μ and ν are bounded, measurable, complex valued coefficients, satisfying

$$|\mu| + |\nu| \le k < 1 \quad \text{almost everywhere in } \Omega, \tag{3.11}$$

with k depending on h only (see [A-M] and also [A-R1, Proposition 3.1]).

We recall that, for any k, $0 \le k < 1$, a function f is called *k-quasiregular* in a domain D if it is an H^1 solution to (3.10), (3.11), see for instance [L, B-J-S].

We observe that $\tilde{\Phi}$ is Hölder continuous on Γ_N, with constants depending on the *a priori* data only. We also recall that $\partial\Omega$ is Lipschitz. Using these facts and a reflection argument along Γ_D, from standard regularity estimates for elliptic equations we have that w, weak solution to (3.7), satisfies the following Hölder condition on $\overline{\Omega}$

$$|w(z_0) - w(z_1)| \le K_4 |z_0 - z_1|^{\alpha_1}, \quad \text{for any } z_0, z_1 \in \overline{\Omega}, \tag{3.12}$$

where $K_4 > 0$, α_1, $0 < \alpha_1 < 1$, depend on the *a priori* data only.

Since w is the stream function of v, that is the weak solution to (3.4), we have that also v satisfies a Hölder condition on $\overline{\Omega}$

$$|v(z_0) - v(z_1)| \le K_5 |z_0 - z_1|^{\alpha_2}, \quad \text{for any } z_0, z_1 \in \overline{\Omega}, \tag{3.13}$$

where $K_5 > 0$, α_2, $0 < \alpha_2 < 1$, depend on the *a priori* data only. This is a rather straightforward consequence of the representation theorem for quasiregular mappings ([B-N, p. 116]) and of Privaloff's theorem ([B-J-S, Part II, Chapter 6, Theorem 5]).

We say that a mapping χ is *bi-Lipschitz* if it is a homeomorphism such that χ and its inverse are Lipschitz continuous.

By locally deforming $\partial\Omega$, we can construct a bi-Lipschitz, orientation preserving mapping χ_1 from Ω onto the square $\Omega_1 = [-1, 0] \times [0, 1]$, such that the four subarcs decomposing $\partial\Omega$ are sent onto the four sides of the square, respectively. We denote $\tilde{\Gamma} = \chi_1(\Gamma) = \{z = x + iy \in \mathbb{C} : \; x = 0, \; 0 < y < 1\}$. From the prior assumptions on Ω, we can find such χ_1 so that $\|\chi_1\|_{C^{0,1}(\Omega)}$ and $\|\chi_1^{-1}\|_{C^{0,1}(\Omega_1)}$ are bounded by a constant depending on δ, M and L only.

Let us reflect Ω_1 in $\tilde{\Gamma}$ and consider the so obtained rectangle $\tilde{\Omega} = [-1, 1] \times [0, 1]$. On $\tilde{\Omega}$, we define the complex valued function $\tilde{f} = \tilde{v} + i\tilde{w}$ as follows

$$\tilde{f}(x, y) = \begin{cases} f \circ \chi_1^{-1}(x, y) & \text{if } -1 < x \leq 0, \\ \overline{f} \circ \chi_1^{-1}(-x, y) + 2ci & \text{if } 0 < x < 1. \end{cases}$$

We have that $\tilde{f}$ satisfies $\tilde{f}_{\overline{z}} = \tilde{\mu}\tilde{f}_z + \tilde{\nu}\overline{\tilde{f}_z}$ almost everywhere in $\tilde{\Omega}$, where $\tilde{\mu}$ and $\tilde{\nu}$ are bounded, measurable, complex valued coefficients, satisfying $|\tilde{\mu}| + |\tilde{\nu}| \leq \tilde{k} < 1$ almost everywhere in $\tilde{\Omega}$, with $\tilde{k}$ depending on the *a priori* data only. From (3.12) and (3.13) we also obtain that $\tilde{f}$ is Hölder continuous on $\overline{\tilde{\Omega}}$.

Now, let us denote, for any $d > 0$,

$$\tilde{\Omega}_d = \{z \in \tilde{\Omega} : \ \mathrm{dist}(z, \partial\tilde{\Omega}) > d\} \quad \text{and} \quad \tilde{\Gamma}_d = \tilde{\Omega}_d \cap \tilde{\Gamma}.$$

We prove an estimate from below for $|\tilde{f}|$ on $\tilde{\Gamma}_d$.

Proposition 3.3 *Under the previous assumptions, there exists a constant $d_0 > 0$, depending on the* a priori *data only, such that for any $z_0 \in \overline{\tilde{\Omega}}$ and for any d, $0 < d \leq d_0$, there exists a finite number K of points $z_1, \ldots, z_K \in \tilde{\Omega}$ such that*

$$|\tilde{f}(z) - \tilde{f}(z_0)| \geq c'(d) \prod_{k=1}^{K} \left(\frac{|z - z_k|}{C'} \right)^{b_k/\alpha'}, \quad \text{for any } z \in \tilde{\Omega}_d, \tag{3.14}$$

where $b_1, \ldots, b_K$ are positive integers such that

$$\sum_{k=1}^{K} b_k \leq C'(d), \tag{3.15}$$

and where $C' > 0$ and α', $0 < \alpha' < 1$, depend on the a priori *data only, and $c'(d) > 0$ and $C'(d)$ depend on the* a priori *data and on d only.*

Proof. One can verify that $\log |\tilde{f} - \tilde{f}(z_0)|$ is a solution to an elliptic equation with isolated singularities at the points $z_1, \ldots, z_K$. The estimates (3.14), (3.15) are then obtained by the use of Harnack's inequality, the maximum principle and Lemma 3.2. See for details the proof of Theorem 3.3 in [A-R2]. □

Now we have what we need to prove the estimate from below for $|u|$ on Γ_d.
Proof of Proposition 2.3. Let $u \in H^1(\Omega)$ be the weak solution to (1.1)–(1.4), and $\tilde{f}$ the function constructed above. Take $z_0 = 0$; clearly $z_0 \in \tilde{\Gamma}$ and then, by Proposition 3.3, since $\tilde{w}|_{\tilde{\Gamma}} = c$ for some constant c, and $\tilde{v}(0) = 0$, we obtain that for any d, $0 < d \leq d_0$, there exists a finite number K of points $z_1, \ldots, z_K \in \tilde{\Omega}$ such that

$$|\tilde{v}(z)| \geq c'(d) \prod_{k=1}^{K} \left(\frac{|z - z_k|}{C'} \right)^{b_k/\alpha'}, \quad \text{for any } z \in \tilde{\Gamma}_d, \tag{3.16}$$

where $b_1, \ldots, b_K$ are positive integers such that $\sum_{k=1}^{K} b_k \leq C'(d)$. Here the constants used are equal to the ones appearing in Proposition 3.3.

Then the proof immediately follows from (3.16) by noticing that $v = \tilde{v} \circ \chi_1$, where χ_1 is a bi-Lipschitz mapping, and $u = Uv$, with $U \geq h > 0$. □

4 Stability for the Cauchy problem

Under the assumptions of Theorem 2.2, let $u_i \in H^1(\Omega)$ be the weak solution to (1.1)–(1.4), with data $\phi \in L^2(\Gamma_N)$, $\gamma = \gamma_i \in C^\beta(\Gamma)$, and $\psi_i = u_i|_{\Gamma_N}$, $i = 1, 2$.

Let $\tilde{u}_i \in H^1(\Omega)$, $i = 1, 2$, be a harmonic conjugate of u_i. We have that $\tilde{u}_i$, $i = 1, 2$, is a weak solution to the following problem:

$$\begin{cases} \Delta \tilde{u}_i = 0 & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}_i}{\partial \nu} = 0 & \text{on } \Gamma_D, \\ \tilde{u}_i = \Phi & \text{on } \Gamma_N, \\ \tilde{u}_i = \Psi_i & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where $\Phi = \int \phi(s)\mathrm{d}s$ and $\Psi_i = -\int \gamma_i(s) u_i(s)\mathrm{d}s$, $i = 1, 2$. We recall that $\tilde{u}_i$, $i = 1, 2$, is defined up to an additive constant.

Now, let us denote $F = u_1 - u_2 + i(\tilde{u}_1 - \tilde{u}_2)$; F is a holomorphic function, defined on $\overline{\Omega}$, and we choose the additive constants of $\tilde{u}_1$ and $\tilde{u}_2$ so that $\Im F = 0$ on Γ_N and thus, from (2.9), we have that

$$\sup_{\Gamma_N} |F| \le \varepsilon.$$

We are interested in the following Cauchy problem:

$$\begin{cases} F_{\overline{z}} = 0 & \text{almost everywhere in } \Omega, \\ |F| \le \varepsilon & \text{on } \Gamma_N. \end{cases} \tag{4.2}$$

By recalling the prior assumptions and (2.6), we have that Φ and Ψ_i, $i = 1, 2$, are Hölder continuous, hence by arguments already used to prove (3.12) and (3.13), we obtain that u_i and $\tilde{u}_i$, $i = 1, 2$, are Hölder continuous on $\overline{\Omega}$, with constants depending on the *a priori* data only. So we have that F satisfies

$$|F(z_0) - F(z_1)| \le R_1 |z_0 - z_1|^{\beta_1}, \qquad \text{for any } z_0, z_1 \in \overline{\Omega}, \tag{4.3}$$

where $R_1 > 0$ and β_1, $0 < \beta_1 < 1$, depend on the *a priori* data only. Then it is easy to infer that there exists a constant $R_2 > 0$, depending on the *a priori* data only, such that

$$\sup_{\overline{\Omega}} |F| = R_2. \tag{4.4}$$

Proof of Proposition 2.4. The stability of the Cauchy type problem (4.2), with the a priori bounds (4.3), (4.4), can be derived by the method of harmonic measure (see for example [L-R-S]). Use is also made of the fact that Ω satisfies a uniform interior cone condition. This property of Ω is an easy consequence of the fact that its boundary is Lipschitz with constants δ, M. Details can be found, for instance, in [A]. $\square$

Lemma 4.1 *Let $u \in H^1(\Omega)$ solve (1.1)–(1.4) and let d be a positive constant. Then ∇u is Hölder continuous on $\overline{(\Omega_\Gamma)_d}$*

$$|\nabla u(z_0) - \nabla u(z_1)| \le R_3 |z_0 - z_1|^{\beta_2}, \qquad \textit{for any } z_0, z_1 \in \overline{(\Omega_\Gamma)_d}, \tag{4.5}$$

where $R_3 > 0$, β_2, $0 < \beta_2 < 1$, depend on the a priori *data and on d only.*

Proof. From the fact that u is Hölder continuous in $\overline{\Omega}$, $\gamma \in C^{\beta}(\Gamma)$, and from the condition $\frac{\partial u}{\partial \nu} = -\gamma u$ on Γ, it follows that $\frac{\partial u}{\partial \nu}$ is Hölder continuous on Γ. Then, since Γ is a $C^{1,\alpha}$ curve, we have that $\tilde{u}$, the harmonic conjugate of u, belongs to C^{1,β_3} on Γ, where β_3, $0 < \beta_3 < 1$, depends on the *a priori* data only. So (4.5) follows from standard boundary regularity results for the Dirichlet problem. □

Proof of Proposition 2.5. We recall the notations introduced in the proof of Theorem 2.2, page 5. We assume, without loss of generality, that d is so that $0 < d \leq \tilde{d}$ and we consider the $C^{1,\alpha}$ domain $\hat{\Omega}_d$ so that $(\Omega_\Gamma)_d \subset \hat{\Omega}_d \subset (\Omega_\Gamma)_{M_1 d}$, $0 < M_1 < 1$ depending on the *a priori* data and on d only.

From Lemma 4.1 we can find constants $R_4 > 0$ and β_4, $0 < \beta_4 < 1$, depending on the *a priori* data and on d only, such that

$$|\nabla(u_1 - u_2)|_{C^{\beta_4}(\overline{\hat{\Omega}_d})} \leq R_4. \tag{4.6}$$

We apply the following interpolation inequality (see for related inequalities [G-T, Section 6.8])

$$\begin{aligned}\|\nabla(u_1 - u_2)\|_{L^\infty(\hat{\Omega}_d)} \leq R_5 \Bigg(& |\nabla(u_1 - u_2)|^{\frac{1}{1+\beta_4}}_{C^{\beta_4}(\overline{\hat{\Omega}_d})} \|u_1 - u_2\|^{\frac{\beta_4}{1+\beta_4}}_{L^\infty(\hat{\Omega}_d)} + \\ & + \|u_1 - u_2\|_{L^\infty(\hat{\Omega}_d)} \Bigg),\end{aligned} \tag{4.7}$$

where R_5 depends on the *a priori* data and on d only.

Since $\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{L^\infty(\Gamma_d)} \leq \|\nabla u_1 - \nabla u_2\|_{L^\infty(\hat{\Gamma}_d)}$, where $\hat{\Gamma}_d = \partial \hat{\Omega}_d \cap \Gamma$, the proof can be concluded by inserting (4.6) into (4.7) and by using Proposition 2.4 to estimate $\|u_1 - u_2\|_{L^\infty(\Omega)}$. □

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