# Three-dimensional Green's functions for two-dimensional quasi-crystal bimaterials 

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Owing to their specific structure, which can neither be classified as crystalline nor amorphous, quasi-crystals (QCs) exhibit properties that are interesting to both material science and mathematical physics or continuum mechanics. Within the framework of a mathematical theory of elasticity, one major focus is on features evolving from the coupling of phonon and phason fields, which is not observed in classical crystalline or amorphous materials. This paper deals with the problems of combinations of point phonon forces and point phason forces, which are applied to the interior of infinite solids and bimaterial solids of two-dimensional hexagonal QCs. By using the general solution of QCs, a series of displacement functions is adopted to obtain the analytical results when the two half-spaces are supposed to be ideally bonded or to be in smooth contact. In the final expressions, we provide three-dimensional Green's functions for infinite bimaterial QC solids in the closed form, which are very convenient to be used in the study of dislocations, cracks and inhomogeneities of the new solid phase. Furthermore, the paper is concluded by a discussion of some special cases, in which Green's functions for infinite transversely isotropic solids and Green's functions for a half-space with free or fixed boundary are given.

Keywords: Green's functions; two-dimensional hexagonal quasi-crystal; bimaterials; interface

## 1. Introduction

A two-dimensional quasi-crystal (QC) is defined as a three-dimensional body in which the atom arrangement is quasi-periodic in a plane and periodic in the orthogonal direction. Therefore, there are two kinds of displacement fields in the elastic theory of QCs. One is a phonon displacement field, corresponding to the displacement field of classical crystals. The other is the phason displacement field, which is diffusive due to the elementary excitation associated with the phason mode and describes the local rearrangements of the unit cells. Both types of fields are coupled with each other. Accordingly, there are phonon and phason stresses and strains, the phason appearing just in QC elasticity.

From the theoretical point of view, QCs are an interesting subject of investigation since they exhibit a clearly defined anisotropy and the phenomenon of field coupling. It is a fundamental concern to extend the theory of elasticity towards this extraordinary class of solids. From the technological point of

[^0]view, there is a variety of one-and two-dimensional quasi-crystalline materials (Gao \& Ricoeur 2011, submitted). For engineering applications, a theoretical basis for stress and strain analyses has to be elaborated. Here, Green's functions supply a perfect background since arbitrary loading situations can be reproduced by superposition. Hence, even cracks and other defects can be included in a stress analysis based on Green's functions. This is particularly important to assess fracture loads and the lifetime of engineering structures composed of quasi-crystalline materials.

QCs have become the focus of theoretical and experimental studies in the physics of condensed matter since the first discovery of the icosahedral QC in Al-Mn alloys (Levine \& Steinhardt 1984; Shechtman et al. 1984). The physical properties, including elasticity and defects of QCs have been intensively investigated in experimental and theoretical analyses (Wollgarten et al. 1993; Athanasiou et al. 2002; Park et al. $2005 a, b$ ). In particular, the field of linear elastic theory of QCs has been formulated for many years (Levine et al. 1985; Socolar 1989; Ding et al. 1993; Hu et al. 1996). Great progress has been made in the fields of the mechanics involving the elasticity and defects, see review articles for details (Hu et al. 2000; Fan \& Mai 2004).

Many engineering structures are made by binding together two or more materials with different physical properties. The elastic body joined with two dissimilar materials with point forces applied at an arbitrary point is fundamental to the development of three-dimensional elastic theory and is of vital significance in the structural design. For isotropic materials, Vijayakumar \& Cormack (1987) and Huang \& Wang (1991) studied the fundamental solutions for the case when a point force is applied at one of the two bonded semi-infinite solids. With regard to transversely isotropic elastic materials, Pan \& Chou (1979) conducted a systematic study on three-dimensional Green's functions for point forces applied to the interior of a two-phase infinite space. In the Fourier transformed domain, Ting (1996) obtained Green's functions of point forces in a two-phase anisotropic elastic solid. In the case of transversely isotropic piezoelectric materials, Ding et al. (1997a) developed effectively a novel method determining the closed-form point force and point charge solutions for two-phase piezoelectric media using the analysis technique of the image source. In a similar way, they considered plane problems (Ding et al. 1997c). For plane problems of cubic QCs with imperfect interface, Gao \& Ricoeur (2010) obtained two-dimensional Green's functions for line forces applied at the interior of a two-phase infinite plane. The significance of fundamental solutions and Green's functions in constructing solutions to various kinds of boundary-value problems has been well recognized in the mechanics literature (Eshelby 1957; Mura 1987; Ting 1996, 2000).

Obviously, three-dimensional Green's functions play an important role in the analysis, because they not only have theoretical merits themselves, but also can be benchmarks to clarify various approximate methods, such as the finite element method and the boundary element method as well as in the study of cracks, defects and inclusions. However, relevant three-dimensional Green's functions for two-dimensional QCs have not been attempted. The purpose of this paper is to develop the previous work (Gao \& Ricoeur 2010), and to study the problem of point forces applied to the interior of an infinite bimaterial two-dimensional hexagonal QC. To achieve this, the general solution of two-dimensional hexagonal QC (Gao \& Zhao 2009) is used to uncouple the system of equations of equilibrium.

Similar to the derivation of Green's functions for transversely isotropic elastic and piezoelectric materials (Ding et al. $1997 a, b, c$ ), Green's functions for infinite QC solids and two half-spaces with point forces are obtained in the closed-form by generalizing the analysis technique of the image source to two-dimensional QC media.

## 2. Basic equations and the general solution

In a fixed rectangular coordinate system $\left(x_{1}, x_{2}, x_{3}\right)$, a two-dimensional QC refers to a three-dimensional solid structure with quasi-periodic arrangement in a plane ( $x_{1}-x_{2}$ plane) and periodic arrangement in the orthogonal direction ( $x_{3}$-direction). From two-dimensional QC elastic theory ( Hu et al. 1996), in the absence of body forces, the general equations governing the three-dimensional theory of two-dimensional hexagonal QC can be written as:

$$
\begin{align*}
\varepsilon_{p q} & =\frac{\partial_{q} u_{p}+\partial_{p} u_{q}}{2} \quad \text { and } \quad w_{\alpha q}=\partial_{q} w_{\alpha},  \tag{2.1}\\
\partial_{q} \sigma_{p q} & =0 \quad \text { and } \quad \partial_{q} H_{\alpha q}=0, \tag{2.2}
\end{align*}
$$

and

$$
H_{13}=2 R_{4} \varepsilon_{31}+K_{4} w_{13}, \quad H_{21}=2 R_{6} \varepsilon_{12}+K_{6} w_{12}+K_{3} w_{21},
$$

$$
\sigma_{11}=C_{11} \varepsilon_{11}+C_{12} \varepsilon_{22}+C_{13} \varepsilon_{33}+R_{1} w_{11}+R_{2} w_{22}
$$

$$
\sigma_{22}=C_{12} \varepsilon_{11}+C_{11} \varepsilon_{22}+C_{13} \varepsilon_{33}+R_{2} w_{11}+R_{1} w_{22}
$$

$$
\sigma_{33}=C_{13} \varepsilon_{11}+C_{13} \varepsilon_{22}+C_{33} \varepsilon_{33}+R_{3} w_{11}+R_{3} w_{22}
$$

$$
\sigma_{23}=\sigma_{32}=2 C_{44} \varepsilon_{23}+R_{4} w_{23}, \quad \sigma_{31}=\sigma_{13}=2 C_{44} \varepsilon_{31}+R_{4} w_{13}
$$

$$
\begin{equation*}
\sigma_{12}=\sigma_{21}=2 C_{66} \varepsilon_{12}+R_{6} w_{12}+R_{6} w_{21} \tag{2.3}
\end{equation*}
$$

$$
H_{11}=R_{1} \varepsilon_{11}+R_{2} \varepsilon_{22}+R_{3} \varepsilon_{33}+K_{1} w_{11}+K_{2} w_{22}
$$

$$
H_{22}=R_{2} \varepsilon_{11}+R_{1} \varepsilon_{22}+R_{3} \varepsilon_{33}+K_{2} w_{11}+K_{1} w_{22}
$$

$$
H_{23}=2 R_{4} \varepsilon_{23}+K_{4} w_{23}, \quad H_{12}=2 R_{6} \varepsilon_{12}+K_{3} w_{12}+K_{6} w_{21}
$$

where the subscripts $i, p, q=1,2,3, j=4,5$ and $\alpha, \beta=1,2$ are used throughout this paper. $u_{p}$ and $w_{q}$ denote phonon and phason displacements in the physical and perpendicular spaces, respectively; $\sigma_{p q}$ and $\varepsilon_{p q}$ are phonon stresses and strains, respectively; $H_{\alpha q}$ and $w_{\alpha q}$ represent phason stresses and strains, respectively; $C_{k l}$, $K_{k}$ and $R_{k}$ stand for the elastic constants in the phonon, phason and phononphason coupling fields, respectively, with the relationships $2 C_{66}=C_{11}-C_{12}, K_{6}=$ $K_{1}-K_{2}-K_{3}, 2 R_{6}=R_{1}-R_{2}$.

For the sake of compactness, the notations $U_{1 \beta}=u_{\beta}, \quad U_{2 \beta}=w_{\beta}, \quad T_{11}=\sigma_{11}$, $T_{21}=H_{11}, T_{12}=\sigma_{22}, T_{22}=H_{22}, T_{13}=\sigma_{23}, T_{23}=H_{23}, T_{14}=\sigma_{13}$ and $T_{24}=H_{13}$ are applied in the present paper. According to the general solution of two-dimensional hexagonal QC with distinct eigenvalues (Gao \& Zhao 2009), the components of displacements take the form:

$$
\begin{equation*}
U_{\alpha 1}=k_{\alpha i} \partial_{1} \psi_{i}+k_{\alpha j} \partial_{2} \psi_{j}, \quad U_{\alpha 2}=k_{\alpha i} \partial_{2} \psi_{i}-k_{\alpha j} \partial_{1} \psi_{j} \quad \text { and } \quad u_{3}=k_{3 i} \partial_{3} \psi_{i} . \tag{2.4}
\end{equation*}
$$

The constants $k_{\alpha i}, k_{\alpha j}$ and $k_{3 i}$ are defined as

$$
\begin{aligned}
k_{1 i} & =\delta_{I i}, k_{2 i}=\frac{a_{4} s_{i}^{2}-a_{3}}{a_{2} s_{i}^{2}-a_{1}}, \quad k_{3 i}=\frac{a_{7} s_{i}^{4}-a_{6} s_{i}^{2}+a_{5}}{a_{2} s_{i}^{4}-a_{1} s_{i}^{2}}, \quad k_{1 j}=\delta_{J j}, \quad k_{2 j}=\frac{R_{6}-R_{4} s_{j}^{2}}{K_{4} s_{j}^{2}-K_{3}}, \\
a_{1} & =R_{1}\left(R_{3}+R_{4}\right)-K_{1}\left(C_{13}+C_{44}\right), \quad a_{2}=R_{4}\left(R_{3}+R_{4}\right)-K_{4}\left(C_{13}+C_{44}\right), \\
a_{3} & =R_{1}\left(C_{13}+C_{44}\right)-C_{11}\left(R_{3}+R_{4}\right), \quad a_{4}=R_{4}\left(C_{13}+C_{44}\right)-C_{44}\left(R_{3}+R_{4}\right), \\
a_{5} & =C_{11} K_{1}-R_{1}^{2}, a_{6}=C_{11} K_{4}+C_{44} K_{1}-2 R_{1} R_{4} \quad \text { and } \quad a_{7}=C_{44} K_{4}-R_{4}^{2},
\end{aligned}
$$

where $\delta_{k l}$ is the Kronecker delta symbol, and the following summation convention has been used throughout this paper: the Einstein summation over repeated lower case indices is applied, while upper case indices take on the same numbers as the corresponding lower case ones but are not summed. Besides, the potential functions $\psi_{i}$ and $\psi_{j}$ satisfy the equations

$$
\begin{equation*}
\nabla_{I}^{2} \psi_{i}=\left(\Lambda+\frac{1}{s_{I}^{2}} \partial_{3}^{2}\right) \psi_{i}=0 \quad \text { and } \quad \nabla_{J}^{2} \psi_{j}=\left(\Lambda+\frac{1}{s_{J}^{2}} \partial_{3}^{2}\right) \psi_{j}=0 \tag{2.5}
\end{equation*}
$$

in which $\Lambda=\partial_{1}^{2}+\partial_{2}^{2}$ is the planar Laplacian; $s_{I}^{2}$ are three eigenvalues of the following cubic algebraic equation of $s^{2}, a s^{6}-b s^{4}+c s^{2}-d=0$; and $s_{J}^{2}$ are two eigenvalues of the following quadratic algebraic equation of $s^{2},\left(C_{44} K_{4}-R_{4}^{2}\right) s^{4}-$ $\left(C_{66} K_{4}+C_{44} K_{3}-2 R_{4} R_{6}\right) s^{2}+C_{66} K_{3}-R_{6}^{2}=0$, where

$$
\begin{aligned}
a= & C_{33}\left(C_{44} K_{4}-R_{4}^{2}\right), \\
b= & C_{44}\left(C_{44} K_{4}-R_{4}^{2}\right)+C_{33}\left(C_{11} K_{4}+C_{44} K_{1}-2 R_{1} R_{4}\right) \\
& +2 R_{4}\left(C_{13}+C_{44}\right)\left(R_{3}+R_{4}\right)-C_{44}\left(R_{3}+R_{4}\right)^{2}-K_{4}\left(C_{13}+C_{44}\right)^{2}, \\
c= & C_{33}\left(C_{11} K_{1}-R_{1}^{2}\right)+C_{44}\left(C_{11} K_{4}+C_{44} K_{1}-2 R_{1} R_{4}\right) \\
& +2 R_{1}\left(C_{13}+C_{44}\right)\left(R_{3}+R_{4}\right)-C_{11}\left(R_{3}+R_{4}\right)^{2}-K_{1}\left(C_{13}+C_{44}\right)^{2}
\end{aligned}
$$

and $\quad d=C_{44}\left(C_{11} K_{1}-R_{1}^{2}\right)$.
The components of stresses obtained from equations (2.3) and (2.4) can be shown to be
and

$$
\begin{align*}
& T_{\alpha 1}=a_{\alpha i} \partial_{3}^{2} \psi_{i}+2 b_{\alpha i} \partial_{1}^{2} \psi_{i}+2 b_{\alpha j} \partial_{1} \partial_{2} \psi_{j}, \\
& T_{\alpha 2}=a_{\alpha i} \partial_{3}^{2} \psi_{i}+2 b_{\alpha i} \partial_{2}^{2} \psi_{i}-2 b_{\alpha j} \partial_{1} \partial_{2} \psi_{j} \\
& T_{\alpha 3}=c_{\alpha i} \partial_{2} \partial_{3} \psi_{i}-c_{\alpha j} \partial_{1} \partial_{3} \psi_{j}, \quad T_{\alpha 4}=c_{\alpha i} \partial_{1} \partial_{3} \psi_{i}+c_{\alpha j} \partial_{2} \partial_{3} \psi_{j} \\
& \sigma_{33}=d_{i} \partial_{3}^{2} \psi_{i}, \quad \sigma_{12}=2 b_{1 i} \partial_{1} \partial_{2} \psi_{i}+b_{1 j}\left(\partial_{2}^{2}-\partial_{1}^{2}\right) \psi_{j}  \tag{2.6}\\
& H_{12}=e_{i} \partial_{1} \partial_{2} \psi_{i}+e_{j} \partial_{2}^{2} \psi_{j}-a_{2 j} \partial_{1}^{2} \psi_{j} \\
& H_{21}=e_{i} \partial_{1} \partial_{2} \psi_{i}-e_{j} \partial_{1}^{2} \psi_{j}+a_{2 j} \partial_{2}^{2} \psi_{j}
\end{align*}
$$



Figure 1. An infinite QC solid with applied phonon force ( $\left.Q_{1}, P_{1}, F\right)$ and phason force $\left(Q_{2}, P_{2}\right)$.
where

$$
\begin{aligned}
& a_{1 i}=-\left(C_{12} k_{1 i}+R_{2} k_{2 i}\right) \frac{1}{s_{I}^{2}}+C_{13} k_{3 i}, \quad a_{2 i}=-\left(R_{2} k_{1 i}+K_{2} k_{2 i}\right) \frac{1}{s_{I}^{2}}+R_{3} k_{3 i}, \\
& b_{1 i}=C_{66} k_{1 i}+R_{6} k_{2 i}, \quad b_{2 i}=R_{6} k_{1 i}+K_{6} k_{2 i}, \quad b_{1 j}=C_{66} k_{1 j}+R_{6} k_{2 j}, \\
& \quad b_{2 j}=R_{6} k_{1 j}+K_{6} k_{2 j}, \\
& c_{1 i}=C_{44} k_{1 i}+R_{4} k_{2 i}+C_{44} k_{3 i}, \quad c_{2 i}=R_{4} k_{1 i}+K_{4} k_{2 i}+R_{4} k_{3 i}, \\
& c_{1 j}=C_{44} k_{1 j}+R_{4} k_{2 j}, \quad c_{2 j}=R_{4} k_{1 j}+K_{4} k_{2 j}, \\
& d_{i}=-\left(C_{13} k_{1 i}+R_{3} k_{2 i}\right) \frac{1}{s_{I}^{2}}+C_{33} k_{3 i}
\end{aligned}
$$

and

$$
e_{i}=2 R_{6} k_{1 i}+K_{3} k_{2 i}+K_{6} k_{2 i}, \quad e_{j}=R_{6} k_{1 j}+K_{3} k_{2 j}
$$

For the sake of brevity and conciseness, in the following two sections the fundamental solutions for an infinite QC solid and an infinite bimaterial QC solid will be given only for the case of distinct eigenvalues. When equal eigenvalues appear, the fundamental solutions can be obtained by using a similar analysis technique, although for these cases the general solution will take a more complicated form (Gao \& Zhao 2009).

## 3. Three-dimensional Green's functions for infinite QC solids

A two-dimensional hexagonal QC full-space is considered as the domain of the problem. As indicated in figure 1, a point phonon force with components $Q_{1}$, $P_{1}$ and $F$ in $x_{1}, x_{2}$ and $x_{3}$-directions, respectively, and a point phason force with components $Q_{2}, P_{2}$ in $x_{1}$ and $x_{2}$-directions, respectively are applied simultaneously at an arbitrary point in the space. Without loss in generality, we take this point
as the origin of Cartesian coordinates. Based on the theorem of superposition the problem can be divided into two sub-problems: the problem of a point phonon force $F$ in the $x_{3}$-direction and the problem of a combination of point phonon and phason forces $Q_{\alpha}$ in the $x_{1}$-direction or $P_{\alpha}$ in the $x_{2}$-direction. As a classical elastic problem, the fundamental solutions of point phonon forces applied in the $x_{3}$-direction were derived based on the general solution (Gao \& Zhao 2009).
(a) Solution for a point force F in the $\mathrm{x}_{3}$-direction

This is an axisymmetric problem discussed in Gao \& Zhao (2009). The potential functions have the form as

$$
\begin{equation*}
\psi_{i}=\operatorname{sign}\left(x_{3}\right) A_{i} \ln R_{I} \quad \text { and } \quad \psi_{j}=0 \tag{3.1}
\end{equation*}
$$

where

$$
R_{i}=r_{i}+s_{i}\left|x_{3}\right| \quad \text { and } \quad r_{i}=\sqrt{x_{1}^{2}+x_{2}^{2}+s_{i}^{2} x_{3}^{2}}
$$

$A_{i}$ are undetermined constants and $\operatorname{sign}()$ is the signum function.
Since the general solution expressed by the potential functions $\psi_{i}$ and $\psi_{j}$ satisfies the basic equations of QC elasticity, such as the deformation geometry equations, the equilibrium equations and the constitutive equations, the boundary-value problems of QC elasticity are transformed into finding a solution that satisfies the continuity conditions and the prescribed boundary conditions. Substitution of equations (3.1) into equations (2.4) and (2.6) yields the expressions for the phonon and phason fields below

$$
\begin{equation*}
U_{\alpha \beta}=\operatorname{sign}\left(x_{3}\right) k_{\alpha i} A_{i} \frac{x_{\beta}}{R_{I} r_{I}} \quad \text { and } \quad u_{3}=k_{3 i} A_{i} \frac{s_{I}}{r_{I}} \tag{3.2}
\end{equation*}
$$

and
and

$$
\left.\begin{array}{l}
T_{\alpha \beta}=-a_{\alpha i} A_{i} s_{I}^{3} \frac{x_{3}}{r_{I}^{3}}+2 \operatorname{sign}\left(x_{3}\right) b_{\alpha i} A_{i}\left(\frac{1}{R_{I} r_{I}}-\frac{x_{\beta}^{2}}{R_{I} r_{I}^{3}}-\frac{x_{\beta}^{2}}{R_{I}^{2} r_{I}^{2}}\right), \\
T_{\alpha 3}=-c_{\alpha i} A_{i} s_{I} \frac{x_{2}}{r_{I}^{3}}, \quad T_{\alpha 4}=-c_{\alpha i} A_{i} s_{I} \frac{x_{1}}{r_{I}^{3}}  \tag{3.3}\\
\sigma_{33}=-d_{i} A_{i} s_{I}^{3} \frac{x_{3}}{r_{I}^{3}}, \quad \sigma_{12}=-2 \operatorname{sign}\left(x_{3}\right) b_{1 i} A_{i}\left(\frac{x_{1} x_{2}}{R_{I} r_{I}^{3}}+\frac{x_{1} x_{2}}{R_{I}^{2} r_{I}^{2}}\right) \\
H_{12}=H_{21}=-\operatorname{sign}\left(x_{3}\right) e_{i} A_{i}\left(\frac{x_{1} x_{2}}{R_{I} r_{I}^{3}}+\frac{x_{1} x_{2}}{R_{I}^{2} r_{I}^{2}}\right) .
\end{array}\right\}
$$

Two kinds of continuity conditions should be considered in the boundary-value problems to determine the free constants of the potential functions. One kind of continuity condition is the mathematical continuity which means that $\psi_{i}$ and $\psi_{j}$ are continuous functions. The other kind of continuity condition is the mechanical continuity which assumes that the displacements and stresses are continuous. This solution from equations (3.2) and (3.3) shows that the displacement component $u_{3}$ and the stress components $T_{\alpha 3}, T_{\alpha 4}$ and $\sigma_{33}$ are continuous functions except at the origin. However, the continuity of the other displacement and stress components
and the potential functions $\psi_{i}$ in the plane $x_{3}=0$ is unclear and needs further examination. Since the components $U_{\alpha \beta}, T_{\alpha \beta}, \sigma_{12}, H_{12}, H_{21}$ and $\psi_{i}$ are odd functions of $x_{3}$, they must vanish across the plane $x_{3}=0$. This implies that

$$
\begin{equation*}
k_{\alpha i} A_{i}=0, \quad b_{\alpha i} A_{i}=0 \quad \text { and } \quad e_{i} A_{i}=0 \tag{3.4}
\end{equation*}
$$

In view of equations (3.4), the second and third equations of equations (3.4) are satisfied automatically. Besides, the equilibrium condition of a layer between any two planes, such as $x_{3}= \pm h$, must meet the requirement stipulated as follows:

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty}\left[\sigma_{33}\left(x_{1}, x_{2}, h\right)-\sigma_{33}\left(x_{1}, x_{2},-h\right)\right] \mathrm{d} x_{1} \mathrm{~d} x_{2}+F=0 \tag{3.5}
\end{equation*}
$$

Substituting equations (3.3) into equation (3.5), one obtains

$$
\begin{equation*}
4 \pi d_{i} A_{i} s_{I}^{2}=F \tag{3.6}
\end{equation*}
$$

Thus, the three unknown constants $A_{i}$ can be calculated from the first equation of equations (3.4) and (3.6), written down matrix notation yielding

$$
\left[\begin{array}{l}
A_{1}  \tag{3.7a}\\
A_{2} \\
A_{3}
\end{array}\right]=\frac{F}{4 \pi}\left[\begin{array}{ccc}
1 & 1 & 1 \\
k_{21} & k_{22} & k_{23} \\
d_{1} s_{1}^{2} & d_{2} s_{2}^{2} & d_{3} s_{3}^{2}
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

or in another form

$$
\begin{equation*}
A_{i}=\frac{\varepsilon_{i p q} \delta_{P p} k_{2 q}}{4 \pi \varepsilon_{i p q} \delta_{I i} k_{2 p} d_{q} s_{Q}^{2}} F \tag{3.7b}
\end{equation*}
$$

(b) Solution for point forces $\mathrm{Q}_{\alpha}$ in the $\mathrm{x}_{1}$-direction or $\mathrm{P}_{\alpha}$ in the $\mathrm{x}_{\mathcal{D}}$-direction

When the point forces $Q_{\alpha}$ are applied at the origin along the $x_{1}$-direction, it can be assumed that

$$
\begin{equation*}
\psi_{i}=\frac{B_{i} x_{1}}{R_{I}} \quad \text { and } \quad \psi_{j}=\frac{B_{j} x_{2}}{R_{J}} \tag{3.8}
\end{equation*}
$$

where $B_{i}$ and $B_{j}$ are constants to be determined. Following the same manipulation as in the previous case, the expressions for the phonon and phason fields are summarized as follows:
and

$$
\left.\begin{array}{rl}
U_{\alpha 1} & =k_{\alpha i} B_{i}\left(\frac{1}{R_{I}}-\frac{x_{1}^{2}}{R_{I}^{2} r_{I}}\right)+k_{\alpha j} B_{j}\left(\frac{1}{R_{J}}-\frac{x_{2}^{2}}{R_{J}^{2} r_{J}}\right)  \tag{3.9}\\
U_{\alpha 2} & =-k_{\alpha i} B_{i} \frac{x_{1} x_{2}}{R_{I}^{2} r_{I}}+k_{\alpha j} B_{j} \frac{x_{1} x_{2}}{R_{J}^{2} r_{J}} \\
u_{3} & =-\operatorname{sign}\left(x_{3}\right) k_{3 i} B_{i} s_{I} \frac{x_{1}}{R_{I} r_{I}}
\end{array}\right\}
$$

$$
\text { and } H_{21}=e_{i} B_{i} x_{2}\left(2 \frac{x_{1}^{2}}{R_{I}^{3} r_{I}^{2}}+\frac{x_{1}^{2}}{R_{I}^{2} r_{I}^{3}}-\frac{1}{R_{I}^{2} r_{I}}\right)+a_{2 j} B_{j} x_{2}\left(2 \frac{x_{2}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{2}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{3}{R_{J}^{2} r_{J}}\right)
$$

$$
\begin{equation*}
-e_{j} B_{j} x_{2}\left(2 \frac{x_{1}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{1}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{1}{R_{J}^{2} r_{J}}\right) \tag{3.10}
\end{equation*}
$$

As before, the continuity of the components $u_{3}, T_{\alpha 3}$ and $T_{\alpha 4}$ on $x_{3}=0$ demands

$$
\begin{equation*}
k_{3 i} B_{i} s_{I}=0 \quad \text { and } \quad c_{\alpha i} B_{i} s_{I}-c_{\alpha j} B_{j} s_{J}=0 \tag{3.11}
\end{equation*}
$$

$$
\begin{aligned}
& T_{\alpha 1}=a_{\alpha i} B_{i} s_{I}^{2} \frac{x_{1}}{r_{I}^{3}}+2 b_{\alpha i} B_{i} x_{1}\left(2 \frac{x_{1}^{2}}{R_{I}^{3} r_{I}^{2}}+\frac{x_{1}^{2}}{R_{I}^{2} r_{I}^{3}}-3 \frac{1}{R_{I}^{2} r_{I}}\right) \\
& +2 b_{\alpha j} B_{j} x_{1}\left(2 \frac{x_{2}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{2}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{1}{R_{J}^{2} r_{J}}\right), \\
& T_{\alpha 2}=a_{\alpha i} B_{i} s_{I}^{2} \frac{x_{1}}{r_{I}^{3}}+2 b_{\alpha i} B_{i} x_{1}\left(2 \frac{x_{2}^{2}}{R_{I}^{3} r_{I}^{2}}+\frac{x_{2}^{2}}{R_{I}^{2} r_{I}^{3}}-\frac{1}{R_{I}^{2} r_{I}}\right) \\
& -2 b_{\alpha j} B_{j} x_{1}\left(2 \frac{x_{2}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{2}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{1}{R_{J}^{2} r_{J}}\right), \\
& T_{\alpha 3}=\operatorname{sign}\left(x_{3}\right) c_{\alpha i} B_{i} s_{I}\left(\frac{x_{1} x_{2}}{R_{I}^{2} r_{I}^{2}}+\frac{x_{1} x_{2}}{R_{I} r_{I}^{3}}\right) \\
& -\operatorname{sign}\left(x_{3}\right) c_{\alpha j} B_{j} s_{J}\left(\frac{x_{1} x_{2}}{R_{J}^{2} r_{J}^{2}}+\frac{x_{1} x_{2}}{R_{J} r_{J}^{3}}\right), \\
& T_{\alpha 4}=\operatorname{sign}\left(x_{3}\right) c_{\alpha i} B_{i} s_{I}\left(\frac{x_{1}^{2}}{R_{I}^{2} r_{I}^{2}}+\frac{x_{1}^{2}}{R_{I} r_{I}^{3}}-\frac{1}{R_{I} r_{I}}\right) \\
& +\operatorname{sign}\left(x_{3}\right) c_{\alpha j} B_{j} s_{J}\left(\frac{x_{2}^{2}}{R_{J}^{2} r_{J}^{2}}+\frac{x_{2}^{2}}{R_{J} r_{J}^{3}}-\frac{1}{R_{J} r_{J}}\right), \\
& \sigma_{33}=d_{i} B_{i} s_{I}^{2} \frac{x_{1}}{r_{I}^{3}}, \\
& \sigma_{12}=2 b_{1 i} B_{i} x_{2}\left(2 \frac{x_{1}^{2}}{R_{I}^{3} r_{I}^{2}}+\frac{x_{1}^{2}}{R_{I}^{2} r_{I}^{3}}-\frac{1}{R_{I}^{2} r_{I}}\right) \\
& +b_{1 j} B_{j} x_{2}\left(2 \frac{x_{2}^{2}-x_{1}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{2}^{2}-x_{1}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{2}{R_{J}^{2} r_{J}}\right) \\
& H_{12}=e_{i} B_{i} x_{2}\left(2 \frac{x_{1}^{2}}{R_{I}^{3} r_{I}^{2}}+\frac{x_{1}^{2}}{R_{I}^{2} r_{I}^{3}}-\frac{1}{R_{I}^{2} r_{I}}\right)+e_{j} B_{j} x_{2}\left(2 \frac{x_{2}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{2}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{3}{R_{J}^{2} r_{J}}\right) \\
& -a_{2 j} B_{j} x_{2}\left(2 \frac{x_{1}^{2}}{R_{J}^{3} r_{J}^{2}}+\frac{x_{1}^{2}}{R_{J}^{2} r_{J}^{3}}-\frac{1}{R_{J}^{2} r_{J}}\right)
\end{aligned}
$$

Also, the equilibrium condition of the layer bounded by the planes $x_{3}= \pm h$ becomes

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty}\left[T_{\alpha 4}\left(x_{1}, x_{2}, h\right)-T_{\alpha 4}\left(x_{1}, x_{2},-h\right)\right] \mathrm{d} x_{1} \mathrm{~d} x_{2}+Q_{\alpha}=0 \tag{3.12}
\end{equation*}
$$

Substituting the expression for $T_{\alpha 4}$ in equations (3.10) into equation (3.12) yields

$$
\begin{equation*}
c_{\alpha i} B_{i} s_{I}+c_{\alpha j} B_{j} s_{J}=\frac{Q_{\alpha}}{2 \pi} . \tag{3.13}
\end{equation*}
$$

From the integral result in equation (3.13), we know that the integral in equation (3.12) is independent of the value of $h$. The five unknown constants $B_{i}$ and $B_{j}$ can be determined from equations (3.11) and (3.13) as follows:

$$
\begin{equation*}
B_{i}=l_{\alpha i} Q_{\alpha} \quad \text { and } \quad B_{j}=l_{\alpha j} Q_{\alpha}, \tag{3.14}
\end{equation*}
$$

where

$$
\begin{align*}
& \quad\left[\begin{array}{l}
l_{11} s_{1} \\
l_{12} s_{2} \\
l_{13} s_{3}
\end{array}\right]=\frac{1}{4 \pi}\left[\begin{array}{lll}
k_{31} & k_{32} & k_{33} \\
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23}
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
l_{21} s_{1} \\
l_{22} s_{2} \\
l_{23} s_{3}
\end{array}\right]=\frac{1}{4 \pi}\left[\begin{array}{lll}
k_{31} & k_{32} & k_{33} \\
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23}
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] \\
& \text { and } \quad\left[\begin{array}{l}
l_{14} s_{4} \\
l_{15} s_{5}
\end{array}\right]=\frac{1}{4 \pi}\left[\begin{array}{ll}
c_{14} & c_{15} \\
c_{24} & c_{25}
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right],\left[\begin{array}{l}
l_{24} s_{4} \\
l_{25} s_{5}
\end{array}\right]=\frac{1}{4 \pi}\left[\begin{array}{ll}
c_{14} & c_{15} \\
c_{24} & c_{25}
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
1
\end{array}\right] . \tag{3.15}
\end{align*}
$$

When the problem of infinite QC solids subjected to point forces $P_{\alpha}$ in the $x_{2}$-direction is considered, we should replace $x_{1}$ by $x_{2}$ and $x_{2}$ by $-x_{1}$ in equations (3.8), respectively, and replace $Q_{\alpha}$ by $P_{\alpha}$ in equations (3.14). Therefore, if the problem of combination of point forces $Q_{\alpha}$ in the $x_{1}$-direction and point forces $P_{\alpha}$ in the $x_{2}$-direction is considered, the potential functions $\psi_{i}$ and $\psi_{j}$ take the form

$$
\begin{equation*}
\psi_{i}=\frac{l_{\alpha i}\left(Q_{\alpha} x_{1}+P_{\alpha} x_{2}\right)}{R_{I}} \quad \text { and } \quad \psi_{j}=\frac{l_{\alpha j}\left(Q_{\alpha} x_{2}-P_{\alpha} x_{1}\right)}{R_{J}} . \tag{3.16}
\end{equation*}
$$

By using the general solution in equations (2.4) and heuristic functions with simple forms, the expressions of the displacements and stresses can be obtained. It is important to note that these expressions are continuous except at the origin. The only approximation is introduced by the approximate specification of the boundary conditions of the layer, i.e. the boundary conditions in equations (3.5) and (3.12) are specified in terms of the stress resultants, instead of the stress distribution in the vicinity of the origin. Therefore, in the cases where SaintVenant's principle holds, Green's function solutions should be very accurate ones.

## (c) The degenerated form of infinite quasi-crystal solids

Determination of the independent elastic constants $C_{i j}, K_{i}$ and $R_{i}$ for different kinds of QCs depends on their symmetries with the group representation theory (Ding et al. 1993; Hu et al. 1996, 2000). It is noted that, although $C_{i j}$ in QCs can be measured by some experimental methods, $K_{i}$ are difficult to measure (Tanaka et al. 1996). Significant progress in this area has been made by Jeong \& Steinhardt (1993), who evaluated $K_{i}$ of decagonal QCs by Monte Carlo simulation. The
values of $K_{i}$ are of the same order of magnitude as $C_{i j}$ obtained by resonant ultrasound spectroscopy (Chernikov et al. 1998). There are no data available for $R_{i}$ which, based on the estimation of some experts (Edagawa 2007; Takeuchi \& Edagawa 2007) working in the field of QCs, hold lower values than $K_{i}$.

However, the relevant data such as $K_{i}$ and $R_{i}$ associated with the present paper are still lacking. Alternatively, we will discuss a degenerated form of infinite QC solids to investigate its validity, i.e. a two-dimensional QC body reduces to a transversely isotropic elastic body. In this case, it can be shown that $R_{i}=0$. Hence the governing equations (2.1)-(2.3) reduce to two groups of equations for uncoupled phonon and phason field problems, respectively. Then, the cubic algebraic equation $a s^{6}-b s^{4}+c s^{2}-d=0$ can be reformulated as

$$
\begin{equation*}
\left[C_{33} C_{44} s^{4}+\left(C_{13}^{2}+2 C_{13} C_{44}-C_{11} C_{33}\right) s^{2}+C_{11} C_{44}\right]\left(K_{4} s^{2}-K_{1}\right)=0 \tag{3.17}
\end{equation*}
$$

Let $s_{\alpha}^{2}$ be the roots of the first multiplier of equation (3.17), and $s_{3}^{2}=K_{1} / K_{4}$ be the root of the second multiplier of equation (3.17) with no loss of generality. The quadratic algebraic equation $\left(C_{44} K_{4}-R_{4}^{2}\right) s^{4}-\left(C_{66} K_{4}+C_{44} K_{3}-\right.$ $\left.2 R_{4} R_{6}\right) s^{2}+C_{66} K_{3}-R_{6}^{2}=0$ is rewritten as

$$
\begin{equation*}
\left(C_{44} s^{2}-C_{66}\right)\left(K_{4} s^{2}-K_{3}\right)=0 \tag{3.18}
\end{equation*}
$$

We assume $s_{4}^{2}=C_{66} / C_{44}$ and $s_{5}^{2}=K_{3} / K_{4}$. Then it can be seen that $s_{\alpha}^{2}$ and $s_{4}^{2}$ relate only to elastic constants in the phonon field, while $s_{3}^{2}$ and $s_{5}^{2}$ associate only with elastic constants in the phason field.

For the transversely isotropic elastic body, the constants $k_{3 i}, k_{2 i}$ and $k_{2 j}$ take the following form inserting $s_{\alpha}^{2}$ from equation (3.17):

$$
\begin{equation*}
k_{3 \alpha}=\frac{C_{11}-C_{44} s_{\alpha}^{2}}{\left(C_{13}+C_{44}\right) s_{\alpha}^{2}}=\frac{C_{13}+C_{44}}{C_{33} s_{\alpha}^{2}-C_{44}} \quad \text { and } \quad k_{33}=k_{2 \alpha}=k_{24}=0 \tag{3.19}
\end{equation*}
$$

and $k_{23}, k_{25} \neq 0$ which associate with $s_{3}^{2}$ and $s_{5}^{2}$. Since the analysis in the following calculation does not involve $k_{33}, k_{23}, k_{24}$ and $k_{25}$ except the requirement $k_{23}, k_{25} \neq 0$, it suffices to discuss only $k_{3 \alpha}$ and $s_{\alpha}^{2}$.

For the case of the point force $F$ in the $x_{3}$-direction, the solution of equations $(3.7 a, b)$ reduces to

$$
\begin{equation*}
A_{1}=-A_{2}=\frac{\left(C_{13}+C_{44}\right) F}{4 \pi C_{33} C_{44}\left(s_{2}^{2}-s_{1}^{2}\right)} \tag{3.20}
\end{equation*}
$$

Substitution of equation (3.20) into the degenerated equation of equations (3.2) leads to the expressions of displacements in transversely isotropic elastic solids
and

$$
\left.\begin{array}{l}
u_{\alpha}=\frac{\left(C_{13}+C_{44}\right) x_{\alpha} x_{3}}{4 \pi C_{33} C_{44} r_{1} r_{2}\left(s_{2} r_{1}+s_{1} r_{2}\right)} F  \tag{3.21}\\
u_{3}=\frac{C_{44}\left(x_{1}^{2}+x_{2}^{2}\right)+C_{33} s_{1} s_{2}\left[x_{1}^{2}+x_{2}^{2}+\left(s_{1}^{2}+s_{2}^{2}\right) x_{3}^{2}\right]}{4 \pi C_{33} C_{44} r_{1} r_{2}\left(s_{2} r_{1}+s_{1} r_{2}\right)} F .
\end{array}\right\}
$$

From equations (3.2) and (3.7a,b), the expressions of displacements in twodimensional QC solids are
and

$$
\left.\begin{array}{l}
u_{\alpha}=\operatorname{sign}\left(x_{3}\right) \frac{\delta_{I i} \varepsilon_{i p q} \delta_{P p} k_{2 q}}{4 \pi \varepsilon_{i p q} \delta_{I i} k_{2 p} d_{q} s_{Q}^{2}} \frac{x_{\alpha}}{R_{I} r_{I}} F,  \tag{3.22}\\
u_{3}=\frac{k_{3 i} \varepsilon_{i p q} \delta_{P p} k_{2 q}}{4 \pi \varepsilon_{i p q} \delta_{I i} k_{2 p} d_{q} s_{Q}^{2}} \frac{s_{I}}{r_{I}} F \\
w_{\alpha}=\operatorname{sign}\left(x_{3}\right) \frac{k_{2 i} \varepsilon_{i p q} \delta_{P p} k_{2 q}}{4 \pi \varepsilon_{i p q} \delta_{I i} k_{2 p} d_{q} s_{Q}^{2}} \frac{x_{\alpha}}{R_{I} r_{I}} F .
\end{array}\right\}
$$

In comparison with Green's function of transversely isotropic solids, the configuration and deformation in two-dimensional QC solids are strongly related with not only the phonon elastic constants, but also phason and phonon-phason coupling elastic constants, even if the point phonon force is applied alone. Owing to the introduction of the phason field, a theoretical description of the deformed state of QCs requires a combined consideration of interrelated phonon and phason fields, so the elasticity of QCs is much more complex than that of conventional crystals. Green's function solutions provide important information for studying the mechanical behaviours of the new solid phase and understanding clearly the interplay of the interaction between the phonon and phason activity.

For the case of the point forces $Q_{1}$ in the $x_{1}$-direction, the solution of equations (3.14) becomes

$$
\begin{equation*}
B_{1}=\frac{s_{1}\left(C_{11}-C_{44} s_{2}^{2}\right) Q_{1}}{4 \pi C_{11} C_{44}\left(s_{1}^{2}-s_{2}^{2}\right)}, \quad B_{2}=-\frac{s_{2}\left(C_{11}-C_{44} s_{1}^{2}\right) Q_{1}}{4 \pi C_{11} C_{44}\left(s_{1}^{2}-s_{2}^{2}\right)} \quad \text { and } \quad B_{4}=\frac{Q_{1}}{4 \pi C_{44} s_{4}} \tag{3.23}
\end{equation*}
$$

The above results in equations (3.20) and (3.23) are the same as the corresponding results deduced by Ding et al. (1997b). To illustrate the applications of Green's functions, two numerical examples, isotropic elastic solids with linear distributed loads on one surface and transversely isotropic elastic solids containing a spheroidal cavity under uniform tension along the $x_{3}$ direction, were examined by the boundary element method (Ding et al. 1997b). Comparing the results with the known solutions, these examples show that the exact or accurate solutions may be obtained by applying Green's function. When the relevant material data are available, the same numerical examples of two-dimensional QCs can be also obtained by performing similar derivations.

As Ding et al. (1997b) mentioned, Green's function solutions for transversely isotropic elastic solids are valid for two cases in which two eigenvalues are distinct or equal to each other. Thus, these are the unified solutions for both isotropic elasticity and transversely isotropic elasticity. This property means that there is no need to worry about the eigenvalue conditions. This is particularly important for materials with very close eigenvalues. By setting $R_{i}=0$, the coefficients in equations (3.7a,b) and (3.14) are available for both the two-dimensional QC and degenerated cases (transversely isotropic elastic solids), so Green's


Figure 2. An infinite QC solid composed of two half-spaces.
function solutions are expressed in the united form for both two-dimensional QC elasticity and pure elasticity. Therefore, the same codes can be used to perform the numerical computation for different cases implementing an appropriate algorithm.

## 4. Three-dimensional Green's functions for infinite QC solids composed of two half-spaces

An infinite solid composed of two two-dimensional hexagonal QC half-spaces with different material constants is shown in figure 2 . Let the lower half-space $x_{3} \geq 0$ be occupied by material 1 and the upper half-space $x_{3} \leq 0$ be occupied by material 2 . The interface of the two half-spaces is parallel to their quasi-periodic plane and the point forces are applied at the point $(0,0, h)$.

If the two half-spaces are rigidly bonded together along the interface $x_{3}=0$ such that the components of stresses and displacements are continuous across the interface, we have the following boundary conditions:

$$
\begin{equation*}
U_{\alpha \beta}^{-}=U_{\alpha \beta}^{+}, \quad u_{3}^{-}=u_{3}^{+}, \quad T_{\alpha 3}^{-}=T_{\alpha 3}^{+}, \quad T_{\alpha 4}^{-}=T_{\alpha 4}^{+} \quad \text { and } \quad \sigma_{33}^{-}=\sigma_{33}^{+}, \tag{4.1}
\end{equation*}
$$

where superscripts ()$^{-}$and ()$^{+}$denote the variables in the lower half-space $x_{3} \geq 0$ and the upper half-space $x_{3} \leq 0$, respectively. On the other hand, if the halfspaces are in smooth contact, i.e. in a complete contact with relative frictionless movement of both half-spaces along the $x_{1}-x_{2}$ plane, then

$$
\begin{equation*}
u_{3}^{-}=u_{3}^{+}, \quad \sigma_{33}^{-}=\sigma_{33}^{+} \quad \text { and } \quad T_{\alpha 3}^{-}=T_{\alpha 3}^{+}=T_{\alpha 4}^{-}=T_{\alpha 4}^{+}=0 \tag{4.2}
\end{equation*}
$$

In the following discussion, the components of displacements in the lower halfspace can be decomposed into two parts:

$$
\begin{equation*}
U_{\alpha \beta}^{-}=U_{\alpha \beta}+U_{\alpha \beta}^{\prime} \quad \text { and } \quad u_{3}^{-}=u_{3}+u_{3}^{\prime}, \tag{4.3}
\end{equation*}
$$

where $U_{\alpha \beta}$ and $u_{3}$ are the displacements owing to Green's functions for infinite QC solids, and $U_{\alpha \beta}^{\prime}$ and $u_{3}^{\prime}$ are the displacements which make equations (4.3) satisfy the boundary conditions (4.1) or (4.2) on the interface. In the lower half-space, $U_{\alpha \beta}$ and $u_{3}$ can be obtained simply by replacing $x_{3}$ by $x_{3}-h$ in equations (3.2), (3.3), (3.9) and (3.10).

## (a) Point force $F$ in the $x_{3}$-direction

When only a point phonon force $F$ is applied at the origin along the $x_{3}$-axis, for the case of the lower half-space, namely, $x_{3} \geq 0$, we assume that

$$
\begin{equation*}
\psi_{i}^{-}=\operatorname{sign}\left(x_{3}-h\right) A_{i} \ln \bar{R}_{I}+A_{i m}^{-} \ln R_{I m}^{-} \quad \text { and } \quad \psi_{j}^{-}=0 \tag{4.4}
\end{equation*}
$$

where $m=1,2,3$, and

$$
R_{i m}^{-}=r_{i m}^{-}+s_{i}^{-} x_{3}+s_{m}^{-} h, \quad r_{i m}^{-}=\sqrt{x_{1}^{2}+x_{2}^{2}+\left(s_{i}^{-} x_{3}+s_{m}^{-} h\right)^{2}}
$$

and

$$
\bar{R}_{i}=\bar{r}_{i}+s_{i}^{-}\left|x_{3}-h\right|, \quad \bar{r}_{i}=\sqrt{x_{1}^{2}+x_{2}^{2}+\left(s_{i}^{-}\right)^{2}\left(x_{3}-h\right)^{2}}
$$

$A_{i}$ have been obtained in equations $(3.7 a, b)$, and $A_{i m}^{-}$are nine constants to be determined. On substituting equations (4.4) into equations (2.4) and (2.6), we can obtain the corresponding components of displacements and stresses as follows:
and

$$
\left.\begin{array}{rl}
U_{\alpha \beta}^{-} & =\operatorname{sign}\left(x_{3}-h\right) k_{\alpha i}^{-} A_{i} \frac{x_{\beta}}{\bar{R}_{I} \bar{r}_{I}}+k_{\alpha i}^{-} A_{i m}^{-} \frac{x_{\beta}}{R_{I m}^{-} r_{I M}^{-}}  \tag{4.5}\\
u_{3}^{-} & =k_{3 i}^{-} A_{i} s_{I}^{-} \frac{1}{\bar{r}_{I}}+k_{3 i}^{-} A_{i m}^{-} s_{I}^{-} \frac{1}{r_{I m}^{-}},
\end{array}\right\}
$$

and

$$
\left.\begin{array}{l}
T_{\alpha 3}^{-}=-c_{\alpha i}^{-} A_{i} s_{I}^{-} \frac{x_{2}}{\bar{r}_{I}^{3}}-c_{\alpha i}^{-} A_{i m}^{-} s_{I}^{-} \frac{x_{2}}{\left(r_{I m}^{-}\right)^{3}}, \quad T_{\alpha 4}^{-}=-c_{\alpha i}^{-} A_{i} s_{I}^{-} \frac{x_{1}}{\bar{r}_{I}^{3}}-c_{\alpha i}^{-} A_{i m}^{-} s_{I}^{-} \frac{x_{1}}{\left(r_{I m}^{-}\right)^{3}} \\
\sigma_{33}^{-}=-d_{i}^{-} A_{i}\left(s_{I}^{-}\right)^{3} \frac{x_{3}-h}{\bar{r}_{I}^{3}}-d_{i}^{-} A_{i m}^{-}\left(s_{I}^{-}\right)^{2} \frac{\left(s_{I}^{-} x_{3}+s_{M}^{-} h\right)}{\left(r_{I m}^{-}\right)^{3}} \tag{4.6}
\end{array}\right\}
$$

For the case of the upper half-space, i.e. $x_{3} \leq 0, \psi_{i}^{+}$and $\psi_{j}^{+}$take the form

$$
\begin{equation*}
\psi_{i}^{+}=A_{i m}^{+} \ln R_{I m}^{+} \quad \text { and } \quad \psi_{j}^{+}=0 \tag{4.7}
\end{equation*}
$$

where

$$
R_{i m}^{+}=r_{i m}^{+}-\left(s_{i}^{+} x_{3}-s_{m}^{-} h\right) \quad \text { and } \quad r_{i m}^{+}=\sqrt{x_{1}^{2}+x_{2}^{2}+\left(s_{i}^{+} x_{3}-s_{m}^{-} h\right)^{2}}
$$

and $A_{i m}^{+}$are also nine constants to be determined. By using equations (2.4), (2.6) and (4.7), we can obtain the displacements and stresses in the upper half-space

$$
\begin{equation*}
U_{\alpha \beta}^{+}=k_{\alpha i}^{+} A_{i m}^{+} \frac{x_{\beta}}{R_{I m}^{+} r_{I M}^{+}} \quad \text { and } \quad u_{3}^{+}=-k_{3 i}^{+} A_{i m}^{+} s_{I}^{+} \frac{1}{r_{I m}^{+}} \tag{4.8}
\end{equation*}
$$

and

$$
\left.\begin{array}{rl}
T_{\alpha 3}^{+} & =c_{\alpha i}^{+} A_{i m}^{+} s_{I}^{+} \frac{x_{2}}{\left(r_{I m}^{+}\right)^{3}}, \quad T_{\alpha 4}^{+}=c_{\alpha i}^{+} A_{i m}^{+} s_{I}^{+} \frac{x_{1}}{\left(r_{I m}^{+}\right)^{3}}  \tag{4.9}\\
\sigma_{33}^{+} & =d_{i}^{+} A_{i m}^{+}\left(s_{I}^{+}\right)^{2} \frac{\left(s_{I}^{+} x_{3}-s_{M}^{-} h\right)}{\left(r_{I m}^{+}\right)^{3}}
\end{array}\right\}
$$

The undetermined constants can be obtained using the contact conditions on the interface. When the two half-spaces are ideally bonded, the conditions in equations (4.1) lead to
and

$$
\left.\begin{array}{rl}
-k_{\alpha i}^{-} A_{I}+k_{\alpha m}^{-} A_{m i}^{-} & =k_{\alpha m}^{+} A_{m i}^{+}, \\
k_{3 i}^{-} A_{I} s_{I}^{-}+k_{3 m}^{-} A_{m i}^{-} s_{M}^{-} & =-k_{3 m}^{+} A_{m i}^{+} s_{M}^{+}, \\
-c_{\alpha i}^{-} A_{I} s_{I}^{-}-c_{\alpha m}^{-} A_{m i}^{-} s_{M}^{-} & =c_{\alpha m}^{+} A_{m i}^{+} s_{M}^{+}  \tag{4.10}\\
d_{i}^{-} A_{I}\left(s_{I}^{-}\right)^{3}-d_{m}^{-} A_{m i}^{-} s_{I}^{-}\left(s_{M}^{-}\right)^{2} & =d_{m}^{+} A_{m i}^{+} s_{I}^{+}\left(s_{M}^{+}\right)^{2} .
\end{array}\right\}
$$

When the two half-spaces are in smooth contact as defined by equations (4.2), we get
and

$$
\left.\begin{array}{rl}
k_{3 i}^{-} A_{I} s_{I}^{-}+k_{3 m}^{-} A_{m i}^{-} s_{M}^{-} & =-k_{3 m}^{+} A_{m i}^{+} s_{M}^{+}, \\
d_{i}^{-} A_{I}\left(s_{I}^{-}\right)^{3}-d_{m}^{-} A_{m i}^{-} s_{I}^{-}\left(s_{M}^{-}\right)^{2} & =d_{m}^{+} A_{m i}^{+} s_{I}^{+}\left(s_{M}^{+}\right)^{2},  \tag{4.11}\\
-c_{\alpha i}^{-} A_{I} s_{I}^{-}-c_{\alpha m}^{-} A_{m i}^{-} s_{M}^{-} & =0 \\
c_{\alpha m}^{+} A_{m i}^{+} s_{M}^{+} & =0 .
\end{array}\right\}
$$

It can be seen that there are totally 18 independent linear algebraic equations involved in equations (4.10) or (4.11), from which the 18 unknown constants $A_{i m}^{-}$and $A_{i m}^{+}$can be uniquely determined. However, the solutions will be lengthy and we do not list them here. When calculating the displacements and stresses numerically, it is easier to solve the equations listed in equations (4.10) or (4.11) directly. According to the interface conditions (4.1), only some components ( $U_{\alpha \beta}$, $u_{3}, T_{\alpha 3}, T_{\alpha 4}$ and $\sigma_{33}$ ) are continuous across the interface, while the others ( $T_{\alpha \beta}$, $\sigma_{12}, H_{12}$ and $H_{21}$ ) are discontinuous if the two bonded half-spaces are really dissimilar. This continuous property is in line with the physical understanding of two half-spaces with perfect interface. Therefore, more work has to be done if these Green's function solutions are to be adopted to analyse the coupling fields at interfaces. One example is using Green's function as fundamental solutions in integral kernels of the boundary element method.

## (b) Point forces $\mathrm{Q}_{\alpha}$ in the $\mathrm{x}_{1}$-direction or $\mathrm{P}_{\alpha}$ in the $\mathrm{x}_{2}$-direction

When point forces $Q_{\alpha}$ are applied at the origin along the $x_{1}$-direction, for material 1 in $x_{3} \geq 0$, we take $\psi_{i}^{-}$and $\psi_{j}^{-}$as

$$
\begin{equation*}
\psi_{i}^{-}=B_{i} \frac{x_{1}}{\bar{R}_{I}}+B_{i m}^{-} \frac{x_{1}}{R_{I m}^{-}} \quad \text { and } \quad \psi_{j}^{-}=B_{j} \frac{x_{2}}{\bar{R}_{J}}+B_{j n}^{-} \frac{x_{2}}{R_{J n}^{-}} \tag{4.12}
\end{equation*}
$$

where $n=4,5 . \bar{R}_{j}$ and $R_{j n}^{-}$have the same structures as $\bar{R}_{i}$ and $R_{i m}^{-} . B_{i}$ and $B_{j}$ have been obtained in equations (3.14), yet $B_{i m}^{-}$and $B_{j n}^{-}$are 13 undetermined constants. Substitution of equations (4.12) into equations (2.4) and (2.6) leads to

$$
\left.\begin{array}{rl}
U_{\alpha 1}^{-}= & k_{\alpha i}^{-} B_{i}\left(\frac{1}{\bar{R}_{I}}-\frac{x_{1}^{2}}{\bar{R}_{I}^{2} \bar{r}_{I}}\right)+k_{\alpha i}^{-} B_{i m}^{-}\left[\frac{1}{R_{I m}^{-}}-\frac{x_{1}^{2}}{\left(R_{I m}^{-}\right)^{2} r_{I M}^{-}}\right] \\
& +k_{\alpha j}^{-} B_{j}\left(\frac{1}{\bar{R}_{J}}-\frac{x_{2}^{2}}{\bar{R}_{J}^{2} \bar{r}_{J}}\right)+k_{\alpha j}^{-} B_{j n}^{-}\left[\frac{1}{R_{J n}^{-}}-\frac{x_{1}^{2}}{\left(R_{J n}^{-}\right)^{2} r_{J N}^{-}}\right], \\
U_{\alpha 2}^{-}= & -k_{\alpha i}^{-} B_{i} \frac{x_{1} x_{2}}{\bar{R}_{I}^{2} \bar{r}_{I}}-k_{\alpha i}^{-} B_{i m}^{-} \frac{x_{1} x_{2}}{\left(R_{I m}^{-}\right)^{2} r_{I M}^{-}}+k_{\alpha j}^{-} B_{j} \frac{x_{1} x_{2}}{\bar{R}_{J}^{2} \bar{r}_{J}}+k_{\alpha j}^{-} B_{j n}^{-} \frac{x_{1} x_{2}}{\left(R_{J n}^{-}\right)^{2} r_{J N}^{-}}
\end{array}\right\}
$$

and $\quad u_{3}^{-}=-\operatorname{sign}\left(x_{3}-h\right) k_{3 i}^{-} B_{i} s_{I}^{-} \frac{x_{1}}{\bar{R}_{I} \bar{r}_{I}}-k_{3 i}^{-} B_{i m}^{-} s_{I}^{-} \frac{x_{1}}{R_{I m}^{-} r_{I m}^{-}}$,

$$
\begin{align*}
T_{\alpha 3}^{-}= & \operatorname{sign}\left(x_{3}-h\right) c_{\alpha i}^{-} B_{i} s_{I}^{-}\left(\frac{x_{1} x_{2}}{\bar{R}_{I}^{2} \bar{r}_{I}^{2}}+\frac{x_{1} x_{2}}{\bar{R}_{I} \bar{r}_{I}^{3}}\right)  \tag{4.13}\\
& +c_{\alpha i}^{-} B_{i m}^{-} s_{I}^{-}\left[\frac{x_{1} x_{2}}{\left(R_{I m}^{-} r_{I M}^{-}\right)^{2}}+\frac{x_{1} x_{2}}{R_{I m}^{-}\left(r_{I M}^{-}\right)^{3}}\right] \\
& -\operatorname{sign}\left(x_{3}-h\right) c_{\alpha j}^{-} B_{j} s_{I}^{-}\left(\frac{x_{1} x_{2}}{\bar{R}_{J}^{2} \bar{r}_{J}^{2}}+\frac{x_{1} x_{2}}{\bar{R}_{J} \bar{r}_{J}^{3}}\right) \\
& -c_{\alpha j}^{-} B_{j n}^{-} s_{J}^{-}\left[\frac{x_{1} x_{2}}{\left(R_{j n}^{-} r_{J N}^{-}\right)^{2}}+\frac{x_{1} x_{2}}{R_{J n}^{-}\left(r_{J N}^{-}\right)^{3}}\right], \\
T_{\alpha 4}^{-}= & \operatorname{sign}\left(x_{3}-h\right) c_{\alpha i}^{-} B_{i} s_{I}^{-}\left(\frac{x_{1}^{2}}{\bar{R}_{I}^{2} \bar{r}_{I}^{2}}+\frac{x_{1}^{2}}{\bar{R}_{I} \bar{r}_{I}^{3}}-\frac{1}{\bar{R}_{I} \bar{r}_{I}}\right) \tag{4.14}
\end{align*}
$$

$$
+c_{\alpha i}^{-} B_{i m}^{-} s_{I}^{-}\left[\frac{x_{1}^{2}}{\left(R_{I m}^{-} r_{I M}^{-}\right)^{2}}+\frac{x_{1}^{2}}{R_{I m}^{-}\left(r_{I M}^{-}\right)^{3}}-\frac{1}{R_{I m}^{-} r_{I M}^{-}}\right]
$$

$$
+\operatorname{sign}\left(x_{3}-h\right) c_{\alpha j}^{-} B_{j} s_{J}^{-}\left(\frac{x_{1}^{2}}{\bar{R}_{J}^{2} \bar{r}_{J}^{2}}+\frac{x_{1}^{2}}{\bar{R}_{J} \bar{r}_{J}^{3}}-\frac{1}{\bar{R}_{J} \bar{r}_{J}}\right)
$$

$$
+c_{\alpha j}^{-} B_{j n}^{-} s_{J}^{-}\left[\frac{x_{1}^{2}}{\left(R_{J n}^{-} r_{J N}^{-}\right)^{2}}+\frac{x_{1}^{2}}{R_{J n}^{-}\left(r_{J N}^{-}\right)^{3}}-\frac{1}{R_{J n}^{-} r_{J N}^{-}}\right]
$$

and

$$
\sigma_{33}^{-}=d_{i}^{-} B_{i}\left(s_{I}^{-}\right)^{2} \frac{x_{1}}{\bar{r}_{I}^{3}}+d_{i}^{-} B_{i m}^{-}\left(s_{I}^{-}\right)^{2} \frac{x_{1}}{\left(r_{I M}^{-}\right)^{3}}
$$

For material 2 in $x_{3} \leq 0$, we assume that

$$
\begin{equation*}
\psi_{i}^{+}=B_{i m}^{+} \frac{x_{1}}{R_{I m}^{+}} \quad \text { and } \quad \psi_{j}^{+}=B_{j n}^{+} \frac{x_{2}}{R_{J n}^{+}} \tag{4.15}
\end{equation*}
$$

where $B_{i m}^{+}$and $B_{j n}^{+}$are 13 constants to be determined again. $R_{j n}^{+}$has the same structure as $R_{i m}^{+}$. With the use of equations (4.15), we rewrite equations (2.4) and (2.6) as

$$
\left.\begin{array}{rl}
U_{\alpha 1}^{+} & =k_{\alpha i}^{+} B_{i m}^{+}\left[\frac{1}{R_{I m}^{+}}-\frac{x_{1}^{2}}{\left(R_{I m}^{+}\right)^{2} r_{I M}^{+}}\right]+k_{\alpha j}^{+} B_{j n}^{+}\left[\frac{1}{R_{J n}^{+}}-\frac{x_{1}^{2}}{\left(R_{J n}^{+}\right)^{2} r_{J N}^{+}}\right] \\
U_{\alpha 2}^{+} & =-k_{\alpha i}^{+} B_{i m}^{+} \frac{x_{1} x_{2}}{\left(R_{I m}^{+}\right)^{2} r_{I M}^{+}}+k_{\alpha j}^{+} B_{j n}^{+} \frac{x_{1} x_{2}}{\left(R_{J n}^{+}\right)^{2} r_{J N}^{+}} \tag{4.16}
\end{array}\right\}
$$

and

$$
u_{3}^{+}=k_{3 i}^{+} B_{i m}^{+} s_{I}^{+} \frac{x_{1}}{R_{I m}^{+} r_{I m}^{+}}
$$

$$
\left.\begin{array}{rl}
T_{\alpha 3}^{+}= & -c_{\alpha i}^{+} B_{i m}^{+} s_{I}^{+}\left[\frac{x_{1} x_{2}}{\left(R_{I m}^{+} r_{I M}^{+}\right)^{2}}+\frac{x_{1} x_{2}}{R_{I m}^{+}\left(r_{I M}^{+}\right)^{3}}\right]+c_{\alpha j}^{+} B_{j n}^{+} s_{J}^{+}\left[\frac{x_{1} x_{2}}{\left(R_{j n}^{+} r_{J N}^{+}\right)^{2}}+\frac{x_{1} x_{2}}{R_{J n}^{+}\left(r_{J N}^{+}\right)^{3}}\right], \\
T_{\alpha 4}^{+}= & -c_{\alpha i}^{+} B_{i m}^{+} s_{I}^{+}\left[\frac{x_{1}^{2}}{\left(R_{I m}^{+} r_{I M}^{+}\right)^{2}}+\frac{x_{1}^{2}}{R_{I m}^{+}\left(r_{I M}^{+}\right)^{3}}-\frac{1}{R_{I m}^{+} r_{I M}^{+}}\right] \\
& -c_{\alpha j}^{+} B_{j n}^{+} s_{J}^{+}\left[\frac{x_{1}^{2}}{\left(R_{J n}^{+} r_{J N}^{+}\right)^{2}}+\frac{x_{1}^{2}}{R_{J n}^{+}\left(r_{J N}^{+}\right)^{3}}-\frac{1}{R_{J n}^{+} r_{J N}^{+}}\right] \\
\sigma_{33}^{+}= & d_{i}^{+} B_{i m}^{+}\left(s_{I}^{+}\right)^{2} \frac{x_{1}}{\left(r_{I M}^{+}\right)^{3}} . \tag{4.17}
\end{array}\right\}
$$

If the two half-spaces are ideally bonded, similarly, the conditions listed in equations (4.1) lead to
and

$$
\left.\begin{array}{rl}
k_{\alpha i}^{-} B_{I}+k_{\alpha m}^{-} B_{m i}^{-} & =k_{\alpha i}^{+} B_{i m}^{+}, \\
k_{\alpha j}^{-} B_{J}+k_{\alpha n}^{-} B_{n j}^{-} & =k_{\alpha n}^{+} B_{n j}^{+}, \\
k_{3 i}^{-} B_{I} s_{I}^{-}-k_{3 m}^{-} B_{m i}^{-} s_{M}^{-} & =k_{3 m}^{+} B_{m i}^{+} s_{M}^{+}, \\
-c_{\alpha i}^{-} B_{I} s_{I}^{-}+c_{\alpha m}^{-} B_{m i}^{-} s_{M}^{-} & =-c_{\alpha m}^{+} B_{m i}^{+} s_{M}^{+},  \tag{4.18}\\
c_{\alpha j}^{-} B_{J} s_{J}^{-}-c_{\alpha n}^{-} B_{n j}^{-} s_{N}^{-} & =c_{\alpha n}^{+} B_{n j}^{+} s_{N}^{+} \\
d_{i}^{-} B_{I}\left(s_{I}^{-}\right)^{2}+d_{m}^{-} B_{m i}^{-}\left(s_{M}^{-}\right)^{2} & =d_{m}^{+} B_{m i}^{+}\left(s_{M}^{+}\right)^{2} .
\end{array}\right\}
$$

Again, when the two half-spaces are in smooth contact on their interface, the conditions specified by equations (4.2) lead to
and

$$
\begin{align*}
k_{3 i}^{-} B_{I} s_{I}^{-}-k_{3 m}^{-} B_{m i}^{-} s_{M}^{-} & =k_{3 m}^{+} B_{m i}^{+} s_{M}^{+}, \\
d_{i}^{-} B_{I}\left(s_{I}^{-}\right)^{2}+d_{m}^{-} B_{m i}^{-}\left(s_{M}^{-}\right)^{2} & =d_{m}^{+} B_{m i}^{+}\left(s_{M}^{+}\right)^{2}, \\
-c_{\alpha i}^{-} B_{I} s_{I}^{-}+c_{\alpha m}^{-} B_{m i}^{-} s_{M}^{-} & =0 \\
c_{\alpha m}^{+} B_{m i}^{+} s_{M}^{+} & =0  \tag{4.19}\\
c_{\alpha j}^{-} B_{J} s_{J}^{-}-c_{\alpha n}^{-} B_{n j}^{-} s_{N}^{-} & =0 \\
c_{\alpha n}^{+} B_{n j}^{+} s_{N}^{+} & =0
\end{align*}
$$

It can be seen again that there are totally 26 independent linear algebraic equations involved in equations (4.18) or (4.19), from which the 26 unknown constants $B_{i m}^{-}, B_{j n}^{-}, B_{i m}^{+}$and $B_{j n}^{+}$can be solved.

If the problem of the QC half-spaces subjected to point forces $P_{\alpha}$ in the $x_{2}$ direction is considered, in a similar manner to infinite solids we replace $x_{1}$ by $x_{2}$ and $x_{2}$ by $-x_{1}$ in equations (4.12) and (4.15), respectively, and replace $Q_{\alpha}$ by $P_{\alpha}$ in equations (4.18) and (4.19). Thus, the potential functions $\psi_{i}^{-}, \psi_{j}^{-}, \psi_{i}^{+}$and $\psi_{j}^{+}$ for the problem of combination of point forces $Q_{\alpha}$ in the $x_{1}$-direction and point forces $P_{\alpha}$ in the $x_{2}$-direction are as follows:

$$
\begin{align*}
& \psi_{i}^{-}=l_{\alpha i} \frac{Q_{\alpha} x_{1}+P_{\alpha} x_{2}}{\bar{R}_{I}}+l_{\alpha i m}^{-} \frac{Q_{\alpha} x_{1}+P_{\alpha} x_{2}}{R_{I m}^{-}} \text {and } \\
& \psi_{j}^{-}=l_{\alpha j} \frac{Q_{\alpha} x_{2}-P_{\alpha} x_{1}}{\bar{R}_{J}}+l_{\alpha j n}^{-} \frac{Q_{\alpha} x_{2}-P_{\alpha} x_{1}}{R_{J n}^{-}},  \tag{4.20}\\
& \psi_{i}^{+}=l_{\alpha i m}^{+} \frac{Q_{\alpha} x_{1}+P_{\alpha} x_{2}}{R_{I m}^{+}} \text {and } \psi_{j}^{+}=l_{\alpha j n}^{+} \frac{Q_{\alpha} x_{2}-P_{\alpha} x_{1}}{R_{J n}^{+}} . \tag{4.21}
\end{align*}
$$

In equations (4.20) and (4.21), $l_{\alpha i}$ and $l_{\alpha j}$ have been determined in equations (3.15). Besides, the algebraic equations to determine $l_{\text {aim }}^{-}, l_{\text {ajn }}^{-}, l_{\text {aim }}^{+}$and $l_{\text {ajn }}^{+}$are the same as equations (4.18) or (4.19), only $B_{i m}^{-}, B_{j n}^{-}, B_{i m}^{+}$and $B_{j n}^{+}$in these equations are replaced by the corresponding constants $l_{\alpha i m}^{-}, l_{\alpha j n}^{-}, l_{\alpha i m}^{+}$and $l_{\alpha j n}^{+}$, respectively. When the QC materials of the two half-spaces are the same, that is to say, $A_{i m}^{-}, A_{i m}^{+}, B_{i m}^{-}, B_{j n}^{-}, B_{i m}^{+}$and $B_{j n}^{+}$in these equations are equal to zero, the results obtained above reduce directly to those in the previous section for infinite QC solids, provided that $h=0$.

## 5. Three-dimensional Green's functions for half-space QC solids

For classical elastic problems, Lorentz (1907) developed the fundamental solutions for the displacement boundary-value problem of the elastic half-space. Phanthien (1983) showed that Lorentz's results could be further extended to point forces applied in the interior of a half-space with a fixed boundary. By superposing a complementary part of the solution to Kelvin's full-space function, Mindlin (1936) gave the fundamental solutions of the same problem under traction-free boundary conditions. These two solutions construct the basis of solving elastic problems in the half-space. According to the above results, we can directly extend Mindlin's and Lorentz's results for half-space QC solids with either free or fixed boundaries.

For the generalized Lorentz problem of half-space QC solids, the boundary condition at the surface is that the displacement components in the lower halfspace are zero, i.e. $U_{\alpha \beta}^{+}=u_{3}^{+}=0$. As a result, the right-hand sides of the first two
equations of equations (4.10) vanish, and we can get nine algebraic equations to determine the constants $A_{i m}^{-}$.

$$
\left[\begin{array}{c}
A_{1 i}^{-}  \tag{5.1}\\
A_{2 i}^{-} \\
A_{3 i}^{-}
\end{array}\right]=\left[\begin{array}{ccc}
k_{11}^{-} & k_{12}^{-} & k_{13}^{-} \\
k_{21}^{-} & k_{22}^{-} & k_{23}^{-} \\
-k_{31}^{-} s_{1}^{-} & -k_{32}^{-} s_{2}^{-} & -k_{33}^{-} s_{3}^{-}
\end{array}\right]^{-1}\left[\begin{array}{c}
k_{1 I}^{-} \\
k_{2 I}^{-} \\
k_{3 I}^{-} s_{I}^{-}
\end{array}\right] A_{i} .
$$

Similarly, the constants $B_{i m}^{-}$and $B_{j n}^{-}$can be determined from the first three equations of equations (4.18), so they can be conveniently solved as follows:
and

$$
\begin{align*}
& {\left[\begin{array}{l}
B_{1 i}^{-} \\
B_{2 i}^{-} \\
B_{3 i}^{-}
\end{array}\right]=-\left[\begin{array}{ccc}
k_{11}^{-} & k_{12}^{-} & k_{13}^{-} \\
k_{21}^{-} & k_{22}^{-} & k_{23}^{-} \\
-k_{31}^{-} s_{1}^{-} & -k_{32}^{-} s_{2}^{-} & -k_{33}^{-} s_{3}^{-}
\end{array}\right]^{-1}\left[\begin{array}{c}
k_{1 I}^{-} \\
k_{2 I}^{-} \\
k_{3 I}^{-} s_{I}^{-}
\end{array}\right] B_{i}}  \tag{5.2}\\
& {\left[\begin{array}{l}
B_{4 j}^{-} \\
B_{5 j}^{-}
\end{array}\right]=-\left[\begin{array}{ll}
k_{44}^{-} & k_{45}^{-} \\
k_{54}^{-} & k_{55}^{-}
\end{array}\right]^{-1}\left[\begin{array}{c}
c_{4 J}^{-} \\
c_{5 J}^{-}
\end{array}\right] B_{j} .}
\end{align*}
$$

For the generalized Mindlin problem of half-space QC solids, the boundary condition at the surface of the lower half-space is traction-free, i.e. $T_{\alpha 3}^{+}=T_{\alpha 4}^{+}=$ $\sigma_{33}^{+}=0$. As a result, the right-hand sides of the last two equations of equations (4.10) and the last three equations of equations (4.18) also vanish, so we can get the corresponding algebraic equations to determine the constants $A_{i m}^{-}, B_{i m}^{-}$and $B_{j n}^{-}$. It is useful to list them here:
and

$$
\begin{align*}
& {\left[\begin{array}{l}
A_{11}^{-} s_{I}^{-} \\
A_{2 i}^{-} s_{I}^{-} \\
A_{3 i}^{-} s_{I}^{-}
\end{array}\right]=\left[\begin{array}{lll}
-c_{11}^{-} s_{1}^{-} & -c_{12}^{-} s_{2}^{-} & -c_{13}^{-} s_{3}^{-} \\
-c_{21}^{-} s_{1}^{-} & -c_{22}^{-} s_{2}^{-} & -c_{23}^{-} s_{3}^{-} \\
d_{1}^{-}\left(s_{1}^{-}\right)^{2} & d_{2}^{-}\left(s_{2}^{-}\right)^{2} & d_{3}^{-}\left(s_{3}^{-}\right)^{2}
\end{array}\right]^{-1}\left[\begin{array}{c}
c_{1 I}^{-} s_{I}^{-} \\
c_{2 I}^{-} s_{I}^{-} \\
d_{I}^{-}\left(s_{I}^{-}\right)^{3}
\end{array}\right] A_{i},}  \tag{5.3}\\
& \left.\left.\left[\begin{array}{l}
B_{1 i}^{-} \\
B_{2 i}^{-} \\
B_{3 i}^{-}
\end{array}\right]=-\left[\begin{array}{lll}
-c_{11}^{-} s_{1}^{-} & -c_{12}^{-} s_{2}^{-} & -c_{13}^{-} s_{3}^{-} \\
-c_{21}^{-} s_{1}^{-} & -c_{22}^{-} s_{2}^{-} & -c_{23}^{-} s_{3}^{-} \\
d_{1}^{-}\left(s_{1}^{-}\right)^{2} & d_{2}^{-}\left(s_{2}^{-}\right)^{2} & d_{3}^{-}\left(s_{3}^{-}\right)^{2}
\end{array}\right]^{c_{1 I}^{-} s_{I}^{-}}\right] \begin{array}{c}
c_{2 I}^{-} s_{I}^{-} \\
d_{I}^{-}\left(s_{I}^{-}\right)^{2}
\end{array}\right] B_{i}  \tag{5.4}\\
& {\left[\begin{array}{l}
B_{4 j}^{-} \\
B_{5 j}^{-}
\end{array}\right]=\left[\begin{array}{ll}
c_{44}^{-} s_{4}^{-} & c_{45}^{-} s_{5}^{-} \\
c_{54}^{-} s_{4}^{-} & c_{55}^{-} s_{5}^{-}
\end{array}\right]^{-1}\left[\begin{array}{c}
c_{4 J}^{-} s_{J}^{-} \\
c_{5 J}^{-} s_{J}^{-}
\end{array}\right] B_{j} .}
\end{align*}
$$

When the point forces are applied to the surface of the half-space, i.e. $h=0$, the generalized Mindlin problem degenerates to the generalized Boussinesq problem (Gurtin 1972).

Up to here, these three generalized problems of half-space QC solids can be solved directly from the fundamental solutions of two infinite half-spaces. Results show that the fundamental solutions developed in this paper are reliable and help to describe problems in an incisive way, so they can serve as a basis for further applications.

## 6. Conclusions

The present paper studies the problems of combination of point phonon forces and point phason forces applied in infinite spaces and bimaterials, which consist of two half-spaces of dissimilar two-dimensional QC materials bonded together. By using the general solution of two-dimensional hexagonal QC, a series of displacement functions is established, and the fundamental solutions of infinite spaces and two half-spaces with point forces applied in the interior of QC solids are obtained. Furthermore, the present solutions are reduced to the solutions of infinite transversely isotropic solids, the fundamental solutions of the generalized Lorentz problem, the generalized Mindlin problem and the generalized Boussinesq problem of half-space QC solids with either free or fixed boundary. The method for deriving the displacements and stresses presented here has some merits. One of the key features of the method is that the physical quantities can be readily calculated without the need of performing any transformation operations.

Three-dimensional Green's functions can be extended to a wide range of engineering problems, for instance interlaminar stresses and surface responses in a laminated structure composed of different material layers (Gao \& Noda 2005), interfacial cracks and contact in earthquake/rock engineering (Gao \& Wang 2001), and inverse evaluation of materials properties using experimental approaches. Moreover, the solutions can be directly applied to the study of strained quantum dot semiconductor devices (Pan \& Yang 2001), since a semiconductor device commonly appears as an anisotropic heterostructure, i.e. a layered structure.

The performance of QC materials is influenced by the presence of defects such as dislocations, inclusions, cracks, etc. Such solutions are very convenient to be used in the study of point defects and inhomogeneities in the materials. These provide important information for further studying the deformation and fracture of the new solid phase and understanding clearly the interplay of the interaction between the phonon and phason activity. They also play an important role in numerical simulations such as the finite element method and the boundary element method. The straightforward but tedious formulation process can be simplified if a computer symbolic manipulator software package, such as Mathematica or Maple, is used.

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