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THE SET OF BALANCED ORBITS OF MAPS OF S^1 AND S^3 ACTIONS

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ABSTRACT. Suppose that the group $G = S^1$ or $G = S^3$ acts freely on a space X and on a representation space V for G. Let $f\colon X \to V$. The paper studies the size of the subset of X consisting of orbits over which the average of f is zero. The result can be viewed as an extension of the Borsuk-Ulam theorem.

1. The average of a map. Let f be a map from S^n to $\mathbf{R}^n$. The classical Borsuk-Ulam theorem says that the set $A_f = \{x \in S^n \mid fx = f(-x)\}$ is nonempty. The formula $f(-x) - fx$ may be viewed as the average of f at the point x, with respect to the antipodal $\mathbf{Z}_2$-actions on the source space S^n, and on the target space $\mathbf{R}^n$. Thus the Borsuk-Ulam theorem can be expressed by saying that for any map f: $S^n \to \mathbf{R}^n$ there is a point where the average of f (with respect to the antipodal actions) is zero.

The average can be defined for any map of a G-space into a representation space, provided that the transformation group G admits a Haar integral, as is the case for compact groups. A theorem proved by Liulevicius in [5] can be expressed as follows: If G is a nontrivial compact Lie group acting freely on S^m and freely and orthogonally on the unit sphere in a representation space V of $\dim_{\mathbf{R}} V \leqslant m$ then for any map $f\colon S^m \to V$ there exists an $x \in S^m$ where the average of f is zero.

(1.1) DEFINITION. Let X be a G-space and let V be a finite-dimensional representation space of G. Let $f\colon X \to V$ be a (continuous) map.Then *the average of* f is the map $\mathrm{Av}f\colon X \to V$ defined by

$$(\mathrm{Av}f)x = \int g^{-1}f(gx)\,dg.$$

We note the following properties:

(1.2) For any map $f\colon X \to V$, $\mathrm{Av}f\colon X \to V$ is an equivariant map.

(1.3) If $f\colon X \to V$ is equivariant, then $\mathrm{Av}f = f$.

2. The set of balanced points.

(2.1) DEFINITION. Let X be a G-space and let V be a finite-dimensional representation space of G. A map $f\colon X \to V$ is said to be *balanced* at a point $x \in X$ if $(\mathrm{Av}f)x = 0$. (We will also say then that x is a *balanced point* of f.) Let A_f denote the set of points of X where f is balanced. Then A_f is an invariant subset of X; it is

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the union of orbits consisting of balanced points. Note that

$$A_f = A_{(\mathrm{Av}f)} = (\mathrm{Av}f)^{-1}0. \tag{2.2}$$

(2.3) EXAMPLE. Let α: $X \to X$ be an involution of X and f: $X \to \mathbf{R}^n$ be a map of X into $\mathbf{R}^n$, with the antipodal involution on $\mathbf{R}^n$. Then $A_f = \{x \in S \mid fx = f(\alpha x)\}$.

Thus the Borsuk-Ulam theorem says that any map f: $S^n \to \mathbf{R}^n$ is balanced at some point: $A_f \neq \varnothing$. Various extensions of the Borsuk-Ulam theorem have been concerned with the size of the set A_f of balanced points of f for $\mathbf{Z}_2$-actions.

3. The index. A useful invariant of a free involution α: $X \to X$ on a space X is its characteristic class, $u(X) \in H^1(X/\alpha; \mathbf{Z}_2)$; it is the 1st Stiefel-Whitney class of the orbit map $X \to X/\alpha$, which is a double covering. The index of X, $\mathrm{Ind}(X)$, is the largest integer n such that $u^n(X) \neq 0$. The index of a free involution was defined by Yang [7] and Conner and Floyd [1]. Fadell, Husseini and Rabinowitz [3, 4] extended the concept of index to actions of compact Lie groups G other than $\mathbf{Z}_2$, including nonfree actions. In this paper we are concerned with the cases $G = S^1$ or $G = S^3$, i.e., G is the unit sphere in $\mathbf{F}$ where $\mathbf{F}$ is the field of complex numbers, $\mathbf{C}$, or quaternions, $\mathbf{H}$. Let d be the dimension of $\mathbf{F}$ over $\mathbf{R}$, that is, $d = 2$ for $\mathbf{F} = \mathbf{C}$, and $d = 4$ for $\mathbf{F} = \mathbf{H}$.

The universal space E_G for these groups is the infinite dimensional sphere and the classifying space $E_G/G = B_G$ is the infinite projective space $P_\infty\mathbf{F}$. The cohomology of B_G is a polynomial algebra over $\mathbf{Z}$ on a single generator $u_\mathbf{F} \in H^d(P_\infty\mathbf{F})$.

If G acts freely on a space X, then X admits an equivariant map ϕ: $X \to E_G$. The characteristic class of the action is $u_\mathbf{F}(X) = (\phi/G)^* u_\mathbf{F} \in H^d(X/G)$. We define the index, $\mathrm{Ind}_\mathbf{F}(X)$, to be the highest integer n such that $u_\mathbf{F}^n(X)$ is of an infinite order in $H^{nd}(X/G)$. If $S_\mathbf{F} = S^{d(n+1)-1}$ is the unit sphere in $\mathbf{F}^{n+1}$ with the standard (scalar multiplication) action of G, we will simply write $u_\mathbf{F}(S_\mathbf{F}) = u_\mathbf{F}$. The index of $S_\mathbf{F}$ is n.

The following proposition can be proved in the same way as Proposition 2, part (ii), in Dold [2]:

(3.1) PROPOSITION. *If $S_\mathbf{F}$ is the sphere with the standard action and $\tilde{S}_\mathbf{F}$ denotes that sphere with an arbitrary free action, then there exists an equivariant map ϕ: $S_\mathbf{F} \to \tilde{S}_\mathbf{F}$.*

Such a map can be constructed as in [2] because $P_n\mathbf{F}$ is a cell complex whose dimension does not exceed the dimension of the sphere $\tilde{S}_\mathbf{F}$.

(3.2) COROLLARY. $\mathrm{Ind}_\mathbf{F}(\tilde{S}_\mathbf{F}) = \mathrm{Ind}_\mathbf{F}(S_\mathbf{F}) = n$.

PROOF. The inequality $\mathrm{Ind}_\mathbf{F}(\tilde{S}_\mathbf{F}) \geqslant \mathrm{Ind}_\mathbf{F}(S_\mathbf{F})$ follows from (3.1). On the other hand, $\mathrm{Ind}_\mathbf{F}(\tilde{S}_\mathbf{F})$ cannot exceed n since the covering dimension of $\tilde{S}_\mathbf{F}/G$ is at most dn as the fibre of the orbit map $\tilde{S}_\mathbf{F} \to \tilde{S}_\mathbf{F}/G$ is S^{d-1}, a manifold.

4. Main result. If X is a G-space, we will usually denote by $\overline{X}$ the orbit space X/G. If ϕ: $X \to Y$ is a G-map from X to some space Y, $\bar{\phi} = \phi/G$: $\overline{X} \to Y$ will denote the induced map of the orbit space.

We will be using the Alexander-Spanier cohomology with integer coefficients.

The main result of this paper is the following theorem. It may be viewed as an extension of the theorems of Borsuk-Ulam and Yang to actions of S^1 and S^3.

(4.1) THEOREM. *Let $G = S^1$ or $G = S^3$, respectively, and let G act freely on a space X and orthogonally and freely outside the origin on a representation space V for G over $\mathbf{F}$. Let $f\colon X \to V$ be a map. Then* $\operatorname{Ind}_{\mathbf{F}}(A_f) \geqslant \operatorname{Ind}_{\mathbf{F}}(X) - \dim_{\mathbf{F}} V$.

By (3.2) we have

(4.2) COROLLARY. *If $\tilde{S}_{\mathbf{F}}$ is the unit sphere in $\mathbf{F}^{n+1}$ with any free action of G and $f\colon \tilde{S}_{\mathbf{F}} \to V$ is a map of $\tilde{S}_{\mathbf{F}}$ into an orthogonal representation space V for G over $\mathbf{F}$, free outside the origin, then* $\operatorname{Ind}_{\mathbf{F}}(A_f) \geqslant n - \dim_{\mathbf{F}} V$.

(4.3) COROLLARY. *The covering dimension of A_f is at least $d(n-k)+d-1$, where $k = \dim_{\mathbf{F}} V$.*

This is because $H^{d(n-k)}\bar{A}_f \neq 0$ and $A_f \to \bar{A}_f$ is a bundle with fibre S^{d-1}.
Actually, a theorem more general than (4.1) will be proved in §6.

5. Comments on the equivariant cohomology. In the proof we will be using the equivariant cohomology H_G^*. If X is a G-space then $H_G^*X = H^*(E_G \times_G X)$, where G acts on $E_G \times X$ by $g(e,x) = (ge, gx)$ and $E_G \times_G X = (E_G \times X)/G$. The projection $E_G \times X \to E_G$ induces a map $E_G \times_G X \to E_{G/G} = B_G$ which is a bundle with fibre X. If G acts trivially on X, then $E_G \times_G X \cong B_G \times X$. If G acts freely on X, then the projection $E_G \times X \to X$ induces a map $E_G \times_G X \to X/G = \bar{X}$ which is a bundle with a contractible fibre E_G; in this case $H_G^*X \cong H^*\bar{X}$.

If $\cdot$ denotes a single point space then $H_G^*(\cdot) \cong H^*B_G$; in fact, the constant map $E_G \to \cdot$ induces an isomorphism $H_G^*(\cdot) \cong H_G^*E_G \cong H^*B_G$. This ring (in our case of $G = S^1$ or $G = S^3$) is polynomial algebra on a generator $u_{\mathbf{F}} \in H^dP_\infty\mathbf{F}$.

Let V be a representation space for G with $\dim_{\mathbf{R}} V = m$ and let $V_0 = V - (0)$. Since the map $E_G \times_G V \to B_G$ induced by the first projection is an orientable bundle with fibre V, it has its Thom class $U(V) \in H^m(E_G \times_G V, E_G \times_G V_0) = H_G^m(V, V_0)$. The restriction of $U(V)$ to V will be denoted by $U'(V)$. The isomorphism $\pi^*\colon H^mB_G \cong H_G^mV$ induced by the bundle projection $\pi\colon E_G \times_G V \to B_G$ maps the Euler class $e(\pi)$ to $U'(V)$: $U'(V) = \pi^*e(\pi)$. The class $e(\pi)$ will also be called the Euler *class of the action on* V and will be denoted by $e(V)$.

(5.1) PROPOSITION. *Let X be a free G-space, let $\phi\colon X \to E_G$ be a classifying map for X and let $f\colon X \to V$ be any equivariant map. Then $f^*\pi^* = \phi^*$.*

PROOF. Let $c\colon E_G \to \mathbf{F}^k$ be the constant map to 0. Consider the diagram

$$\begin{tikzcd}
E_G \times_G X \arrow[r, "1\times_G f"] \arrow[d, "1\times_G \phi"'] & E_G \times_G V \arrow[d, "\pi"] \\
E_G \times_G E_G \arrow[ur, "1\times_G c"] \arrow[r, "p"] & B_G
\end{tikzcd}$$

Since the fibre V of π is contractible, and the fibre E_G of p is contractible, the two triangles are homotopy commutative. Applying the cohomology, we have $\phi^* = (p(1 \times_G \phi))^* = f^*\pi^*$.

(5.2) PROPOSITION. *If G ($= S^1$ or S^3) acts on $V = \mathbf{F}^k$ by scalar multiplication, then $e(\mathbf{F}^k) = u_{\mathbf{F}}^k \in H^{dk}P_k\mathbf{F}$.*

PROOF. In $H_G^{dk}(\mathbf{F}^k, \mathbf{F}_0^k) \xrightarrow{i^*} H_G^{dk}\mathbf{F}^k \xleftarrow{\pi^*} H^{dk}B_G = H^{dk}P_k\mathbf{F}$, the first arrow is an isomorphism since $H_G^{dk}\mathbf{F}_0^k \cong H^{dk}P_{k-1}\mathbf{F} = 0$ and $H_G^{dk-1}\mathbf{F}_0^k \cong H_G^{dk-1}P_{k-1}\mathbf{F} = 0$. It follows that $e(\mathbf{F}^k) = \pi^{*-1}U'(\mathbf{F}^k) = u_{\mathbf{F}}^k$.

(5.3) PROPOSITION. *Let $\tilde{V}$ be an orthogonal representation space for G over $\mathbf{F}$, free outside the origin. Then the Euler class $e(\tilde{V}) \neq 0$.*

PROOF. Let $V = \mathbf{F}^k$ be the representation space with the standard (scalar multiplication) action of G and let $S(V) = S_{\mathbf{F}}$ and $S(\tilde{V})$ denote the unit sphere with the corresponding free actions. By (3.1), there is an equivariant map ϕ: $S(V) \to S(\tilde{V})$ which extends to an equivariant map ψ: $V \to \tilde{V}$. It follows that $\psi^*U'(\tilde{V}) = U'(V)$ which is nonzero by (5.2). Therefore $U'(\tilde{V}) \neq 0$ and $e(\tilde{V}) = \pi^{*-1}U'(\tilde{V}) \neq 0$.

6. Proof of the Theorem. We will prove the following theorem and show that (4.1) is a consequence of it.

(6.1) THEOREM. *Suppose that G ($= S^1$ or S^3, respectively) acts freely on a space X and orthogonally on a representation space V for G over $\mathbf{F}$. Let f: $X \to V$ be a map. If the Euler class $e(V) \neq 0$, then $\operatorname{Ind}_{\mathbf{F}}(A_f) \geqslant \operatorname{Ind}_{\mathbf{F}}(X) - \dim_{\mathbf{F}} V$.*

PROOF. By (1.2), (1.3) and (2.3) we can assume that f is equivariant; otherwise we can replace f by Avf. Let $\operatorname{Ind}_{\mathbf{F}}(X) = n$ so that $u^n(X)$ is of infinite order and let $k = \dim_{\mathbf{F}} V$. We want to show that $u_{\mathbf{F}}^{n-k}(A_f) = u_{\mathbf{F}}^{n-k}(X) \mid A_f$ is of infinite order. By the continuity of the Alexander-Spanier cohomology, it suffices to show that for every invariant neighborhood N of A_f in X, the restriction $u_{\mathbf{F}}^{n-k}(X) \mid N$ is of infinite order.

The map f can be viewed as an equivariant map of pairs f: $(X, X - A_f) \to (V, V_0)$. Let f_N: $(N, N - A_f) \to (V, V_0)$ be the restriction of f, let i denote the inclusion $X \to (X, X - A_f)$ or $V \to (V, V_0)$ and let e: $(N, N - A_f) \to (X, X - A_f)$ be the excision map. Then

$$\begin{aligned} i^*e^{*-1}&\big(\big(u_{\mathbf{F}}^{n-k}(X) \mid N\big) \cup f_N^*\big(U(V) \mid (N, N - A_f)\big)\big) \\ &= i^*\big(u_{\mathbf{F}}^{n-k}(X) \cup f^*U(V)\big) = u_{\mathbf{F}}^{n-k}(X) \cup f^*i^*U(V) \\ &= u_{\mathbf{F}}^{n-k}(X) \cup f^*U'(V) = u_{\mathbf{F}}^{n-k}(X) \cup f^*\pi^*e(V). \end{aligned}$$

Since $H^kB_G = H^kP_k\mathbf{F}$ is freely generated by $u_{\mathbf{F}}^k$, $e(V) = mu_{\mathbf{F}}^k$, where m is a nonzero integer since $e(V) \neq 0$.

Now, by (5.1),

$$\begin{aligned} u_{\mathbf{F}}^{n-k}(X) \cup f^*\pi^*e(V) &= u_{\mathbf{F}}^{n-k}(X) \cup \phi^*\big(mu_{\mathbf{F}}^k\big) \\ &= m \cdot u_{\mathbf{F}}^{n-k}(X) \cup \phi^*u_{\mathbf{F}}^k = m \cdot u_{\mathbf{F}}^{n-k}(X) \cup u_{\mathbf{F}}^k(X) \\ &= m \cdot u_{\mathbf{F}}^n(X) \end{aligned}$$

is of infinite order.

Finally, Theorem (6.1) and Proposition (5.3) imply Theorem (4.1).

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