# EXTENSION OF A THEOREM OF GUDDER AND SCHELP TO POLYNOMIALS OF ORTHOMODULAR LATTICES 

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#### Abstract

Consider a polynomial expression $p(b, c, \ldots, d)=e$ where any two of the elements $b, c, \ldots, d$ commute. If an element $a$ commutes with $e$, then $b$ commutes with $p(a, c, \ldots, d)$.


1. Introduction. In this paper, $\mathcal{L}=\left(L, \vee, \wedge,^{\prime}, 0,1\right)$ always means an orthomodular lattice. If $a, b \in L$ and $a=(a \wedge b) \vee\left(a \wedge b^{\prime}\right)$, then $a, b$ are said to commute; in this case we write $a C b$.

First recall some useful properties of the relation $C$ (see e.g. [2, pp. 52-53]).
1.1. Lemma. (i) If $a, b, c \in L$ are such that $a C b$ and $a C c$, then $a C b \wedge c$ and $a C b \vee c$.
(ii) If $\{a, b, c\}=\{x, y, z\}$ is a subset of $L$, and if $x C y$ and $x C z$, then $a \wedge(b \vee c)$ $=(a \wedge b) \vee(a \wedge c)$ and $a \vee(b \wedge c)=(a \vee b) \wedge(a \vee c)$.

A well-known result states that if $a, b \in L$ and $a C b$, then $b C a$. For orthomodular lattices a result of Gudder and Schelp [5, p. 235] can be simplified and formulated in the following way: If $a C b \wedge c$ and $b C c$, then $b C a \wedge c$. The author's paper arose as the result of attempts to generalize these two special situations.
2. Preliminaries. In what follows, $\left\{b, c_{2}, \ldots, c_{n}\right\}$ denotes a subset of $L$ where any two elements commute.
2.1. Lemma. Let $p$ be an n-ary lattice polynomial. Then
(1) $p\left(b, c_{2}, \ldots, c_{n}\right)=B_{1}\left(b, c_{2}, \ldots, c_{n}\right) \wedge B_{2}\left(b, c_{2}, \ldots, c_{n}\right)$, where
(2) $B_{1}\left(y_{1}, x_{2}, \ldots, x_{n}\right)=\bigwedge\left(y_{1} \vee \vee\left(x_{i} \mid i \in I_{j}\right) \mid j \in J\right)$,
(3) $B_{2}\left(y_{1}, x_{2}, \ldots, x_{n}\right)=\bigwedge\left(\bigvee\left(x_{k} \mid k \in K_{t}\right) \mid t \in T\right)$
and where $I_{j}, J, K_{t}, T$ are finite sets.
Proof. By Lemma 1(ii), the sublattice generated by $b, c_{2}, \ldots, c_{n}$ is distributive and the assertion follows from [1, Theorem 12, p. 145].
2.2. Corollary. Let $e_{1}=d, e_{2}, \ldots, e_{n} \in L$ be such that $e_{i} C e_{j}$ for $1 \leqslant i, j \leqslant n$ and suppose (1) is valid. Then

$$
p\left(d, e_{2}, \ldots, e_{n}\right)=B_{1}\left(d, e_{2}, \ldots, e_{n}\right) \wedge B_{2}\left(d, e_{2}, \ldots, e_{n}\right)
$$

where $B_{1}$ and $B_{2}$ are the same polynomials as in (2) and (3).
2.3. Proposition. If $a, b \in L$ are such that $a C b \wedge c$ and $b C c$, then $b C a \wedge c$.

Proof. Since $a=[a \wedge(b \wedge c)] \vee\left[a \wedge(b \wedge c)^{\prime}\right]$, we have

$$
a^{\prime}=\left(a^{\prime} \vee b^{\prime} \vee c^{\prime}\right) \wedge\left[a^{\prime} \vee(b \wedge c)\right]
$$

Thus

$$
a^{\prime} \vee c^{\prime}=\left\{\left(a^{\prime} \vee b^{\prime} \vee c^{\prime}\right) \wedge\left[a^{\prime} \vee(b \wedge c)\right]\right\} \vee c^{\prime}
$$

However, $a^{\prime} \vee b^{\prime} \vee c^{\prime} \geqslant a \wedge\left(b^{\prime} \vee c^{\prime}\right)=\left[a^{\prime} \vee(b \wedge c)\right]^{\prime}$ and it follows $a^{\prime} \vee(b \wedge$ c) $C a^{\prime} \vee b^{\prime} \vee c^{\prime}$. Similarly, $c^{\prime} C a^{\prime} \vee b^{\prime} \vee c^{\prime}$. By Lemma 1.1(ii),

$$
a^{\prime} \vee c^{\prime}=\left(a^{\prime} \vee b^{\prime} \vee c^{\prime}\right) \wedge\left[a^{\prime} \vee c^{\prime} \vee(b \wedge c)\right]
$$

But $c^{\prime} \vee(b \wedge c)=c^{\prime} \vee b$, by Lemma 1.1(ii). Therefore

$$
a^{\prime} \vee c^{\prime}=\left(a^{\prime} \vee c^{\prime} \vee b^{\prime}\right) \wedge\left(a^{\prime} \vee c^{\prime} \vee b\right)
$$

and, finally, $a \wedge c=[(a \wedge c) \wedge b] \vee\left[(a \wedge c) \wedge b^{\prime}\right]$.
3. Key lemma and main theorem. We begin this section with the following technical lemma.
3.1. Lemma. Let p be an n-ary lattice polynomial and let (1) be valid. Then for every $a \in L$

$$
D_{1} \wedge D_{2} \leqslant p\left(a, c_{2}, \ldots, c_{n}\right)<D_{2}
$$

where

$$
\begin{aligned}
& D_{1}=\Lambda\left(\bigvee\left(c_{i} \mid i \in I_{j}\right) \mid j \in J\right) \\
& D_{2}=\wedge\left(\bigvee\left(c_{k} \mid k \in K_{t}\right) \mid t \in T\right)
\end{aligned}
$$

Proof. Since $p$ is a lattice polynomial, by [3, Lemma 6, p. 33] we have

$$
p\left(0, c_{2}, \ldots, c_{n}\right) \leqslant p\left(a, c_{2}, \ldots, c_{n}\right) \leqslant p\left(1, c_{2}, \ldots, c_{n}\right) .
$$

By Corollary 2.2,

$$
p\left(0, c_{2}, \ldots, c_{n}\right)=B_{1}\left(0, c_{2}, \ldots, c_{n}\right) \wedge B_{2}\left(0, c_{2}, \ldots, c_{n}\right)=D_{1} \wedge D_{2}
$$

Moreover,

$$
p\left(1, c_{2}, \ldots, c_{n}\right)=B_{1}\left(1, c_{2}, \ldots, c_{n}\right) \wedge B_{2}\left(1, c_{2}, \ldots, c_{n}\right)=D_{2}
$$

3.2. Theorem. Let $c_{1}, c_{2}, \ldots, c_{n}$ and a be elements of an orthomodular lattice. If $c_{i} C c_{j}$ for any $1 \leqslant i, j \leqslant n$ and if $a C p\left(c_{1}, c_{2}, \ldots, c_{n}\right)$ where $p$ is an $n$-ary lattice polynomial, then

$$
c_{k} C p\left(c_{1}, c_{2}, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_{n}\right)
$$

for every $k=1,2, \ldots, n$.
Proof. We may assume that $k=1$. Let $b=c_{1}$. By assumption, $a C p\left(b, c_{2}, \ldots, c_{n}\right)$. From Lemma 1.1(i), $c_{i} C p\left(b, c_{2}, \ldots, c_{n}\right)$ for every $i=2, \ldots, n$. By the same lemma we also have $p\left(a, c_{2}, \ldots, c_{n}\right) C p\left(b, c_{2}, \ldots, c_{n}\right)$. That is,

$$
B_{1}\left(b, c_{2}, \ldots, c_{n}\right) \wedge B_{2}\left(b, c_{2}, \ldots, c_{n}\right) C p\left(a, c_{2}, \ldots, c_{n}\right)
$$

Since $b C c_{i}$ for every $i=2, \ldots, n$,

$$
B_{1}\left(b, c_{2}, \ldots, c_{n}\right)=b \vee\left(\wedge\left(\bigvee\left(c_{i} \mid i \in I_{j}\right) \mid j \in J\right)\right)=b \vee D_{1}
$$

Replacing $B_{2}\left(b, c_{2}, \ldots, c_{n}\right)$ by $D_{2}$ we see that

$$
\left(b \vee D_{1}\right) \wedge D_{2} C p\left(a, c_{2}, \ldots, c_{n}\right)
$$

By Lemma 1.1(ii), $b C D_{1}$ and $b C D_{2}$. Hence $\left(b \wedge D_{2}\right) \vee\left(D_{1} \wedge D_{2}\right) C p\left(a, c_{2}, \ldots, c_{n}\right)$ and also $b \wedge D_{2} C D_{1} \wedge D_{2}$. In view of the assertion dual to Proposition 2.3, $b \wedge D_{2} C p\left(a, c_{2}, \ldots, c_{n}\right) \vee\left(D_{1} \wedge D_{2}\right)$. Then, by Lemma 3.1, $b \wedge$ $D_{2} C p\left(a, c_{2}, \ldots, c_{n}\right)$. Similarly, $b C p\left(a, c_{2}, \ldots, c_{n}\right) \wedge D_{2}=p\left(a, c_{2}, \ldots, c_{n}\right)$.


Figure 1

Remark. We conclude by observing that the assertion of Theorem 3.2 is optimal in the following sense: If $q$ is a polynomial in $\vee, \wedge$ and ', then an analogue of Theorem 3.2 need not be valid. This can easily be seen by considering the orthomodular lattice constructed by Greechie's classical method [4] from two copies of $2^{3}$ (cf. Figure 1). Let $q(x, y)=(x \wedge y) \vee x^{\prime}$. Here we have $a C q\left(d^{\prime}, e\right)=$ 1 but $d^{\prime}$ and $q(a, e)=b$ do not commute.

## References

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