$$\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}\frac{B_jB_k}{m+n-j-k+1} \tag{3}$$
$$= (-1)^{m-1}\frac{m!n!}{(m+n)!}B_{m+n} \qquad (m+n \text{ even}, mn>0).$$

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MINKOWSKI'S THEOREM ON NONHOMOGENEOUS APPROXIMATION

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For θ irrational, let γ be any real number such that $\gamma=\theta a+b$ has no solution in integers a and b. We give a short proof of Minkowski's classic result that there are infinitely many pairs of integers satisfying $F<1/4$, where $F=F(\theta,\gamma,x,y)=|x|\cdot|\theta x+y+\gamma|$. First we prove that given any real numbers α and β there exists an integer u such that $|u-\beta|<1$ and such that at least one of the following holds:

(A) $|u-\alpha|\cdot|u-\beta| \leqq 1/4;\ |u-\alpha|\cdot|u-\beta| \leqq |\beta-\alpha|/2.$

If β is an integer, set $u=\beta$. Otherwise define the integer n by $n<\beta<n+1$. If $n\leqq\alpha\leqq n+1$ then $|n-\alpha|\cdot|n+1-\alpha|\leqq 1/4$ and similarly for β, and so

$$|n-\alpha|\cdot|n-\beta|\cdot|n+1-\alpha|\cdot|n+1-\beta| \leqq 1/16.$$

Hence $u=n$ or $u=n+1$ gives inequality (A_1). The cases $n>\alpha$ and $\alpha>n+1$ are symmetric, and we treat $n>\alpha$. We note that

$$2(n-\alpha)^{1/2}(n+1-\beta)^{1/2}(\beta-n)^{1/2}(n+1-\alpha)^{1/2}$$
$$\leqq (n-\alpha)(n+1-\beta)+(\beta-n)(n+1-\alpha)=\beta-a$$

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and so (A_2) must hold for $u=n$ or $u=n+1$.

Now by the pigeon-hole method [1, p. 1 or 2, p. 42] it is known that there exist infinitely many pairs of integers h, k such that $|k|\cdot|k\theta-h|<1$. For each such pair choose integers r, s such that $|rh-sk+\gamma k|\leqq 1/2$. Apply (A) with

$$\alpha = r/k$$

and

$$\beta = (r\theta - s + \gamma)/(k\theta - h),$$

and define $x=r-uk$, $y=-s+uh$. Then we get $|\theta x+y+\gamma|<|k\theta-h|$ and $F<1/4$ from (A_1), $F\leqq 1/4$ from (A_2). Since $k\theta-h$ can be made arbitrarily small, and since $\theta x+y+\gamma\neq 0$, we get infinitely many pairs x, y satisfying $F\leqq 1/4$. But at most one pair can give $F=1/4$, because

$$\theta x_1 + y_1 + \gamma = \pm (4x_1)^{-1} \quad \text{and} \quad \theta x_2 + y_2 + \gamma = \pm (4x_2)^{-1}$$

would imply the rationality of $\theta(x_1-x_2)+y_1-y_2$ and hence of θ. This proof can be readily extended to Minkowski's theorem on the product of two linear forms, as will be shown elsewhere. A proof that $1/4$ is the best possible constant is given in [1, p. 49].

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