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A NOTE ON FUZZY SEMI-IRRESOLUTE AND STRONGLY IRRESOLUTE FUNCTIONS

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Abstract. In this paper, we have some characterizations of fuzzy semi-irresolute and strongly irresolute functions.

1. Introduction

In 1989, M. N. Mukherjee and S. P. Sinha [12] introduced the notion of fuzzy irresolute function and investigated some of its properties. Since then, modified fuzzy continuous functions has been extensively studied.

S. Malaker [10] introduced the concepts of fuzzy semi-irresolute function and fuzzy strongly irresolute function, and obtained several characterizations of these functions.

The purpose of this paper is to give several characterizations of fuzzy semi-irresolute and fuzzy strongly irresolute functions. We obtain some of these characterizations through the introduction of type of convergences for fuzzy nets that we call RN-convergence and SN-convergence. Moreover, we obtain a characterization of RN-convergence through the concept of fuzzy regular semi-open sets, and then we obtain a characterization of fuzzy semi-irresolute functions.

2. Preliminaries

For an ordinary set X and the closed unit interval $I = [0, 1]$ of the real line, a fuzzy set A of X is a mapping from X into I. Throughout the paper, by (X, T) and (Y, T^*) (or simply X and Y) we shall mean fuzzy topological spaces in Chang's sense [2].

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A fuzzy singleton [13] with support x and value α, where $0 < \alpha \leq 1$, is denoted by x_α. A fuzzy singleton x_α is said to belong to a fuzzy set A(written $x_\alpha \in A$) if $\alpha \leq A(x)$. By 0_X and 1_X we will mean the constant fuzzy sets taking on respectively the values 0 and 1 on X.

For fuzzy sets A and B in X, we say that B includes A(written $A \leq B$) if $A(x) \leq B(x)$ for each $x \in X$. For a fuzzy set A in X, the notations $Cl(A)$, $Int(A)$ and $1_X - A$ will respectively stand for the fuzzy closure, fuzzy interior and complement of A. For fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident [13] with B. B is a quasi-neighborhood [13](simply, q-nbd) of A if there exists a fuzzy open set U such that $AqU \leq B$.

A fuzzy set A in X is called fuzzy semi-open set [1] if there exists a fuzzy open set B such that $B \leq A \leq Cl(B)$, or equivalentely $A \leq Cl(Int(A))$. The complement of a fuzzy semi-open set is called a fuzzy closed set. The intersection of all fuzzy semi-closed sets containing A is called the fuzzy semi-closure [7] of A and is denoted by ${}_sCl(A)$. The union of all fuzzy semi-open sets contained in a fuzzy set A in X is called the fuzzy semi-interior of A and is denoted by ${}_sInt(A)$.

A fuzzy set A is fuzzy semi-closed if and only if $A = {}_sCl(A)$ and is fuzzy semi-open if and only if $A = {}_sInt(A)$.

A fuzzy set A in X is called a semi-q-nbd [7] of a fuzzy singleton x_α in X if there exists a fuzzy semi-open set V in X such that $x_\alpha qV \leq A$. It is well know that [7] the semi-closure ${}_sCl(A)$ of a fuzzy set A in X is the union of all fuzzy singleton x_α such that every fuzzy semi-open semi-q-nbd of x_α is q-coincident with A. A fuzzy singleton x_α is said to be a fuzzy semi-θ-cluster point [14] of a fuzzy set A in X if the fuzzy semi-closure of every fuzzy semi-open semi-q-nbd of x_α is q-coincident with A. The union of all fuzzy semi-θ-cluster points of A is called the semi-θ-closure [14] of A and is denoted by ${}_sCl_\theta(A)$. A fuzzy set A in X is called fuzzy semi-θ-closed [14] if $A = {}_sCl_\theta(A)$, and complement of a fuzzy semi-θ-closed set is fuzzy semi-θ-open set.

For a fuzzy set A in X, $A \leq {}_sCl(A) \leq {}_sCl_\theta(A)$ and hence each fuzzy semi-θ-closed set is a fuzzy semi-closed.

A fuzzy set A in X is called fuzzy regular semi-open [11] if there exists a fuzzy regular open set U such that $U \leq A \leq Cl(U)$. It is clear that every fuzzy regular semi-open set is a fuzzy semi-open set.

In the following, we provide some Lemmas which are going to be used in the sequel.

LEMMA 2.1. *For a fuzzy set A in X, we have that*

(1) ${}_sInt_\theta(1_X - A) = 1_X - {}_sCl_\theta A$,
(2) ${}_sCl_\theta(1_X - A) = 1_X - {}_sInt_\theta A$.

LEMMA 2.2 [10]. *For a fuzzy semi-open set A in X, ${}_sCl(A) = {}_sCl_\theta(A)$.*

LEMMA 2.3 [3]. *For a fuzzy set A in X, the following statements are equivalent:*
(1) *A is fuzzy regular semi-open;*
(2) *$1_X - A$ is fuzzy regular semi-open;*
(3) *$A = {}_sCl({}_sInt(A))$;*
(4) *A is fuzzy semi-clopen;*
(5) *there exists a fuzzy regular open set U in X such that $U \leq A \leq Cl(U)$.*

From Lemma 2.1, 2.2 and 2.3, we get the following result.

LEMMA 2.4. *A fuzzy set A in X is a fuzzy regular semi-open set if and only if A is a fuzzy semi-θ-clopen set in X.*

LEMMA 2.5 [3]. *For a fuzzy semi-open set A in X, the fuzzy set ${}_sCl(A)$ is a fuzzy regular semi-open set in X.*

3. Characterizations

DEFINITION 3.1 [10]. A function $f : X \to Y$ is said to be fuzzy semi-irresolute(resp. fuzzy strongly irresolute) if for any fuzzy singleton x_α in X and each fuzzy semi-open set V containing $f(x_\alpha)$, there exists a fuzzy semi-open set U containing x_α such that $f(U) \leq {}_sCl(V)$ (resp. $f({}_sCl(U)) \leq V$).

In this section, we obtain several characterizations of fuzzy semi-irresolute and fuzzy strongly irresolute functions.

THEOREM 3.1. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) *f is fuzzy semi-irresolute;*
(2) *for each fuzzy singleton $x_\alpha \in X$ and each fuzzy regular semi-open set V containing $f(x_\alpha)$, there exists a fuzzy regular semi-open set U containing x_α such that $f(U) \leq V$;*
(3) *for each fuzzy singleton $x_\alpha \in X$ and each fuzzy regular semi-open set V containing $f(x_\alpha)$, there exists a fuzzy semi-open set U containing x_α such that $f({}_sCl(U)) \leq V$.*

PROOF. (1) $\Longrightarrow$ (2). Let x_α be a fuzzy singleton in X and V be a fuzzy regular semi-open V containing $f(x_\alpha)$. Then V is a fuzzy semi-open set. By Theorem 2.9 [10], there exists fuzzy semi-open set G containing x_α such that $f({}_sCl(G)) \leq {}_sCl(V) = V$. Then $U = {}_sCl(G)$ is a fuzzy regular semi-open set and $f(U) \leq V$.

(2) $\Longrightarrow$ (3). Let x_α be a fuzzy singleton in X and V be a fuzzy regular semi-open set in Y containing $f(x_\alpha)$. By (2), there exists a regular semi-open set U containing x_α such that $f(U) \leq V$. Then U is a fuzzy semi-clopen set and $f({}_sCl(U)) \leq V$.

(3) $\Longrightarrow$ (1). Let x_α be a fuzzy singleton in X and V be a fuzzy semi-open set V containing $f(x_\alpha)$. Then ${}_sCl(V)$ is a fuzzy regular semi-open set containing $f(x_\alpha)$, and so there is a fuzzy semi-open set U containing x_α such that $f({}_sCl(U)) \leq {}_sCl(V)$. From Theorem 2.9 [10], f is semi-irresolute. □

THEOREM 3.2. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) *f is fuzzy semi-irresolute;*
(2) *for each fuzzy singleton x_α in X and each fuzzy semi-θ-clopen set V containing $f(x_\alpha)$, there exists a fuzzy semi-θ-clopen set U containing x_α such that $f(U) \leq V$;*
(3) *$f^{-1}(V)$ is fuzzy regular semi-open for every fuzzy regular semi-open set V in Y;*
(4) *$f^{-1}(V) \leq {}_sInt_\theta(f^{-1}({}_sCl_\theta(V)))$ for every fuzzy semi-open set V in Y;*
(5) *${}_sCl_\theta(f^{-1}({}_sInt_\theta(V))) \leq f^{-1}(V)$ for any fuzzy semi-closed set V in Y;*
(6) *${}_sCl_\theta(f^{-1}(V)) \leq f^{-1}({}_sCl_\theta(V))$ for any fuzzy semi-open set V in Y.*

PROOF. (1) $\Longleftrightarrow$ (2). It follows from Lemma 2.4 and Theorem 3.1.

(2) $\Longrightarrow$ (3). Let V be a fuzzy regular semi-open set in Y. By Theorem 2.6 [10], ${}_sCl(f^{-1}(V)) \leq f^{-1}({}_sCl_\theta(V)) = f^{-1}({}_sCl(V)) = f^{-1}(V)$ and hence $f^{-1}(V)$ is fuzzy semi-closed. Since $1_Y - V$ is fuzzy regular semi-open, $1_X - f^{-1}(V) = f^{-1}(1_Y - V)$ is also fuzzy semi-closed. Therefore, $f^{-1}(V)$ is a fuzzy regular semi-open set.

(3) $\Longrightarrow$ (4). Let V be a fuzzy semi-open set in Y. Then ${}_sCl_\theta(V)$ is fuzzy regular semi-open set, and so $f^{-1}({}_sCl_\theta(V))$ is a fuzzy regular semi-open set and fuzzy semi-θ-open set. Since $f^{-1}(V) \leq f^{-1}({}_sCl_\theta(V))$, $f^{-1}(V) \leq {}_sInt_\theta f^{-1}({}_sCl_\theta(V))$.

(4) $\Longrightarrow$ (5). Let V be a fuzzy semi-closed set in Y. Then $1_Y - V$ is

fuzzy semi-open set in Y. By (4) and Lemma 2.1 , we have that

$$\begin{aligned}
&1_X - f^{-1}(V)\\
&= f^{-1}(1_Y - V)\\
&\leq {}_sInt_\theta(f^{-1}({}_sCl_\theta(1_Y - V)))\\
&= {}_sInt_\theta(f^{-1}(1_Y - {}_sInt_\theta(V)))\\
&= {}_sInt_\theta(1_X - f^{-1}({}_sInt_\theta(V)))\\
&= 1_X - {}_sCl_\theta(f^{-1}({}_sInt_\theta(V))).
\end{aligned}$$

Therefore, ${}_sCl_\theta(f^{-1}({}_sInt_\theta(V))) \leq f^{-1}(V)$.

(5) $\Longrightarrow$ (6). Let V be a fuzzy semi-open set in Y. Then ${}_sCl(V)$ is fuzzy regular semi-open and so ${}_sCl(V)$ is semi-θ-clopen. By (5),

$${}_sCl_\theta(f^{-1}(V)) \leq {}_sCl_\theta(f^{-1}({}_sCl(V))$$

$$= {}_sCl_\theta(f^{-1}({}_sInt_\theta({}_sCl(V)))) \leq f^{-1}({}_sCl(V)) = f^{-1}({}_sCl_\theta(V)).$$

(6) $\Longrightarrow$ (3). Let V be a fuzzy regular semi-open set in Y. Then V is fuzzy semi-clopen. By (6), ${}_sCl_\theta(f^{-1}(V)) \leq f^{-1}({}_sCl_\theta(V)) = f^{-1}({}_sCl(V)) = f^{-1}(V)$. So $f^{-1}(V) = {}_sCl_\theta(f^{-1}(V))$ and $f^{-1}(V)$ is a semi-θ-closed set. Since V is a fuzzy regular semi-open set, $1_Y - V$ is a fuzzy regular semi-open set. Thus $f^{-1}(1_Y - V) = 1_X - f^{-1}(V)$ is a semi-θ-closed set, and hence $f^{-1}(V)$ is a semi-θ-open set. Therefore, $f^{-1}(V)$ is a semi-θ-clopen set, and hence regular semi-open set.

(3) $\Longrightarrow$ (1). This implication immediately follows from Theorem 2.9 [10], Lemma 2.3 and 2.4. □

From Lemma 2.2, 2.4, 2.5 and Theorem 2.9 [10], we get the following result.

Corollary 3.1. *A function $f : X \to Y$ is semi-irresolute if and only if for each singleton x_α in X and each semi-open set V containing $f(x_\alpha)$, there exists a fuzzy semi-θ-open set U containing x_α such that $f({}_sCl_\theta(U)) \leq {}_sCl(V)$.*

From Theorem 2.2 and 2.9 of [10], we have the following result.

Proposition 3.3. *A function $f : X \to Y$ is semi-irresolute if and only if for each singleton x_α in X and each semi-q-nbd V of $f(x_\alpha)$, there exists a fuzzy semi-q-nbd U of x_α in X such that $f({}_sCl(U)) \leq {}_sCl(V)$.*

Theorem 3.4. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) f is fuzzy semi-irrsolute;
(2) ${}_sCl_\theta(f^{-1}(B)) \leq f^{-1}({}_sCl_\theta(B))$ for any fuzzy set B in Y;
(3) $f({}_sCl_\theta(A)) \leq {}_sCl_\theta(f(A))$ for any fuzzy set A in X;
(4) $f^{-1}(F)$ is fuzzy semi-θ-closed for every fuzzy semi-θ-closed F in Y;
(5) $f^{-1}(V)$ is fuzzy semi-θ-open for every fuzzy semi-θ-open V in Y.

Proof. (1) $\Longrightarrow$ (2). Let B be any fuzzy set in Y and $x_\alpha \notin f^{-1}({}_sCl_\theta(B))$. Then $f(x_\alpha) \notin {}_sCl_\theta(B)$, and so there is a fuzzy semi-open semi-q-nbd V of $f(x_\alpha)$ such that ${}_sCl(V) \not{q} B$. By Proposition 3.3, there exists a semi-q-nbd U of x_α such that $f({}_sCl(U)) \leq {}_sCl(V)$. Thus $f({}_sCl(U)) \not{q} B$, and so $f({}_sCl(U)) \leq 1_Y - B$ and ${}_sCl(U) \leq f^{-1}(1_Y - B)$. Therefore, ${}_sCl(U) \not{q} f^{-1}(B)$ and $x_\alpha \notin {}_sCl_\theta(f^{-1}(B))$.

(2) $\Longrightarrow$ (3). Let A be a fuzzy set in X. Then ${}_sCl_\theta(A) \leq {}_sCl_\theta(f^{-1}(f(A)))$. By (2), ${}_sCl_\theta(f^{-1}(f(A))) \leq f^{-1}({}_sCl_\theta(f(A)))$. Thus $f({}_sCl_\theta(A)) \leq f(f^{-1}({}_sCl_\theta(f(A)))) \leq {}_sCl_\theta(f(A))$.

(3) $\Longrightarrow$ (4). Let F be a fuzzy semi-θ-closed set in Y. Then

$$f({}_sCl_\theta(f^{-1}(F))) \leq {}_sCl_\theta(f(f^{-1}(F))) \leq {}_sCl_\theta(F) = F.$$

Thus ${}_sCl_\theta(f^{-1}(F)) \leq f^{-1}(F)$ and $f^{-1}(F)$ is a fuzzy semi-θ-closed set in X.

(4) $\Longrightarrow$ (5). This implication is obvious.

(5) $\Longrightarrow$ (1). It follows from Theorem 2.6 [10], because each semi-θ-open set is a semi-open set. □

Theorem 3.5. *A function $f : X \to Y$ is fuzzy strongly irresolute if and only if for each fuzzy singleton x_α in X and each fuzzy semi-open set V containing $f(x_\alpha)$, there exists a fuzzy regular semi-open set U containing x_α such that $f(U) \leq V$.*

Proof. It follows immediately from Lemma 2.3 and 2.5. □

Definition 3.2 [9]. Let $(D, \geq)$ be a directed set. A fuzzy net in a fuzzy space X is a map $\phi : D \to \mathcal{B}_F(X)$, where $\mathcal{B}_F(X)$ is the collection of all fuzzy singletons in X. We also denote ϕ by $\{\phi(d) : d \in D\}$ or $(\phi(d))$.

Definition 3.3. A fuzzy net $(\phi(d))$ in a fuzzy space X is said to θN- converges to a fuzzy singleton x_α in X if for each fuzzy open set U containing x_α, there exists d_0 such that $\phi(d) \in Cl(U)$ for all $d \geq d_0$.

Definition 3.4. A fuzzy net $(\phi(d))$ in a fuzzy space X is said to RN-converges to a fuzzy singleton x_α in X if for each fuzzy semi-open set U containing x_α, there exists d_0 such that $\phi(d) \in {}_sCl(U)$ for all $d \geq d_0$.

Definition 3.5. A fuzzy net $(\phi(d))$ in a fuzzy space X is said to SN-converges(resp. $S^{'}N$-converges) to a fuzzy singleton x_α in X if for each fuzzy semi-open(resp. semi-θ-open) set U containing x_α, there exists d_0 such that $\phi(d) \in U$(resp. $\phi(d) \in {}_sCl_\theta(U)$) or all $d \geq d_0$.

It is easy to see that the following Lemma holds.

Lemma 3.1. *For a fuzzy net $(\phi(d))$ in a fuzzy space X,*
(1) *if $(\phi(d))$ RN-converges to x_α, then $(\phi(d))$ θN-converges to x_α.*
(2) *if $(\phi(d))$ SN-converges to x_α, then $(\phi(d))$ $S^{'}N$-converges to x_α.*
(3) *if $(\phi(d))$ $S^{'}N$-converges to x_α, then $(\phi(d))$ RN-converges to x_α.*

Theorem 3.6. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) *f is fuzzy semi-irresolute;*
(2) *for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which RN-converges to x_α, the net $(f(\phi(d)))$ RN-converges to $f(x_\alpha)$;*
(3) *for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which $S^{'}N$-converges to x_α, the net $(f(\phi(d)))$ RN-converges to $f(x_\alpha)$.*

Proof. (1) $\Longrightarrow$(2). Let x_α be a fuzzy singleton in X and let$(\phi(d))$ be a fuzzy net in X such that $(\phi(d))$ RN-converges to x_α. Let V be a fuzzy semi-open set containing $f(x_\alpha)$. Since f is semi-irresolute, there exists a fuzzy semi-open set U containing x_α such that $f({}_sCl(U)) \leq {}_sCl(V)$. Since $(\phi(d))$ RN-converges to x_α, there exists d_0 such that $\phi(d) \in {}_sCl(U)$ for all $d \geq d_0$. Hence $f(\phi(d)) \in {}_sCl(V)$ for all $d \geq d_0$. Thus $(f(\phi(d)))$ RN-converges to $f(x_\alpha)$.

(2) $\Longrightarrow$(3). Let x_α be a fuzzy singleton in X and let $(\phi(d))$ be a fuzzy net in X such that $(\phi(d))$ $S^{'}N$-converges to x_α. By Lemma 3.1, $(\phi(d))$ RN -converges to x_α. By (2), $(f(\phi(d)))$ RN-converges to $f(x_\alpha)$.

(3) $\Longrightarrow$(1). Suppose that f is not fuzzy semi-irresolute. Then there exist a fuzzy singleton x_α in X and a fuzzy semi-open set V containing $f(x_\alpha)$ such that $f(U) \not\leq {}_sCl(V)$ for all fuzzy semi-θ-clopen sets U containing x_α. Thus there exists a fuzzy singleton $x_{\alpha_U} \in U$ such that $f(x_{\alpha_U}) \notin {}_sCl(V)$. Then the fuzzy net (x_{α_U}) $S^{'}N$-converges to x_α but $(f(x_{\alpha_U}))$ does not RN-converges to $f(x_\alpha)$. □

By Lemma 3.1 and Theorem 3.6, we get the following results.

COROLLARY 3.2. *If a function $f : X \to Y$ is fuzzy semi-irresolute, then for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which SN-converges to x_α, the fuzzy net $(f(\phi(d)))$ θN-converges to $f(x_\alpha)$.*

PROPOSITION 3.7. *A fuzzy net $(\phi(d))$ in a fuzzy space X RN-converg -es to x_α if and only if for each fuzzy regular semi-open set U containing x_α, there exists d_0 such that $\phi(d) \in U$ for all $d \geq d_0$.*

PROOF. It follows from Lemma 2.5 and Definition. □

By Theorem 3.6 and Proposition 3.7, we have the following Corollary.

COROLLARY 3.3. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) f is semi-irresolute;
(2) If, for each fuzzy singleton x_α in X, a fuzzy net $(\phi(d))$ in X RN-converges to x_α, then for each fuzzy regular semi-open set V containing $f(x_\alpha)$, there exists d_0 such that $f(\phi(d)) \in V$ for all $d \geq d_0$;
(3) If, for each fuzzy singleton x_α in X, a fuzzy net $(\phi(d))$ in X $S'N$-converges to x_α, then for each fuzzy regular semi-open set V containing $f(x_\alpha)$, there exists d_0 such that $f(\phi(d)) \in V$ for all $d \geq d_0$.

THEOREM 3.8. *For a function $f : X \to Y$, the following statements are equivalent:*
(1) f is fuzzy strongly irresolute;
(2) for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which RN-converges to x_α, the fuzzy net $(f(\phi(d)))$ SN-converges to $f(x_\alpha)$;
(3) for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which $S'N$-converges to x_α, the fuzzy net $(f(\phi(d)))$ SN-converges to $f(x_\alpha)$.

PROOF. The proof is similar to that of Theorem 3.6. □

COROLLARY 3.4. *If a function $f : X \to Y$ is fuzzy strongly irresolute, then for each fuzzy singleton x_α in X and each fuzzy net $(\phi(d))$ in X which SN-converges to x_α, the fuzzy net $(f(\phi(d)))$ RN-converges to $f(x_\alpha)$.*

Therefore, $(f(\phi(d)))$ θN-converges to $f(x_\alpha)$.

4. Some properties

DEFINITION 4.1. A function $f : X \to Y$ is said to be fuzzy semi-θ-open if for each fuzzy semi-θ-open set U in X, $f(U)$ is fuzzy semi-θ-open in Y.

DEFINITION 4.2. A fuzzy space X is said to be semi-θ-T_2 if for each fuzzy singleton x_α and y_β in X with different support, there exist two fuzzy semi-open semi-q-neighborhoods U and V of x_α and y_β, respectively such that ${}_sCl(U) \bigwedge {}_sCl(V) = 0_X$.

THEOREM 4.1. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions.*
(a) *If f is fuzzy semi-θ-open surjection and $g \circ f$ is fuzzy semi-irresolute, then g is fuzzy semi-irresolute.*
(b) *f is fuzzy strongly irresolute and g is fuzzy semi-irresolute, then $g \circ f$ is fuzzy semi-irresolute.*

PROOF. (a) Let V be a fuzzy semi-θ-open set in Z. Since $g \circ f$ is fuzzy semi-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is fuzzy semi-θ-open in X by Theorem 3.4. Since f is semi-θ-open and surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is fuzzy semi-θ-open in Y. By Theorem 3.4, g is fuzzy semi-irresolute.

(b) Let x_α be a fuzzy singleton in X and $y_\alpha = f(x_\alpha)$. Let V be a fuzzy regular semi-open in Z containing $g(y_\alpha) = g(f(x_\alpha))$. Since g is fuzzy semi-irresolute, there exists a fuzzy regular semi-open set U containing y_α such that $g(U) \le V$. Since f is fuzzy strongly irresolute, there exists a fuzzy semi-open set G containing x_α such that $f({}_sCl(G)) \le U$. Hence $(g \circ f)({}_sCl(G)) = g(f({}_sCl(G))) \le g(U) \le V$. By Theorem 3.1, $g \circ f$ is fuzzy semi-irresolute. □

THEOREM 4.2. *If $f : X \to Y$ is fuzzy semi-irresolute injection and Y is fuzzy semi-θ-T_2, then X is fuzzy semi-θ-T_2.*

PROOF. Let x_α and y_β be a pair of fuzzy singletons in X with different support. Since f is injection, $f(x_\alpha) \ne f(y_\beta)$. Then there exist two fuzzy semi-open semi-q-nbds U and V of $f(x_\alpha)$ and $f(y_\beta)$ such that ${}_sCl(U) \bigwedge {}_sCl(V) = 0_Y$. Since ${}_sCl(U)$ and ${}_sCl(V)$ are fuzzy regular semi-open, $f^{-1}({}_sCl(U))$ and $f^{-1}({}_sCl(V))$ are fuzzy regular semi-open in X by Theorem 3.2. Moreover, $f^{-1}({}_sCl(U))$ and $f^{-1}({}_sCl(V))$ are semi-open semi-q-nbds of x_α and y_β, respectively and $f^{-1}({}_sCl(U)) \bigwedge f^{-1}$ $({}_sCl(V)) = 0_X$. Therefore, X is fuzzy semi-θ-T_2. □

Definition 4.3. A fuzzy space X is said to be semi-θ-disconnected if there exist two fuzzy semi-open sets V_1 and V_2 with $V_1 \neq 0_X$ and $V_2 \neq 0_X$ such that ${}_sCl(V_1) \bigwedge {}_sCl(V_2) = 0_X$ and $1_X = {}_sCl(V_1) \bigvee {}_sCl(V_2)$. A fuzzy space X is called semi-θ-connected if it is not semi-θ-disconnected.

Theorem 4.3. *If $f : X \to Y$ is fuzzy semi-irresolute surjection and X is fuzzy semi-θ-connected, then Y is fuzzy semi-θ-connected.*

Proof. Suppose that Y is not fuzzy semi-θ-connected. Then there exist two fuzzy semi-open sets V_1 and V_2 in Y with $V_1 \neq 0_Y$ and $V_2 \neq 0_Y$ such that

$$ {}_sCl(V_1) \bigwedge {}_sCl(V_2) = 0_Y \quad \text{and} \quad 1_Y = {}_sCl(V_1) \bigvee {}_sCl(V_2). $$

Since ${}_sCl(V_1) \neq 0_Y$ and ${}_sCl(V_2) \neq 0_Y$, $f^{-1}({}_sCl(V_1) \neq 0_X$ and f^{-1} $({}_sCl(V_2)) \neq 0_X$. Since ${}_sCl(V_1)$ and ${}_sCl(V_2)$ are fuzzy regular semi-open, $f^{-1}({}_sCl(V_1))$ and $f^{-1}({}_sCl(V_2))$ are fuzzy semi-θ-clopen. Moreover,

$$ {}_sCl(f^{-1}({}_sCl(V_2))) \bigwedge {}_sCl(f^{-1}({}_sCl(V_2))) = 0_X $$

and

$$ 1_X = {}_sCl(f^{-1}({}_sCl(V_1))) \bigvee {}_sCl(f^{-1}({}_sCl(V_2))). $$

Therefore, X is not fuzzy semi-θ-connected. □

Definition 4.4. A fuzzy space X is said to be S^*-closed [14](resp. S-closed [11]) if for each fuzzy semi-open cover $\{V_\alpha \mid \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $1_X = \bigvee_{\alpha \in \Delta_0} {}_sCl(V_\alpha)$(resp.$1_X = \bigvee_{\alpha \in \Delta_0} Cl(V_\alpha)$).

Theorem 4.4. *If $f : X \to Y$ is fuzzy semi-irresolute surjection and X is S^*-closed, then Y is S^*-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a fuzzy semi-open cover of Y. Then $\{{}_sCl(V_\alpha) \mid \alpha \in \Delta\}$ is fuzzy regular semi-open cover of Y. By Theorem 3.2, $\{f^{-1}({}_sCl(V_\alpha)) \mid \alpha \in \Delta\}$ is fuzzy regular semi-open cover of X, and hence semi-open cover of X. Since X is S^*-closed, there exists a finite subset Δ_0 of Δ such that $1_X = \bigvee_{\alpha \in \Delta_0} {}_sCl(f^{-1}({}_sCl(V_\alpha)))$. Since each $f^{-1}({}_sCl(V_\alpha))$ is fuzzy semi-θ-clopen by Theorem 3.2 and Lemma 2.4, ${}_sCl(f^{-1}({}_sCl(V_\alpha))) = {}_sCl_\theta(f^{-1}({}_sCl(V_\alpha))) = f^{-1}({}_sCl(V_\alpha))$. Thus $1_X = \bigvee_{\alpha \in \Delta_0} f^{-1}({}_sCl(V_\alpha))$. Since f is surjection, $1_Y = f(1_X) = \bigvee_{\alpha \in \Delta_0} {}_sCl(V_\alpha)$. Therefore, Y is S^*-closed. □

COROLLARY 4.1. *If $f : X \to Y$ is fuzzy semi-irresolute surjection and X is S^*-closed, then Y is S-closed.*

DEFINITION 4.5. A fuzzy space X is s-regular if for each fuzzy singleton x_α in X and each fuzzy semi-open set V containing x_α, there exists a fuzzy open set U containing x_α such that ${}_sCl(U) \leq V$.

THEOREM 4.5. *If $f : X \to Y$ is fuzzy semi-irresolute and Y is fuzzy s-regular, then f is strongly irresolute.*

PROOF. Let x_α be a fuzzy singleton in X and let V be a fuzzy semi-open set containing $f(x_\alpha)$. Then there exists a fuzzy open set G containing $f(x_\alpha)$ such that $f(x_\alpha) \in G \leq {}_sCl(G) \leq V$. By Theorem 2.9 [10], there exists a fuzzy semi-open U containing x_α such that $f({}_sCl(U)) \leq {}_sCl(G)$. Thus $f({}_sCl(U)) \leq V$ and f is fuzzy strongly irresolute. □

DEFINITION 4.6. A function $f : X \to Y$ is said to be fuzzy semi-continuous [1](resp. fuzzy irresolute [12]) if $f^{-1}(V)$ is fuzzy semi-open in X for each fuzzy open(resp. fuzzy semi-open) set V in Y.

THEOREM 4.6. *If $f : X \to Y$ is fuzzy semi-continuous and Y is fuzzy s-regular, then f is fuzzy irresolute.*

PROOF. Let x_α be a fuzzy singleton in X and let V be a fuzzy semi-open set containing $f(x_\alpha)$. Then there is a fuzzy open set G containing $f(x_\alpha)$ such that ${}_sCl(G) \leq V$. Since f is semi-continuous, $U = f^{-1}(G)$ is fuzzy semi-open in X and $x_\alpha \in U$. Thus $f(U) \leq V$ and f is fuzzy irresolute. □

By Remark 3.1 [10], Theorem 4.5 and Theorem 4.6, we get the following result.

COROLLARY 4.2. *If $f : X \to Y$ is a function and Y is fuzzy s-regular, then the following statements are equivalent:*
(1) *f is fuzzy strongly irresolute;*
(2) *f is fuzzy irresolute;*
(3) *f is fuzzy semi-irresolute;*
(4) *f is fuzzy semi-continuous.*

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