Jaakko Hintikka and Gabriel Sandu

A REVOLUTION IN LOGIC?

Logic might at first sight seem an unlikely arena for revolutions, even for revolutions in Jefferson's sense rather than Lenin's. Kant maintained that Aristotelian logic had not changed in two thousand years and could never change.[1] Even though Kant's view of Aristotle's logic has been thoroughly discredited (even by Aristotle's own standards) both systematically and historically,[2] most contemporary philosophers and linguists are adopting the same Kantian attitude to Frege's logic, or strictly speaking rather to that part of Frege's logic that has come to be known variously as first-order logic, quantification theory, or lower predicate calculus. When a critic once suggested to a prominent philosopher of language that one of the cornerstones of ordinary first-order logic, Frege's treatment of verbs for being like *is*, does not capture the true *Sprachlogik*, he looked at the speaker with an expression of horror—mock horror, we hope—and said, 'Nothing is sacred in philosophy any longer!'

Yet Frege's formulation of first-order logic contains a fundamental error. It is manifested already in his formation rules. And this same virus has infected all the subsequent formulations and versions of first-order logic. In order to see what Frege's mistake is, we have to go back to the basics and ask what first-order logic is all about. The answer is obvious. First-order logic is about quantifiers. It is not for nothing called quantification theory. But what is often overlooked is that quantification theory is not a study of quantifiers in splendid isolation from each other. If you use quantifiers one by one unrelated to each other, the only logic you end up with is monadic first-order

[1] Immanuel Kant, *Kritik der reinen Vernunft* B 7–8.

[2] Cf. Jaakko Hintikka, "Aristotle's Incontinent Logician", *Ajatus* vol. 37 (1978), pp. 48–65, where Hintikka argues that Aristotle's system of logic was an unhappy compromise between the different guiding ideas he was trying to implement.

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logic or some other mild generalization of Aristotelian syllogistic. The real power of first-order logic lies in the use of *dependent quantifiers*, as in

(1) $$(\forall x)(\exists y)S[x, y]$$

where the truth-making choice of a value of y depends on the value of x. Without dependent quantifiers, one cannot even express the functional dependence of a variable on another. Quantifier dependence is therefore the veritable secret of the strength of first-order logic. One can almost say that to understand first-order logic *is* to understand the notion of quantifier dependence. Several of the typical applications of first-order logic in philosophical analysis, such as uncovering quantifier-switch fallacies, exploit directly the idea of quantifier dependence.

But to understand quantifier dependence is *ipso facto* to understand the notion of quantifier *in*dependence. The latter is simply the same concept as the former in a different dress. Hence it does not mean transcending in the least the conceptual repertoire of ordinary quantification theory if we introduce into it an explicit independence indicator by means of which the independence of (Q_1x) of another quantifier (Q_2x) is expressed by writing it (Q_1x/Q_2x).

The same slash notation can be applied to propositional connectives. It is intuitively obvious what is meant by such formulas as

(2) $$(\forall x)(A(x)(\vee/\forall x)B(x))$$

or

(3) $$(\forall x)(\exists y)(\forall z)(S_1[x, y, z](\vee/\forall x)S_2[x, y, z])$$

It turns out that if we limit our attention to formulas in negation normal form (i.e. in a form where all negation-signs precede immediately atomic formulas or identities), it suffices to consider only two kinds of uses of the slash, viz. $(\exists x/\forall y)$ and $(\vee/\forall y)$.

The result of adding the slash to the conceptual arsenal of the received first-order logic results in what will be called *independence-friendly* (IF) *first-order logic.*

What is IF first-order logic like? Does it really go beyond from received first-order logic? In simplest cases, the new notation does not yield anything new. For instance,

(4) $$(\forall x)(\exists y/\forall x)S[x, y]$$

is obviously logically equivalent with

(5) $$(\exists y)(\forall x)S[x, y]$$

Likewise (2) is equivalent with

$$(6) \qquad (\forall x)A(x) \vee (\forall x)B(x)$$

But in more complex cases something remarkable happens. Consider, for instance, the sentence

$$(7) \qquad (\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)S[x, y, z, u]$$

which is equivalent to

$$(8) \qquad (\forall x)(\exists y)(\forall z)(\exists u/\forall x)S[x, y, z, u]$$

Can (7) be expressed without the slash notation? If so, there would be a linear ordering of the four initial quantifiers $(\forall x), (\exists y), (\forall z), (\exists u)$ that would result in the same logical force as (7). But what kind of ordering might that be? Since $(\exists y)$ depends in (7) on $(\forall x)$ but not on $(\forall z)$, the relative order of the three would have to be

$$(9) \qquad (\forall x)(\exists y)(\forall z)$$

But since $(\exists u)$ depends on $(\forall z)$ but not on $(\forall x)$, their relative order would have to be $(\forall z)(\exists u)(\forall x)$, which is incompatible with (9).

Likewise, it can be shown that (3) cannot be expressed without the slash notation (or equivalent).

Here we are beginning to see the whole horror of Frege's mistake. The notation he introduced (like the later notation of Russell and Whitehead) arbitrarily rules out certain perfectly possible patterns of dependence and independence between quantifiers or between connectives and quantifiers. These patterns are the ones represented by irreducible slashed quantifier combination (and similar combinations involving quantifiers and connectives). Anyone who understands received first-order logic understands sentences which involve such patterns, for instance understands sentences of the form (7) or (8), in the concrete sense of understanding the precise conditions that their truth imposes on reality. Hence they ought to be expressible in the language of our basic logic. And the only way of doing so is to build our true logic of quantification so as to dispense with the artificial restrictions Frege imposed on the received first-order logic. The real logic of quantification, in other words the real ground floor of the edifice of logic, is not the ordinary first-order logic. It is IF first-order logic. Terminologically speaking, we are doing ourselves an injustice by calling the received first-order logic "ordinary". The limitations we have been discussing make Fregean logic systematically speaking quite extraordinary, as is also reflected by the fact that the properties of the received first-order logic do not reflect at all faithfully what one can expect to happen in logic in general.[3]

In order to reach IF first-order logic, it is not necessary to introduce any new notation like our slash notation. All that is needed is to formulate the rules for parentheses (scope) more liberally than Frege and Russell. If you reflect on the matter (and even if you don't), there is absolutely no reason why the so-called scope of a quantifier, expressed by the associated pair of parentheses, should comprise a continuous segment of a formula adjacent to the quantifier and following it.[4] Expressions violating this requirement (and consequently violating the usual formation rules) can have a perfectly understandable semantical interpretation. Indeed, (8) might as well be written as follows:

$$(\forall x)[(\exists y)(\forall z)](\exists u)[S[x,y,z,u]] \tag{10}$$

where the two outer pairs of square brackets indicate the (discontinuous) scope of $(\forall x)$. In more complicated cases, alas, the liberated parentheses notation turns out to be unintuitive to the point of being unreadable. Hence for practical reasons we prefer the slash notation.

A fragment of IF first-order logic is known to the cognoscendi as the logic of partially ordered ("branching") quantifiers.[5] This logic is nevertheless not entirely representative of the conceptual situation in general. For one thing, propositional connectives can exhibit the same independence phenomena as quantifiers.[6] Moreover, even though quantifier dependencies and independencies can always be dealt with in terms of partial ordering, this is not true of other concepts. For there is in general no reason why the dependence of logical notions on each other should be transitive.

[3]Cf. Jaakko Hintikka, *The Principles of Mathematics Revisited.* Cambridge University Press, Cambridge, 1996.

[4]Note that the usual scope notation for propositional connectives already violates some of these requirements.

[5]Branching quantifiers were introduced by Leon Henkin, "Some Remarks on Infinitely Long Formulas", in *Infinitistic Methods,* Warsaw, 1959, pp. 167–183. For their theory, see e.g. W. Walkoe, "Finite Partially Ordered Quantification", in *Journal of Symbolic Logic,* vol. 35 (1970), pp. 535–550; Herbert Enderton, "Finite Partially-Ordered Quantifiers", in *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* vol. 16 (1970), pp. 393–397; and Jon Barwise, "Some Applications of Henkin Quantifiers", in *Israel Journal of Mathematics*, 25, 1976, pp. 47–80.

[6]See Gabriel Sandu and Jouko Väänänen, "Partially Ordered Connectives", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* vol. 38 (1992), pp. 361–372.

Furthermore, the treatment of negation in the context of independent quantifiers requires more attention than it has been paid in the theory of partially ordered quantifiers.[7]

By excluding the possibility of genuine independence between quantifiers (and by implication between other concepts) Frege excluded from the purview of logic a wealth of important subjects. Many logicians are still apt to consider the logic of branching quantifiers and also IF first-order logic as a marginal curiosity. Nothing could be further from the truth. One could write an entire book about the impact of IF first-order logic on the foundations of mathematics. (Indeed, Jaakko Hintikka has done just that. See note 3.) But the impact of this new logic extends much more widely. Far from being a recondite linguistic phenomenon, independence of the sort IF first-order logic deals with is a frequent and important feature of natural-language semantics.[8] Without the notion of independence, we cannot fully understand the logic of such concepts as belief, knowledge, questions and answers, or the *de dicto* vs. *de re* contrast.[9]

In first-order logic itself, the phenomenon of independence occasions changes in the most basic laws of logical inference. (Such laws were of course precisely what Frege was trying to capture by means of his *Begriffsschrift.*) Existential quantifiers cannot any longer be treated (along the lines of familiar *tableau* or Gentzen-type methods)[10] by an ordinary rule of instantiation for formula-initial quantifiers. In order to see this, consider an existential formula

$$(\exists x/\forall z_1, \forall z_2, \ldots)S_2[x] \tag{11}$$

occurring in a wider context

$$S_1[(\exists x/\forall z_1, \forall z_2, \ldots)S_2[x]] \tag{12}$$

(We are of course assuming that (12) is in a negation normal form, i.e. that all its negation-signs precede immediately atomic formulas or identities.) If we try to set up instantiation rules that deal only with

[7] For the behaviour of negation in IF logic, see Hintikka, *op. cit.*, note 3, chapter 7.

[8] See Jaakko Hintikka and Gabriel Sandu, "Informational Independence as a Semantical Phenomenon", in J. E. Fenstad *et al.* (eds.), *Logic, Methodology and Philosophy of Science,* Elsevier, Amsterdam, 1989, pp. 571–589.

[9] See here Jaakko Hintikka, "Knowledge Acknowledged", *Philosophy and Phenomenological Research,* forthcoming, 1996.

[10] The classical formulation of the *tableau* method is E. W. Beth, "Semantical Entailment and Formal Derivability", *Mededelingen van de Koninklijke Nederlandse Akademie van Wetenschappen,* Afd. Letterkunde, N.R. vol. 18, 1953, 309–342.

formulas of the form (11), we are in trouble. For in the transition from (11) to (12) in, say, a *tableau*-type procedure, individual constants will have to be substituted for $z_1, z_2, \ldots$ and more generally for all variables bound to quantifiers whose scope includes (11). But the result of carrying out all these substitutions in (11) is ill-formed (since there are no longer any quantifiers $(\forall z_1), (\forall z_2), \ldots$ for $(\exists x)$ to be independent of), and has to be replaced by a formula of the form

$$(13) \qquad (\exists x)S^*[x]$$

where all the free variables of (11) have been replaced by individual constants. But then the difference between the outer quantifiers in (12) which $(\exists x)$ depends or doesn't depend on will be erased.

What has to be done is obviously to formulate a rule of instantiation that allows us to move from (12) to

$$(14) \qquad S_1[S_2[f(y_1, y_2, \ldots)]]$$

where $(\forall y_1), (\forall y_2), \ldots$ are all the universal quantifiers other than $(\forall z_1), (\forall z_2), \ldots$ within the scope of which $(\exists x/\forall z_1, \forall z_2, \ldots)$ occurs in (12). In (14), f must be a new function symbol. Notice that in spite of the introduction of a new function symbol, (14) is first-order. It does not involve any quantification over functions or other higher-order entities.

Frege's rejection of independent quantifiers can be motivated by one or both of two general theoretical ideas, both of which Frege espoused in so many words. One of them is Frege's interpretation of quantifiers as higher-order predicates. On this view, what an existential statement like (13) says is that the (usually complex) predicate $S^*[x]$ is not empty. Now for the same reasons as were seen to prevent an adequate treatment of existential quantifiers by the sole means of the usual rule of existential instantiation, the Fregean treatment of existence fails to do justice to independent quantifiers. At bottom, the interpretation of quantifiers as higher-order predicates is but an example of philosophers' unfortunate tendency to try to deal without quantifiers without taking into account their interaction. Another one is the presumption that we can understand quantifiers in the sole terms of their "ranging over" a class of values.

In this department, the sins of one's intellectual ancestors are visited on the theories of their scholarly children and grandchildren. The most intensively cultivated recent treatment of quantifiers in general has been the so-called theory of generalized quantifiers,[11] which relies

[11] For a survey of the theory of generalized quantifiers, see M. Krynicki, M.

crucially on the Fregean interpretation of quantifiers as higher-order predicates. Small wonder, therefore, that this theory has not been general enough to deal with independent quantifiers. The best that the representatives of this tradition have been able to do is to give interpretations of certain particular combinations of quantifiers involving independent ones. Such a treatment fails to be a genuine theory of independent quantifiers by a long shot. For instance, it is helpless to say anything about independence relations between quantifiers and propositional connectives.

But the possibility of independent quantifiers militates against even more received general semantical principles. This can be seen most vividly by changing our notation and associating the independence indicator (changed now to a double slash) to the quantifier that another one is independent of. Thus instead of (4) we could now write

(15) $$(\forall x//\exists y)(\exists y)S[x,y]$$

and instead of (7)

(16) $$(\forall x//\exists u)(\forall z//\exists y)(\exists y)(\exists u)S[x,y,z,u]$$

The double-slash notation makes explicit what is less conspicuous in our original notation. One of the most striking things conceptually about IF first-order logic is that it violates what is often known as the principle of compositionality.[12] This principle can be formulated by saying that according to it the crucial semantical attributes of an expression are functions of the semantical attributes of its constituent expressions. Another, somewhat more illuminating, way of putting it is to say that the principle is an assertion of semantical context-independence. According to it, the semantical interpretation of an expression must not depend on its context. The reason is clear. If it did, the semantical properties of the expression in question could not be completely determined by those of its constituent expressions.

But in IF first-order logic we do find semantical context-dependence. For instance, the very same expressions $(\exists y)S[x,y]$ and $(\exists y)(\exists u)S[x,y,z,u]$ occur not only as subformulas of (15) and (16) but as subformulas of (1) and of

(17) $$(\forall x)(\forall z)(\exists y)(\exists u)S[x,y,z,u]$$

Mostowski and L. K. Szczerba (eds.), *Quantifiers: Logics, Models and Computation,* Kluwer Academic Publishers, 1995.

[12] For this principle, see Barbara Partee, "Compositionality", in F. Landman and F. Veltman (eds.), *Varieties of Formal Semantics,* Dordrecht, Foris, 1984, pp. 281–312 and Francis J. Pelletier, "The Principle of Semantic Compositionality" *Topoi*, vol. 13 (1994), pp. 11–24.

respectively. Yet their semantical behaviour depends on the context, for only from the context can one see which other quantifiers $(\exists y)$ and $(\exists u)$ depend and do not depend on. Hence IF first-order logic offers a clear-cut counter-example to the principle of compositionality.

Even though there may be some room for doubt—or at least need for some further explanations—Frege accepted the principle of compositionality, so much so that it is sometimes referred to as the Frege principle.[13] Small wonder, then, that Frege did not countenance independent quantifiers. They would have forced him to give up the principle of compositionality.

The properties of IF first-order logic need (and deserve) a much more extensive discussion than can be presented in the confines of one paper.[14] Independence-friendly first-order logic has many of the same pleasant metalogical properties as the received first-order logic. It is compact, and such results as the Löwenheim-Skolem theorem and the separation theorem hold in it. Notice, however, that negation in this logic is strong, dual negation, and not classical contradictory negation. Thus the fact that this logic has the above stated properties does not contradict Lindström's theorem. IF first-order logic also admits of a complete disproof procedure, assuming of course a reformulation of the rule of the existential instantiation as a rule of functional instantiation explained above. In other words, the set of inconsistent formulas is recursively enumerable. But there is one massive difference between IF first-order logic and its predecessors. *IF first-order logic does not admit of a complete axiomatization.* It is inevitably incomplete.[15] The class of valid IF first-order logic formulas is not recursively enumerable.

[13] See Michael Dummett, *Frege: Philosophy of Language,* Duckworth, London, 1973.

[14] For a tentative exploration, see Jaakko Hintikka, "What Is True Elementary Logic?", in Kostas Gavroglu, John Stachel and Marx Wartofsky (eds.), *Physics, Philosophy and the Scientific Community,* Kluwer Academic Publishers, Dordrecht, 1994, pp. 301–326.

[15] The easiest way to see this incompleteness is the following: The so-called Henkin quantifier H acting on four variables is defined as follows:

$$M \models H\ xyzw\varphi \Leftrightarrow M \models \exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(z)).$$

The Henkin quantifier is definable in IF logic in an obvious way:

$$M \models H\ xyzw\varphi \Leftrightarrow M \models \forall x \forall z (\exists y/\forall z)(\exists w/\forall x)\varphi.$$

It is well known (cf. M. Krynicki and A. Lachlan, "On the Semantics of Henkin Quantifier", *Journal of Symbolic Logic,* vol. 44, (1979), pp. 184–200) that the logic which extends first-order logic with the Henkin quantifier is incomplete. From this it also follows that IF logic is incomplete.

The title of our paper speaks of a revolution. Indeed, IF first-order logic has many revolutionary features. It will take more than one monograph to spell them out. However, there does not seem any doubt that the semantical incompleteness of our basic logic—for that's what IF first-order logic was just found to be—is the most profoundly revolutionary feature of the new logic. Here we can only briefly indicate some of the main consequences of this particular aspect of IF first-order logic.

The historical reason for the revolutionary consequences of the incompleteness of IF first-order logic is that our basic logic has traditionally been assumed to be complete in the familiar sense that we can give a complete set of purely formal rules of logical inference. Without this assumption, the main foundational projects in the last hundred and fifty years make little sense. Frege's enterprise in logic consisted precisely in the presentation of a complete and completely formal axiom system of logic.[16] Indeed, he believed that no other approach to logic than a purely formal (symbolic) and axiomatic one is possible, because the semantics of a language cannot according to his lights be expressed in that language. Hence his mission in logic was doomed to remain unaccomplished. Furthermore, in a sense the completeness of his logic was also a presupposition of Frege's foundational project. For the form in which he conceived of the reduction of all mathematical concepts and modes of reasoning to logic was a reduction of mathematics to a system of logic. If that system is not complete, there will not be any one system to reduce mathematics to. There will always be modes of reasoning whose status is not yet determined. Are they logical or irreducibly mathematical? A Poincaré would have been all too quick to opt for the second alternative.

Even more blatantly, Hilbert's proof-theoretical project was predicated on the possibility of a complete axiomatization of the rules of logical inference. For the way he wanted to prove the consistency of various mathematical theories was to show that one cannot formally prove a contradictory conclusion from their axioms. Without the completeness of our rules of formal proof this project is vacuous. For no matter how formal rules of proof are formulated, a contradiction may be hidden in those consequences of the axiom system that are not reducible by the rules of inference so far formulated.

That major earthquake of twentieth-century logic, Gödel's first incompleteness theorem, has unfortunately served only to strengthen the illusion of the completeness of our basic logic. For it was generally

[16]See here Jean van Heijenoort, "Logic as Calculus and Logic as Language", *Synthese,* vol. 17 (1967), pp. 324–330.

assumed that what Gödel proved was the inevitable incompleteness of relatively simple *mathematical* theories in every interesting sense of the term.

But if our logic itself is incomplete, we have to reconsider the entire situation in the foundations of logic and mathematics. For one thing, as Jaakko Hintikka has pointed out,[17] absolutely speaking Gödel's incompleteness theorem establishes only the deductive incompleteness of elementary arithmetic. In other words, we cannot formally derive by means of underlying logic S or $\neg S$ for each closed sentence S. This deductive incompleteness implies the descriptive incompleteness (nonaxiomatizability) of elementary arithmetic only if the underlying logic is semantically complete. Hence the incompleteness of IF first-order logic opens up a realistic possibility that we might be able to formulate descriptively (model-theoretically) complete axiom systems for various nontrivial mathematical theories already on the first-order level without violating Gödel's incompleteness theorem. All we need to do is to use IF first-order logic or some suitable extension of it instead of traditional first-order logic as our basic logic. This does not necessitate any excuses for it was argued earlier in this paper that we have to use such a logic in any case. We can even study such descriptively complete axiom systems model-theoretically and ascertain in this way some of their properties without having to worry about the rules of logical proof at all.

It is well known that descriptively complete but deductively incomplete axiomatizations of practically all mathematical theories can be formulated in higher-order logic. But such formulations involve all the dangers, uncertainties and other problems that pertain to the existence assumptions concerning higher-order entities, including sets. Since the kind of descriptively complete axiomatization envisaged here would be first-order, it could be free of all difficulties concerning the existence of sets or other higher-type entities.

Our observations also imply that the notion of logical proof has to be reconsidered. It is in these days often asserted or alleged that the notion of proof has lost its central position in mathematics as a consequence of Gödel's results.[18] Nothing is further from the truth.

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[17] Jaakko Hintikka, *The Principles of Mathematics Revisited* (note 3), chapter 6.

[18] In the last few years there has been an extensive discussion of the status of rigor and proofs in the columns of journals addressed by mathematicians to mathematicians. See, e.g., Arthur Jaffe and Frank Quinn, "'Theoretical Mathematics': Toward a Cultural Synthesis of Mathematics and Theoretical Physics", *Bulletin (New Series) of the American Mathematical Society,* vol. 29 (1993), pp. 1–13; William P. Thurston, "On Proof and Progress in Mathematics", *ibid.*, vol. 30 (1994), pp. 161–177, and the other responses to Jaffe and Quinn, *loc. cit.* pp. 178–211; Bonnie Gold,

Because we can formulate descriptively complete axiomatic theories, the task of a mathematical or scientific theorist can still be thought of as a derivation of logical consequences from the axioms. And such a derivation can still be considered as a purely logical proof. The requirement that characterizes its different steps is still the same as of old, viz. truth preservation. All the former rules concerning such truth preservation remain valid, and new ones can in principle always be discovered model-theoretically. Each legitimate step in a proof is still an instance of a purely formal, truth-preserving pattern of inference.

What is new is merely that a full set of purely formal rules for such proofs cannot be specified once and for all. In proving genuinely new results one must always be ready in principle to countenance steps which are not in accordance with the previously acknowledged rules of inference. But as long as they are truth-preserving, such new inference patterns must of course be considered as legitimate rules of logical proof. This legitimacy means the same as of old, viz. that all steps of the same form preserve truth. If by the completeness of formal logic one means the possibility of expressing each and every valid logical inference as an instance of truth-preserving formal inference pattern, then all valid logical inferences are formal and formal logic is in this sense complete. Perhaps the generally accepted terminology is not as descriptive as it could be. Perhaps we ought to change it and to say that IF first-order logic is "formally complete" but not "computable". To put the same point in different terms, the inevitable incompleteness that is found in logic and in the foundations of mathematics is not a symptom of any intrinsic limitation to what can be done by means of logic or mathematics. It is a limitation to what can be done by means of computers.

What the recent and current unease with the traditional conception of proof in mathematics is a symptom of is not any flaw in the notion of logical proof, but the need of making the notion of proof more realistic along the lines just adumbrated, which merely means not restricting the legitimate steps in a logical argument to some mechanically specified list of inference patterns.

But the most serious and most widespread philosophical mistake in this direction still remains to be diagnosed. It can be brought out by asking: How are the new inferential principles (new logical truths) discovered? Here a perverse form of the belief in the completeness of logic has led to the idea that, since the introduction of new principles of

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"What Is the Philosophy of Mathematics and What Should It Be?" *The Mathematical Intelligencer* vol. 16, no. 3 (1994), pp. 20-24, and cf. John Horgan, "The Death of Proof", *Scientific American*, vol. 269, no. 4 (October 1993), pp. 92–103.

proof cannot be done by means of preprogrammable rules, their introduction must be in principle a nonlogical matter, perhaps a matter of a special intuition, of an arbitrary decision, or of a conjecture later to be tested and possibly modified. The need of such constant amplification of the inferential basis of logical and mathematical proof has even been alleged to cast doubts on the uses of the notion of truth in logic and in mathematics.

These views are wrong, and invidiously so. A particularly pernicious variant is the postulation of a special mental capacity, sometimes referred to as mathematical intuition, as a source of new mathematical truths.[19] Now an unanalysed postulation of such a mysterious capacity amounts to an object lowering of intellectual standards. If you look at the earlier, respectable uses of the notion of intuition in the history of philosophy, you will find that its role as a source of insight or knowledge was always supposed to be backed up by some philosophical or psychological theory. For Aristotle, the backing was his theory of thinking as an authentic realization of forms in one's soul, for Descartes it was the doctrine of innate ideas, and for Kant it was his transcendental assumption that the forms of space and time have been imposed on object by our faculty of sense-perception, which makes these forms recoverable in intuition. But in recent discussion mathematical intuition has all too often been resorted to also when no such theoretical backing is in the offing. Measured by the standards of serious philosophy, such a freestanding notion of intuition is on the level of ESP.

In sober reality, the ways of introducing and justifying new inference patterns and new logical truths are much more mundane than an appeal to an extralogical intuition. In fact, we suspect that most appeals to the so-called mathematical intuition are merely tacit appeals to model-theoretical or proof-theoretical considerations. Such arguments can sometimes even be formulated in the very same theory whose axioms and rules of inference one is studying. If an example is needed, Gödel's very proof of the incompleteness of any given axiomatization AX of elementary arithmetic can serve as a case in point. In it, an argument expressible in elementary arithmetic itself leads us to a truth of elementary arithmetic which is not derivable from AX and hence can serve to strenghten it. Admittedly, Gödel's argument is proof-theoretical rather than model-theoretical. For this reason, philosophers

[19]The alleged role of mathematical intuition was emphasized especially strongly by Kurt Gödel, and philosophers' worshipful attitude to him has made the subject of intuition a popular one in the philosophy of mathematics. Unfortunately this is nothing but a painful reminder of the fact that a great logician can hold primitive philosophical views.

have not been tempted to think of it as an appeal to intuition. But it is not impossible to construct model-theoretical arguments which help to discover new inferential principles or to vindicate old ones under criticism. In fact, IF first-order logic opens up new possibilities in this direction. The reason is that several crucial model-theoretical concepts, including the concepts of truth and satisfaction, for a suitable first-order logic can be expressed in the very same language.

An instructive object lesson in the utter futility of trying to appeal to mathematical or logical intuition is offered by the curious history of the axiom of choice.[20] Different competent mathematicians have had contradictory "intuitions" about it. Zermelo for one introduced it as an obviously valid set-theoretical axiom. Hilbert's intuition seems to have told him that the axiom of choice is as obviously valid as $2+2=4$, and valid in the same sense.[21] Wisely, he treated this view as a hunch to be established through a suitable formulation of the axiom rather than an infallible dictum. Others, notably Brouwer and his fellow intuitionists rejected the axiom. Worse still, several of the mathematicians whose intuitions (or perhaps semi-intuitions) told them to reject the axiom subsequently turned out to have used the axiom themselves in their own mathematical reasoning. Which intuitions of these mathematicians are genuine ones, the ones that they had tacitly appealed to in their working reasoning, or the ones prompted by their reflections on the general and abstract formulation of the axiom of choice?

This whole mare's nest of problems can be clarified, not by an appeal to intuitions, but by logical, especially model-theoretical, analyses. The basic ideas of these analyses have been explained by Hintikka.[22] Without trying to summarize the whole argument here, it can be mentioned that several of our *prima facie* intuitions have to be re-educated in the light of the model-theoretical analyses. For instance, the most popular "intuitions" that have led to the rejection of the axiom of choice turn out to be based on outright confusion, viz. based on giving different quantifiers in a common formulation of the axiom a different interpretation. The formulation we have in mind is the second-order axiom schema

$$(\forall x)(\exists y)S[x,y] \supset (\exists f)(\forall x)S[x,f(x)] \tag{18}$$

Critics have turned out to have implicitly given the first-order quanti-

[20] The history of the axiom of choice is studied by Gregory H. Moore, *Zermelo's Axiom of Choice,* Springer-Verlag, Berlin, 1982.

[21] David Hilbert, "Neubegründung der Mathematik", in *Gesammelte Abhandlungen*, vol. 3, Springer-Verlag, Berlin, 1935, pp. 157–177 (see especially p. 157).

[22] See, *op. cit.,* note 3 above, chapters 2, 8 and 11.

fiers $(\forall x)$, $(\exists y)$ a classical interpretation but the second-order quantifier $(\exists f)$ a nonclassical one. If all quantifiers are given the same interpretation the axiom of choice holds. This has apparently been noted by Dummett, whose mature intuitions in this matter hence differ from those of other intuitionists.[23] In other cases, too, an alleged intuitive justification for rejecting the axiom has proved to be fallacious. Most importantly, the intuitions of the intuitionists can be shown to pertain to an altogether different subject matter than the intuitions of the typical nonintuitionist.[24] The latter are concerned with our knowledge of mathematical *facts* (truths) while the former are concerned with our knowledge of mathematical *objects*. Hence the two parties have been arguing at cross-purposes. For instance, on the case of the axiom of choice the intuitionists are asking whether we know the choice function while the classical mathematicians are asking whether we know that such choice function exists. The confusion between these two issues thus exemplifies a confusion between two kinds of knowledge, viz. knowledge of truths and knowledge of objects.

But even after all the alleged intuitive testimony against the axiom of choice has been discredited, don't we still need intuition to decide whether it is acceptable or not? No, what we need is a grasp of the meaning of quantifiers. Earlier in this paper, it was pointed out that an essential component of the logical behaviour of quantifiers is the way they depend on each other. For instance, it is part and parcel of the meaning of quantifiers that in a sentence like

$$(\forall x)(\exists y)S[x,y] \tag{1}$$

the truth-making choice of the value of y depends on the choice of the value of x. Hence (1) is true if and only if there is a function that implements that choice. In other words, (1) is true if and only if the following second-order sentence is true:

$$(\exists f)(\forall x)S[x,f(x)] \tag{19}$$

but the implication from (1) to (19) that is (18), is a form of the axiom of choice. This axiom is in other words but a manifestation of the interaction between different quantifiers that is a part of their meaning. The axiom of choice is therefore not only valid, but a valid *logical* principle, just as Hilbert surmised.

The fact that the validity of the axiom of choice has not been more universally recognized has a *prima facie* tempting but fallacious reason.

[23] Michael Dummett, *Elements of Intuitionism,* Clarendon Press, 1977, pp. 52–54.
[24] See, *op. cit.,* note 3, chapter 11.

This usually tacit reason is a belief in the principle of compositionality. According to it, the meaning of a quantifier must not depend on what there is outside its scope. But in a perfectly good sense, the truth-making choice of a variable bound to an existential quantifier, say to $(\exists y)$ in (1), depends on something outside its scope, viz. on the value of the universal quantifiers within the scope of which it occurs, in (1), on the value of x. Thus in the last analysis the interplay of quantifiers amounts to living (albeit oblique) testimony against the principle of compositionality. This testimony should be enough to put to rest all realistic hopes of maintaining compositionality in any strict sense of the word in one's logical semantics.

When this is realized, the last objections to the axiom of choice evaporate and it is seen that the axiom of choice is valid. If it is construed as a rule of inference, that rule is truth-preserving.

Thus in this entire matter of the axiom of choice, it is the appeals to intuition that have led mathematicians to various wild goose chases, leaving the resulting mess to be clarified by model-theoretical analyses. Generally speaking, the time is ripe to say to the philosophers who have been discussing the mythical faculty of mathematical intuition: Put up or shut up. So far they have not come up with any unequivocal insight into the subjects they have promised us intuitions about.

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