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# ON A CLASS OF GORENSTEIN IDEALS OF GRADE FOUR

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**Abstract.** We provide a minimal free resolution for a class of Gorenstein ideal of grade 4 which is the sum of an almost complete intersection $J$ of grade 3 and a perfect ideal $I$ of grade 3 with type 2 and $\lambda(I) > 0$ geometrically linked by a regular sequence, where $I$ is generated by odd elements.

## 1. Introduction

Let $R$ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and let $I$ be a proper ideal of $R$ with finite projective dimension. The type of a perfect ideal $I$ of grade $g$ is defined to be the dimension of $R/\mathfrak{m}$-vector space $\operatorname{Ext}^g_R(R/\mathfrak{m}, R/I)$. We denote it by type $I$. Equivalently, if

$$\mathbb{F} : 0 \longrightarrow F_g \longrightarrow F_{g-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R$$

is the minimal free resolution of $R/I$, then $\operatorname{type} I = \operatorname{rank} F_g$. A perfect ideal $I$ of grade $g$ is a complete intersection if $I$ is generated by a regular sequence $x_1, x_2, \cdots, x_g$, an almost complete intersection if it is minimally generated by $g+1$ elements, and a Gorenstein ideal if $I$ has type 1.

In 1987, Brown [2] gave a structure theorem for a class of perfect ideals of grade 3 with type 2 and $\lambda(I) = \dim_k \Lambda^2_1 > 0$, where $\lambda(I)$ is the numerical invariant introduced by Kustin and Miller [11] to distinguish classes of Gorenstein ideals $I$ of grade 4 in term of free resolutions of $R/I$. In [9], Kang and Ko introduce the complete matrix $f$ of grade 4 and the ideal $\mathcal{K}_3(f)$ associated with it to give a structure theorem for complete intersection of grade 4. Similarly, the almost complete matrix $f$ of grade 3 with type $r$ determined by an $r \times 3$ matrix $A$ and an $r \times r$ alternating matrix $Y$ [10, Definition 3.3] for a positive integer $r$ with

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$r > 1$ and the ideal $\mathcal{K}_3(f)$ associated with it enable us to characterize almost complete intersections of grade 3 and to give a structure theorem for these classes. These almost complete matrices distinguish the even and odd types of almost complete intersections of grade 3. In [9], they define an $n \times n$ skew-symmetrizable matrix (or generalized alternating matrix) to be a square matrix $X$ such that $DXD'$ is an alternating matrix for some diagonal matrices $D$ and $D'$ [see Definition 3.1, 9]. We call this an alternating matrix induced by $X$ and denote it by $\mathcal{A}(X)$. In [5], let $r$ be an odd integer with $r > 1$. For two elements $v, w \in \mathfrak{m}$, Choi, Kang and Ko define an $(r+2) \times (r+2)$ skew-symmetrizable matrix $G_1$ by

$$G_1 = \left[\begin{array}{c|c} B & vA \\ \hline -A^t & Y(1,2) \end{array}\right], \text{ where } B = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}$$

and $Y(1,2)$ is the $r \times r$ alternating matrix of $Y$ obtained by deleting the first two rows and columns of $Y$. They observe that there exist a class of perfect ideals of grade 3, type 2 which is generated by certain quotients of the submaximal order pfaffians of the alternating matrix induced by the skew-symmetrizable matrix $G_1$. In [5], similarly, let $r$ be an even integer with $r \geq 4$ and $v \in \mathfrak{m}$. They define an $(r+3) \times (r+3)$ skew-symmetrizable matrix $G_2$ by

$$G_2 = \left[\begin{array}{c|c} \mathbf{0} & \bar{F} \\ \hline -\bar{F}^t & Y \end{array}\right], \text{ where } \bar{F} = \begin{bmatrix} va_{11} & va_{21} & \cdots & va_{r1} \\ -va_{12} & -va_{22} & \cdots & -va_{r2} \\ va_{13} & va_{23} & \cdots & va_{r3} \end{bmatrix}.$$

They observe that there exist a class of perfect ideals of grade 3, type 3 which is generated by certain quotients of the submaximal order pfaffians of the alternating matrix induced by the skew-symmetrizable matrix $G_2$. In [6], let $r$ be an odd integer with $r > 1$, $s$ be a regular element, $A$ be a $r \times 4$ matrix and $U$ be a $4 \times 4$ alternating matrix. We define an $(r+4) \times (r+4)$ skew-symmetrizable matrix $G_3$ by

$$G_3 = \left[\begin{array}{c|c} U & sA^t \\ \hline -A & Y \end{array}\right].$$

We observe that there exist a class of perfect ideals of grade 3, type 2 and $\lambda(I) = 0$ which is generated by certain quotients of the submaximal order pfaffians of the alternating matrix induced by the skew-symmetrizable matrix $G_3$.

In this paper, we construct a class of Gorenstein ideals of grade 4 determined by a skew-symmetrizable matrix $G_1$ and an induced almost complete matrix $\widetilde{f}$ [10]. So we present the minimal free resolution for a class of Gorenstein ideals of grade 4, and illustrate the structure of the minimal free resolution. In section 2, we review linkage theory and some structure theorems for perfect ideals of grade 3. In section 3, we define an $(r+4) \times (r+3)$ augmented matrix $\widetilde{G}$ determined by an $(r+3)\times(r+3)$ skew-symmetrizable matrix $G_1$ [5] and a $4\times(r+3)$ matrix $\widetilde{f}$ extracted from an almost complete matrix $f$ [10]. These play a key role in constructing a minimal free resolution for a class of Gorenstein ideals of grade 4 which is the sum of an almost complete intersection $J$ of grade 3 and a perfect ideal $I$ of grade 3 with type 2, $\lambda(I) > 0$ geometrically linked by a regular sequence. This is a remarkable difference from [7] in which we give a structure theorem for a class of Gorenstein ideal of grade 4 which is the sum of an almost complete intersection of grade 3 and a Gorenstein ideal of grade 3 geometrically linked by a regular sequence.

## 2. Preliminaries

In this section, we review linkage theory and the structure theorems for some classes of perfect ideals of grade 3 which are given by Brown, Buchsbaum, Eisenbud, and Sanchez. To review these structure theorems, first we investigate some properties of an alternating matrix. Let $T = (t_{ij})$ be an $n \times n$ alternating matrix with entries in a commutative ring $R$. It follows from a linear algebra that if $n$ is odd, the determinant of an alternating matrix $T$ is zero and if $n$ is even, it is a square of a homogeneous polynomial of degree $\frac{n}{2}$ in the entries of $T$, which is called the pfaffian of $T$. We will write $\det T = \mathrm{Pf}(T)^2$. Pfaffians can be developed along a row just like the determinants. Denote by $\mathrm{Pf}_s(T)$ the ideal generated by the $s$th order pfaffians of $T$. Let $s < n$ and $(i) = i_1, i_2, \cdots, i_s$ denote the index of integers. Let $\theta(i)$ denote the sign of permutation that rearranges $(i)$ in increasing order. If $(i)$ has a repeated index, then we set $\theta(i) = 0$. Let $\tau(i)$ be the sum of the entries of $(i)$ and $T(i_1, i_2, \cdots, i_s)$ an alternating submatrix of $T$ formed by deleting rows and columns $i_1, i_2, \cdots, i_s$ from $T$. Define

$$T_{(i)} = (-1)^{\tau(i)+1} \cdot \theta(i) \cdot \mathrm{Pf}\,(T(i_1, i_2, \cdots, i_s)) \tag{2.1}$$

If $s = n$, we let $T_{(i)} = (-1)^{\tau(i)+1}\theta(i)$ and if $s > n$, we let $T_{(i)} = 0$. Let $\mathbf{t} = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \end{bmatrix}$ be the row vector of the maximal order pfaffians of $T$, signed appropriately according to the conventions described above. There is a "Laplace expansion" for developing pfaffians in term of ones of lower order.

Now we review the structure theorem for a class of type 2, $\lambda(I) > 0$ perfect ideals $I$ of grade 3 given by Brown [2]. Kustin and Miller introduced the numerical invariant $\lambda(I)$ defined in [11]. Let $I$ be any ideal in a noetherian local ring $R$. Let $(\mathbb{F}, d)$ be a minimal free resolution of $R/I$. Let $C$ be the image of $d_2$ and $K$ the submodule of $C$ which is generated by the Koszul relations on the entries of $d_1$. We note that if $I$ is minimally generated by $r_1, r_2, \cdots, r_n$, and $\{e_1, e_2, \cdots, e_n\}$ is a basis of $F_1$, then $K$ is generated by the set $\{r_j e_i - r_i e_j \,|\, 1 \leq i < j \leq n\}$. Define

$$\lambda(I) = \dim_k(K + \mathfrak{m}C)/\mathfrak{m}C,$$

where $\mathfrak{m}$ is the maximal ideal of $R$ and $k = R/\mathfrak{m}$. Since $\lambda(I)$ is the maximum number of minimal generators of $K$ which can be chosen to be the part of a minimal basis for $C$, we see that $\lambda(I)$ is also the maximum number of Koszul relations which can appear as rows of a matrix for $d_2$. Brown gave a structure theorem for a class of type 2, $\lambda(I) > 0$ perfect ideals $I$ of grade 3. The minimal free resolution $\mathbb{F}$ of $R/I$ is described in [2].

**Theorem 2.1.** [2] *Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $n > 4$ be an integer. Let $I$ be a type 2 perfect ideal of grade 3 minimally generated by $n$ elements. If $\lambda(I) > 0$, then there is an $n \times n$ alternating matrix $T = (t_{ij})$ with $t_{12} = 0$ and $t_{ij} \in \mathfrak{m}$ such that*

(1) *if $n$ is odd, then $I = (T_1, T_2, z_1 T_{12j} + z_2 T_j : 3 \leq j \leq n)$, for some $z_1, z_2 \in \mathfrak{m}$,*

(2) *if $n$ is even, then $I = (Pf(T), T_{12}, z_1 T_{1j} + z_2 T_{2j} : 3 \leq j \leq n)$, for some $z_1, z_2 \in \mathfrak{m}$.*

We observe that there exist a class of perfect ideals of grade 3, type 2 and $\lambda(I) > 0$ which is generated by certain quotients of the submaximal order pfaffians of the alternating matrix induced by the skew-symmetrizable matrix $G_1$.

**Definition 2.2.** Let $\mathcal{A}(G_1)$ be the alternating matrix obtained by multiplying the first two columns of $G_1$ by $v$. Let $x_i$ be an element defined by $x_i {=} \mathcal{A}(G_1)_i/v$ for $i = 1, 2, 3, \cdots, n$. We define $\overline{Pf_{n-1}(G_1)}$ to be the ideal generated by $n$ elements $x_i$.

The following theorem says that if $\overline{Pf_{n-1}(G_1)}$ characterizes a perfect ideal $I$ of grade 3 satisfying the following properties: (1) $I$ has type 2, (2) the number of generators for $I$ is odd, (3) $\lambda(I) > 0$.

**Theorem 2.3.** [5] *Let $R$ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Let $n$ be an odd integer with $n > 3$ and $v, w$ elements in $\mathfrak{m}$. Let $G_1$ be the $n \times n$ skew-symmetrizable matrix defined above. Then*

(1) *If $I = \overline{Pf_{n-1}(G_1)}$ is an ideal of grade 3 with $\lambda(I) > 0$, then $I$ is a perfect ideal of type 2.*

(2) *Every perfect ideal of grade 3, type 2, $\lambda(I) > 0$ minimally generated by $n$ elements arises as in the way of (1).*

We notice that as in [4] or [12], in most cases, linkage is used in the case of perfect ideals in Gorenstein or Cohen-Macaulay local rings. However the result that we use here are true for perfect ideals in any commutative ring, as shown by Golod [8].

**Definition 2.4.** Let $I$ and $J$ be a perfect ideal of grade $g$. An ideal $I$ is linked to $J$, $I \sim J$ if there exists a regular sequence $\mathbf{x} = x_1, x_2, \cdots, x_g$ *in* $I \cap J$ such that $J = (\mathbf{x}) : I$ and $I = (\mathbf{x}) : J$, and *geometrically* linked to $J$ if $I \sim J$ and $I \cap J = (\mathbf{x})$.

A fundamental result is that linkage is a symmetric relation on the set of perfect ideals in a noetherian ring $R$.

**Theorem 2.5.** [12] *Let $R$ be a noetherian ring. If $I$ is a perfect ideal of grade $g$ and $\mathbf{x} = x_1, x_2, \cdots, x_g$ is a regular sequence in $I$, then $J = (\mathbf{x}) : I$ is a perfect ideal of grade $g$ and $I = (\mathbf{x}) : J$.*

The following theorem provides a method of constructing a Gorenstein ideal of grade $g+1$ from perfect ideals of grade $g$.

**Proposition 2.6.** [12] *Let $R$ be a noetherian ring. Let $I$ and $J$ be perfect ideals of grade $g$. If $I$ and $J$ are geometrically linked, then $H = I + J$ is a Gorenstein ideal of grade $g+1$.*

An almost complete intersection of grade $g$ is linked to a Gorenstein ideal of grade $g$ by a regular sequence $\mathbf{x}$.

**Proposition 2.7.** [4] *Let $I$ and $J$ be perfect ideals of the same grade $g$ in a noetherian local ring $R$ and suppose that $I$ is linked to $J$ by a regular sequence $\mathbf{x} = x_1, x_2, \cdots, x_g$. Then*

(1) *If $I$ is Gorenstein, then $J = (\mathbf{x}, w)$ for some $w \in R$ and*

(2) *If $J$ is minimally generated by $\mathbf{x}$ and $w$, then $I$ is Gorenstein.*

In [10], Kang, Cho, and Ko introduce the concept of an almost complete matrix $f$ of grade 3 with type $r$ determined by an $r \times 3$ matrix $A$ and an $r \times r$ alternating matrix $Y$, and define an ideal $\mathcal{K}_3(f)$ associated with an almost complete matrix $f$ of grade 3 to give the structure theorem for almost complete intersections of grade 3. We define $C = (c_i), E = (e_j), S = (s_{ij})$, and $Z = (z_{ij})$ to be a $1 \times 3$ matrix, a $1 \times r$ matrix, a $3 \times r$ matrix, and a $3 \times 3$ matrix, respectively. These matrices are given by the following: For any two integer $m < t$ with $\{i, m, t\} = \{1, 2, 3\}$,

$$c_i = \begin{cases} \displaystyle\sum_{\substack{1 \leq u < v \leq r \\ i \neq m,t}} Y_{uv} \begin{vmatrix} a_{um} & a_{ut} \\ a_{vm} & a_{vt} \end{vmatrix} & \text{if } \ r = \text{even} \\ 0 & \text{if } \ r = \text{odd}, \end{cases}$$

$$e_j = \begin{cases} \displaystyle\sum_{1 \leq u < v < w \leq r} -Y_{juvw} D_{uvw} & \text{if } \ r = \text{even} \\ Y_j & \text{if } \ r = \text{odd}, \end{cases}$$

$$s_{ij} = \begin{cases} (-1)^{i+1} \displaystyle\sum_{1 \leq h \leq r} Y_{jh} a_{hi} & \text{if } \ r = \text{even} \\ (-1)^{i+1} \displaystyle\sum_{1 \leq u < v \leq r} Y_{juv} \begin{vmatrix} a_{um} & a_{ut} \\ a_{vm} & a_{vt} \end{vmatrix} & \text{if } \ r = \text{odd}, \end{cases}$$

$$Z = \operatorname{diag}\{-\operatorname{Pf}(Y), -\operatorname{Pf}(Y), -\operatorname{Pf}(Y)\} \qquad \text{if } \ r = \text{even}$$

$$Z = \begin{bmatrix} 0 & Z_3 & -Z_2 \\ -Z_3 & 0 & Z_1 \\ Z_2 & -Z_1 & 0 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} Z_1 & Z_2 & Z_3 \end{bmatrix} \quad \text{if } \ r = \text{odd},$$

where $D_{uvw}$ is the determinant of a $3 \times 3$ submatrix of $A$ formed by three rows $u, v, w$ of $A$ in this order, and $Z_i = -\displaystyle\sum_{k=1}^{r} Y_k a_{ki}$ for $i = 1, 2, 3$. We also define an element $w$ in $R$ as follow:

$$w = \begin{cases} \operatorname{Pf}(Y) & \text{if } \ r = \text{even} \\ \displaystyle\sum_{1 \leq u < v < w \leq r} Y_{uvw} D_{uvw} & \text{if } \ r = \text{odd}. \end{cases}$$

For the case that $r$ is even, for further use we define $F$ to be a $3 \times r$ matrix given by

$$F = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ -a_{12} & -a_{22} & \cdots & -a_{r2} \\ a_{13} & a_{23} & \cdots & a_{r3} \end{bmatrix} = (f_{ij}), \text{ where } \quad f_{ij} = (-1)^{i+1} a_{ji}.$$

In [10], we assume that $I$ is a perfect ideal of grade 3 minimally generated by $n > 4$ elements and $J = (t_1, t_2, t_3, t_4)$ is an almost complete intersection of grade 3 and $I$ is linked to $J$ by a regular sequence $\mathbf{x} = x_1, x_2, x_3$. Let $J = (t_1, t_2, t_3, t_4)$ be an almost complete intersection of grade 3. Then there exists a $4 \times 3$ matrix $B = (b_{ij})$ such that

$$\mathbf{x} = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix} B.$$

Let $r > 1$ be the type of $J$. And let $\bar{D}_{uvw}$ be the determinant of a submatrix of $B$ formed by rows $u, v, w$ in this order. If $r$ is even, $P = (p_{k1})_{r \times 1}$ given by

$$\begin{aligned} p_{k1} &= \sum_{1 \le u < v < w \le r} -Y_{kuvw} D_{uvw} \bar{D}_{123} \\ &\quad - \sum_{l=1}^{r} (a_{l1} \bar{D}_{234} + a_{l2} \bar{D}_{134} + a_{l3} \bar{D}_{124}) Y_{kl} \\ &= e_k \bar{D}_{123} - (s_{1k} \bar{D}_{234} - s_{2k} \bar{D}_{134} + s_{3k} \bar{D}_{124}). \end{aligned}$$

If $r$ is odd, $P = (p_{k1})_{r \times 1}$ given by

$$p_{k1} = -Y_k \bar{D}_{123} + (s_{1k} \bar{D}_{234} - s_{2k} \bar{D}_{134} + s_{3k} \bar{D}_{124}).$$

**Theorem 2.8.** [10] *Let $A, Y, C, E, S, Z, \mathbf{z}, w$ be defined as above over the noetherian local ring $R$ with maximal ideal $\mathfrak{m}$. Let $f$ be an almost complete matrix of grade 3 determined by $A$ and $Y$. Let $\widetilde{f}$ be a $4 \times (r+3)$ matrix extracted from $f$.*

(1) *If $r$ is even and if $\mathcal{K}_3(f)$ has grade 3, then*

$$\mathbb{F} : 0 \longrightarrow R^r \xrightarrow{f_3} R^{r+3} \xrightarrow{f_2} R^4 \xrightarrow{f_1} R$$

*is a minimal free resolution of $R/\mathcal{K}_3(f)$, where*

$$f_1 = \begin{bmatrix} C & w \end{bmatrix}, \quad f_2 = \widetilde{f} = \left[\begin{array}{c|c} Z & S \\ \hline C & E \end{array}\right], \quad f_3 = \begin{bmatrix} F \\ Y \end{bmatrix}.$$

(2) *If $r$ is odd and if $\mathcal{K}_3(f)$ has grade 3, then*

$$\mathbb{F} : 0 \longrightarrow R^r \xrightarrow{f_3} R^{r+3} \xrightarrow{f_2} R^4 \xrightarrow{f_1} R$$

*is a minimal free resolution of* $R/\mathcal{K}_3(f)$, *where*

$$f_1 = \begin{bmatrix} \mathbf{z} & w \end{bmatrix}, \quad f_2 = \widetilde{f} = \left[\begin{array}{c|c} Z & S \\ \hline C & E \end{array}\right], \quad f_3 = \begin{bmatrix} A & Y \end{bmatrix}^t.$$

In [10], Kang, Cho, and Ko provide the structure theorem for some classes of perfect ideals of grade 3 which are algebraically linked to an almost complete intersection of grade 3 by a regular sequence $\mathbf{x} = x_1, x_2, x_3$. This contains three classes of perfect ideals of grade 3 which were determined by Buchsbaum-Eisenbud, Brown and Sanchez.

**Theorem 2.9.** [10] *Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $J$ and $B$ be an almost complete intersection of grade* 3 *and a matrix defined above, respectively. Let $\mathbf{x} = x_1, x_2, x_3$ be a regular sequence in $J$ defined in* (5.1). *Let $r$ be the type of $J$.*

($i$) *Let $r$ be even. Let $A, E, S$ and $Y$ be matrices defined in* (3.2), (3.3), *with entries in $\mathfrak{m}$, and $p_{k1}$ an element defined in* (5.3) *with $p_{k1}$ in $\mathfrak{m}$, for $k = 1, 2, \cdots, r$.*

($ii$) *Let $r$ be odd. Let $A, Y, S$ and $Z$ be matrices defined in* (3.3) *and* (3.4), *with entries in $\mathfrak{m}$, and $p_{k1}$ an element defined in* (5.4) *with $p_{k1}$ in $\mathfrak{m}$, for $k = 1, 2, \cdots, r$.*

*If $I$ is an ideal generated by $x_1, x_2, x_3, p_{11}, p_{21}, \cdots, p_{r1}$, then $I$ is a perfect ideal of grade* 3 *linked to $J$ by a regular sequence* $\mathbf{x}$ *and is type* $\mu(J/(\mathbf{x}))$.

(2) *Every perfect ideal of grade* 3 *linked to an almost complete intersection $J$ of grade* 3 *by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ arises in the way of* (1).

## 3. Resolution of a class of Gorenstein ideal of grade 4

In this section, we define an $(r+4) \times (r+3)$ augmented matrix $\widetilde{G}$ obtained by adding one row to an $(r+3) \times (r+3)$ skew-symmetrizable matrix $G_1$ [5]. Then we can construct the minimal free resolution for a class of Gorenstein ideal of grade 4 expressed as the sum of an almost complete intersection $J$ of grade 3 and a perfect ideal $I$ of grade 3 with type 2, $\lambda(I) > 0$ geometrically linked by a regular sequence. By the Bass' result, type $J$ is the minimal number of generators in $(\mathbf{x} : J)/(\mathbf{x}) = I/(\mathbf{x})$, where $I$ is linked to $J$ by a regular sequence $\mathbf{x}$. Since the type of $J$ is even, we can construct a perfect ideal of grade

3 with type 2 and $\lambda > 0$ minimally generated by odd elements by giving three types of regular sequence in $I$. Let $s$, $u$ and $v$ be nonzero elements in $R$. For examples, let $\mathbf{x}$ be a regular sequence $uc_1, c_2, c_3$. Then by Theorem 2.9, $I = (uc_1, c_2, c_3, ue_1, ue_2, \cdots, ue_r)$. And secondly, let $\mathbf{x}$ be a regular sequence $uc_1, c_2, c_3 + vw$. Then by Theorem 2.9, $I = (uc_1, c_2, c_3 + vw, ue_1 - uvs_{31}, ue_2 - uvs_{32}, \cdots, ue_r - uvs_{3r})$. And thirdly, let $\mathbf{x}$ be a regular sequence $c_1, c_2, sc_3 + uvw$. Then by Theorem 2.9, $I = (c_1, c_2, sc_3 + uvw, se_1 - uvs_{31}, se_2 - uvs_{32}, \cdots, se_r - uvs_{3r})$. Proposition 2.6 provides a method of constructing a Gorenstein ideal $I + J$ of grade 4 from perfect ideals $I$ and $J$ of grade 3 which are geometrically linked by a regular sequence. This gives us the following theorems and examples.

**Theorem 3.1.** *Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. With notations as above, let $I$ be a perfect ideal of grade 3 with type 2, $\lambda(I) > 0$, and $J = (c_1, c_2, c_3, w)$ an almost complete intersection with the even type $r > 1$. Let $I$ and $J$ be geometrically linked by a regular sequence $uc_1, c_2, c_3$. Then $I + J = (c_1, c_2, c_3, w, ue_1, ue_2, \cdots, ue_r)$ is a Gorenstein ideal of grade 4 and*

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{g_4} R^{r+4} \xrightarrow{g_3} R^{2(r+3)} \xrightarrow{g_2} R^{r+4} \xrightarrow{g_1} R$$

*is a minimal free resolution of $R/(I + J)$, where*

$$g_1 = (c_1, c_2, c_3, w, ue_1, ue_2, \cdots, ue_r),\ g_4 = (ue_1, ue_2, \cdots, ue_r, c_1, c_2, c_3, w)^t,$$

$$g_2 = \left[\begin{array}{c|c|c|c} \mathbf{0} & uF & Z & S \\ \hline \mathbf{0} & \mathbf{0} & C & E \\ \hline -F^t & Y & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{G} & \begin{array}{c} \widetilde{f} \\ \mathbf{0} \end{array} \end{array}\right],$$

$$g_3 = \left[\begin{array}{c|c|c} \mathbf{0} & Z^t & C^t \\ \hline \mathbf{0} & S^t & E^t \\ \hline F & \mathbf{0} & \mathbf{0} \\ \hline Y & -uF^t & \mathbf{0} \end{array}\right].$$

*The matrix $g_2$ contains an $(r+3)\times(r+3)$ skew-symmetrizable submatrix $G_1$* [5].

*Proof.* It follows from lemma 3.6. (1), [10] that $\mathbb{H}$ is a complex. To show that $\mathbb{H}$ is exact, it is sufficient to show that the rank and depth conditions in the Buchsbaum-Eisenbud acyclicity criterion [3] are satisfied. First we prove that the rank condition is satisfied. Clearly, $g_1$ and $g_4$ have rank 1. We show that $g_2$ and $g_3$ have rank $r+3$. We observe that

$$Q = \begin{bmatrix} uF & Z \\ Y & \mathbf{0} \end{bmatrix}$$

is a $(r+3)\times(r+3)$ submatrix of $g_2$ whose determinant is equal to $-w^5$. Since $g_1g_2 = 0$ and $g_1$ generates the ideal of grade 4, the determinant of any $(r+4)\times(r+4)$ submatrix of $g_2$ is equal to zero. So $g_2$ has rank $r+3$. In the similar way we can know that $g_3$ has rank $r+3$. Now we show that the depth condition is satisfied. Clearly, $I_1(g_1)$ and $I_1(g_4)$ have grade 4. Now we want to show that $I_{r+3}(g_2)$ and $I_{r+3}(g_3)$ have grade 4. It is sufficient to show that $I_{r+3}(g_2)$ has grade 4. The similar argument says that $I_{r+3}(g_3)$ has grade 4. For $i = 1, 2, 3$, we define $W_i^{(i)}$ to be a submatrix of $g_2$ obtained by deleting the $i$ row and the corresponding column of $g_2$. Let $m, n$ be positive integers in the set $\{2, 3, 4, \cdots, r+1\}$ with $m < n$ and $p, q$ positive integers in the set $\{3, 4, 5, \cdots, r+2\}$ with $p < q$. Let $w_1, w_2, \cdots, w_{r+3}$ be a sequence of positive integers between 1 and $2r+5$ satisfying the following properties:

(a) $w_1 < w_2 < \cdots < w_{r+3}$.

(b) $w_1 = 1, w_2 = 2$, and $3 \le w_3, w_4, \cdots, w_r \le r+2$ with $w_k \ne p, q$ for $k = 3, 4, \cdots, r$.

(c) $w_{r+1} = 2 + r + i, w_{r+2} = r + 4 + m, w_{r+3} = r + 4 + n$.

Let $W_i^{(i)}(m, n, p, q)$ be a $(r+3)\times(r+3)$ submatrix of $W_i^{(i)}$ formed by $r+3$ columns $w_1, w_2, \cdots, w_{r+3}$ of $W_i^{(i)}$. We can compute the determinant of $W_i^{(i)}(m, n, p, q)$ as follow. Let $T$ be an $(r+3)\times r$ submatrix of $W_i^{(i)}$ formed by the first $r$ columns of $W_i^{(i)}$. For $i = 1, 2, 3$ we define $T_i$ to be a $3 \times r$ submatrix of $T$ obtained by deleting the last $r$ rows of $T$. Let $G_i(m, n)$ and $U_i(p, q)$ be $3\times3$ matrix and $r\times r$ matrix defined in the proof of Theorem 4.1 in [10], respectively. Then we can write $W_i^{(i)}(m, n, p, q)$ as the form

$$W_i^{(i)}(m, n, p, q) = \left[\begin{array}{c|c} T_i & G_i(m, n) \\ \hline U_i(p, q) & \mathbf{0} \end{array}\right].$$

Hence

$$\det W_i^{(i)}(m, n, p, q) = \det G_i(m, n) \cdot \det U_i(p, q).$$

We use this identity to show that $I_{r+3}(g_2)$ contains $c_i^5$ for $i = 1, 2, 3$. If $r = 2$ and $A^{(i)}$ is a $2 \times 2$ submatrix of $A$ obtained by deleting the $i$th column of $A$, then, as we have shown in the proof of Theorem 4.1 in [10], the following identity gives us the desired result:

$$\begin{aligned}&(\det A^{(i)})^2 \det W_i^{(i)}(2,3,3,4)\\&\quad = \{(-1)^i \det A^{(i)} \det G_i(2,3)\} \cdot \{(-1)^i \det A^{(i)} \det U_i(3,4)\}\\&\quad = c_i^3 \cdot c_i^2 = c_i^5.\end{aligned}$$

In the similar way, if $r > 2$ and $d, e \in \{1,2,3\}$ with $d < e$, $d \neq i$, $e \neq i$,

$$A^{(i)}(k-1, l-1) = \begin{bmatrix} a_{k-1\,d} & a_{k-1\,e} \\ a_{l-1\,d} & a_{l-1\,e} \end{bmatrix}$$

is a $2 \times 2$ submatrix of $A$ obtained by deleting the $i$th column of $A$ and $(r-2)$ rows of $A$ except the $(k-1)$th and $(l-1)$th rows of it, then the following identity completes the proof of this part:

$$\begin{aligned}&\sum_{2 \le m < n \le r+1} \sum_{3 \le p < q \le r+2} (-1)^{i+p+q} \det A^{(i)}(m-1, n-1)\\&\qquad\qquad \times \det A^{(i)}(\bar{p}, \bar{q}) \det W_i^{(i)}(m,n,p,q)\\&\quad = \left( \sum_{2 \le m < n \le r+1} (-1)^i \bar{G}_i(m,n) \right) \times \left( \sum_{3 \le p < q \le r+2} (-1)^{p+q} \bar{U}_i(p,q) \right)\\&\quad = c_i^3 \cdot c_i^2 = c_i^5,\\&\quad \text{where } \bar{p} = p-2, \;\; \bar{q} = q-2, \;\; \text{and}\\&\quad \bar{G}_i(m,n) = \det A^{(i)}(m-1,n-1) \det G_i(m,n),\\&\quad \bar{U}_i(p,q) = \det A^{(i)}(\bar{p},\bar{q}) \det (U_i)_{pq}.\end{aligned}$$

Hence $I_{r+3}(g_2)$ contains $c_i^5$ for $i = 1,2,3$. Finally we complete our proof of Theorem 3.1 by showing that $I_{r+3}(g_2)$ contains $(ue_1)^3, (ue_2)^3, \cdots, (ue_r)^3$. Let $v_1, v_2, \cdots, v_{r+3}$ be a sequence of positive integers between 1 and $2r+6$ satisfying the following properties:

(a) $v_1 < v_2 < \cdots < v_{r+3}$.

(b) $v_i = i$ for $i = 1,2,3$, and $4 \le v_4, v_5, \cdots, v_{r+2} \le r+3$ with $v_k \neq 3+i$ for $k = 4,5,\cdots,r+2$ and for $i = 1,2,3$.

(c) $v_{r+3} = r+6+i$.

Let $V$ be a $(r+4) \times (r+3)$ submatrix of $g_2$ formed by the columns $v_1, v_2, \cdots, v_{r+3}$ of it. For $i = 1,2,\cdots,r$, we let $V_i$ be a $(r+3) \times (r+3)$ submatrix of $V$ obtained by deleting the $(i+4)$th row of $V$. Let $(uF)^{(i)}$ be submatrice of $uF$ obtained by deleting the $i$th column of $uF$. Let $-F_i^t$ be a $(r-1) \times 3$ submatrix of $-F^t$ obtained by deleting the $i$th row of it. Then we can write $V_i$ as the form

$$V_i = \begin{bmatrix} \mathbf{0} & (uF)^{(i)} & \mathbf{c}_i(S) \\ \mathbf{0} & \mathbf{0} & e_i \\ -F_i^t & Y(i) & \mathbf{0} \end{bmatrix},$$

where $\mathbf{c}_i(S)$ is the $i$th column of $S$ and $Y(i)$ is an $(r-1) \times (r-1)$ alternating submatrix of $Y$ obtained by deleting the $i$th row and the corresponding column of $Y$. Let $(V_i)_j^{(r+3)}$ be a $(r+2)\times(r+2)$ submatrix of $V_i$ obtained by deleting the $j$th row and the $(r+3)$th column of $V_i$. Then we have

$$\det V_i = s_{1i} \det(V_i)_1^{(r+3)} - s_{2i} \det(V_i)_2^{(r+3)} + s_{3i} \det(V_i)_3^{(r+3)} - e_i \det(V_i)_4^{(r+3)}.$$

Direct computations show that for $j = 1, 2, 3, 4$,

$$\det(V_i)_j^{(r+3)} = d_j p_{i1}^2 \text{ , where } d_j = \bar{D}_{uvw} \text{ and } \{j, u, v, w\} = \{1, 2, 3, 4\}.$$

Hence for $i = 1, 2, 3, 4$, we have

$$\det V_i = -p_{i1}^2(e_i\bar{D}_{123} - s_{1i}\bar{D}_{234} + s_{2i}\bar{D}_{134} - s_{3i}\bar{D}_{124}) = -p_{i1}^3 = -(ue_i)^3.$$

So $I_{r+3}(g_2)$ contains $c_i^5, w^5$, and $(ue_i)^3$ for $i = 1, 2, 3$ and for $k = 1, 2, 3, \cdots, r$. Thus if $I_1(g_1) = (c_1, c_2, c_3, w, ue_1, ue_2, \cdots, ue_r)$ has grade 4, then the complex $\mathbb{H}$ satisfies the depth condition of the Buchsbaum-Eisenbud acyclicity criterion. Thus $\mathbb{H}$ is the resolution of $R/H$. Since every entry in the $g_i$'s is contained in the maximal ideal $\mathfrak{m}$, it is minimal. $\square$

Now we illustrate Theorem 3.1 by investigating the example of Gorenstein ideal of grade 4 which is the sum of an almost complete intersection of grade 3 of type 4 and a perfect ideal of grade 3 with type 2, $\lambda > 0$ geometrically linked by a regular sequence $\mathbf{x}$.

**Example 3.2.** Let $R = \mathbb{C}[[x, y, z, t]]$ be the formal power series ring over the field $\mathbb{C}$ of complex numbers with indeterminates $x, y, z, t$ and $deg\, x = deg\, y = deg\, z = deg\, t = 1$. Let $A = (a_{ij})$ and $Y = (y_{ij})$ be a $4 \times 3$ matrix and a $4 \times 4$ alternating matrix, respectively, given by

$$A = \begin{bmatrix} y & z & x \\ t & x & z \\ z & t & y \\ x & z & y \end{bmatrix}, \; Y = \begin{bmatrix} 0 & x & z & t \\ -x & 0 & y & x \\ -z & -y & 0 & z \\ -t & -x & -z & 0 \end{bmatrix}.$$

Then we define

$$\begin{array}{rcl} c_1 & = & -x^2z - 4xyz + y^2z + 2z^3 + x^2t + 2xyt - zt^2, \\ c_2 & = & -2x^2y - xy^2 + y^3 + x^2z + xyz + xz^2 + yz^2 - xzt - yzt \\ & & -z^2t + yt^2, \\ c_3 & = & x^2z + y^2z + 2xz^2 - x^2t - xyt - xzt - 2z^2t + t^3, \\ w & = & yt, \\ e_1 & = & -x^2y + xyz - z^3 + xzt + yzt - yt^2, \\ e_2 & = & xyz - y^2z + xz^2 - yz^2 - x^2t + y^2t, \\ e_3 & = & x^3 - xy^2 - xz^2 + yz^2 - xzt + yzt, \\ e_4 & = & xy^2 - x^2z + z^3 - 2yzt + xt^2. \end{array}$$

Let $J = (c_1, c_2, c_3, w)$ be an almost complete intersection of grade 3 of type 4, and let $\mathbf{x}$ be a regular sequence $xc_1, c_2, c_3$. Then $I = (\mathbf{x}) : J$ and $I = (xc_1, c_2, c_3, xe_1, xe_2, xe_3, xe_4)$ is a perfect ideal of grade 3 of type 2 by the Bass' result. Since $(\mathbf{x}) : I = J$, and $I \cap J = \mathbf{x}$, $I$ and $J$ are geometrically linked by a regular sequence $\mathbf{x}$. By Proposition 2.6, an ideal $L = I + J = (c_1, c_2, c_3, w, xe_1, xe_2, xe_3, xe_4)$ is a Gorenstein ideal of grade 4. So the minimal free resolution $\mathbb{H}$ of $R/L$ is given by

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{g_4} R^8 \xrightarrow{g_3} R^{14} \xrightarrow{g_2} R^8 \xrightarrow{g_1} R,$$

where

$$g_1 = \begin{bmatrix} c_1 & c_2 & c_3 & w & xe_1 & xe_2 & xe_3 & xe_4 \end{bmatrix},$$

$$g_2 = \left[\begin{array}{c|c|c|c} \mathbf{0} & xF & Z & S \\ \hline \mathbf{0} & \mathbf{0} & C & E \\ \hline -F^t & Y & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{G} & \begin{array}{c} \widetilde{f} \\ \mathbf{0} \end{array} \end{array}\right],$$

$$g_3 = \left[\begin{array}{c|c|c} \mathbf{0} & Z^t & C^t \\ \hline \mathbf{0} & S^t & E^t \\ \hline F & \mathbf{0} & \mathbf{0} \\ \hline Y & -xF^t & \mathbf{0} \end{array}\right],$$

*where* $w = Pf(Y)$, *and* $g_4 = \begin{bmatrix} xe_1 & xe_2 & xe_3 & xe_4 & c_1 & c_2 & c_3 & w \end{bmatrix}^t$.

**Theorem 3.3.** *Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. With notations as above, let $I$ be a perfect ideal of grade 3 with type 2, $\lambda(I) > 0$, and $J = (c_1, c_2, c_3, w)$ an almost complete intersection with* *the even type $r > 1$. Let $I$ and $J$ be geometrically linked by a regular sequence $uc_1, c_2, c_3 + vw$. Then $I + J = (c_1, c_2, c_3, w, ue_1 - uvs_{31}, ue_2 - uvs_{32}, \cdots, ue_r - uvs_{3r})$ is a Gorenstein ideal of grade 4 and*

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{g_4} R^{r+4} \xrightarrow{g_3} R^{2(r+3)} \xrightarrow{g_2} R^{r+4} \xrightarrow{g_1} R$$

*is a minimal free resolution of $R/(I+J)$, where*

$$g_1 = (c_1, c_2, c_3, w, ue_1 - uvs_{31}, ue_2 - uvs_{32}, \cdots, ue_r - uvs_{3r}),$$

$$g_2 = \left[\begin{array}{c|c|c|c} B & uF & Z & S \\ \hline \mathbf{0} & uv[a_{i3}]_{i=1,2,\cdots,r} & C & E \\ \hline -F^t & Y & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{G} & \begin{matrix} \widetilde{f} \\ \mathbf{0} \end{matrix} \end{array}\right],$$

$$g_3 = \left[\begin{array}{c|c|c} \mathbf{0} & Z^t & C^t \\ \hline \mathbf{0} & S^t & E^t \\ \hline F & B^t & \mathbf{0} \\ \hline Y & -uF^t & uv[a_{i3}]^t_{i=1,2,\cdots,r} \end{array}\right], \quad B = \begin{bmatrix} 0 & uv & 0 \\ -uv & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_4 = (-ue_1 + uvs_{31}, -ue_2 + uvs_{32}, \cdots, -ue_r + uvs_{3r}, c_1, c_2, c_3, w)^t.$$

*The matrix $g_2$ is determined by an augmented matrix $\widetilde{G}$ induced by the skew-symmetrizable matrix $G_1$ and an induced almost complete matrix $\widetilde{f}$.*

*Proof.* The same argument in Theorem 3.1 gives us the proof. □

**Example 3.4.** Let $R = \mathbb{C}[[x, y, z, t]]$ be the formal power series ring over the field $\mathbb{C}$ of complex numbers with indeterminates $x, y, z, t$ and $deg\, x = deg\, y = deg\, z = deg\, t = 1$. Let $A = (a_{ij})$ and $Y = (y_{ij})$ be a $4 \times 3$ matrix and a $4 \times 4$ alternating matrix, respectively, given by

$$A = \begin{bmatrix} z & y & t \\ y & x & z \\ x & t & y \\ t & z & x \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & t & x & y \\ -t & 0 & x & t \\ -x & -x & 0 & z \\ -y & -t & -z & 0 \end{bmatrix}.$$

Then we define

$$\begin{aligned}
c_1 &= -x^3+x^2y+xy^2+xz^2+yz^2-y^2t-2xzt-2yzt+xt^2+t^3,\\
c_2 &= -x^2y+y^3+x^2z-xyz+z^3+x^2t+xzt-2yzt-yt^2,\\
c_3 &= -x^2y-xyz-y^2z+2xz^2+x^2t+y^2t+xzt-zt^2-t^3,\\
w &= xy-xt+zt,\\
e_1 &= x^3+y^2z-xz^2-2xyt+zt^2,\\
e_2 &= -x^2y-yz^2+y^2t+2xzt-t^3,\\
e_3 &= xy^2-x^2z+z^3-2yzt+xt^2,\\
e_4 &= -y^3+2xyz-x^2t-z^2t+yt^2,\\
s_{31} &= x^2+z^2-yt,\ s_{32}=-x^2+y^2-zt,\\
s_{33} &= -yz+xt+t^2,\ s_{34}=xz-xt-yt.
\end{aligned}$$

Let $J=(c_1,c_2,c_3,w)$ be an almost complete intersection of grade 3 of type 4, and let $\mathbf{x}$ be a regular sequence $yc_1,c_2,c_3+yw$. Then $I=(\mathbf{x}):J$ and $I=(yc_1,c_2,c_3,ye_1-y^2s_{31},ye_2-y^2s_{32},ye_3-y^2s_{33},ye_4-y^2s_{34})$ is a perfect ideal of grade 3 of type 2 by the Bass' result. Since $(\mathbf{x}):I=J$, and $I\cap J=\mathbf{x}$, $I$ and $J$ are geometrically linked by a regular sequence $\mathbf{x}$. By Proposition 2.6, an ideal $L=I+J=(c_1,c_2,c_3,w,ye_1-y^2s_{31},ye_2-y^2s_{32},ye_3-y^2s_{33},ye_4-y^2s_{34})$ is a Gorenstein ideal of grade 4. So the minimal free resolution $\mathbb{H}$ of $R/L$ is given by

$$\mathbb{H}:0\longrightarrow R\xrightarrow{g_4}R^8\xrightarrow{g_3}R^{14}\xrightarrow{g_2}R^8\xrightarrow{g_1}R,$$

where

$$g_1=\begin{bmatrix}c_1 & c_2 & c_3 & w & ye_1-y^2s_{31} & ye_2-y^2s_{32} & ye_3-y^2s_{33} & ye_4-y^2s_{34}\end{bmatrix},$$

$$g_2=\left[\begin{array}{c|c|c|c} B & yF & Z & S\\ \hline \mathbf{0} & y^2[a_{i3}]_{i=1,2,3,4} & C & E\\ \hline -F^t & Y & \mathbf{0} & \mathbf{0}\end{array}\right]=\left[\begin{array}{c|c} \widetilde{G} & \begin{matrix}\widetilde{f}\\ \mathbf{0}\end{matrix}\end{array}\right],$$

$$g_3=\left[\begin{array}{c|c|c} \mathbf{0} & Z^t & C^t\\ \hline \mathbf{0} & S^t & E^t\\ \hline F & B^t & \mathbf{0}\\ \hline Y & -yF^t & y^2[a_{i3}]^t_{i=1,2,3,4}\end{array}\right],\quad B=\begin{bmatrix}0 & y^2 & 0\\ -y^2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$g_4=\begin{bmatrix}-ye_1+y^2s_{31} & -ye_2+y^2s_{32} & -ye_3+y^2s_{33} & -ye_4+y^2s_{34} & c_1 & c_2 & c_3 & w\end{bmatrix}^t.$$

**Theorem 3.5.** *Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. With notations as above, let $I$ be a perfect ideal of grade 3 with type 2, $\lambda(I) > 0$, and $J = (c_1, c_2, c_3, w)$ an almost complete intersection with the even type $r > 1$. Let $I$ and $J$ be geometrically linked by a regular sequence $c_1, c_2, sc_3 + uvw$. Then $I + J = (c_1, c_2, c_3, w, se_1 - uvs_{31}, se_2 - uvs_{32}, \cdots, se_r - uvs_{3r})$ is a Gorenstein ideal of grade 4 and*

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{g_4} R^{r+4} \xrightarrow{g_3} R^{2(r+3)} \xrightarrow{g_2} R^{r+4} \xrightarrow{g_1} R$$

*is a minimal free resolution of $R/(I + J)$, where*

$$g_1 = (c_1, c_2, c_3, w, se_1 - uvs_{31}, se_2 - uvs_{32}, \cdots, se_r - uvs_{3r}),$$

$$g_2 = \left[\begin{array}{c|c|c|c} B & sF & Z & S \\ \hline \mathbf{0} & uv[a_{i3}]_{i=1,2,\cdots,r} & C & E \\ \hline -F^t & Y & \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{G} & \begin{matrix}\widetilde{f} \\ \mathbf{0}\end{matrix} \end{array}\right],$$

$$g_3 = \left[\begin{array}{c|c|c} \mathbf{0} & Z^t & C^t \\ \hline \mathbf{0} & S^t & E^t \\ \hline F & B^t & \mathbf{0} \\ \hline Y & -sF^t & uv[a_{i3}]^t_{i=1,2,\cdots,r} \end{array}\right], \quad B = \begin{bmatrix} 0 & uv & 0 \\ -uv & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_4 = (-se_1 + uvs_{31}, -se_2 + uvs_{32}, \cdots, -se_r + uvs_{3r}, c_1, c_2, c_3, w)^t.$$

*Proof.* The same argument in Theorem 3.1 gives us the proof. □

**Example 3.6.** Let $R = \mathbb{C}[[x, y, z, t]]$ be the formal power series ring over the field $\mathbb{C}$ of complex numbers with indeterminates $x, y, z, t$ and $deg\, x = deg\, y = deg\, z = deg\, t$. Let $A = (a_{ij})$ and $Y = (y_{ij})$ be a $4 \times 3$ matrix and a $4 \times 4$ alternating matrix, respectively, given by

$$A = \begin{bmatrix} y & x & z \\ t & y & x \\ x & z & t \\ z & t & y \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & z & y & t \\ -z & 0 & y & z \\ -y & -y & 0 & x \\ -t & -z & -x & 0 \end{bmatrix}.$$

Then we define

$$
\begin{aligned}
c_1 &= x^3+xy^2-y^3-xyz+yz^2+z^3+xyt-2xzt-yzt+yt^2-zt^2,\\
c_2 &= x^2y+y^3+2xyz+xz^2-yz^2-x^2t-y^2t-xzt-yzt-z^2t+t^3,\\
c_3 &= xy^2+x^2z-xyz+y^2z-yz^2-z^3-x^2t-xyt+y^2t+xzt\\
&\quad -yt^2+zt^2,\\
w &= xz-yz+yt,\\
e_1 &= xy^2+xz^2-x^2t-2yzt+t^3,\\
e_2 &= -x^2y+y^2z-z^3+2xzt-yt^2,\\
e_3 &= -y^3-x^2z+yz^2+2xyt-zt^2,\\
e_4 &= x^3-2xyz+y^2t+z^2t-xt^2,\\
s_{31} &= x^2+y^2-zt,\ s_{32}=-y^2-xz+t^2,\\
s_{33} &= yz+z^2-xt,\ s_{34}=xy-yz-zt.
\end{aligned}
$$

Let $J=(c_1,c_2,c_3,w)$ be an almost complete intersection of grade 3 of type 4, and let $\mathbf{x}$ be a regular sequence $c_1,c_2,xc_3+z^2w$. Then $I=(\mathbf{x}):J$ and $I=(c_1,c_2,xc_3+z^2w,xe_1-z^2s_{31},xe_2-z^2s_{32},xe_3-z^2s_{33},xe_4-z^2s_{34})$ is a perfect ideal of grade 3 of type 2 by the Bass' result. Since $(\mathbf{x}):I=J$, and $I\cap J=\mathbf{x}$, $I$ and $J$ are geometrically linked by a regular sequence $\mathbf{x}$. By Proposition 2.6, an ideal $L=I+J=(c_1,c_2,c_3,w,xe_1-z^2s_{31},xe_2-z^2s_{32},xe_3-z^2s_{33},xe_4-z^2s_{34})$ is a Gorenstein ideal of grade 4. So the minimal free resolution $\mathbb{H}$ of $R/L$ is given by

$$\mathbb{H}: 0 \longrightarrow R \xrightarrow{g_4} R^8 \xrightarrow{g_3} R^{14} \xrightarrow{g_2} R^8 \xrightarrow{g_1} R,$$

where $g_1,g_2,g_3,g_4$ are similarly determined by the above theorem 3.5.

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