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Subdivision of Wheel is Graceful and Cordial

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Abstract

An edge uv is said to be subdivided if the edge uv is replaced by the path $P: uwv$, where w is the new vertex. A graph obtained by subdividing each edge of a graph G is called *subdivision* of the graph G, and is denoted by $S(G)$. In this paper, we show that the subdivision of wheel graph $S(W_n)$ is graceful for even values of $n \geq 4$ and cordial for all $n \geq 4$.

Keywords: Graph labeling, graceful labeling, cordial labeling, subdivision of graphs

1. Introduction

Rosa [9] introduced the graceful labeling as a tool to attack the Ringel's conjecture that the complete graph with $2m+1$ vertices can be decomposed into $2m+1$ copies of a given tree with m edges.

A function f is called a graceful labeling of a graph G with m edges, if f is an injection from the set of vertices of G to the set $\{0,1,2,...,m\}$ such that when each edge uv is assigned the label $|f(u)-f(v)|$ then the resulting edge labels are distinct. A graph which admits a graceful labeling is called graceful graph.

The cordial labeling was introduced by Cahit [2] as a variation of graceful labeling. A function f from the set of vertices of a graph G to the set $\{0,1\}$ and for each edge uv assign the label $|f(u)-f(v)|$ is called cordial labeling, if the number of vertices labeled 0's and the number of vertices labeled 1's differ by at most 1, and the number of edges labeled 0's and the number of edges labeled 1's differ by at most 1.

Graceful or Cordial labeling are not only useful in theoretical studies, but also play an important role in applications. The characterization of graceful and cordial graphs is NP-complete. Due to inherent difficulties of these labeling, researchers have investigated these labeling on various families of graphs. A wheel W_n of order n is the graph $C_n + K_1$ for $n \geq 3$. In [5], Hoede have shown that all wheels are graceful, for $n \geq 3$. A gear graph is obtained from the wheel by subdividing the edges of the cycle in the wheel. Ma and Feng [7] have proved that all the gear graphs are graceful. Liu [6] have shown that if two or more vertices are inserted between every pair of vertices of the cycle of the wheel, the resulting graph is graceful.

In this paper, we show that $S(W_n)$ is graceful for even values of $n \geq 4$ and cordial for all $n \geq 4$.

2. Subdivision of wheels are graceful and Cordial

In this section, we show that the graph $S(W_n)$ is graceful, for even $n \geq 4$, and cordial, for all $n \geq 4$. For the convenience of labeling we describe $S(W_n)$ as shown in the following Figure 1.

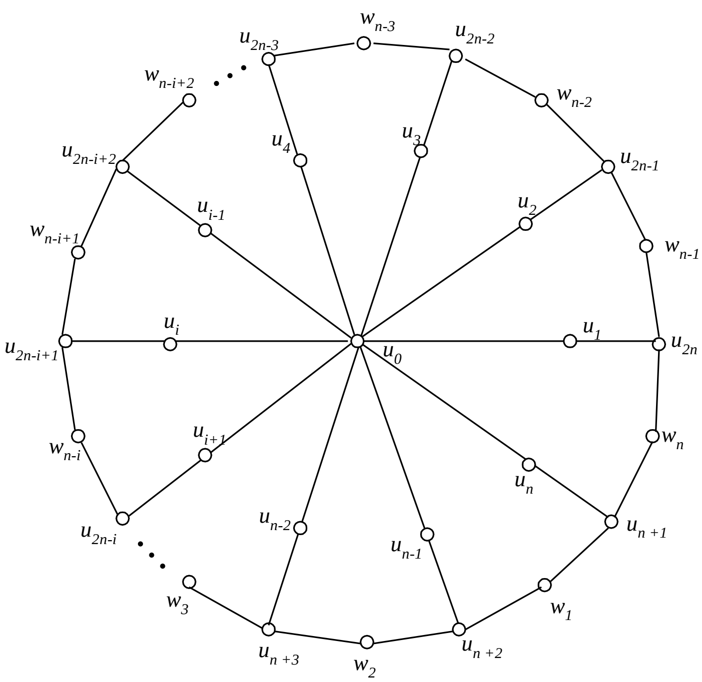

Figure 1. The graph $S(W_n)$

Theorem 1. For even $n \geq 4$, the graph $S(W_n)$ is graceful.

Proof. Consider $S(W_n)$, the subdivision of the wheel W_n of order n, whose vertices are described as shown in Figure 1.

Observe that

$$|V(S(W_n))| = 3n + 1,$$

$$|E(S(W_n))| = 4n.$$

Let $m = |E(S(W_n))|$.

Define $\phi : V(S(W_n)) \rightarrow \{0, 1, 2, ..., m\}$ by

$\phi(u_0) = 0$,

$$\phi(u_i) = \begin{cases} m + 1 - i, & \text{for } 1 \leq i \leq n \\ 2i - 1 - 2n, & \text{for } n + 1 \leq i \leq 2n \end{cases},$$

$$\phi(w_i) = \begin{cases} 6i - 2, & \text{for } 1 \leq i \leq \frac{n}{2} \\ 4n + 1 - 2i, & \text{for } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

It follows that the labels $\phi(u_i)$'s, for $0 \leq i \leq 2n$, and $\phi(w_i)$'s, for $1 \leq i \leq n$ are distinct.

Let A be the set of edges of $S(W_n)$ which are adjacent to u_0. Let B be the set of edges of $S(W_n)$ which are incident with any vertex of edges in the set A. Let C be the set of edges of $S(W_n)$ which are not in A and not in B.

Let A', B', C' be the edge labels of the edges in the sets A, B, C respectively.

Observe that

$A' = \{m, \cdots, m + 1 - n\}$,

$B' = \{m - n, m - n - 1, m - n - 2, \cdots, m + 1 - 2n\}$,

$C' = \{m - 2n, m - 2n - 1, m - 2n - 2, \cdots, 3, 2, 1\}$.

It follows that the edge labels in the sets A', B', C' are distinct and

$A' \cup B' \cup C' = \{1, 2, 3, ..., m-1, m\}$.

Hence the graph $S(W_n)$ is graceful, for even $n \geq 4$.

Theorem 2. For $n \geq 4$, the graph $S(W_n)$ is cordial.

Proof. We consider the vertex sequence of $S(W_n)$ for the matching with certain 0-1 sequence. This matching defines the required cordial labeling. Consider the description of the vertices of $S(W_n)$ as shown in the Figure 1.

We arrange the vertices of $S(W_n)$ as the following sequence.

$u_0, u_1, u_2, ..., u_n, u_{n+1}, w_1, u_{n+2}, w_2, u_{n+3}, ..., u_{2n-i}, w_{n-i}, u_{2n-i+1}, w_{n-i+1}, u_{2n-i+2}, ...,$

$u_{2n-2}, w_{n-2}, u_{2n-1}, w_{n-1}, u_{2n}, w_n$.

Table 1. The vertex and edge labels of the graph $S(W_n)$

Nature of n [in the cycle C_n]	Nature of the vertices of $S(W_n)$	The 0-1 sequence for term wise matching with the sequence of vertices of $S(W_n)$	Relation between $\lvert V_0\rvert$ and $\lvert V_1\rvert$	The edge label sequence for the set			Relation between $\lvert E_0\rvert$ and $\lvert E_1\rvert$
				A'	B'	C'	
$4r$	$4t+1$	$0(1100)^t$	$\lvert V_0\rvert = \lvert V_1\rvert + 1$	$(1100)^r$	$(1001)^r$	$(01)^{4r}$	$\lvert E_0\rvert = \lvert E_1\rvert$
$4r+1$	$4t$	$0(1100)^{t-2}$ 11 10001	$\lvert V_0\rvert = \lvert V_1\rvert$	$(1100)^r 1$	$(0011)^r 0$	$(10)^{4r-2}010010$	$\lvert E_0\rvert = \lvert E_1\rvert$
$4r+2$	$4t+3$	$0(1100)^{t-1}$ 011 101	$\lvert V_0\rvert = \lvert V_1\rvert - 1$	$(1100)^r 11$	$(1001)^{r-1}100011$	$(01)^{4r-2}00100111$	$\lvert E_0\rvert = \lvert E_1\rvert$
$4r+3$	$4t+2$	$0(1100)^{t-1}11010$	$\lvert V_0\rvert = \lvert V_1\rvert$	$(1100)^r 110$	$(0011)^r 000$	$(10)^{4r+1}1110$	$\lvert E_0\rvert = \lvert E_1\rvert$

The particular 0-1 sequence of length $3n+1$ for the corresponding term wise matching with the sequence of vertices of $S(W_n)$ is given in the Table 1.

Let A, A', B, B', C, C' are the sets as defined in Theorem 1.

Let V_0 and V_1 denotes the set of vertices of $S(W_n)$ were assigned the labels 0's and 1's respectively.

Let E_0 and E_1 denote the set of edges of $S(W_n)$ having the labels 0's and 1's respectively.

From the Table 1., it is clear that the number of vertices labeled 0's and the number of vertices labeled 1's are differ by at most 1 and the number of edges labeled 0's and the number of edges labeled 1's are differ by at most 1.

Hence, $S(W_n)$ is cordial.

3. Conclusion

It appears that proving the gracefulness of $S(W_n)$, for odd values of $n \geq 4$, seem to be hard to establish. So we conclude this paper with the following question.

Is the graph $S(W_n)$ graceful, for odd values of $n \geq 4$?

References

[1] Bloom G. S. and Golomb S. W. (1977), 'Applications of numbered undirected graphs'. Proc., IEEE, 65, pp. 562-570. http://dx.doi.org/10.1109/proc.1977.10517

[2] Cahit I. (1987), 'Cordial graphs: a weaker version of graceful and harmonious graphs', Ars Combin., 23, pp. 201-207.

[3] Cahit I. (1990), 'Recent results and open problems on cordial graphs' Contemporary methods in Graph theory, R. Bodendiek (ed.), Wissenschaft-sverlag, Mannheim, pp. 209-230.

[4] Gallian J. A. (2005), 'A dynamic survey of graph labeling', Electro. J. Combin. 5, # DS6, http://www.combinatorics.org.

[5] Hoede C. and Kuiper H. (1987), 'All wheels are graceful', Utilitas Math., 14, pp. 311.

[6] Liu Y. (1996), 'Crown graphs Q_{2n} are harmonious graphs', Hunan Annals Math., 16, pp. 125-128.

[7] Ma K. J. and Feng C. J. (1984), 'On the gracefulness of gear graphs', Math. Practice Theory, pp. 72-73.

[8] Ringel G. (1964), 'Problem 25, in theory of Graphs and its Applications', Proc. Symposium Smolenice 1963, Prague, pp. 162.

[9] Rosa A. (1967), 'On certain valuations of vertices of a graph', in; Theory of graphs, International symposium, Rome, July 1966 Gordon and Breach, New York, Dunod, Paris, pp. 349-355.

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