What makes bowling balls hook?

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This article presents exact equations of motion for a rotating bowling ball in a form that explicitly separates contributions due to nonequal principal moments of inertia, center-of-mass offset, and friction between the ball and lane. A computer program that solves the equations demonstrates that all of these factors are important for a realistic analysis of bowling. These factors significantly affect how much balls hook, that is, deflect sideways and approach the pins at an oblique angle. Simulations that approximate real bowling conditions indicate that the largest contribution comes from variable friction along the lane, that is, bowling lanes are generally prepared so that lane friction is higher by a factor of 2 or more along the last one-third of the ball's trajectory. The analysis supports most (but not all) of the guidelines that bowlers have developed for predicting ball performance.
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I. INTRODUCTION

Most people are familiar with the basics of bowling. In the United States about four million people compete in sanctioned leagues, about 50 million bowl at least once each year, and the networks regularly televise professional tenpins matches. Thus bowling is a fruitful source of examples for teaching about rotational mechanics because the vagaries of a bowling ball's path are familiar to most students and available on videotape. The rotation of the ball is an essential feature of bowling, because higher scores are achievable if the ball takes a path that curves or "hooks" and thus approaches the pins at an oblique angle.

Unfortunately for those who consider both physics and bowling serious pastimes, the available literature about bowling is inadequate.[1] Articles by physicists and engineers make overly simplistic assumptions about the mechanical properties of bowling balls or lanes. For example, Hopkins and Patterson[2] assume that the ball is a uniform sphere and that lane friction is a constant along its path; they thus conclude that the ball's curved path is parabolic and does not curve more sharply as it approaches the pins. Zecchini and Foutch[3] discuss the path for a ball with two distinct principal moments of inertia; however, they present equations only for the case where the three principal moments of inertia are identical, and do not consider balls for which the center of mass is offset from the geometrical center of the ball. Finally, although Refs. 4 and 5 treat balls with distinct principal moments and offset centers of mass, both focus on computational issues and do not provide much insight into how variations in the properties of the ball affect the path.

There also are numerous publications written by coaches or people associated with ball drilling or manufacture.[6] These articles are confusing to physicists because bowlers and drillers have their own technical language to describe the properties of bowling balls, and it is not immediately obvious how to translate this language into physics (see Table I). Moreover, these articles never include equations, and some that purport to explain the principles of rotational mechanics are (to a physicist) just plain horrible. However, do not be fooled; the better bowlers do understand how changing the mechanical properties of a bowling ball affects performance and have developed guidelines for predicting its behavior (Table II).

The purpose of this paper is to present a realistic framework for evaluating the motion of bowling balls. Section II and the appendices present the equations of motion with separate terms describing the influence of nonidentical principal moments, the position of the center of mass, and friction. Section III presents examples of ball trajectories for a variety of idealized situations. Finally, I discuss the validity of various assertions such as those in Table II.

II. EQUATIONS OF MOTION

In tenpin bowling, the bowler releases the ball with both rotational and translational motion. Initially the ball slides along the lane and, before it reaches the pins, it may or may not begin rolling without slipping.

Consider a coordinate system (see Fig. 1) where the x axis extends from the foul line $(x=0)$ to the pins, the y axis extends from the right gutter $(y=0)$ toward the left, and the z axis extends upward from the center of the ball $(z=0)$. Let $\vec{r}$ be the position of the ball's center of mass, and let $\vec{r}_\Delta$ and $\vec{R}_{\text{con}}$ be vectors extending respectively from the center of mass to the center of the ball, and from the center of the ball to the point of contact on the lane (Fig. 2).

If the ball has mass M, moment of inertia tensor $\mathbf{I}$, and rotates with angular velocity $\vec{\omega}$, the force and torque equations about the center of mass are

$$M\ddot{\vec{r}}=\vec{F}_{\text{con}}+\vec{F}_g\,, \tag{1}$$

$$\frac{d}{dt}(\mathbf{I}\vec{\omega})=(\vec{r}_\Delta+\vec{R}_{\text{con}})\times\vec{F}_{\text{con}}\,, \tag{2}$$

where $\vec{F}_g$ and $\vec{F}_{\text{con}}$ are the gravitational and the contact force applied by the lane to the ball, respectively. In the simplest case where $\mathbf{I}$ is diagonal and $\vec{r}_\Delta$ is zero, Eq. (2) becomes

$$\mathbf{I}\vec{\alpha}=\tau, \tag{3}$$

where $\vec{\alpha}$ is the angular acceleration and τ is the frictional torque $\vec{R}_{\text{con}}\times\vec{F}_{\text{con}}$.

Table I. Explanation of language used by drillers and coaches to describe mechanical features of bowling balls. CMO is the center-of-mass offset, and $\mathbf{R}_G$ is the radius of gyration.

Bowlers' language	*Physicists' language*
	Mass distribution
Pin	Weight block symmetry axis—intersection with ball surface.
Label	Center-of-mass offset axis before drilling—intersection with ball surface.
Pin in (out)	Center-of-mass axis before drilling and weight block symmetry axis form an angle of 25° or less (or more)
Top (bottom) weight	Center-of-mass is toward (opposite) holes.
Finger (thumb) weight	Center-of-mass is $\perp$ to holes, in direction toward finger (thumb) holes.
Left or positive (right or negative) side weight	Center-of-mass is $\perp$ to holes, toward the left (right) when facing holes and thumb hole is at 6 o'clock.
	Behavior of ball
Length	Distance ball slides before rolling.
Backend reaction	Hook that causes ball to arrive at pins at a highly oblique angle.
Positive axis point	Initial rotation axis—intersection with ball surface.
Axis drilling	Initial rotation axis is relatively close to weight block symmetry axis.
Label or high RG drilling	Initial rotation axis is nearly 90° from weight block symmetry axis.
Leverage drilling	Initial rotation axis is 30°–60° from weight block symmetry axis.
High (low) RG ball	$\mathbf{R}_G$ is closer to 7.11 cm (6.17 cm).
Skid-flip ball	Large difference between $\mathbf{R}_G$ for different axes; properly drilled ball thus slides further before rolling.
Spare ball	$\mathbf{R}_G$ is nearly the same along all axes and the ball has low friction on both oiled and unoiled lanes.
Reactive ball	Friction is very low on oil, very high when there is no oil.

In general, $\mathbf{I}$ is not diagonal and $\vec{r}_\Delta$ is not zero. For example, for nondiagonal $\mathbf{I}$ the left-hand side of Eq. (2) becomes[7]

$$\frac{d}{dt}(\mathbf{I}\vec{\omega}) = \mathbf{I}\vec{\alpha} + \vec{\omega}\times(\mathbf{I}\vec{\omega}), \tag{4}$$

and Eq. (1) is coupled to Eq. (2) because the contact force $\vec{F}_{\rm con}$ changes as the ball rotates; that is, if $\vec{r}_\Delta$ is nonzero, $\vec{F}_{\rm con}$ depends on $\vec{\alpha}$. Thus to determine the ball's motion, we first solve Eq. (2) to find $\vec{\alpha}$, then substitute $\vec{F}_{\rm con}$ into Eq. (1), and solve for the ball's position.

Table II. Assertions about bowling balls, stated in both bowlers' (B) and physicists' (P) language. Assertions like these appear explicitly or implicitly in bowling journals. The simulations in this paper do not confirm assertions B4/P4 or B5/P5.

B1: High-RG balls get better length and good backend reaction. P1: Increasing the moment of inertia (that is, the radius of gyration $\mathbf{R}_G$) makes a ball slide further and reach pins at a more oblique angle.
B2: Increasing the top or finger weight increases the length and backend reaction. P2: If the center-of-mass offset r_Δ remains mostly on the left side of the ball during its trajectory, the ball slides further and reaches the pins at a more oblique angle.
B3: Balls with leverage drilling get more length and more backend reaction than balls with label or axis drilling. P3: A ball drilled so that the initial rotation axis is midway between the principal rotation axes will slide further and reach pins at a more oblique angle.
B4: Balls with high differential RG get more backend reaction. P4: A ball where the radii of gyration are not all equal will reach the pins at a more oblique angle.
B5: Two balls may behave differently if their weight blocks have different shapes, even though they have identical coverstocks, positive weights, and RG. P5: Balls with identical surface friction, center-of-mass offset, and moments of inertia may have different rotational properties.

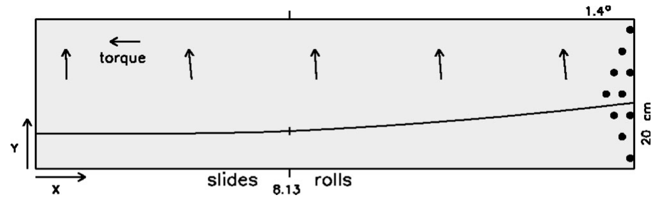

Fig. 1. Coordinate system. Path for a ball (case 4 in Table III) with center-of-mass offset $r_\Delta = 1$ mm, delivered with rotation $\vec{\omega}$ and $\vec{r}_\Delta$ parallel to the y axis, initial velocity 8 m/s, and $\omega = 30$ rad/s. The lane friction μ is 0.12. The ball slides until it reaches the point indicated by a tic mark, and rolls thereafter. The torque due to the center-of-mass offset is initially perpendicular to $\vec{\omega}$ so that the ball deflects to the left and approaches the pins at an oblique angle (the pocket angle). Labels on this and all subsequent trajectory figures indicate the pocket angle (here, 1.4°), the deflection from a straight path (here, 20 cm), and (if appropriate) the distance traveled before rolling begins (here 8.13 m). The scales of the y axes (widths of the lanes) are exaggerated 4.5 times to make hook more visible.

Because we wish to evaluate how the off-center center of mass and nondiagonal components of $\mathbf{I}$ influence the motion, we write Eq. (2) so that it preserves the form of Eq. (3), that is, we separate the influence of the nondiagonal components of $\mathbf{I}$ and nonzero $\vec{r}_\Delta$. We rewrite $\mathbf{I}$ as $\mathbf{I}_o + \mathbf{I}_{\text{dev}}$, where $\mathbf{I}_o$ is diagonal with elements $\mathbf{I}_{\text{ave}}$ equal to the mean of $\mathbf{I}$'s three principal moments. Then Eq. (4) becomes

$$\frac{d}{dt}(\mathbf{I}\vec{\omega}) = (\mathbf{I}_o + \mathbf{I}_{\text{dev}})\vec{\alpha} + \vec{\omega} \times (\mathbf{I}_{\text{dev}}\vec{\omega}). \tag{5}$$

There is no $\vec{\omega} \times (\mathbf{I}_o\vec{\omega})$ term because $\vec{\omega}$ and $\mathbf{I}_o\vec{\omega}$ are parallel. Thus the term $\vec{\omega} \times (\mathbf{I}_{\text{dev}}\vec{\omega})$ expresses how the nondiagonal components of $\mathbf{I}$ affect the ball's motion. Because this term does not involve $\vec{\alpha}$ explicitly, we can bring it to the right-hand side of Eq. (2) and think of it as a "pseudo-torque" which makes the ball "roll funny" because $\mathbf{I}$ is nondiagonal. We will call this term τ_{dev} so that $\tau_{\text{dev}} = (\mathbf{I}_{\text{dev}}\vec{\omega}) \times \vec{\omega}$.

Similarly, if $\vec{r}_\Delta$ is nonzero, the ball's center of mass experiences up and down motion as the ball rotates, and thus the contact force $\vec{F}_{\text{con}}$ depends on $\vec{\alpha}$ and $\vec{\omega}$. To preserve the form of Eq. (3), we write the terms that depend explicitly on $\vec{\alpha}$ on the left and the other terms on the right. Appendix A derives an exact, explicit expression for Eq. (2) in this form for the case when the ball is sliding:

$$(\mathbf{I}_o + \mathbf{I}_{\text{dev}} + \mathbf{I}^s_\Delta + \mathbf{I}^s_{\Delta\Delta})\vec{\alpha} = \vec{\tau}_{\text{fric}} + \vec{\tau}_{\text{dev}} + \vec{\tau}^s_\Delta + \vec{\tau}^s_{\Delta\Delta}. \tag{6}$$

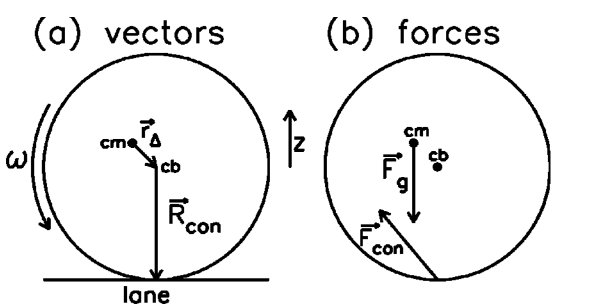

Fig. 2. The definition of the vectors $\vec{r}_\Delta$ and $\vec{R}_{\text{con}}$ (a) and force diagram (b). While the ball rotates, both $\vec{r}_\Delta$ and $\vec{R}_{\text{con}}$ have constant lengths (r_Δ and R_{ball}); however, $\vec{r}_\Delta$ rotates with the ball while $\vec{R}_{\text{con}}$ is always directed downward. If r_Δ is nonzero, the difference between gravity and the vertical component of the contact force $\vec{F}_{\text{con}}$ provides the force allowing vertical acceleration of the center of mass.

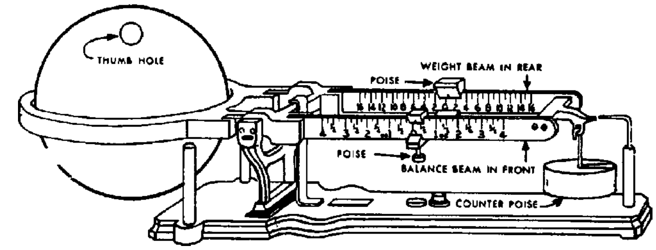

Fig. 3. Determination of the center-of-mass offset. The sketch shows the balance that the American Bowling Congress and Women's International Bowling Congress instructions recommend for determining the center-of-mass offset. The device holds the ball in a fixed position, and thus the amount of mass necessary to balance the ball depends on the distance L between the ball center and the fulcrum, and whether the center of mass is to the left or right of the ball's center. Thus, if counterweights are adjusted at right to balance the ball, but an additional mass δm is required to balance the scale after the ball is rotated 180°, the component of center-of-mass offset along the axis parallel to the lever arm is $L\delta m/(2M)$, where M is the ball's mass.

Here $\mathbf{I}^s_\Delta$ and $\mathbf{I}^s_{\Delta\Delta}$ are (nondiagonal) matrices that account for the effects of the offset center of mass; they are respectively first and second order in $\vec{r}_\Delta$; the superscript "s" designates terms specific to the sliding case. On the right, $\vec{\tau}_{\text{fric}}$ is the friction torque $\mu R_{\text{con}} M g$ for a uniform sliding sphere, τ_{dev} is the pseudo-torque because the ball isn't uniform (see above), and $\vec{\tau}^s_\Delta$ and $\vec{\tau}^s_{\Delta\Delta}$ are correction torques which are first and second order in $\vec{r}_\Delta$.

Appendix B derives the corresponding equation for the case where the ball is rolling:

$$(\mathbf{I}_o + \mathbf{I}_{\text{dev}} + \mathbf{I}_{\text{Roll}} + \mathbf{I}^r_\Delta)\vec{\alpha} = \vec{\tau}_{\text{dev}} + \vec{\tau}^r_\Delta + \vec{\tau}^r_{\Delta\Delta}, \tag{7}$$

the superscript "r" designates rolling. Here there is no sliding friction, but the constraint that the ball rolls without slipping appears as a (nondiagonal) matrix $\mathbf{I}_{\text{Roll}}$ on the left.

III. EXAMPLES OF BALL TRAJECTORIES

What is the relative importance of the various terms in Eqs. (6) and (7)? Can any be safely ignored if we want to simulate the motion of a bowling ball? In this section I evaluate their importance using a program that implements Eqs. (1), (6), and (7).

A. Typical parameters

According to the rules of bowling,[8] a drilled bowling ball must weigh no more than 16 lbs and have a diameter between 8.500 and 8.595 in. ($R_{\text{ball}} \sim 10.85$ cm). It need not be uniform or symmetric; indeed, balls are generally manufactured by casting a resin about a "weight block" that may have all manner of shapes. However, the drilled ball's radius of gyration $\mathbf{R}_G$ about any axis must fall between 2.43 and 2.80 in. (6.17–7.11 cm), with the maximum difference δ about any two axes not exceeding 0.08 in. (0.20 cm).

If the drilled ball is clamped in different orientations and balanced with counterweights on a special balance with a lever arm with length L of 16.2 cm (Fig. 3),[9] the maximum difference in mass necessary to counterbalance opposite orientations may not exceed 3 ounces. The difference of the remaining two axes may not exceed 1 ounce. In effect, this requirement means r_Δ must be 1 mm or less.[10] To aid drillers, there is a pencil-sized "pin" visible on the surface of the

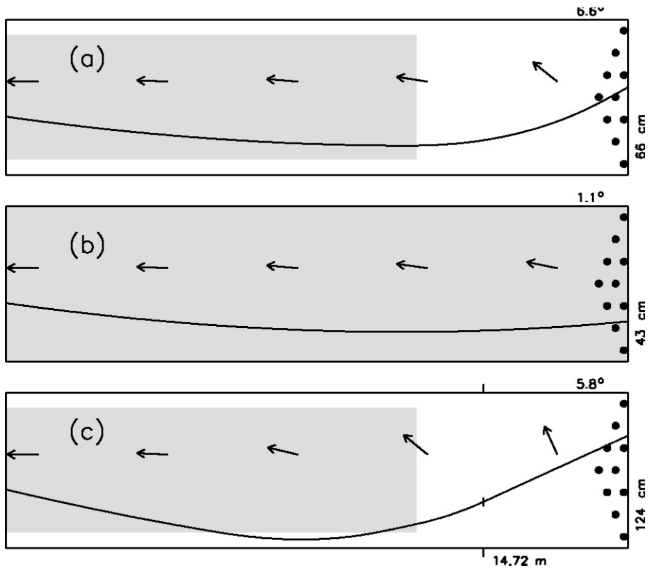

Fig. 4. Typical oil pattern and trajectory for a ball with sideways rotation. Bowling lanes are dressed with oil (darker color) covering the central four-fifths of the lane over all but the last 20 feet before the pins. Thus, when a ball delivered with sideways rotation leaves the oil, it deflects more strongly to the left [(a) case 7 in Table III] than does the same ball delivered on an evenly oiled lane [(b) case 6 in Table III]. Because there is typically no oil near the edges, a ball delivered further toward the right [(c) case 8 in Table III] deflects the farthest to the left. However, because it stops sliding and begins after traveling 14.72 m, it still arrives at the pins with a smaller pocket angle (5.8°) than the ball in the (a) (6.6°). Arrows along the path indicate the orientation of the rotation $\vec{\omega}$.

ball that indicates the orientation of the weight block. During manufacture the ball is placed on a low friction spinner to locate where the center of mass offset vector $\vec{r}_\Delta$ intersects the ball's surface. This point is indicated by a special mark, which is called the "label" because it generally is some feature of the manufacturer's identifying label.

Bowling lanes are 42 in. wide and extend 60 ft from a line (the foul line) where the bowler delivers the ball. The rules state that the friction coefficient μ cannot exceed 0.39; however, generally μ isn't more than about 0.20 even for "dry" (unoiled) portions of the lane. In practice, selected portions of lanes are dressed with a thin layer of oil (Fig. 4); on oiled areas μ is typically about 0.04. Oil is an essential feature of contemporary bowling; different oil patterns may be applied to make competition either easier or more difficult, and, over the course of a match, patterns change as balls move the oil over the lane.

There is, of course, considerable individual variation in the initial speed V_o and angular velocity ω_o with which bowlers deliver a ball. Studies indicate that V_o and ω_o tend to be higher for higher than average bowlers.[11] For all calculations in this paper I will assume that V_o is 8.0 m/s and ω_o is 30 rad/s, values that are typical for bowlers who average about 200.

B. Simulation of motion

What bowlers care about is whether something makes their ball hook a few inches more or less.[12] I wrote a program that simulates the motion of a bowling ball with specified initial orientation, moment of inertia tensor **I** and center-of-mass offset $\vec{r}_\Delta$, delivered with velocity V_o and initial rotation $\vec{\omega}_o$ on a lane with one of several specified friction patterns. The program solves Eq. (6) or (7) and Eq. (1) to determine the position, orientation, and angular velocity of the ball using a time step of 0.001 s.

Because both **I** and $\vec{r}_\Delta$ change as the ball rotates, it is necessary to determine the rotation matrix at each time step. This determination presents some computational difficulties. If the time step is too large, errors arise because the principal axes change too much between increments, while if the time step is too small, the numerical solution is overcome by errors arising from taking small differences. The approach I used was to parametrize the ball's orientation using Euler angles. Because both the first and second time derivatives of the Euler angles can be written as linear combinations of $\vec{\omega}$ and $\vec{\alpha}$,[13] the solution to Eq. (6) or (7) makes it possible to recalculate the Euler angles at each time step and construct the rotation matrix. This calculation becomes unstable when the ball's orientation coincides with certain singular values in the Euler angles. When this problem occurred, it was necessary to redo the calculation, after rotating the coordinate system to a different reference frame. This singularity afflicts many common problems in rotational dynamics, and various different computational schemes have been proposed to overcome it.[14]

1. Pure forward and pure sideways rotation

To illustrate the effect of various parameters on the ball trajectory, first consider the trajectory of a ball delivered with purely forward rotation ($\vec{\omega}_o$ horizontal, parallel to the y axis). Many accomplished bowlers deliver balls that rotate somewhat like this; the rotation comes about as the ball slips off the bowler's fingers. The bowler's hand and fingers remain directly behind and under the ball during delivery, and for this reason and because there is no sideways twisting motion, the delivery can be repeated with minimal variation, even as the bowler tires.

How much does such a ball hook? There is zero deflection if there is no center-of-mass offset and **I** is a scalar, that is, if it has three identical principal moments (case 1 in Table III); in this case sliding friction makes ω increase until the ball is some distance from the foul line, then it rolls without slipping thereafter (see Fig. 5). There also is no deflection in some special cases where **I** is not a scalar and r_Δ is nonzero (cases 2 and 3 in Table III), although the ball slides further before rolling. However, if $\vec{r}_\Delta = 1$ mm and is initially parallel to $\vec{\omega}_o$, the deflection is significant (20 cm for case 4). Moreover, when **I** is not a scalar, the ball may deflect a significant amount if the principal axes are offset even a few degrees from $\vec{\omega}_o$ (case 5). These examples demonstrate clearly why ball drillers can't ignore either $\vec{r}_\Delta$ or the orientation of the weight block, because an improperly drilled ball will hook in the wrong direction.

Many accomplished bowlers deliver the ball with considerable sideways rotation, that is, with $\vec{\omega}_o$ approximately horizontal and aligned with the negative x axis. Such a ball (see Fig. 4) hooks to the left, although the path is not parabolic—the friction torque causes the angular momentum vector to swing around until $\vec{\omega}_o$ is perpendicular to the ball's path; if the friction is high enough, the ball ultimately stops sliding and rolls to the pins. Of course, the amount and distribution of friction μ affects the amount of deflection and whether the ball begins rolling before it reaches the pins.

Table III. Parameters for bowling simulations. TOA (take-off angle) is the angle between the velocity $\vec{V}_o$ and the x axis at release; slide is the distance the ball slides before it begins rolling; hook is leftward deflection in comparison to a ball that travels straight; pocket angle is the angle between the velocity and the x axis when the ball reaches the pins; positive TOA and pocket angles correspond to balls heading toward the left gutter. For all the simulations, the initial speed V_o is 8 m/s and ω_o is 30 rad/s. To specify the initial orientation of $\mathbf{I}$, the long axis $\vec{a}_{\text{long}}$ is the ball's principal axis corresponding to the smallest principal moment. If the friction μ is labeled as constant, then the friction over the entire lane surface is constant with the value specified; if it is labeled as "typical," then the friction pattern is as in Fig. 4, with μ=0.04 and 0.20 on different portions of the lane.

Case	TOA	$\mathbf{R}_G$ (cm)	Offset $\vec{r}_\Delta$ (cm)	Friction μ	Slide (m)	Hook (cm)	Pocket angle
			Forward rotation: $\vec{\omega}_o$ *horizontal toward left*				
1	0°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	0	constant 0.12	8.10	0	0°
2	0°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	1 mm; $\vec{r}_\Delta \perp \vec{\omega}_o$	constant 0.12	8.16	0	0°
3	0°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$=6.7, 6.9, 6.9 along the three axes; $\vec{a}_{\text{long}} \perp \vec{\omega}_o$	0	constant 0.12	8.41	0	0°
4	0°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	1 mm; $\vec{r}_\Delta \parallel \vec{\omega}_o$	constant 0.12	8.13	20	1.4°
5	0°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$=6.7, 6.9, 6.9 along the three axes; $\vec{a}_{\text{long}}$ horizontal, 75° from $\vec{\omega}_o$	0	constant 0.12	8.60	7	0.7°
			Sideways rotation: $\vec{\omega}_o$ *horizontal toward rear*				
6	−1.8°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	0	constant 0.04	to pins	43	1.1°
7	−1.8°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	0	typical	to pins	66	6.6°
8	−3°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.7	0	typical	14.72	124	5.8°
			Combination (low RG): $\vec{\omega}_o$ *35° to rear of left, inclined 15° above horizontal*; $\vec{a}_{\text{long}}$ *horizontal, 35° to rear of left*				
9	−1°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.2	0	typical	15.60	51	3.1°
10	−1°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$ 6.2, 6.4, 6.4 along the three axes;	0	typical	15.64	48	3.0°
11	−1°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=6.2	0.7 mm; $r_\Delta \parallel$ to $\vec{a}_{\text{long}}$	typical	15.83	56	4.0°
12	−1°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$ 6.2, 6.4, 6.4 along the three axes	0.7 mm; $r_\Delta \parallel$ to $\vec{a}_{\text{long}}$	typical	15.84	53	3.8°
			Combination (high RG): $\vec{\omega}_o$ *35° to rear of left, inclined 15° above horizontal*; $\vec{a}_{\text{long}}$ *horizontal, 35° to rear of left*				
13	−1°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=7.1	0	typical	16.40	54	3.9°
14	−1°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$ 6.9, 7.1, 7.1 along the three axes;	0	typical	16.41	52	3.6°
15	−1°	$\mathbf{I}$ scalar: $\mathbf{R}_G$=7.1	0.7 mm; $r_\Delta \parallel$ to $\vec{a}_{\text{long}}$	typical	16.61	58	4.8°
16	−1°	$\mathbf{I}$ nonscalar: $\mathbf{R}_G$ 6.9, 7.1, 7.1 along the three axes	0.7 mm; $r_\Delta \parallel$ to $\vec{a}_{\text{long}}$	typical	16.62	56	4.6°
17	−1°			constant 0.04	to pins	37	1.5°
18	−3°			constant 0.12	10.91	120	2.7°

2. More realistic examples

Most serious bowlers deliver a ball that has a combination of forward and sideways rotation and with the rotation $\vec{\omega}_o$ inclined with respect to the horizontal. We can assess how the center-of-mass offset $\vec{r}_\Delta$, nonscalar moments of inertia, and radius of gyration $\mathbf{R}_G$ affect the ball trajectory by simulating motion on a lane with a typical oil pattern as in Fig. 4.

For the parameters chosen in Table III the simulation allows us to test the first four assertions in Table II. First, the simulation demonstrates that balls with larger $\mathbf{R}_G$ slide further, deflect more, and arrive at the pins with larger pocket angles (see Fig. 6; in Table III compare cases 9 to 13; 10 to 14, etc.). Furthermore, a nonzero $\vec{r}_\Delta$ enhances slide, deflection, and pocket angle (Fig. 6 and compare cases 9 to 11, 10 to 12, etc. in Table III) as long as the center of mass stays mostly on the left side of the ball's path as it moves along its trajectory. Thus the second assertion is also true: top and finger weight (defined in Table I) increase the slide and deflection because, for typical deliveries, the bowler's hand and fingers are somewhat to the left of the ball's center.

The simulations indicate that variable friction along the lane is essential for achieving a trajectory that arrives at the headpin with a large pocket angle. For example, if case 16 is altered so that the lane friction is constant but low (μ=0.04 in case 17), the ball experiences very little deflection and has a pocket angle of only 1.5°. However, if the lane friction is constant but high (μ=0.12 in case 18), the ball deflects but starts rolling about halfway down the lane. If the ball is to reach the pins in the middle of the lane, one must throw it 3° toward the right, that is, deliver it so that the velocity vector is at an angle of 3° to the right of the x axis. When it reaches the pins, the pocket angle is 2.7°, or about 2° less than for case 16.

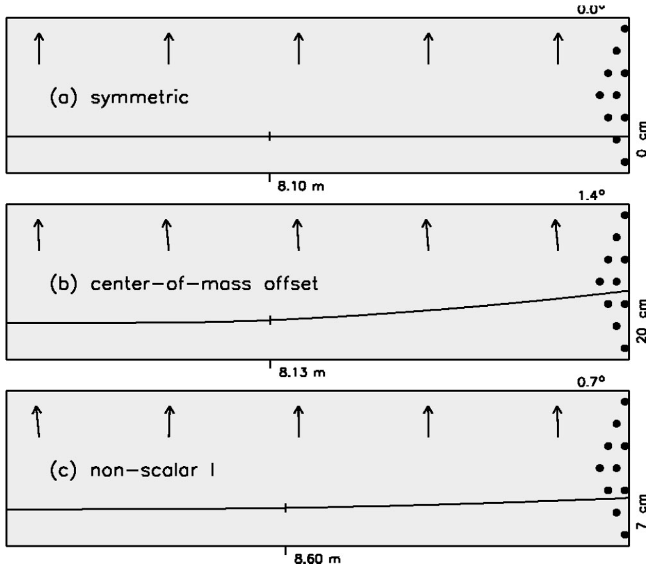

Fig. 5. Forward rotation. A ball delivered with forward rotation only will not hook if it is sufficiently symmetric [(a) case 1 in Table III]. But, in general, the ball will hook if there is either center-of-mass offset [(b) case 4 in Table III] or nonscalar moments of inertia [(c) case 5 in Table III]. Arrows indicate the direction of $\vec{\omega}$.

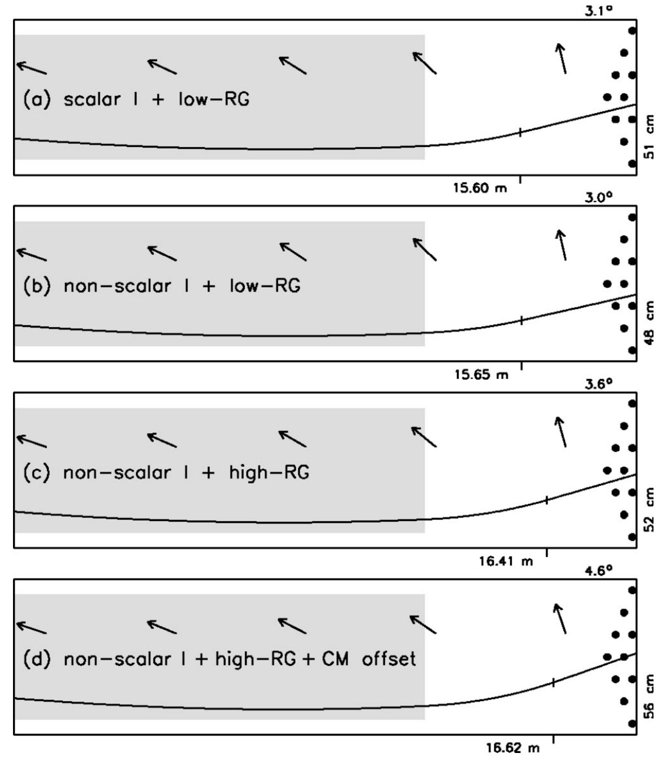

Fig. 6. Realistic ball trajectories. A ball delivered with both sideways and forward rotation on a typically oiled lane will slide more than 15 m before rolling, and deflecting about 0.5 m before reaching the pins with pocket angle of 3°–5°. Increasing the radius of gyration $\mathbf{R}_G$ and the center-of-mass offset increases the pocket angle; whether the ball has scalar or nonscalar $\mathbf{I}$ makes little difference. (a) Case 9 in Table III: $\mathbf{I}$ scalar, low $\mathbf{R}_G$; (b) case 10: $\mathbf{I}$ nonscalar, low $\mathbf{R}_G$; (c) case 14: $\mathbf{I}$ nonscalar, high $\mathbf{R}_G$; and (d) case 16: $\mathbf{I}$ nonscalar, high $\mathbf{R}_G$ and center-of-mass offset.

IV. DISCUSSION

We have presented the equations of motion for a bowling ball in a form that makes it straightforward to evaluate the importance of nonscalar moments of inertia, a center-of-mass offset, and lane friction. These equations are new and are exact for the given assumptions, that is, (1) the ball surface is a rigid sphere and the lane is flat, (2) the sliding friction acts along the direction of slip and is proportional to the normal force, (3) the rolling friction is zero, and (4) the air resistance is zero.[15] The last two assumptions are easily relaxed, but because they produce forces that act along the direction of motion, they are unlikely to affect the results in this paper. The analysis here also indicates that it probably isn't necessary to include the very small terms in Eqs. (6) and (7) (for example, $\mathbf{I}^s_{\Delta\Delta}$, $\mathbf{I}^s_{\Delta}$, $\vec{\tau}^s_{\Delta}$, and $\vec{\tau}^s_{\Delta\Delta}$); under real conditions minor spatial and temporal variations in the lane friction almost certainly affect ball motion more than these terms.

So, what makes bowling balls hook, that is, deflect sharply leftward and reach the pins at an oblique pocket angle? The simulations demonstrate that under conditions that resemble real bowling, variable friction along the lane is the principal factor making it possible for balls to arrive at the headpin with a large pocket angle. Bowling lanes are oiled so that friction increases sharply as the ball approaches the pins. If, instead, the friction is constant everywhere but low, the ball doesn't deflect enough to achieve large pocket angles. If the friction is constant and high, the ball deflects but rolls too soon to allow large pocket angles. Although friction is the largest factor influencing hook, the simulations show that a ball's center-of-mass offset and nonscalar moments of inertia also significantly affect its motion. Of these factors, the center-of-mass offset is more important, but simulations that ignore either factor do not accurately determine ball trajectories.

None of these conclusions will surprise bowlers, and although few bowlers understand or care about the language of physics, the analysis here confirms that many bowlers generally do understand bowling balls. In particular, most of the guidelines they use for drilling balls and predicting their trajectories are in accord with my simulations.[16]

However, two assertions that bowlers commonly make merit some discussion. First, is it true that a ball hooks more if it has different radii of gyration along different axes (assertion 4 in Table II)? Clearly this behavior is not always found (for example, in Table III, compare case 9 to 10). I suspect that when it is observed, it is largely because the precession of the ball around the principal axes leaves less oil on the ball surface at the point of contact with the lane[17] and thus increases the friction when the ball reaches the unoiled part of the lane. This increased friction from a constantly "clean" ball surface causes the ball to hook more, but was not considered in the simulations.

Second, does the shape and type of material in a bowling ball's core affect the motion (assertion 5 in Table II)? That is, if we manufacture two balls with the same surface material, the same center-of-mass offset, and the same moments of inertia, is it possible that they will behave differently just because their cores are different? My answer—the physicist's answer—is no. However, most bowlers would probably answer yes, or at least ball manufacturers must think so. Otherwise, why do advertisements commonly show pictures of the core block and focus on exotic materials, like titanium, in core components? It is conceivable (but unlikely) that two balls with different core materials could have different coefficients of restitution, thus affecting their ability to knock down pins. But if they have the same moments of inertia and experience the same surface friction, their rotation and motion along the lane will be identical.

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APPENDIX A: EQUATIONS OF MOTION: SLIDING

For either sliding or rolling, we can combine Eqs. (2) and (4) to obtain

$$\mathbf{I}\vec{\alpha}+\vec{\omega}\times(\mathbf{I}\vec{\omega})=(\vec{r}_\Delta+\vec{R}_{\text{con}})\times\vec{F}_{\text{con}}. \tag{A1}$$

However, the force equation (1) and the torque equation (A1) are not independent, because both depend on the contact force $\vec{F}_{\text{con}}$, which depends on whether the ball is rolling or sliding.[18]

If the ball slides, the relative motion of the floor past the ball will be

$$\vec{s}=\vec{R}_{\text{con}}\times\vec{\omega}-\dot{\vec{r}}_{\text{cb}}, \tag{A2}$$

where $\vec{r}_{\text{cb}}$ is the position of the center of the ball. We assume the horizontal component of the contact force $\vec{F}_{\text{con}}$ is due to ordinary Coulomb friction, and thus is proportional to a friction coefficient μ times the normal component F_n of the contact force:

$$\vec{F}_{\text{con}}=F_n(\mu_x,\mu_y,1)=F_n\vec{\mu}. \tag{A3}$$

We have defined the scalars μ_x and μ_y such that $\mu\vec{s}/|\vec{s}| =(\mu_x,\mu_y,0)$.

What is the normal force F_n? If $\vec{r}_\Delta$ is zero, $F_n=Mg$, but otherwise the difference between F_n and Mg will provide the force that allows vertical motion of the center of mass. Thus:

$$F_n=M(g-\ddot{r}_{\Delta,z}), \tag{A4}$$

where $\ddot{r}_{\Delta,z}$ is the z component of $\ddot{\vec{r}}_\Delta$. However, we can rewrite Eq. (A4) because[19]

$$\ddot{\vec{r}}_\Delta=\frac{d\dot{\vec{r}}_\Delta}{dt}=\frac{d(\vec{\omega}\times\vec{r}_\Delta)}{dt}=\vec{\alpha}\times\vec{r}_\Delta+\vec{\omega}\times(\vec{\omega}\times\vec{r}_\Delta). \tag{A5}$$

Thus Eq. (A4) becomes

$$F_n=M(g+a_\alpha+a_\omega), \tag{A6}$$

where

$$a_\alpha=[\vec{r}_\Delta\times\vec{\alpha}]_z \quad \text{and} \quad a_\omega=[(\vec{\omega}\times\vec{r}_\Delta)\times\vec{\omega}]_z, \tag{A7}$$

that is, a_α and a_ω are the z components of the indicated vectors.

If we substitute the results from Eqs. (A7), (A6), and (A3) into Eq. (1), the right-hand side now depends only on $\vec{\alpha}$ and the parameters of the motion, such as the ball's velocity and $\vec{\omega}$. Thus, after solving Eq. (2) and finding $\vec{\alpha}$, we can solve Eq. (1) to find the change in the ball's velocity and position.

As explained in Sec. II, it is useful to separate terms describing the contributions of various effects. For convenience, we also normalize all tensors and torques by dividing by the ball's mass M. If we write $\mathbf{I}=\mathbf{I}_o+\mathbf{I}_{\text{dev}}$, where $\mathbf{I}_o$ and $\mathbf{I}_{\text{dev}}$ are the diagonal and a non-diagonal component (see Sec. II), and apply Eqs. (A3) and (A6) to Eq. (A1), the latter becomes

$$(\mathbf{I}_o+\mathbf{I}_{\text{dev}})\vec{\alpha}+\vec{\omega}\times(\mathbf{I}_{\text{dev}}\vec{\omega})=(\vec{r}_\Delta+\vec{R}_{\text{con}})\times(g+a_\alpha+a_\omega)\vec{\mu}. \tag{A8}$$

We can further expand the right-hand side, and use matrices to rewrite the a_α terms that involve $\vec{\alpha}$. That is, if $\vec{r}_\Delta =(r_{\Delta x},r_{\Delta y},r_{\Delta z})$ and we define

$$\mathbf{I}_\Delta^s=R_{\text{ball}}\begin{bmatrix} r_{\Delta y}\mu_y & -r_{\Delta x}\mu_y & 0\\ r_{\Delta y}\mu_x & r_{\Delta x}\mu_x & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{A9}$$

then $\mathbf{I}_\Delta^s\vec{\alpha}=-a_\alpha\vec{R}_{\text{con}}\times\vec{\mu}$. And if

$$\mathbf{I}_{\Delta\Delta}^s=\begin{bmatrix} r_{\Delta y}(r_{\Delta y}-r_{\Delta z}\mu_y) & -r_{\Delta x}(r_{\Delta y}-r_{\Delta z}\mu_y) & 0\\ r_{\Delta y}(r_{\Delta z}\mu_x-r_{\Delta x}) & -r_{\Delta x}(r_{\Delta z}\mu_x-r_{\Delta x}) & 0\\ r_{\Delta y}(r_{\Delta x}\mu_y-r_{\Delta y}\mu_x) & -r_{\Delta x}(r_{\Delta x}\mu_y-r_{\Delta y}\mu_x) & 0\end{bmatrix}, \tag{A10}$$

then $\mathbf{I}_{\Delta\Delta}^s\vec{\alpha}=-a_\alpha\vec{r}_\Delta\times\vec{\mu}$. For the remaining terms we can define various torques:

$$\begin{aligned}&\vec{\tau}_{\text{fric}}=g\vec{R}_{\text{con}}\times\vec{\mu}, \quad \vec{\tau}_{\text{dev}}=(\mathbf{I}_{\text{dev}}\vec{\omega})\times\vec{\omega},\\ &\vec{\tau}_\Delta^s=g\vec{r}_\Delta\times\vec{\mu}+a_\omega\vec{R}_{\text{con}}\times\vec{\mu}, \quad \vec{\tau}_{\Delta\Delta}^s=a_\omega\vec{r}_\Delta\times\vec{\mu}.\end{aligned} \tag{A11}$$

As noted in Sec. II, $\vec{\tau}_{\text{dev}}$ is not strictly a torque, because it comes from the left side of Eq. (A1). If we collect the $\vec{\alpha}$ terms on the left and all others on the right, the equation for rotational motion, Eq. (A8), becomes

$$(\mathbf{I}_o+\mathbf{I}_{\text{dev}}+\mathbf{I}_\Delta^s+\mathbf{I}_{\Delta\Delta}^s)\vec{\alpha}=\vec{\tau}_{\text{fric}}+\vec{\tau}_{\text{dev}}+\vec{\tau}_\Delta^s+\vec{\tau}_{\Delta\Delta}^s, \tag{A12}$$

which is the same as Eq. (6).

APPENDIX B: EQUATIONS OF MOTION: ROLLING

What is $\vec{F}_{\text{con}}$ if the ball is rolling? In this case the slip $\vec{\mathbf{s}}$, Eq. (A2), is zero and so $\dot{\vec{r}}_{\text{cb}}=\vec{R}_{\text{con}}\times\vec{\omega}$. If we take the derivative, we have

$$\ddot{\vec{r}}_{\text{cb}}=\vec{R}_{\text{con}}\times\vec{\alpha}. \tag{B1}$$

If we rewrite Eq. (1) using the fact that $\vec{r}=\vec{r}_{\text{cb}}-\vec{r}_\Delta$, we obtain

$$M(\ddot{\vec{r}}_{\text{cb}}-\ddot{\vec{r}}_\Delta)=\vec{F}_{\text{con}}+\vec{F}_g. \tag{B2}$$

We substitute Eqs. (A5) and (B1) in Eq. (B2) and solve for $\vec{F}_{\text{con}}$ to find

$$\vec{F}_{\text{con}}/M=-g\vec{n}+(\vec{r}_\Delta+\vec{R}_{\text{con}})\times\vec{\alpha}+(\vec{\omega}\times\vec{r}_\Delta)\times\vec{\omega}, \tag{B3}$$

where $\vec{n}$ is an upward-directed unit vector.

Thus for rolling, the torque equation, Eq. (A1), becomes:

$$(\mathbf{I}_o+\mathbf{I}_{\rm dev})\vec{\alpha}+\vec{\omega}\times(\mathbf{I}_{\rm dev}\,\vec{\omega}) =(\vec{r}_\Delta+\vec{R}_{\rm con})\times[-g\vec{n}+(\vec{r}_\Delta+\vec{R}_{\rm con})\times\vec{\alpha}+(\vec{\omega}\times\vec{r}_\Delta)\times\vec{\omega}]. \tag{B4}$$

As with sliding, the solution for $\vec{\alpha}$ lets us know $\vec{F}_{\rm con}$, Eq. (B3), and we can solve the force equation (1) to find the position.

To cast Eq. (B4) in a form comparable to Eq. (A12), we define

$$\mathbf{I}_{\rm Roll}=\begin{bmatrix} R_{\rm ball}^2 & 0 & 0 \\ 0 & R_{\rm ball}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{I}_\Delta^r=\begin{bmatrix} r_{\Delta y}^2+r_{\Delta z}^2-2r_{\Delta z}R_{\rm ball} & -r_{\Delta x}r_{\Delta y} & -r_{\Delta x}r_{\Delta z}+r_{\Delta x}R_{\rm ball} \\ -r_{\Delta x}r_{\Delta y} & r_{\Delta x}^2+r_{\Delta z}^2-2r_{\Delta z}R_{\rm ball} & -r_{\Delta y}r_{\Delta z}+r_{\Delta y}R_{\rm ball} \\ -r_{\Delta x}r_{\Delta z}+r_{\Delta x}R_{\rm ball} & -r_{\Delta y}r_{\Delta z}+r_{\Delta y}R_{\rm ball} & r_{\Delta x}^2+r_{\Delta y}^2 \end{bmatrix}. \tag{B5}$$

If we apply the identity $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$ and define

$$\vec{\tau}_\Delta^r=g\vec{r}_\Delta\times\vec{n}+R_{\rm ball}[\omega^2\vec{r}_\Delta\times\vec{n}+(\vec{\omega}\,\vec{r}_\Delta)\vec{n}\times\vec{\omega}],$$
$$\vec{\tau}_{\Delta\Delta}^r=(\vec{\omega}\,\vec{r}_\Delta)\vec{\omega}\times\vec{r}_\Delta, \tag{B6}$$

then the torque equation, Eq. (B4), becomes

$$(\mathbf{I}_o+\mathbf{I}_{\rm dev}+\mathbf{I}_{\rm Roll}+\mathbf{I}_\Delta^r)\vec{\alpha}=\vec{\tau}_{\rm dev}+\vec{\tau}_\Delta^r+\vec{\tau}_{\Delta\Delta}^r. \tag{B7}$$

As with sliding, $\mathbf{I}_{\rm dev}$ and $\vec{\tau}_{\rm dev}$ are zero if $\mathbf{I}$ has three identical principal moments; $\mathbf{I}_{\rm Roll}$ expresses the main rolling constraint, and the remaining terms are first and second order in $\vec{r}_\Delta$.

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[1]The literature is also out of date. Since about 1990, manufacturers began competing to produce balls with much greater variation in core geometry and cover material. This competition caused a proliferation in the varieties of balls available to ordinary bowlers, and has spawned numerous articles by coaches and drillers as they struggle to best utilize these new balls.

[2]D. C. Hopkins and J. D. Patterson, "Bowling frames: Paths of a bowling ball," Am. J. Phys. **45**, 263–266 (1977).

[3]E. J. Zecchini and G. L. Foutch, "The bowling ball's path," CHEMTECH **21**, 731–735 (1991).

[4]R. L. Huston, C. Passerello, J. M. Winget, and J. Sears, "On the dynamics of a weighted bowling ball," J. Appl. Mech. **46**, 937–943 (1979).

[5]Y.-X. Xu and D. Kohli, "The dynamics of the trajectory of the bowling ball-the Eulerian model," Technical report commissioned by the Amer. Bowling Congress and Womens' Int. Bowling Congress, Univ. Wisconsin, Milwaukee (1992).

[6]Columns in each issue of the journal *Bowling This Month* review the properties of balls by all the major manufacturers and discuss how different drilling patterns affect their motion. Two examples of books by drillers that try to explain how ball properties influence performance are B. Taylor, *Balance* (BT Bowling Products, 1988), and C. Zielke, *Revolutions* (Revolutions International, Matteson, IL, 1995).

[7]Because the derivative is taken in an inertial reference frame, the angular momentum changes both because the angular velocity $\vec{\omega}$ changes [first term on the right in Eq. (4)] and also because the moment of inertia tensor **I** is rotating (second term on the right). See, for example, H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*, 3rd ed. (Prentice Hall, Englewood Cliffs, NJ, 2002).

[8]ABC/WIBC Equipment Specification Manual (2000).

[9]The rules don't explicitly specify the length of the lever arm, but all regulation balances appear to have a lever arm of this length.

[10]The limitations on r_Δ also serve to prevent the ball from losing contact with the lane. In particular, $\omega^2 r_\Delta<g$ means that for r_Δ of one mm, ω would have to exceed 100 rad/s or 15 rev/s. Real bowlers seldom if ever exceed ω of about 8 rev/s.

[11]D. Speranza and L. Vezina, *C. A. T. S (Computer Aided Tracking System) Information Manual* (ABC/WIBC/USA Bowling, 1996).

[12]More precisely, what bowlers want is for the ball's path to approach the headpin at an oblique angle of perhaps 4°–5°. To accomplish this, the ball must slide further down the lane before deflecting, and then deflect a few more inches.

[13]Equations presenting the first and second derivatives of the Euler angles as a function of $\vec{\omega}$ and $\vec{\alpha}$ appear explicitly in Ref. 5, or can be easily derived from the equation relating $\vec{\omega}$ to the Euler angles and their first derivatives which appears in mechanics texts such as L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, New York, 1960).

[14]See Ref. 4, and also W. C. Hassenpflug, "Rotation angles," Comput. Methods Appl. Mech. Eng. **105**, 111–124 (1993).

[15]However, air resistance is not completely negligible. A bowling ball travels with a Reynold's number of about 10^5; thus the drag force is approximately $0.5C_D\rho A v^2$, where ρ is the air density, A is the ball cross section and C_D is about 0.5. For a bowling ball the drag force is about 1 N, which is sufficient to reduce a ball's velocity by a few percent over the length of a bowling lane.

[16]However, real bowlers will note that Table III doesn't include simulations for a large variety of ball properties and initial deliveries; the "realistic" simulations (10–12 and 14–16 in Table III) do not include the full spectrum of label and leverage drilling patterns used by many serious bowlers.

[17]When all radii of gyration are equal, the locus of points on the ball's surface that are in contact with the lane forms a line (actually, a small circle), visible to bowlers because it leaves a track of oil on the ball. However, precession of axes causes this oil track to smear over a wider area, and reduce friction because there is less oil on the part of the ball in contact with the lane. Bowler's use the term "track flare' to describe the widening of the track line when a ball has unequal radii of gyration.

[18]To avoid computational problems, for all the simulations in this paper I assumed that the ball made the transition from sliding to rolling when the slip [Eq. (A2)] became smaller than 1 cm/s.

[19]If any vector $\vec{A}$ with fixed length rotates with angular velocity $\vec{\omega}$, then $d\vec{A}/dt=\vec{\omega}\times\vec{A}$. See Ref. 7.