ON THE

LOCATION OF THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL

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If there are plotted in the plane of the complex variable the roots of a polynomial $f(z)$ and the roots of the derived polynomial $f'(z)$, there are interesting geometric relations between the two sets of points. It was shown by Gauss that the roots of $f'(z)$ are the positions of equilibrium in the field of force due to equal particles situated at each root of $f(z)$, if each particle repels with a force equal to the inverse distance. The derivative vanishes not only at the at the positions of equilibrium but also at the multiple roots of $f(z)$.

From Gauss's theorem follows immediately the theorem of Lucas that the roots of $f'(z)$ lie in any convex polygon in which lie the roots of $f(z)$. In the present note I wish to prove the following theorem, the connection of which with Lucas's theorem will be pointed out later.

THEOREM I. — *If* m_1 *roots of a polynomial* $f(z)$ *lie in or on a circle* C_1 *whose center is* α_1 *and radius* r_1, *and if all the remaining roots of* $f(z)$, m_2 *in number, lie in or on a circle* C_2 *whose center is* α_2 *and radius* r_2, *then all the roots of* $f'(z)$ *lie in or on* C_1, C_2, *and a third circle* C_3 *whose center is*

$$\frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2} \text{ and radius } \frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}.$$

If the circles C_1, C_2, C_3 *are mutually external they contain respectively the following numbers of roots of* $f'(z)$: $m_1 - 1$, $m_2 - 1$, 1.

The centers of the circles C_1, C_2, C_3 are collinear. If $r_1 = r_2$, the three circles have the same radius. If $r_1 \neq r_2$, the point $\frac{r_1\alpha_2 - r_2\alpha_1}{r_1 - r_2}$ is a center of similitude for any pair of the circles C_1, C_2, C_3.

Theorem I is a generalization of the trivial theorem where $f(z)$ has but two roots, α_1 and α_2, of respective multiplicities m_1 and m_2. Then α_1 and α_2 are roots of $f'(z)$ of respective multiplicities $m_1 - 1$ and $m_2 - 1$, and there is a root of $f'(z)$ at the point $\dfrac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2}$ which divides the segment (α_1, α_2) in the ratio $m_1 : m_2$.

We shall prove Theorem I from Gauss's theorem, making use of several lemmas.

Lemma I. — *If m particles lie on or within a circle C, the corresponding resultant force at any point P exterior to C is equivalent to the force at P due to m coincident particles on or within C.*

The force at P due to a particle at any point Q is in magnitude, direction, and sense Q′P, where Q′ denotes the inverse of Q in the unit circle whose center is P. To replace m points Q by m coincident points we replace the m vectors Q′P by m coincident vectors, and hence replace their terminals Q′ by the center of gravity of the m points Q′. Denote by C′ the inverse of the circle C in the unit circle whose center is P. Then under the conditions of the lemma all the points Q lie in or on C, all the points Q′ lie in or on C′, their center of gravity lies in or on C′, and hence the inverse of the center of gravity lies in or on C. This completes the proof of the lemma.

Lemma II. — *Let the points z_1 and z_2 be allowed to assume independently all positions on or within the respective circles C_1 and C_2 whose centers are α_1 and α_2 and radii r_1 and r_2 respectively. Then the locus of the point $\dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$ which divides the segment (z_1, z_2) in the constant ratio $m_1 : m_2 (m_1, m_2 > 0)$ is the interior (including the boundary) of the circle C_3 whose center is*

$$\frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2} \text{ and radius } \frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}.$$

Under our hypothesis, the point $\dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$ lies on or within C_3. For we have

$$|z_1 - \alpha_1| \leqq r_1,$$
$$|z_2 - \alpha_2| \leqq r_2,$$

and hence

$$\left|\frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} - \frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2}\right| = \left|\frac{m_1(z_2 - \alpha_2)}{m_1 + m_2} + \frac{m_2(z_1 - \alpha_1)}{m_1 + m_2}\right| \leqq \frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}.$$

Any point z on or within C_3 corresponds to some pair of points z_1 and z_2 on or within C_1 and C_2 respectively. For let us set

$$z_1 - \alpha_1 = \left(z - \frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2}\right) \frac{r_1}{\frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}},$$

$$z_2 - \alpha_2 = \left(z - \frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2}\right) \frac{r_2}{\frac{m_1 r_2 + m_2 r_1}{m_1 + m_2}}.$$

We are assuming that

$$\left| z - \frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2} \right| \leqq \frac{m_1 r_2 + m_2 r_1}{m_1 + m_2},$$

and hence

$$|z_1 - \alpha_1| \leqq r_1,$$
$$|z_2 - \alpha_2| \leqq r_2;$$

of course we have

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

This completes the proof of Lemma II.

We are now in a position to prove Theorem I. Suppose a point z external to C_1 and C_2 to be a position of equilibrium in the field of force. The force at z due to the m_1 particles in or on C_1 is equivalent to the force at z due to m_1 particles coinciding at some point z_1 in or on C_1, and the force at z due to the m_2 particles in or on C_2 is equivalent to the force at z due to m_2 particles coinciding at some point z_2 in C_2. Then z divides the segment (z_1, z_2) in the ratio $m_1 : m_2$ and hence lies in or on C_3. Therefore the three circles C_1, C_2, C_3 contain all positions of equilibrium. They contain also all multiple roots of $f(z)$ and hence they contain all roots of $f'(z)$.

If the circles C_1, C_2, and C_3 are mutually external, we allow the m_1 roots of $f(z)$ in C_1 to move continuously in C_1 and to coalesce at a point in C_1, and similarly allow the m_2 roots of $f(z)$ in C_2 to move continuously in C_2 and to coalesce in C_2. In the final position the numbers of roots of $f'(z)$ in C_1, C_2, and C_3 are $m_1 - 1$, $m_2 - 1$, and 1 respectively. Throughout the motion of the roots of $f(z)$, the roots of $f'(z)$ move continuously, none enters or leaves any of the three circles, and hence the final number of roots of $f(z)$ in each of those circles is the same as the initial number. The proof of Theorem I is now complete.

If no point of the circle C_2 is exterior to C_1, no point of C_3 is exterior to C_1, so we are led to the following theorem which is equivalent to the theorem of Lucas quoted above :

If all the roots of a polynomial lie on or within a circle, then all the roots of the derived polynomial lie on or within that circle.

For polynomials all of whose roots are real the following theorem is easily proved either from Theorem I or directly by a proof similar to the proof of that theorem :

THEOREM II. — *If m_1 roots of a polynomial $f(z)$ lie in a closed interval l_1 of the axis of reals whose center is α_1 and length l_1, and if all the remaining roots of $f(z)$, m_2 in number, lie in a closed interval l_2 of the axis of reals whose center is α_2 and length l_2, then all the roots of $f'(z)$ lie in l_1, l_2, and a third interval l_3 of the axis of reals whose center is*

$$\frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2} \text{ and length } \frac{m_1 l_2 + m_2 l_1}{m_1 + m_2}.$$

If the intervals l_1, l_2, l_3 are mutually external, they contain respectively the following numbers of roots of $f'(z)$: $m_1 - 1$, $m_2 - 1$, 1.

There are various generalizations of Theorems I and II. It is entirely incidental in Theorem I that we considered the *interiors* of two circles. By slight changes in the formulas used we obtain a similar theorem for the interior of one circle and the exterior of the other. The theorems can easily be extended to give results concerning the roots of the derivative of a rational function, or the roots of the jacobian of two binary forms. The last generalization is particularly interesting because the results are invariant under linear transformation of the complex variable. For these and other results the reader is referred to a paper by the present writer which has appeared and several others which are expected to appear in the *Transactions of the American Mathematical Society.*