Proceedings of the International Congress of Mathematicians
Helsinki, 1978

# The Work of Gregori Aleksandrovitch Margulis

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The work of Margulis belongs to combinatorics, differential geometry, ergodic theory, the theory of dynamical systems and the theory of discrete subgroups of real and $p$-adic Lie groups. In this report, I shall concentrate on the last aspect which covers his main results.

**1. Discrete subgroups of Lie groups. The origin.** Discrete subgroups of Lie groups were first considered by Poincaré, Fricke and Klein in their work on Riemann surfaces: if $M$ is a Riemann surface of genus $\geqslant 2$, its universal covering is the Lobatchevski plane (or Poincaré half-plane), therefore the fundamental group of $M$ can be identified with a discrete subgroup $\Gamma$ of $\mathrm{PSL}_2(\boldsymbol{R})$; the problem of uniformization and the theory of differentials on $M$ lead to the study of automorphic forms relative to $\Gamma$.

Other discrete subgroups of Lie groups, such as $\mathrm{SL}_n(\boldsymbol{Z})$ (in $\mathrm{SL}_n(\boldsymbol{R})$) and the group of "units" of a rational quadratic form (in the corresponding orthogonal group) play an essential role in the theory of quadratic forms (reduction theory) developed by Hermite, Minkowski, Siegel and others. In constructing a space of moduli for abelian varieties, Siegel was led to consider the "modular group" $\mathrm{Sp}_{2n}(\boldsymbol{Z})$, a discrete subgroup of $\mathrm{Sp}_{2n}(\boldsymbol{R})$.

The group $\mathrm{SL}_n(\boldsymbol{Z})$, the group of units of a rational quadratic form and the modular group are special instances of "arithmetic groups", as defined by A. Borel and Harish-Chandra. A well-known theorem of those authors, generalizing classical results of Fricke, Klein, Siegel and others, asserts that if $\Gamma$ is an arithmetic subgroup of a semi-simple Lie group $G$, then the volume of $G/\Gamma$ (for any $G$-invariant measure) is finite; we say that $\Gamma$ has *finite covolume* in $G$. The same holds for

$G = \mathrm{PSL}_2(\boldsymbol{R})$ if $\Gamma$ is the fundamental group of a Riemann surface "with null boundary" (for instance, a compact surface minus a finite subset).

Already Poincaré wondered about the possibility of describing all discrete subgroups of finite covolume in a Lie group $G$. The profusion of such subgroups in $G = \mathrm{PSL}_2(\boldsymbol{R})$ makes one at first doubt of any such possibility. However, $\mathrm{PSL}_2(\boldsymbol{R})$ was for a long time the only simple Lie group which was known to contain non-arithmetic discrete subgroups of finite covolume, and further examples discovered in 1965 by Makarov and Vinberg involved only few other Lie groups, thus adding credit to conjectures of Selberg and Pyatetski–Shapiro to the effect that "for most semisimple Lie groups" discrete subgroups of finite covolume are necessarily arithmetic. Margulis' most spectacular achievement has been the complete solution of that problem and, in particular, the proof of the conjectures in question.

**2. The noncocompact case. Selberg's conjecture.** Let $G$ be a semisimple Lie group. To avoid inessential technicalities, we assume that $G$ is the group of real points of a real simply connected algebraic group $\mathscr{G}$ which we suppose embedded in some $\mathrm{GL}_n(\boldsymbol{R})$, and that $G$ has no compact factor. Let $\Gamma$ be a discrete subgroup of $G$ with finite covolume and irreducible in the sense that its projection in any nontrivial proper direct factor of $G$ is nondiscrete. Suppose that the real rank of $G$ is $\geqslant 2$ (this means that $G$ is not a covering group of the group of motions of a real, complex or quaternionic hyperbolic space or of an "octonionic" hyperbolic plane) and that *$G/\Gamma$ is not compact.* Then, Selberg's conjecture asserts that $\Gamma$ is arithmetic which, in this case, means the following: there is a base in $\boldsymbol{R}^n$ with respect to which $\mathscr{G}$ is defined by polynomial equations with rational coefficients and such that $\Gamma$ is commensurable with $\mathscr{G}(\boldsymbol{Z}) = G \cap \mathrm{GL}_n(\boldsymbol{Z})$ (i.e. $\Gamma \cap \mathscr{G}(\boldsymbol{Z})$ has finite index in both $\Gamma$ and $\mathscr{G}(\boldsymbol{Z})$). Selberg himself proved that result in the special case where $G$ is a direct product of (at least two) copies of $\mathrm{SL}_2(\boldsymbol{R})$.

A first important step toward the understanding of noncompact discrete subgroups of finite covolume was the proof by Každan and Margulis [2] of a related, more special conjecture of Selberg: *under the above assumptions* (except that no hypothesis is made on $\mathrm{rk}_{\boldsymbol{R}} G$), *$\Gamma$ contains nontrivial unipotent elements of $G$* (i.e. elements all of whose eigenvalues are 1). This was a vast generalization of results already known for $\mathrm{SL}_2(\boldsymbol{R})$ and products of copies of $\mathrm{SL}_2(\boldsymbol{R})$ (Selberg); in view of a fundamental theorem of Borel and Harish–Chandra ("Godement's conjecture"), it had to be true if $\Gamma$ was to be arithmetic. Let us also note in passing another remarkable byproduct of Každan–Margulis' method: *given $G$, there exists a neighborhood $W$ of the identity in $G$ such that for every $\Gamma$ (cocompact or not), some conjugate of $\Gamma$ intersects $W$ only at the identity*; in particular, *the volume of $G/\Gamma$ cannot be arbitrarily small* (for a given Haar measure in $G$). For $G = \mathrm{SL}_2(\boldsymbol{R})$, the last assertion had been proved by Siegel, who had also given the exact lower bound of $\mathrm{vol}\,(G/\Gamma)$ in that case. A. Borel reported on those results of Každan and Margulis at Bourbaki Seminar [26].

The existence of unipotent elements in $\Gamma$ was giving a hold on its structure. In

[6], Margulis announced, among others, the following result which was soon recognized by the experts as a crucial step for the proof of Selberg's conjecture:

*in the space of lattices in* $\boldsymbol{R}^n$, *the orbits of a one-parameter unipotent subsemigroup of* $\mathrm{GL}_n(\boldsymbol{R})$ *"do not tend to infinity" (in other words, a closed orbit is periodic).*

For a couple of years, Margulis' proof remained unpublished and every attempt by other specialists to supply it failed. When it finally appeared in [9], the proof came as a great surprise, both for being rather short and using no sophisticated technique: it can be read without any special knowledge and gives a good idea of the extraordinary inventiveness shown by Margulis throughout his work.

Using unipotent element, it is relatively easy to show that, $G$ and $\Gamma$ being as above, there is a $\boldsymbol{Q}$-structure $\mathscr{G}$ on $G$ such that $\Gamma\subset\mathscr{G}(\boldsymbol{Q})$. The main point of Selberg's conjecture is then to show that the matrix coefficients of the elements of $\Gamma$ have bounded denominators. In [15], Margulis announced a complete proof of the conjecture and gave the details under the additional assumption that the $\boldsymbol{Q}$-rank of $\mathscr{G}$ is at least 2. Another proof under the same restriction was given independently by M. S. Raghunathan. The much more difficult case of a $\boldsymbol{Q}$-rank one group is treated by Margulis in [19], by means of a very subtle and delicate analysis of the set of unipotent elements contained in $\Gamma$. The main techniques used in [15] and [19] are those of algebraic group theory and $p$-adic approximation.

**3. The cocompact case. Rigidity.** Margulis was invited to give an address at thf Vancouver Congress, no doubt with the idea that he would expose his solution oe Selberg's conjecture. Instead, prevented (as this time) from attending the Congress, he sent a report on completely new and totally unexpected results on the cocompact case [18].

That case, about which nothing was known before, presented two great additional difficulties which nobody knew how to handle. On the one hand, if $G/\Gamma$ is compact, $\Gamma$ contains no unipotent element, so that the main technique used in the other case is not available. But there is another basic difficulty in the very notion of arithmetic group: let $G, \Gamma$ be as in § 2 except that $G/\Gamma$ is no longer assumed to be noncompact; then $\Gamma$ is said to be arithmetic if there exist an algebraic linear semisimple simply connected group $\mathscr{H}$ defined over $\boldsymbol{Q}$ and a homomorphism $\alpha:\mathscr{H}(\boldsymbol{R})\to G$ with compact kernel such that $\Gamma$ is commensurable with $\alpha(\mathscr{H}(\mathbf{Z}))$. The point is chat in the non-cocompact case, $\alpha$ is necessarily an isomorphism. In the general case, there is *a priori* no way of knowing what $\mathscr{H}$ will be (in fact, for a given $G$, $\mathscr{H}$ tan have an arbitrarily large dimension). A conjecture, more or less formulated by Pyatetski-Shapiro at the 1966 Congress in Moscow, to the effect that also in the cocompact case, assuming again $\mathrm{rk}_{\boldsymbol{R}}\, G\geqslant 2$, $\Gamma$ had to be arithmetic, was certainly more daring at the time and seemed completely out of reach. It was the proof of that conjecture that Margulis sent, without warning, to the Vancouver Congress.

Arithmetic subgroups of Lie groups are in some sense "rigid"; intuitively, this follows from the impossibility to alter an algebraic number continuously without destroying the algebraicity. On the other hand, theorems of Selberg, Weil and Mostow showed that in semi-simple Lie groups different from $SL_2(\mathbf{R})$ (up to local isomorphism) cocompact discrete subgroups are rigid, and Selberg had observed that rigidity implies a "certain amount of arithmeticity": in fact, it is readily seen to imply that $\Gamma$ is contained in $\mathcal{G}(K)$ for some algebraic group $\mathcal{G}$ and some number field $K$. As before, the crux of the matter is the proof that the matrix coefficients of the elements of $\Gamma$ have bounded denominators. This is achieved by Margulis through a "superrigidity" theorem which, for groups of real rank at least 2, is a vast generalization of Weil's and Mostow's rigidity theorems:

*Assume* $\mathrm{rk}_{\mathbf{R}}\, G \geqslant 2$, *let* $F$ *be a locally compact nondiscrete field and let* $\varrho\colon \Gamma \to GL_n(F)$ *be a linear representation such that* $\varrho(\Gamma)$ *is not relatively compact and that its Zariski-closure is connected; then* $F=\mathbf{R}$ *or* $\mathbf{C}$ *and* $\varrho$ *extends to a rational representation of* $\mathcal{G}$.

The proof of this theorem is relatively short (considering the power of the result), but is a succession of extraordinarily ingenious arguments using a great variety of very strong techniques belonging to ergodic theory (the "multiplicative ergodic theorem"), the theory of unitary representations, the theory of functional spaces (spaces of measurable maps), algebraic geometry, the structure theory of semi-simple algebraic groups, etc. In 1975–1976, I devoted my course at the Collège de France to those results of Margulis; I believe that I learned more mathematics during that year than in any other year of my life. A summary of the main ideas of that beautiful piece of work is given in **[27]**.

Another, quite different proof of the superrigidity theorem and its application to arithmeticity (both in the cocompact and the noncocompact case)—using the work of H. Furstenberg—can be found in **[20]**.

**4. Other results.**

4.1. *S-arithmetic groups.* Let $K$ be a number field, $S$ a finite set of places of $K$ including all places at infinity, $\mathfrak{o}$ the ring of elements of $K$ which are integral at all finite places not belonging to $S$, $\mathcal{H} \subset \mathcal{GL}_n$ a simply connected semisimple linear algebraic group defined over $K$ and $\mathcal{H}(\mathfrak{o}) = \mathcal{H} \cap GL_n(\mathfrak{o})$. Then, $\mathcal{H}(\mathfrak{o})$ injects as a discrete subgroup of finite covolume in the product $H = \prod_{v \in S} \mathcal{H}(K_v)$, where $K_v$ denotes the completion of $K$ at $v$.

(Example: if $\mathfrak{o}$ is the ring of rational numbers whose denominator is a power of 2, $SL_n(\mathfrak{o})$ is a discrete subgroup of finite covolume of $SL_n(\mathbf{R}) \times SL_n(\mathbf{Q}_2)$). Now let $G$ be a direct product of simply connected semi-simple real or $p$-adic Lie groups. A discrete subgroup $\Gamma$ of $G$ is called $S$-arithmetic if there exist $K, S, \mathcal{H}$ as above and a homomorphism $\alpha\colon H \to G$ with compact kernel such that $\Gamma$ and $\alpha(\mathcal{H}(\mathfrak{o}))$ are commensurable. All results of **[18]**, stated above for ordinary Lie groups and arithmetic groups, are in fact proved by Margulis in the more general framework described here. In particular, he shows that *if* $G$ *is as above, if the* *rank of $G$ (i.e. the sum of the relative ranks of its factors) is at least* 2 *and if $\Gamma$ is a discrete subgroup of finite covolume in $G$, which is irreducible (as defined in $n^\circ 2$), then $\Gamma$ is S-arithmetic.*

4.2. *"Abstract" isomorphisms.* The very general and powerful superrigidity theorem (in the framework of 4.1) has far-reaching consequences besides the arithmeticity. For instance, it enables Margulis to solve almost completely the problem of "abstract isomorphisms" between groups of points of algebraic simple groups over number fields or arithmetic subrings of such fields; his result embodies, in the arithmetic case, all those obtained before on that problem by Dieudonné, O'Meara and his school, A. Borel and me, and goes considerably further.

4.3. *Normal subgroups.* Let $G$ be as in 4.1 and let $\Gamma$ be an irreducible discrete subgroup of $G$ of finite covolume. Margulis was able to show (cf. [27]) that *if* $\mathrm{rk}\, G \geqslant 2$, *then every noncentral normal subgroup of $\Gamma$ has finite index.* (In fact, the conditions of Margulis' theorem are more general: under suitable hypotheses, $G$ is allowed to have factors defined over locally compact local fields of finite characteristic.) So far, the only results known in that direction—results of Mennicke, Bass, Milnor, Serre, Raghunathan—were connected with the congruence subgroup problem and valid only in the cases where that problem has a positive solution.

4.4. *Action on trees.* In a paper which appeared in the Springer Lecture Notes, no 372, Serre showed that the group of integral points of a simple Chevalley group-scheme of rank $\geqslant 2$ cannot act without fixed point on a tree; this also means that such a group is not an amalgam in a nontrivial way. Serre points out that his method of proof does not extend to congruence subgroups and asks whether the result generalizes to such subgroups or to other arithmetic groups. With his own methods, Margulis was able to solve at once the problem in its widest generality: *if $G$ is as in* 4.1, *of rank at least* 2, *and if $\Gamma$ is an irreducible discrete subgroup of finite covolume in $G$, then $\Gamma$ cannot act without fixed point on a tree.*

**5. Conclusion.** Margulis has completely or almost completely solved a number of important problems in the theory of discrete subgroups of Lie groups, problems whose roots lie deep in the past and whose relevance goes far beyond that theory itself. It is not exaggerated to say that, on several occasions, he has bewildered the experts by solving questions which appeared to be completely out of reach at the time. He managed that through his mastery of a great variety of techniques used with extraordinary resources of skill and ingenuity. The new and most powerful methods he has invented have already had other important applications besides those for which they were created and, considering their generality, I have no doubt that they will have many more in the future.

I wish to conclude this report by a nonmathematical comment. This is probably neither the time nor the place to start a polemic. However, I cannot but express my deep disappointment—no doubt shared by many people here—in the absence of Margulis from this ceremony. In view of the symbolic meaning of this city of Helsinki,[1] I had indeed grounds to hope that I would have a chance at last to meet a mathematician whom I know only through his work and for whom I have the greatest respect and admiration.

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