The undercut procedure: an algorithm for the envy-free division of indivisible items

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Abstract

We propose a procedure for dividing indivisible items between two players in which each player ranks the items from best to worst and has no information about the other player's ranking. It ensures that each player receives a subset of items that it values more than the other player's complementary subset, given that such an envy-free division is possible. We show that the possibility of one player's undercutting the other's proposal, and implementing the reduced subset for himself or herself, makes the proposer "reasonable" and generally leads to an envy-free division, even when the players rank items exactly the same. Although the undercut procedure is manipulable, each player's maximin strategy is to be truthful. Applications of the undercut procedure are briefly discussed.

Keywords: fair division, allocation of indivisible items, envy-freeness, ultimatum game

The Undercut Procedure:

An Algorithm for the Envy-Free Division of Indivisible Items[1]

1. Introduction

In this paper we propose a procedure for dividing indivisible items between two players in which each player ranks the items from best to worst. It ensures that each player receives a subset of items that it values at least as much as the other player's complementary subset, given that such a split is possible. Such a split is *envy-free*, because neither player will envy the other player if each obtains a portion it values at 50 percent or more.[2]

Remarkably, such a split is always possible—even if the players rank the items exactly the same—as long as they differ on a *minimal bundle* of items. A minimal bundle for a player is a subset that is worth at least as much as its complement, but if any item in it is eliminated or replaced by a lower ranked item, would be worth less than 50 percent.

As an example of minimal bundles, suppose there are five items that players A and B both rank in the order 1 2 3 4 5. If A thinks a minimal bundle is {1, 2}, which we write as 12, and B thinks a minimal bundle is 2345 (note the players overlap on item 2), there is an envy-free split. In fact, we will show in section 3 that there are three such splits of the five items.

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[1] We thank Paul H. Edelman and Ulle Endriss for valuable comments on an earlier version of this paper.

[2] The concept of "envy-freeness" goes back at least to Foley (1967); for an overview of its use in the fair-division literature, see Brams (2006). Most of the literature we cite in this article relates to algorithmic fair division; there is also an extensive literature on axiomatic fair division, which focuses on characterizing different kinds of fair division—what necessary and sufficient conditions must obtain to ensure, for example, the existence of an envy-free, Pareto-optimal allocation of items—but this literature does not say how to obtain it.

We start by giving a brief description and rationale of the procedure for finding envy-free splits when the players rank a set of items *exactly the same*, the hardest case. Player A proposes to take a subset, such as 12, and to give B the complementary subset. (Henceforth, A will be male and B will be female.) B can respond either by accepting A's offer or undercutting it.

In the latter case, B reduces A's subset to one it values slightly less (e.g., 13) and gives the complement (245) to A. But A, anticipating this reduction if he proposes "too much" for himself, will make an offer such that, should it be undercut by B, yields a complement worth more than 50 percent to himself. Of course, B at the outset may simply accept the complement of A's proposal (i.e., 345) if she thinks it is worth at least 50 percent.

We assume that B chooses the option (accept or undercut) that gives her at least 50 percent. This is always possible as long as the players' minimal bundles are not exactly the same, enabling both players to get at least 50 percent.

In some ways, our procedure is a discrete analogue of the well-known "I cut, you choose" procedure for cake-cutting. However, instead of A's giving B a choice between two perfectly equal portions for himself, which may be impossible with indivisible items, A makes one portion worth at least 50 percent for himself. But A is motivated not to exceed 50 percent by too much so that if B undercuts his portion, A will receive at least 50 percent from the complement. By doing so, A maximizes the value of the minimal bundle he receives, which is the goal we assume the players pursue when they have no information about the preferences of their opponent.[3]

The paper proceeds as follows. In section 2 we give a method for identifying all possible minimal bundles of the contested items, enumerating them for up to 7 items. The number of these bundles increases rapidly with the number of items, making it increasingly likely that A and B will not have the same minimal bundles and enhancing the probability of an envy-free split.

In section 3, we give the assumptions of our algorithm, which we call the *undercut procedure* (UP), and then its rules, which distinguish between items that are "contested" and those that are not. We also illustrate UP, showing, among other things, why there are three possible envy-free splits of the items in our earlier example.

We show, in addition, that if the players are *sincere* (i.e., truthfully rank the items), they can guarantee that the values of the portions of the uncontested items they each receive will be at least 50 percent, and generally more. This also holds for the contested items, except when the players' minimal bundles are exactly the same. We demonstrate that sincerity is a *maximin strategy*—it maximizes the minimum a player can guarantee for himself or herself, regardless of the strategy of the other player, given the players have no information about the ranking of an opponent.

We show in section 4 that sincere rankings are not necessarily optimal if the players have complete information about each other's preferences. Because a player may

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[3] If they do have information, then it is possible to determine equilibrium outcomes, but they will not necessarily be truthful, as we illustrate later. In the absence of information about an opponent's preference ranking, we show that maximin guarantees a player an envy-free allocation—if such an allocation is possible—and it requires truth-telling.

do better by being insincere, sincere strategies need not be in equilibrium, rendering UP manipulable, at least in theory if not in practice.

In section 5 we calculate the expected number of items that will be contested when all possible orderings of the items by the players are equiprobable. It turns out that as the number of items increases, the expected number of contested items also increases, but only very slowly, so the *proportion* of contested items drops rapidly.

This facilitates giving the players more favorable portions from the larger set of uncontested items. To be sure, if the preferences of the players are positively correlated—as they might well be—then a larger proportion of the items will likely be contested. But even if all items are contested because the players rank them exactly the same, it is still likely that there will be an envy-free division if there are more than about five or six items.

Because information is incomplete in most real-life situations, risk-averse players are well-advised to choose sincerely. Not only will the resulting outcome almost always be envy-free if there are more than a few items, but it will also be *Pareto-optimal*—there will be no other split that gives both players at least as much and one player more.

In section 6 we summarize our results and draw some conclusions. From a practical standpoint, UP is relatively easy to apply, because it requires that the players indicate only ordinal rather than cardinal preference information to obtain envy-free splits. Although this is also true of "strict alternation" (Brams and Taylor, 1999, ch. 2), in which the players take turns choosing items, the player going second is at a distinct disadvantage and may envy the first chooser, especially if their preferences are similar.

On other criteria, UP does better than other procedures. For example, divide-and-choose may not give a Pareto-optimal division, and adjusted winner (Brams and Taylor, 1996, 1999) requires that the players provide cardinal information about their preferences.

UP asks of the players only ordinal information, though we assume that each player has underlying cardinal utilities for the items. Moreover, these utilities are additive, which is equivalent to assuming that there are no complementarities or synergies among the items. If there are any and they are different for the players, then this should enable them to obtain even better packages,[4] but demonstrating when this is possible would require a detailed analysis that we do not undertake here.

2. Feasible Subsets When Players Rank Items Exactly the Same

The most difficult case in dividing indivisible items between two players is when they rank them exactly the same, in which case we say that these items are *contested*. Before showing in the next section how UP can effect an envy-free division of them—giving each player a bundle that it values at least as much as its opponent's bundle—we show next how to identify all *possible* envy-free splits of contested items.

We call a subset of contested items *feasible* if it can be part of an envy-free split—if it can be assigned to one player (say, A), and its complement to B, to produce an envy-free split for *some* cardinal valuations of the contested items by the players. Assume that

[4] This would happen if, when new items come up that complement items that the players have already received, the players revise their rankings to try to obtain these complements. Nevertheless, demonstrating the benefits of UP without complements (i.e., where utilities for items are additive), and when players have no information about an opponent, sets a useful benchmark, because if there are complements and there is some information, UP is likely to offer still more benefits.

there are *c* contested items that both players rank 1 2 3 … *c*, with ties possible between items that have adjacent ranks.

If $c = 1$, there is no possible envy-free split, because there is only one item. If $c = 2$, there is also no possible envy-free split, except in the case of a tie, because whoever gets the less-preferred item will envy the player who gets the more-preferred one. If $c = 3$ and both players rank the items 1 2 3, the only possible envy-free split is 1/23, whereby one player gets 1 and the other gets 23. And if $c = 4$ and both players rank the items 1 2 3 4, there are two possible envy-free splits, namely 1/234 and 14/23 (unless some or all of the items are tied in value for at least one player).

The possible envy-free splits for these simple examples are obvious, but the possible envy-free splits when $c = 5$ are not so obvious. Figure 1 shows all subsets that could possibly be part of an envy-free split, assuming that preferences between any two items are always strict.

Figure 1 *about here*

Note that whenever a subset can be part of an envy-free split, so can its complement. In Figure 1, superscripts are used to indicate subsets that are complements—for example, 145 and 23 in the two middle columns, and 12 and 345 in the left-hand and right-hand columns.

Among the subsets shown in Figure 1, some preference relations hold automatically. For example, in the left-hand column, 12 is always preferred to 13, 13 to 14, 14 to 15, and 15 to 1. In fact, Figure 1 is the Hasse diagram (http://en.wikipedia.org/wiki/Hasse_diagram) describing all relations among these subsets. Relations are indicated by lines (arcs) between nodes (subsets); if there is an upward or upward-sloping path from subset *S* to subset *T*, then *T* is always preferred to *S*. For example, 12 is always preferred to 15 and to 23, and 145 is always preferred to 1 and 345. But if there is no upward path between two subsets, then their relative preference can vary; for instance, 234 might be more preferred or less preferred than 145, and 15 might be more or less preferred than 245 (or these pairs may be equally preferred).

We next give necessary and sufficient conditions for subsets to be feasible, which were identified in Brams and Fishburn (2000) but not characterized. Call the set of contested items $I_c = \{1, 2, \ldots, c\}$, for which A and B have the same preference ranking, 1 2 . . . *c*. We assume that each player has a utility vector $u = (u_1, u_2, \ldots, u_c)$, and that his or her total utility for a nonempty subset $S \subseteq I_c$ of the contested items is

$$U(S) = \sum_{i \in S} u_i. \tag{1}$$

Naturally, different players may have different utility vectors, *u*, but these vectors have some common properties. In particular, the preferences of a player imply that a player's utility vector *u* must satisfy $u_1 \geq u_2 \geq \ldots \geq u_c$. We further assume that every item has nonnegative value, which is equivalent to $u_c \geq 0$, and that some item has positive value, which is equivalent to $u_1 > 0$.[5] We adopt the convention that $U(\phi) = 0$.

A nonempty subset $S \subseteq I_c$ is *feasible* iff there exists a utility vector *u* such that

$$W(S, u) = U(S) - U(-S) > 0, \tag{2}$$

[5] If there is a "bad" like nuclear waste that neither player desires, the good (with positive value) can be considered the possession of the bad by the other player. Alternatively, to render the utilities of all items nonnegative if some are bads, one can add a positive constant to the players' utilities.

where $-S = I_c - S$ is the complement of $S \subseteq I_c$. In words, S is feasible if and only if there exists a utility vector, u, according to which S is strictly preferred to its complement. Thus, to a player whose preferences are represented by u, S is worth more than 50 percent. For example, if $c = 5$, then both 12 and 345 are feasible; so is 123, but 45 is not, because $u_4 + u_5 < u_1 + u_2 + u_3$ for any utility vector u. As we will see, some feasible subsets may be parts of envy-free splits (e.g., 12 and 345), but others (e.g., 123) are not because their complements (45) are not feasible.

Lemma 1. *If $S \subseteq T \subseteq I_c$ and S is feasible, then so is T.*

Proof. Assume that $u = (u_1, u_2, \ldots, u_c)$ satisfies (2). From (1) and the assumption that $u_1, u_2, \ldots, u_c$ is nonincreasing, it follows that $U(T) \geq U(S)$ and $U(-T) \leq U(-S)$. Therefore, $W(T, u) = U(T) - U(-T) \geq U(S) - U(-S) > 0$. Q.E.D.

Our first theorem gives a necessary and sufficient condition for S to be feasible—namely, that there exists a value of k such that the number of items in $I_k \cap S$ is greater than $k/2$. For example, if there are $c = 4$ contested items and $S = \{2, 3\}$,

- when $k = 1$, the intersection of $\{1\}$ and $\{2, 3\}$ is ϕ;
- when $k = 2$, the intersection of $\{1, 2\}$ and $\{2, 3\}$ is $\{2\}$;
- when $k = 3$, the intersection of $\{1, 2, 3\}$ and $\{2, 3\}$ is $\{2, 3\}$; and
- when $k = 4$, the intersection of $\{1, 2, 3, 4\}$ and $\{2, 3\}$ is $\{2, 3\}$.

When $k = 3$, the number of items in the intersection, 2, is greater than $k/2 = 3/2$, so the theorem shows that $\{2, 3\}$ is feasible.

Theorem 1. *Let $S \subseteq I_c$ and $S \neq \phi$. Then S is feasible iff*

$|I_k \cap S| > k/2,$ (3)

for some k, k = 1, 2, …, *c*.

Proof. Fix S and suppose that (3) holds for k. Observe that (3) implies that $|I_k - S| < k/2$. For $j = 1, 2, \ldots, c$, define the utility vector $v^j = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$, whose h^{th} component is

$$v_h^j = \begin{cases} 1 & \text{if } h \le j \\ 0 & \text{if } h > j. \end{cases}$$

That is, the first j components of v^j are 1's, and the last $c - j$ components are 0's. For a player with utility vector v^k, $U(S) = |I_k \cap S|$ and $U(\text{-}S) = |I_k - S|$ by (1) and (2), so that

$$W(S, v^k) = |I_k \cap S| - |I_k - S| > k/2 - k/2 = 0.$$

Applying (2) now confirms that (3) is sufficient for S to be feasible.

To prove the converse, we will show that if (3) is false for all values of k, then S cannot be feasible. Assume that $|I_k \cap S| \le k/2$ for all $k = 1, 2, \ldots, c$. It follows that, for any k,

$$\text{W}(S, v^k) = |I_k \cap S| - |I_k - S| \le 0. \quad (4)$$

Now let $u = (u_1, u_2, \ldots, u_c)$ be any utility vector and set $u_{c+1} = 0$. Notice that

$$u = \sum_{j=1}^{c} (u_j - u_{j+1}) v^j = (u_1 - u_2)v^1 + (u_2 - u_3)v^2 + \ldots + (u_c - u_{c+1})v^c,$$

because the k^{th} component of the vector on the right side is

$$\sum_{j=1}^{c}(u_j - u_{j+1})v_k^j = \sum_{j=k}^{c}(u_j - u_{j+1}) = u_k - u_{c+1} = u_k$$

since $v_k^j = 1$ exactly when $j \geq k$. By (1) and (2), we can therefore write

$$W(S, u) = \sum_{k\in S} u_k - \sum_{k\notin S} u_k = \sum_{k\in S}\sum_{j=1}^{c}(u_j - u_{j+1})v_k^j - \sum_{k\notin S}\sum_{j=1}^{c}(u_j - u_{j+1})v_k^j.$$

Reversing the orders of summation on the right side and then extracting the sum with index *j* as a common factor produces

$$W(S, u) = \sum_{j=1}^{c}(u_j - u_{j+1})\left[\sum_{k\in S} v_k^j - \sum_{k\notin S} v_k^j\right] = \sum_{j=1}^{c}\left(u_j - u_{j+1}\right)W(S, v^j) \leq 0,$$

because $u_j - u_{j+1} \geq 0$ for $j = 1, 2, .., c$, since the components of *u* are nonincreasing, while $W(S, v^j) \leq 0$ for all *j* by (4). Because *u* was arbitrary, it follows that $W(S, u) \leq 0$ for any utility vector *u*, proving that *S* cannot be feasible. Q.E.D.

Corollary 1. For any *c*, {1} *is feasible*.

Proof. $S = \{1\}$ satisfies (3) when $k = 1$. Q.E.D.

Any *S* that contains 1 *and* other items is, by Lemma 1, also feasible.

Recall that a subset *S* is feasible if (*S*, -*S*) *could be* an envy-free split. For (*S*, -*S*) to be envy-free, the two players must have different utility vectors, except in the case that (i) they have the same vector and (ii) the items can be divided so that each player receives exactly 50 percent.

Corollary 2. *Suppose that $S \subseteq I_c$ and that $1 \notin S$. If S is feasible, then (S, -S) and (-S, S) are possible envy-free splits.*

Proof. Because $1 \in -S$, $\{1\} \subseteq -S$. By Corollary 1 and Lemma 1, -*S* is feasible. It follows that if *S* is feasible, then both (*S*, -*S*) and (-*S*, *S*) are possible envy-free splits. Q.E.D.

It is not hard to show, using Theorem 1, Corollary 1, and Corollary 2, that the complementary subsets given in Table 1 constitute all possible envy-free splits when $c = 5$. In Table 1, we extend the analysis to $c = 6$ and $c = 7$, but we do not give feasible subsets that include item 1, because they are simply the complements of the feasible subsets that exclude 1.

Table 1 *about here*

In Table 1, we break down our enumeration of the feasible subsets *S* according to the number of items in a feasible subset. Note that the possible envy-free splits are exactly the feasible subsets, *S*, we enumerate in Table 1 together with their complements, -*S*, which always include 1.

Observe that the modal number of feasible subsets in Table 1 occurs in the middle range—either 3 or 4 items when $c = 6$, exactly 4 items when $c = 7$. However, if a player receives just his or her top-ranked item (1)—the complement of 23456 or 234567 in Table 2—or his or her 2nd and 3rd-ranked items (23), these small subsets are also feasible. Thus, there seems ample opportunity for the players to achieve an envy-free split when there are more than a few items in the contested pile.

Brams and Fishburn (2000) derived the following formula, $f(c)$, for the number of possible envy-free splits $(S, -S)$ of a contested pile with c items:

$$f(c) = \begin{cases} 2^{c-1} - \binom{c}{(c-1)/2} & \text{if } c \text{ is odd} \\ 2^{c-1} - \binom{c}{c/2} & \text{if } c \text{ is even.} \end{cases}$$

It is not hard to show that this number increases exponentially in c, which is illustrated in the table below:

***c* = 1**	***c* = 2**	***c* = 3**	***c* = 4**	***c* = 5**	***c* = 6**	***c* = 7**	***c* = 8**	***c* = 9**
0	0	1	2	6	12	29	58	130

Notice that beginning at $c = 3$ and 4, odd and even numbers of possible envy-free splits differ by a factor of 2.

Manifestly, the larger c is, the more likely there will be an envy-free split because of the rapidly increasing number of possibilities. How to find one is the subject of the next section.

3. The Undercut Procedure

To preview UP, we consider the simpler problem of dividing a single divisible good, such as money. In the *ultimatum game*, for example, one player (A) proposes a share of $1 to the other player (B), who may accept or reject A's offer. If A proposes more than a certain amount (e.g., 60¢) for himself, leaving the remainder (40¢) for B, his offer is likely to be rejected in many societies, leaving both players with nothing.

But now give B the option of undercutting A's proposal by 1¢ and implementing the resulting division. It is not hard to see that by offering B exactly 50¢, A can guarantee himself a payoff of at least 50¢:

- If A offers B less (say, only 40¢) and keeps the remainder (60¢) for himself, B will undercut A's proposal by 1¢ and obtain 59¢ for herself, leaving A with only 41¢.
- If A offers B more (say, 60¢) and keeps the remainder (40¢) for himself, B will accept A's proposal and obtain 60¢ for herself, leaving A with only 40¢.

By this reasoning, it is in A's interest to offer B exactly 50¢, and for B to accept, thereby implementing the egalitarian outcome of 50¢ to each player. These strategies are a subgame perfect Nash equilibrium, because neither player can do better by departing from his or her strategy at either the offer stage (for A) or the accept/undercut stage (for B) in this game.[6]

To return to the case of indivisible items, we assume that A and B independently rank n indivisible items from best to worst. These rankings need not be strict, so a player who ranks an item i might be indifferent to an item he or she ranks one step higher ($i - 1$) or lower ($i + 1$). Our objective is to assign a subset of the set of items $\{1, 2, \ldots, n\}$ to A, and its complement to B, so that each player prefers the subset it receives, or is indifferent between the two subsets, yielding an envy-free split.

[6] Provided the game is discrete (amounts must be an integral multiple of 1¢), there is a second subgame perfect equilibrium, whereby A proposes 51¢ for himself and B undercuts this offer by 1¢, implementing the outcome in which both players receive 50¢. This, of course, is the same outcome that the players obtain from the equilibrium described in the text.

We assume that players have utilities for items—on which their rankings are based—and that their utilities for subsets of items are *additive*, so the utility of a player for a subset is the sum of his or her utilities for the individual items in the subset. The latter assumption is unfounded if there are complementarities or synergies among items, as we noted earlier.[7]

To state the rules of UP, we need some definitions. For a fixed nonempty subset of items, *X*, we define a relation on the subsets of *X*. A subset *T* is *ordinally less* than a subset *S* if *T* is a proper subset of *S*, or if *T* can be obtained from *S*, or a proper subset of *S*, by replacing items originally in *S* by equally many lower-ranked (i.e., higher-numbered) items.[8]

As noted in section 2, the complement of *S* (in *X*) is the subset $X – S = -S$ comprising all items in *X* but not in *S*. A player regards a subset *S* as *worth at least* 50 *percent* if he or she finds *S* at least as good as -*S*.[9] A player regards a subset *S* as a *minimal bundle* (from *X*) if (i) *S* is worth at least 50 percent, and (ii) any subset *T* of *X* that is ordinally less than *S* is worth less than 50 percent.

Although we assume the presence of a referee in the rules below, the role of the referee could be played by a computer program since no human judgment is required—the referee compares subsets and makes random choices only. The rules of UP are as follows:

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[7] Possible relaxations of additivity are discussed in Brams and Fishburn (2000) and, more generally, in Barberà, Bossert, and Pattanaik (2004).

[8] The relation *ordinally less* is the transitive closure of the inclusion relation and the right shift (see Taylor and Zwicker, 1999, p. 93). If subset *T* is ordinally less than subset *S*, then either *T* contains only items from *S*, but not all of them, or *T* contains some lower-ranked items in place of the higher-ranked items that *S* contains, or both.

[9] Similarly, a subset is *worth exactly* 50 *percent* to a player if he or she is indifferent between it and its complement, and *worth more than* 50 *percent* if he or she strictly prefers it to its complement.

1. A and B independently name their most-preferred items. If they name different items, each player receives the item he or she names. If they name the same item, this item goes into the *contested pile*.[10]

2. This process is repeated for the next most-preferred items (of those remaining) for each player, and so on, until all the items are either allocated to A or B or assigned to the contested pile.

3. If the contested pile is empty, the procedure ends. Otherwise, both players identify all of their minimal bundles of items in the contested pile and name them to the referee.

4. If both players have exactly the same minimal bundles, there is no envy-free allocation of the contested pile unless one player, say A, has a minimal bundle that is worth exactly 50 percent to him, in which case this minimal bundle is assigned to B, and A receives its complement.[11] Otherwise, the procedure ends without a division of the contested pile.[12] (In section 4, we allow for a randomized division of the contested pile in the latter situation, which enables us to define Nash equilibria in a game that allocates *all* the items and so provides a *full solution*.)

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[10] This is the terminology used for strict alternation in Brams and Taylor (1999, ch. 2), who call rule 1 the "query step" but offer no algorithm for dividing the contested pile. While Brams and Fishburn (2000) suggest a series of steps that can be used to determine a "middle" envy-free allocation—one that does not favor one player or the other—we indicate later why UP is a simpler and more practicable procedure.

[11] The complement of a minimal bundle will also be a minimal bundle if and only if it is worth exactly 50 percent to a player. The referee will be able to identify a minimal bundle that is worth exactly 50 percent to A, because A will have named both it and its complement as minimal bundles.

[12] In this case, we may speak of a *partial solution* that allocates just the uncontested items, even though there may be an envy-free allocation of all the items if the players were to report their utilities for the items. For example, if A's utilities for four items (a, b, c, d) are (60, 20, 15, 5), and B's are (35, 45, 15, 5), UP would give A item a and B item b, but there would be no envy-free allocation of items c and d in the contested pile. But there are two envy-free allocations of all the items: A gets item a and B gets items b, c, and d; and A gets items a and d, and B gets items b and c. However, determining these allocations requires that the players report their utilities, not just their ranks, of items, which may be difficult to elicit (but see Brams and Taylor, 1999, pp. 101-102, for practical ways of obtaining this information).

5. If both players' minimal bundles are not the same, one minimal bundle named by one player (say, A) but not the other is chosen at random. Player A is designated the proposer of this minimal bundle. Then B may respond by (i) accepting the complement of A's proposed minimal bundle (if it is worth at least 50 percent to her) or (ii) proposing for herself a subset that is ordinally less than A's minimal bundle, in which case the complement of B's subset is assigned to A.

6. In case (i), the items are divided according to A's proposal. In case (ii), the items are divided according to B's counterproposal. In either case, the procedure ends with a fair split of the contested pile.

We begin with three observations about the effects of the preceding rules:

Observation 1. If A and B are sincere, they each prefer their items assigned according to rules 1 and 2 to the items assigned to the other player. To guarantee an envy-free split of *all* items, we focus on how to share the contested pile, wherein players have exactly the same rankings. This is because, as reporting proceeds from their most-preferred to their least-preferred items, the item they contest first will be ranked highest in the contested pile, the item they contest second will be ranked second-highest, and so on.

Observation 2. The items in the contested pile may not be ranked the same by the players *initially*. For example, assume initially that A ranks four items 1 2 3 4, and B ranks them 2 3 4 1. After A gets item 1 and B gets item 2 by rule 1, items 3 and 4 are contested. Although they are ranked 1st and 2nd, respectively, in the contested pile, A ranked them 3rd and 4th initially, whereas B ranked them 2nd and 3rd. While both players get their most-preferred items (1 for A, 2 for B), there is no envy-free division of the two items in the contested pile (3 and 4) unless, by rule 4, one player equally prefers them. If so, that player gets item 4, and the other player gets item 3, and this division is envy-free.

Observation 3. Even if the contested pile is empty, the values players place on their own subsets may not be equal, rendering the division *inequitable* (Brams and Taylor, 1996, 1999).[13] This would occur, for example, if A ranked the four items 1 2 3 4, and B ranked them 2 4 1 3. Under UP, A would get 13 (best and 3rd best) and B would get 24 (best and 2nd best), so B would do better in her eyes than A does in his eyes. Nevertheless, A would not envy B's subset, which comprises what A thinks are his 2nd and 4th-best items, and B would not envy A's subset, which comprises what B thinks are her 3rd and 4th-best items.

To illustrate the application of UP to a contested pile, we return to the example we introduced in section 1, in which there are five items in the contested pile, which A and B rank 1 2 3 4 5 from best to worst. We assumed that A proposed 12 as his minimal bundle, and B proposed 2345, according to rule 3. Assume the referee designates A as the proposer according to rule 5.

To undercut his proposal of 12, B can propose 13 for herself—if she thinks 13 is at least 50 percent—which, as will be seen later, is the most valuable subset that is ordinally less than 12. Because 13 is worth less than 50 percent to A (because 12 was a minimal

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[13] The "descending demand procedure" (Herreiner and Puppe, 2002) produces equitable divisions, insofar as possible, but it requires that the players successively name less and less preferred bundles of items until they find bundles that are *compatible* (i.e., have no overlap in items) and are *balanced* (i.e., help the worst-off player as much as possible). Unlike UP, this procedure can be applied to the division of indivisible items among more than two players. Unfortunately, balanced divisions may not be envy-free (Brams and King, 2005; Brams, 2008, ch. 10). For other results on the fair division of indivisible items in which the players rank them but do not assign cardinal utilities to them, see Edelman and Fishburn (2001).

bundle for him), he must think the complement of 13, namely 245, is worth more than 50 percent.

Thereby both players can obtain at least 50 percent if 13 is worth at least 50 percent to B. On the other hand, if B thinks the complement of A's proposal of 12, namely 345, is worth at least 50 percent, then B will accept A's proposal, and again both players will obtain at least 50 percent.

How do we know that player B will value either 13 (her undercut proposal) or 345 (A's proposal to her) at 50 percent or more if 12 is not also B's minimal bundle (as assumed in rule 5)? If 12 is more than a minimal bundle for B (i.e., if there exists a subset that is ordinally less than 12 that is worth at least 50 percent to B), then she can undercut 12 to 13, which must be worth at least 50 percent to her.

Otherwise, 12 is worth less than 50 percent to B, so its complement, 345, must be worth more than 50 percent. One, and only one, of these two statements must be true if 12 is not a minimal bundle for B. More generally, we have the following:

Theorem 2. *In the contested pile, if any player has a minimal bundle that is not a minimal bundle for the other player, then there exists an envy-free split of items, which UP implements.*

Proof. Assume that a subset *S* of the contested pile is a minimal bundle for A but not for B. Clearly, *S* is not worth exactly 50 percent to B; otherwise, it would be a minimal bundle for B. There are now two mutually exclusive possibilities:

1. *S is worth more than* 50 *percent to B.* Then there is a subset that is ordinally less than *S* that, because *S* is not a minimal bundle for B, is worth at least 50 percent to B.

2. *S is worth less than* 50 *percent to B*. Then its complement—the subset that A proposed for B—is worth more than 50 percent to B.

Because these possibilities are mutually exclusive, one of them, and only one—either (1) choosing a reduced *S* or (2) accepting A's proposal—yields B at least 50 percent. At the same time, A also receives at least 50 percent, whether B undercuts his proposal (because the complement of the undercut proposal is worth at least 50 percent to A) or accepts A's proposal, which is worth at least 50 percent to A. Q.E.D.

Note that Theorem 2 not only gives a condition for the existence of an envy-free split (i.e., A's and B's sets of minimal bundles are not identical), but it also establishes that UP will implement an envy-free split.

In our previous example, we assumed that a minimal bundle for A was 12, whereas a minimal bundle for B was 2345. Because the players differ on minimal bundles, we now ask: What are all the *possible* envy-free splits of the players?

Earlier we showed that if A's proposal was selected, B could choose between 13 and 345, whichever gave her at least 50 percent. Because 2345 is the minimal bundle for B, we know that only 13 (not 345) can be worth at least 50 percent for B. We also know that the complement of 13, 245, must be worth at least 50 percent to A, because 13 is worth less than 50 percent to A.

To determine other possible envy-free splits in this example, we reverse the roles of the players and begin with B's minimal bundle of 2345. Now A can either accept the complement, 1, for himself or undercut B by proposing 234. Because 12 is the minimal bundle for A, we know that only 234 (not 1) is at least 50 percent for A. We also know that the complement of 234, 15, must be worth at least 50 percent to B, because 234 is worth less than 50 percent to B.

In summary, we have shown that the A/B splits,

• 245/13 (when A goes first); and

• 234/15 (when B goes first),

give each player at least 50 percent. Moreover, the player who goes second is advantaged—obtaining a more preferred subset than if he or she went first—at least in the case when this player undercuts the proposer's offer (as here). Because, by rule 5, we randomize the selection of the proposer, neither player is favored, in expectation, by UP.[14]

However, if we take into account all information about the players' minimal bundles, then it is apparent that there is a "middle" envy-free split in this example, namely 235/14. This allocation gives each player a portion intermediate between the one obtained by going first and going second.

One problem with algorithms that identify a middle split (Brams and Fishburn, 2000), and thereby produce a result that is arguably more balanced, is that they require that the players rank all their envy-free splits so that the referee can choose a middle one. But under UP, what might be a middle envy-free split, should B accept A's proposal, may not be a middle one should B undercut it. Thus, we think it better to stick with the present rules, which yields *some* envy-free split as long as A's and B's lists of minimal bundles are not the same.

[14] There are many examples of games with a first-mover advantage (Chicken, duopoly games) and others, like the game under UP, with a second-mover advantage.

Finally, we ask whether a player can ever do better by proposing a nonminimal bundle under UP. For example, suppose a minimal bundle of A is 1, but he proposes 12 for himself. If B's minimal bundle is 345, she will accept A's proposal, and A will obtain 12, which is better for him than obtaining only 1 had he proposed just this item for himself, and had B accepted (receiving the complement 2345).

However, if A had proposed 12 and been undercut, B would have obtained 13, leaving the complement of 245 for A, which is less than 50 percent for him (even 2345 is less than 50 percent if A's minimal bundle is 1). Thus, although A could do better by misrepresenting his minimal bundle(s), he also might do worse, not even obtaining a 50-percent portion. As a corollary to Theorem 2, we have

Corollary 3. *A player's maximin strategy under UP is to name all his or her minimal bundles. Given that A's and B's minimal bundles do not all coincide, a player cannot receive less than 50 percent by proposing a minimal bundle, whereas proposing a nonminimal bundle may lead to his or her receiving less than 50 percent.*

Proof. The example in the previous two paragraphs illustrates that one player, say A, may not receive at least 50 percent if he proposes more than a minimal bundle for himself and B undercuts it. Similarly, if A proposes less than a minimal bundle for himself, A will receive less than 50 percent if B accepts its complement. On the other hand, if A proposes a minimal bundle different from B's, Theorem 2 establishes that A will receive at least 50 percent. Thus, proposing a minimal bundle maximizes the minimum—50 percent—that a player can guarantee for himself or herself, given that an envy-free split exists. Moreover, it is in the interest of a player (say, A) to name *all* his minimal bundles, lest one he did not name that B did is selected by the referee, ensuring that A receives less than 50 percent whether he accepts B's proposal or undercuts it. Q.E.D.

4. The Manipulability of UP

We showed in section 3 (Corollary 3) that a player's maximin strategy is to rank items sincerely, which guarantees him or her at least 50 percent of the contested pile as long as the players do not have exactly the same minimal bundles. Because a departure from sincerity can lead to a player's getting less than 50 percent of the contested pile, sincerity is a "safe" strategy, especially when information is incomplete.

But if players have complete information about each other's preference order, then a departure from sincerity may be rational. We show this with a simple example in which sincerity is not a Nash equilibrium.

Assume there are three items to be divided, so there are 3! = 6 possible rankings, or strategies, that each player can report. These are shown in the 6 × 6 outcome matrix in Figure 2, in which the ordered triple (A, C, B) indicates those items that go to A, those that go into the contested pile (C), and those that go to B, respectively.

Figure 2 *about here*

The outcomes shown in Figure 2 define a *game form*, because they are the product of the players' strategies independent of their preferences. The game form becomes a game when we assume the players have particular preferences, from which we can determine Nash equilibria.

Strategy pairs associated with 24 of the 36 outcomes in the resulting payoff matrix are Nash equilibria, indicated by the superscript n, if the players choose strategies that coincide with their sincere rankings of the items. As an example, consider the upper left entry of the outcome matrix, in which A and B choose exactly the same sincere strategy, 1 2 3. Then all three items go into the contested pile, so the outcome is (-, 123, -).

In the absence of information about minimal bundles, assume that each player has a 50-50 chance of getting each item in the contested pile. Then each player on average will get 1 ½ items that he or she values at the mean utility of the three items.

This is better than some allocations and worse than others.[15] In the example, it is not hard to show that neither player can do better by departing from his or her sincere strategy, so these strategies constitute a Nash equilibrium.

Now assume that A continues to choose sincere strategy 1 2 3, but B's sincere strategy is 3 1 2, giving outcome (1, 2, 3) in row 1, column 5, which is underscored in Figure 1 and superscripted B. By switching to strategy 1 3 2 (column 2), B can effect outcome (2, 1, 3), which is starred and which she prefers because a 50-50 chance of getting item 1 is better than a 50-50 chance of getting item 2 (B gets her best item, 3, in either case; by comparison, A gets his next-best item and a 50-50 chance of getting his best item). Because B can do better by switching from her sincere strategy of 3 1 2, the players' sincere strategies are not in equilibrium.

Given the switch by B, can A, in turn, do better by switching to a different strategy? The answer is "no," because A's sincere strategy associated with outcome (2, 1, 3) at row 1, column 2, is part of a Nash equilibrium.

[15] If A's preference is 1 2 3, his ordering from best to worst of the seven possible outcomes satisfies (1, 2, 3) $\succ$ {(1, 3, 2), (2, 1, 3)} $\succ$ (-, 123, -) $\succ$ {(2, 3, 1), (3, 1, 2)} $\succ$ (3, 2, 1). Note that A's relative ranking of the two two-member subsets in this ordering cannot be determined without further information on A's utilities for items 1, 2, and 3. Players with different preferences will have analogous rankings of the outcomes; the pure Nash equilibria of the game in Figure 1 can be determined from such preferences.

This is also true of the best responses by B to the nonNash outcomes superscripted B in each of the other five rows. To these best responses, which are starred, A has no counterresponse. The resulting Nash-equilibrium outcomes are better for B and worse for A than the nonNash outcomes. Similarly, these same starred outcomes show the best responses by A to the nonNash outcomes superscripted A.

To summarize, either A or B can benefit by deviating from his or her sincere strategy associated with the 12 underscored sincere outcomes in Figure 1. Improving on a sincere outcome in this way shows the vulnerability of UP to manipulation.

Are outcomes under UP—either sincere or manipulated—Pareto-optimal? Sincere outcomes are, because

- each uncontested item goes to the player who prefers it; and
- no trade of the contested items can benefit both players since they rank these items the same.

This is also true of manipulated outcomes in our 3-item example, because the manipulator benefits at the expense of the manipulated player vis-à-vis the sincere outcome. But it is unclear whether manipulated outcomes—wherein at least one player is insincere and the resulting outcome is in equilibrium[16]—are always Pareto-optimal when there are more than 3 items, but we conjecture that they are.[17]

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[16] In our 3-item example, there are no instances of equilibria in which both players are insincere, but examples with 4 or more items have such equilibria.

[17] Under "strict alternation," whereby the players choose items in a specified order round by round, Nash-equilibrium outcomes may be Pareto-nonoptimal if there are three or more players (Brams and Straffin, 1979); see also Brams and Kaplan (2004) and Brams (2008, ch. 9). But if, as under UP, there are only two players, equilibrium outcomes under strict alternation are Pareto-optimal.

Because Nash equilibria in games with more than a few items are not trivial to calculate, we think the manipulation of UP, especially when information about player preferences is incomplete, is not a serious practical problem. Coupled with the fact that sincerity is a maximin strategy (Corollary 3), it seems that most players will be sincere—both in ranking items and reporting minimal bundles of the contested items.

5. The Expected Size of the Contested Pile

We turn next to the question of how many of the items to be divided can be expected to be contested. Let the total number of items be n, and assume A's ranking of the items is strict: 1, 2, …, n. Assume the $n!$ possible strict rankings of B are equiprobable.

Let $c(n)$ be the expected size of the contested pile. If $n = 1$, it is clear that this single item must be put in the contested pile, so $c(1) = 1$. If $n = 2$, then B's ranking is either 1 2 or 2 1, each with probability ½. There are two possibilities:

(i) Both players rankings are different, so no items go into the contested pile.

(ii) Both players rankings are the same, so both items go into the contested pile.

Because (i) and (ii) each occur with probability ½, $c(2) = (½)(0) + (½)(2) = 1$.

Now assume that $n \geq 3$. The probability that B most prefers item 1 is $1/n$; in this case, 1 is put into the contested pile, and the players repeat the procedure to divide a set of $n - 1$ items. With complementary probability $(n - 1)/n$, B's most preferred item is not 1; in this case, A gets 1, B gets her most-preferred item, and the players repeat the procedure on the remaining $n - 2$ items. It follows that

$$c(n) = \frac{1}{n}\left[1 + c(n-1)\right] + \frac{n-1}{n}c(n-2). \qquad (5)$$

Lemma 2. *Suppose that* $c(n-1) = c(n-2) = x$. *Then* $c(n) = c(n+1) = x + \frac{1}{n}$.

Proof. The lemma follows from (5), because

$$c(n) = \frac{1}{n}\left[1 + x\right] + \frac{n-1}{n}x = x\left(\frac{1}{n} + \frac{n-1}{n}\right) + \left(\frac{1}{n}\right) = x + \frac{1}{n}.$$

Using (4), with $n + 1$ replacing n, implies

$$c(n+1) = \frac{1}{n+1}\left[1 + x + \frac{1}{n}\right] + \frac{n}{n+1}[x]$$

$$= \frac{n + nx + 1 + n^2x}{n(n+1)} = \frac{(n+1) + n(n+1)x}{n(n+1)} = x + \frac{1}{n},$$

and the lemma follows. Q.E.D.

Theorem 3. *If* $k \geq 1$, *then*

$$c(2k+1) = c(2k+2) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k+1}. \qquad (6)$$

Proof. As noted above, $c(1) = c(2) = 1$. If $k = 1$, then using Lemma 2 with $n = 3$ implies that $c(3) = c(4) = 1 + 1/3$. The proof is completed by induction: If (6) is true for k, then applying Lemma 2 with $n = 2k + 3$ shows that

$$c(2k+3) = c(2k+4) = c(2k+1) + \frac{1}{2k+3}.$$

Q.E.D.

The table below gives some values of $c(n)$ and illustrates Lemma 2—that adjacent odd and even values of n give the same $c(n)$:

$k = 0$	$c(1)$	$c(2)$	1
$k = 1$	$c(3)$	$c(4)$	1 + 1/3 = 1.3333…
$k = 2$	$c(5)$	$c(6)$	1 + 1/3 + 1/5 = 23/5 = 1.5333…
$k = 3$	$c(7)$	$c(8)$	1 + 1/3 + 1/5 + 1/7 = 176/105 = 1.67619…
$k = 4$	$c(9)$	$c(10)$	1 + 1/3 + 1/5 + 1/7 + 1/9 = 563/315 = 1.78730…

Note that the right side of (6) is a variant of the well-known harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...,$$

which is known to be divergent (i.e., the sum does not approach any finite limit as the number of terms increases without bound).[18] The series on the right side of (6) must be unbounded, for if it had a finite sum, say K, then a term-by-term comparison would show

$$0<\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+...\right]<\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]=K$$

Moreover, a positive series with a finite sum must be absolutely convergent, so the terms of the first two summations below can be rearranged, producing

[18] The sum of the first n terms is of the same order as $\ln n = \log_e n$, the natural logarithm of n.

$$0 < \left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\right] + \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots\right] = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots\right] < 2K < \infty,$$

which contradicts the known divergence of the harmonic series. Therefore, the right side of (5) does not approach any limit as k (or $n = 2k + 1$) increases without bound. Q.E.D.

In conclusion, the expected size of the contested pile increases without limit as the number of items to be divided increases, but the increase is very slow. Thus, for 9 or 10 items, the expected size of the contested pile is less than 2 (see the last row of the preceding table).

To be sure, the calculation in this section depends heavily on the equiprobability assumption, which in practice will almost surely be violated because the rankings of A and B are likely to be positively correlated. With 9 or 10 items, the number of items in the contested pile might be 4 or 5 instead of about 2.[19] But this is not an unduly large number for which to name minimal bundles, so we think UP will be applicable in such situations.

6. Summary and Conclusion

The undercut procedure (UP) is an eminently practicable procedure for dividing indivisible items between two players, A and B. They begin by ranking the items they wish to divide from best to worst.

Starting from their top choices, they work down their lists, naming—independently and one at a time—the items they most prefer of those that remain. At each stage, if A

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[19] In the Panama Canal Treaty dispute between Panama and the United States in the 1970s, a ranking of the 10 issues in the dispute shows that 4 would have been contested under UP. We emphasize, however, that most of the issues were divisible, and compromises were, in fact, reached on several (Raiffa, 1982, ch. 12; Brams and Taylor, 1996, ch. 5).

and B differ on their most-preferred items, they each obtain the item they name; if they name the same item, then that item goes into the contested pile.

To divide up the contested pile, both players report their minimal bundles to the referee, a role which could as well be played by a computer.[20] If both players have the same minimal bundles, there is no envy-free division of the contested pile, except for the special case when one player has a bundle that is worth exactly 50 percent.

Otherwise, one minimal bundle not named by both players is selected at random, and the player who offered it, say A, is considered to have proposed this minimal bundle as his subset of the contested pile. B has the option of accepting the proposal (i.e., the complement of A's subset), or counterproposing a reduced bundle that will be worth less than 50 percent to A (if A was sincere). If B makes such a counterproposal, it will be implemented.

In either event, both players get at least 50 percent unless A and B consider exactly the same bundles to be minimal. We compared UP with the ultimatum game—supplemented by a rule that allows B to make a counterproposal that will be implemented—showing that this modified game would, in equilibrium, produce a 50-50 division of $1.

By sincerely naming their minimal bundle(s), players maximize the minimum they guarantee for themselves, which is always an envy-free portion of at least 50 percent if the players differ on at least one minimal bundle. Because the number of possible fair splits increases rapidly with the number of items in the contested pile, an envy-free division of the contested pile is highly probable if there are more than a few contested

[20] To be sure, naming all minimal bundles can be arduous if the contested pile has more than about 6 or 7 items. It might be facilitated by showing the players all feasible subsets (see Table 1) and asking them to indicate which are minimal bundles.

items. Even if there are only one or two items in the contested pile—making an envy-free division of the contested pile impossible—envy-freeness, overall, is still possible if the one or two contested items are randomly assigned, because such an envy-causing split may be counterbalanced by the division of the uncontested items, which will always be envy-free if the players are sincere.

We showed that UP is manipulable in a game of complete information, but this does not undermine the envy-freeness or Pareto-optimality of the resulting division. Indeed, the division at an (insincere) Nash equilibrium may well equalize the value of the portions that A and B receive over that which sincerity gives the players, rendering the division more equitable.

Although the expected size of the contested pile increases with the number of items to be divided, it does so very slowly—at least when all possible rankings are equiprobable—increasing the likelihood that a larger proportion of the items will be uncontested and the overall division will be envy-free. If there are synergies, then this likelihood may be further enhanced by the sequential nature of UP, which allows the players to choose items that complement those uncontested items they already possess, but we did not analyze this situation.

By contrast, under procedures such as divide-and-choose and adjusted winner (Brams and Taylor, 1996, 1999), players must make decisions about all the items at once. With divide-and-choose, after the divider makes a 50-50 division (or one as close as possible to 50-50), the chooser's choice, if it gives each player at least 50 percent, may result in an allocation Pareto-inferior to another allocation.

This is not true of adjusted winner, but this procedure requires that the items be divisible and that the players be able to allocate points across them, which will generally be more difficult than just ranking them and making sequential choices over time. On the other hand, UP does not ensure an equitable allocation, whereby A values his allocation the same as B values hers, whereas adjusted winner does if the players are sincere.

We think UP would be most valuable in the division of physical items, such as the marital property in a divorce (e.g., a house, car, or a boat), which are hard if not impossible to divide or share. Other examples of all-or-nothing division include two teams' choosing players, two political parties choosing cabinet ministries in a coalition government, and the like.

But UP also could be useful in negotiations in which the "items" are issues on which one side or the other might get its way. For example, resolving a border dispute between two countries might require determining which country will have sovereignty over a region, whether there will be a peace-keeping force, and so on, which tend to be all-or-nothing choices and hence not divisible.

Compared to disputes over divisible items like money or land, or disputes in which monetary compensation is possible (Brams and Kilgour, 2001; Brams, 2008, ch. 14), disputes involving indivisible issues are more intractable. We believe that UP, which is relatively easy to apply, could facilitate agreement in many disputes in which at least some of the items or issues are indivisible and must be awarded, in their entirety, to one or the other of the disputants.

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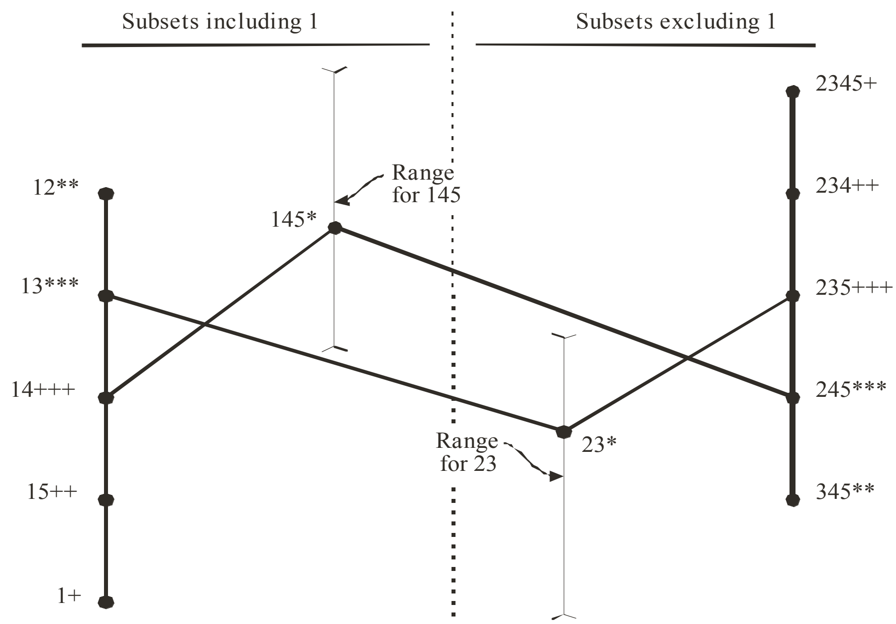
Figure 1

Feasible Subsets *S* for *c* = 5



Key: Subsets with the same superscript are complements.

Table 1

Feasible Subsets *S* for *c* = 6 and *c* = 7 that Do Not Include 1

No. of Items in Subset	*c* = 6	*c* = 7
2	23	23
3	234, 235, 236, 245, 345	234, 235, 236, 237, 245, 345
4	2345, 2346, 2356, 2456, 3456	2345, 2346, 2347, 2356, 2357, 2367, 2456, 2457, 2467, 2567, 3456, 3457, 3467, 3567, 4567
5	23456	23456, 23457, 23467, 23567, 24567, 34567
6		234567

Figure 2

Outcome Matrix for Six Different Rankings of Three Items by A and B

B ⇒ A ⇓	**1 2 3**	**1 3 2**	**2 1 3**	**2 3 1**	**3 1 2**	**3 2 1**
1 2 3	(-, 123, -)^{n}	(2, 1, 3)^{n*}	(1, 3, 2)^{n}	(1, 3, 2)^{A}	(1, 2, 3)^{B}	(1, 2, 3)^{n}
1 3 2	(3, 1, 2)^{n*}	(-, 123, -)^{n}	(1, 3, 2)^{B}	(1, 3, 2)^{n}	(1, 2, 3)^{n}	(1, 2, 3)^{A}
2 1 3	(2, 3, 1)^{n}	(2, 3, 1)^{A}	(-, 123, -)^{n}	(1, 2, 3)^{n*}	(2, 1, 3)^{n}	(2, 1, 3)^{B}
2 3 1	(2, 3, 1)^{B}	(2, 3, 1)^{n}	(3, 2, 1)^{n*}	(-, 123, -)^{n}	(2, 1, 3)^{A}	(2, 1, 3)^{n}
3 1 2	(3, 2, 1)^{A}	(3, 2, 1)^{n}	(3, 1, 2)^{n}	(3, 1, 2)^{B}	(-, 123, -)^{n}	(1, 3, 2)^{n*}
3 2 1	(3, 2, 1)^{n}	(3, 2, 1)^{B}	(3, 1, 2)^{A}	(3, 1, 2)^{n}	(2, 3, 1)^{n*}	(-, 123, -)^{n}

Key: (A, C, B) = items that (go to A, go into the contested pile, go to B)

n = Nash-equilibrium outcome for sincere rankings associated with it

Underscore = nonNash outcome for sincere rankings associated with it

A, B = player who has a best response to nonNash outcome

* = Nash outcome to which a best response from a nonNash oucome (underscored) takes A or B